

## On partial cubes and graphs with convex intervals

BOŠTJAN BREŠAR, SANDI KLAVŽAR

*Abstract.* A graph is called a partial cube if it admits an isometric embedding into a hypercube. Subdivisions of wheels are considered with respect to such embeddings and with respect to the convexity of their intervals. This allows us to answer in negative a question of Chepoi and Tardif from 1994 whether all bipartite graphs with convex intervals are partial cubes. On a positive side we prove that a graph which is bipartite, has convex intervals, and is not a partial cube, always contains a subdivision of  $K_4$ .

*Keywords:* isometric embeddings, hypercubes, partial cubes, convex intervals, subdivisions

*Classification:* 05C12, 05C75

### 1. Introduction

Isometric subgraphs of Hamming graphs (called *partial Hamming graphs*) and related classes of graphs have been considered by several authors over the last years. Isometric subgraphs of hypercubes (called *partial cubes*), which are precisely bipartite partial Hamming graphs, have been first investigated in the seventies by Graham and Pollak [10] who used them as a model for a communication network. Djoković [8], Avis [3], Winkler [19], Chepoi [4], and Wilkeit [18] followed with nice characterizations of these graphs. Recognition algorithms for partial cubes and for partial Hamming graphs of complexity  $O(mn)$ , where  $m$  is the number of edges and  $n$  the number of vertices, were developed in [2] and [1], respectively. Interestingly, no faster algorithms are known by now, cf. [11], [12], so it seems that even more insight into the structure of these graphs is needed in order to either improve this complexity or to prove an appropriate lower bound. Partial cubes have also found several applications, cf. [5], [6], [9], [14], [15].

Clearly, partial cubes are bipartite and it is not difficult to see that they have convex intervals. (In fact, just observe that hypercubes have convex intervals, and use the definition of partial cubes.) During the 1994 Bielefeld conference on “Discrete Metric Spaces”, Chepoi and Tardif [7] asked whether the converse could also be true. This question appeared as Conjecture 2.45 in [13] under the name “Chepoi-Tardif conjecture”, and it was the main motivation for the present paper. More precisely, calling graphs with convex intervals *interval monotone graphs*, the

question was whether every bipartite interval monotone graph is a partial cube. (Interval monotonicity versus interval-regularity was studied in [16].)

In order to answer it we first study subdivisions of wheels and the convexity of their intervals. Let  $W_k$  be the  $k$ -wheel, and let  $W_k(m, n)$  be the graph obtained from  $W_k$  by subdividing every edge incident to the central vertex of  $W_k$  by  $n$  vertices and every other edge by  $m$  vertices. Then we characterize interval monotone graphs and partial cubes among the family of graphs  $W_k(m, n)$ ,  $k \geq 3$ ,  $n, m \geq 0$ . As a consequence we obtain that  $W_3(m, n)$  is a bipartite, interval monotone graph, which does not admit an isometric embedding into a hypercube, provided that  $n \geq 2$ ,  $m$  is an odd integer, and  $m \leq 2n$ . We also prove that a graph which is bipartite, has convex intervals, and is not a partial cube, contains a subdivision of  $K_4$ .

For a graph  $G$ , the *distance*  $d_G(u, v)$  (or briefly  $d(u, v)$ ) between vertices  $u$  and  $v$  is defined as the number of edges on a shortest  $u, v$ -path. The *interval*  $I(u, v)$  between vertices  $u$  and  $v$  consists of all vertices on shortest paths between  $u$  and  $v$ . A subgraph  $H$  of  $G$  is *convex*, if for any  $u, v \in V(H)$ ,  $I(u, v) \subseteq V(H)$ . A subgraph  $H$  of  $G$  is called *isometric* if  $d_H(u, v) = d_G(u, v)$  for all  $u, v \in V(H)$ . Isometric subgraphs of hypercubes are called *partial cubes*. An important subclass of partial cubes are *median graphs*, that is, the graphs  $G$  in which for every triple of vertices  $u, v$ , and  $w$  of  $G$  we have  $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$ . For an edge  $ab$  of a graph  $G$  let

$$W_{ab} = \{x \in V(G) : d(x, a) < d(x, b)\}.$$

We will also use  $W_{ab}$  to denote the corresponding induced subgraph of  $G$ . Djoković [8] characterized partial cubes in the following way.

**Theorem 1.1.** *A graph  $G$  is a partial cube if and only if it is bipartite and if for any edge  $ab$  of  $G$  the subgraph  $W_{ab}$  is convex.*

In [8] relation  $\Theta$  was defined as follows: Edges  $xy, ab \in E(G)$  are in relation  $\Theta$ , if  $x \in W_{ab}$  and  $y \in W_{ba}$ . For bipartite graphs this is equivalent to the next definition:  $xy, ab \in E(G)$  are in relation  $\Theta$  if

$$d(x, a) + d(y, b) \neq d(x, b) + d(y, a).$$

Using this definition Winkler proved in [19]:

**Theorem 1.2.** *A graph  $G$  is a partial cube if and only if it is bipartite and  $\Theta$  is transitive.*

Besides the above two characterizations of partial cubes we will also make use of the following one due to Wilkeit [18]:

**Theorem 1.3.** *A graph  $G$  is a partial cube if and only if it is bipartite and if for any edges  $ab$  and  $xy$ ,  $xy\Theta ab$  implies  $W_{ab} = W_{xy}$ .*

Let  $W_k$  be the  $k$ -wheel, that is, the graph obtained as a join of the one vertex graph  $K_1$  and the  $k$ -cycle  $C_k$ . In the rest of the paper we will denote the central vertex of  $W_k$  by  $u$  and the remaining vertices by  $w_1, w_2, \dots, w_k$ , where adjacencies are defined in a natural way. The cycle of  $W_k$  induced by the vertices  $w_1, w_2, \dots, w_k$  will also be called the *outer cycle* of  $W_k$ . These notions will also be used for subdivided wheels, in particular for the graphs  $W_k(m, n)$ .

**2. Interval monotone subdivisions of wheels**

In this section we characterize interval monotone graphs among the subdivided wheels  $W_k(m, n)$ . We begin with graphs that are obtained from  $W_3$  (i.e. from  $K_4$ ). Let  $W(m_1, m_2, m_3; n_1, n_2, n_3)$  be the graph obtained by subdividing edges of  $K_4$ , where  $m_i$  is the number of vertices added on the edges of the outer cycle, and  $n_i$  the number of vertices added on the inner edges, so that numbers  $n_i$  and  $m_i$  correspond to two nonincident edges of  $K_4$  ( $i = 1, 2, 3$ ). Then we have:

**Lemma 2.1.** *Assume that for  $W(m_1, m_2, m_3; n_1, n_2, n_3)$  the following properties hold:*

- (A)  $m_i + m_j \geq m_k$ , for all permutations  $(i, j, k)$  of  $(1, 2, 3)$ ;
- (B)  $m_i \leq n_j + n_k$ , for all permutations  $(i, j, k)$  of  $(1, 2, 3)$ ;
- (C)  $m_i - m_j = n_j - n_i$ , for all  $1 \leq i < j \leq 3$ ;
- (D)  $n_i \leq m_j + n_k$ , for all permutations  $(i, j, k)$  of  $(1, 2, 3)$ .

*Then  $W(m_1, m_2, m_3; n_1, n_2, n_3)$  is interval monotone.*

PROOF: Note first that if  $x$  and  $y$  are vertices of the outer cycle then (B) implies that  $I(x, y)$  is contained in the outer cycle, hence convex.

Let  $x$  and  $y$  be two vertices of the inner subdivided edges. If they both lie on a path between some  $w_i$  and  $u$ , then from (D) we deduce that  $I(x, y)$  is a path. Suppose next that they are in different subdivided inner edges, say on  $w_iu$  and  $w_ju$ . Then  $I(x, y)$  is a path, if they are both close enough to  $u$  (e.g., if they are both neighbors of  $u$ ). If they lie far from  $u$ , then use (A) and (D) to observe that  $I(x, y)$  can be either a path which goes through  $w_i$  and  $w_j$ , or the cycle  $w_i \rightarrow \dots \rightarrow w_j \rightarrow \dots \rightarrow u \rightarrow \dots \rightarrow w_i$ . Again use (A) and (D) for the convexity of this cycle.

It remains to check the case when  $x$  is on the outer cycle, say between  $w_1$  and  $w_2$ , and  $y$  is one of the inner vertices. The case when  $y$  is on  $w_1u$  or  $w_2u$  is essentially the same as above. Thus the last case to consider is when  $y$  is on  $w_3u$ . If  $I(x, w_3)$  is not equal to the outer cycle we can argue as above. So let  $I(x, w_3)$  be the outer cycle, i.e.,  $x$  is the vertex on the outer cycle at the largest distance from  $w_3$ . But then we deduce from (C) that  $I(x, u)$  is the whole cycle  $w_1 \rightarrow \dots \rightarrow w_2 \rightarrow \dots \rightarrow u \rightarrow \dots \rightarrow w_1$ , hence  $I(x, y)$  is either the whole graph

or one of the cycles together with some short path from  $u$  to  $y$ , or from  $w_3$  to  $y$ . □

**Theorem 2.2.** *Let  $k \geq 3$ . Then  $W_k(m, n)$  is interval monotone if and only if*

- (i)  $k = 3$  and  $m \leq 2n$ ; or
- (ii)  $k \geq 3, m \geq n = 0$ .

PROOF: We distinguish several cases.

**Case 1:**  $k \geq 4, m > n \geq 1$ .

Let  $x_1, x_2, \dots, x_m$  be the vertices of  $W_k(m, n)$  with which the edge  $w_1w_2$  is subdivided and let  $y_m, y_{m-1}, \dots, y_1$  be the vertices with which  $w_2w_3$  is subdivided, see Figure 1.

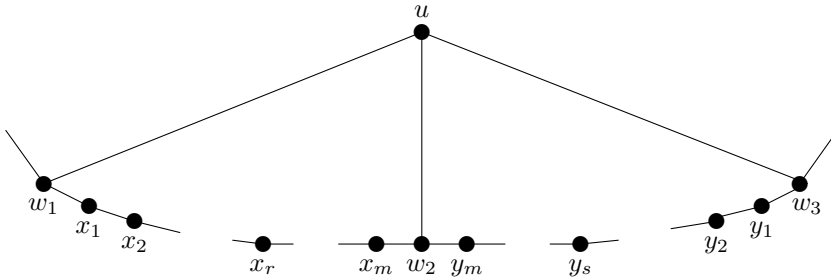


Figure 1: Subdivided vertices of Case 1.

Denote  $x_0 = w_1$ , set  $r = \lfloor (m - n)/2 \rfloor$ ,  $s = \lceil (m - n)/2 \rceil$ , and consider the following paths between  $x_r$  and  $y_s$ :

- $P_1 : x_r, x_{r+1}, \dots, x_m, w_2, y_m, \dots, y_s$ ;
- $P_2 : x_r, x_{r-1}, \dots, w_1, \dots, u, \dots, w_3, y_1, \dots, y_s$ ;
- $P_3 : x_r, x_{r-1}, \dots, w_1, \dots, u, \dots, w_2, y_m, \dots, y_s$ ;
- $P_4 : x_r, x_{r+1}, \dots, w_2, \dots, u, \dots, w_3, y_1, \dots, y_s$ .

Then the lengths of these paths are:

- $P_1 : (m + 1 - \lfloor (m - n)/2 \rfloor) + (m + 1 - \lceil (m - n)/2 \rceil) = m + n + 2$ ;
- $P_2 : \lfloor (m - n)/2 \rfloor + 2(n + 1) + \lceil (m - n)/2 \rceil = m + n + 2$ ;
- $P_3 : \lfloor (m - n)/2 \rfloor + 2(n + 1) + (m + 1 - \lceil (m - n)/2 \rceil) \geq 2n + m + 2$ ;
- $P_4 : (m + 1 - \lfloor (m - n)/2 \rfloor) + 2(n + 1) + \lceil (m - n)/2 \rceil \geq 2n + m + 2$ .

It follows that  $d(x_r, y_s) = m + n + 2$ . Moreover,  $u, w_2 \in I(x_r, y_s)$ , but no interior vertex on the  $w_2, u$ -path of length  $n + 1$  belongs to  $I(x_r, y_s)$ . Hence,  $I(x_r, y_s)$  is not convex.

**Case 2:**  $k \geq 4, 1 \leq m \leq n$  or  $0 = m < n$ .

Let  $z_1, z_2, \dots, z_n$  be the vertices of  $W_k(m, n)$  with which the edge  $wu_3$  is subdivided, see Figure 2.

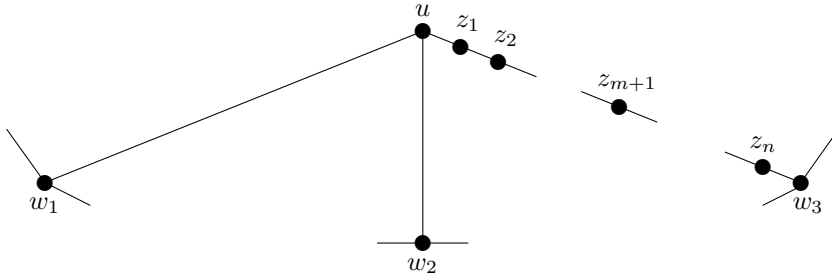


Figure 2: Subdivided vertices of Case 2.

Then the following two paths between  $w_1$  and  $z_{m+1}$  (note that we allow  $z_{m+1} = w_3$ ):

$$P_1 : w_1, \dots, u, z_1, \dots, z_{m+1};$$

$$P_2 : w_1, \dots, w_2, \dots, w_3, z_n, \dots, z_{m+1},$$

are shortest  $w_1, z_{m+1}$ -paths of length  $n + m + 2$ . Hence  $u, w_2 \in I(w_1, z_{m+1})$ . Since no interior vertex of the  $w_2, u$ -path of length  $n + 1$  belongs to this interval, we are done also in this case.

**Case 3:**  $k = 3, m \geq 2n + 1 \geq 3$ .

Let  $x_1, x_2, \dots, x_m, y_m, y_{m-1}, \dots, y_1, r$ , and  $s$ , be defined as in Case 1, cf. Figure 1. In addition, let  $P_1, P_2, P_3$  and  $P_4$  be the  $x_r, y_s$ -paths as defined in Case 1. Then the length of  $P_1$  as well as of  $P_2$  is  $m + n + 2$ . Also, the lengths of  $P_3$  and  $P_4$  are at least  $2n + m + 2$ . Finally, let  $P_5$  be the path  $x_r, x_{r-1}, \dots, w_1, \dots, w_3, y_1, \dots, y_s$ . Its length is  $r + (m + 1) + s = 2m - n + 1$  and since  $m \geq 2n + 1$  the length of  $P_5$  is at least  $m + n + 2$ . We conclude that no interior vertex of the  $w_2, u$ -path of length  $n + 1$  belongs to  $I(x_r, y_s)$ . ■

**Case 4:**  $k = 3, 1 \leq m \leq 2n$ .

Note that  $W_3(m, n)$  is isomorphic to  $W(m, m, m; n, n, n)$ . Hence in this case  $W_3(m, n)$  is interval monotone by Lemma 2.1.

The last case to consider is  $n = 0$ , more precisely:

**Case 5:**  $k \geq 3, m \geq n = 0$ .

We claim that  $W_k(m, 0)$  is interval monotone for  $k \geq 3, m \geq 0$ . For wheels, that is, for  $m = 0$ , this is clear. Also, it is well known that  $W_k(1, 0)$  are median graphs. Let  $m \geq 2$ . It clearly suffices to check interval monotonicity for vertices  $x, y$  of the outer cycle. In this case  $d(w_i, w_j) = 2$  for  $i \neq j$ , while the distance between  $w_i$  and  $w_j$  in the outer cycle is at least three. From here we conclude by a simple case analysis that the claim is true also in the subcase  $m \geq 2$ . □

### 3. Subdivisions of wheels and partial cubes

In this section we locate partial cubes among the graphs  $W_k(m, n)$ . Together with Theorem 2.2 this allows us to answer the question of Chepoi and Tardif. At the end we also show that the structure of bipartite, interval monotone graphs, which are not partial cubes, must be similar as in our examples.

We begin with the following two straightforward observations.

**Lemma 3.1.** *The graph  $W_k(m, n)$  is bipartite if and only if  $m$  is odd.*

**Lemma 3.2.** *Let  $C$  be an isometric even cycle of a graph  $G$  and  $e$  an edge of  $C$ . Then the antipodal edge of  $e$  is the unique edge (different from  $e$ ) of  $C$  that is in relation  $\Theta$  with  $e$ .*

**Theorem 3.3.** *Let  $k \geq 3$ . Then  $W_k(m, n)$  is a partial cube if and only if*

- (i)  $k = 3, m = 1,$  and  $n = 1;$  or
- (ii)  $k \geq 3, n = 0,$  and  $m$  is odd.

PROOF: First we consider the case  $n > 0$ . Since partial cubes are interval monotone, Theorem 2.2 implies that then  $k = 3$  and  $m \leq 2n$ . Let  $m \geq 3$ . Then  $n \geq 2$ , and let  $x$  be the neighbor of  $w_1$  on the subdivision of  $w_1w_3$ . Since the outer cycle is isometric, by Lemma 3.2 there is a unique edge  $ab$  on the outer cycle which is in relation  $\Theta$  with  $w_1w_3$ . In addition, it is clear that  $ab$  belongs to the subdivision of  $w_2w_3$  and that  $a \neq w_2, b \neq w_2$ . There exists an isomorphism  $\varphi$  between the isometric cycles

$$C_1 : w_3 \rightarrow \dots \rightarrow u \rightarrow \dots \rightarrow w_1 \rightarrow \dots \rightarrow w_3 \quad \text{and}$$

$$C_2 : w_3 \rightarrow \dots \rightarrow u \rightarrow \dots \rightarrow w_2 \rightarrow \dots \rightarrow w_3$$

which preserves the shortest  $w_3, u$ -path, so that  $\varphi(w_1) = w_2$ . Note that  $xw_1$  (resp.  $ab$ ) is in relation  $\Theta$  with precisely one edge  $e$  of  $C_1$  (resp.  $f$  of  $C_2$ ) whose endvertices are the unique vertices at the largest distance from  $x$  and  $w_1$  (resp.  $a$  and  $b$ ). Both  $e$  and  $f$  are on subdivided edge of  $uw_3$ , but  $e \neq f$  because  $\varphi(xw_1) \neq ab$ . Hence  $\Theta$  is not transitive and by Theorem 1.2 we infer that  $W_3(m, n), n > 0, m \geq 3$ , is not a partial cube.

Let  $m = 1$  and  $n \geq 2$ . We define cycles  $C_1$  and  $C_2$  as above, but this time let  $x$  be a neighbor of  $w_1$  on the subdivision of  $w_1u$ . Applying Lemma 3.2 we infer that  $w_1x$  is in relation  $\Theta$  with precisely two edges  $e$  and  $f$  of the cycle  $C_2$ , where  $e$  is on the subdivision of  $w_2u$  and  $f$  on the subdivision of  $w_3u$ , and endvertices of  $e$  and  $f$  are neither  $w_2$  nor  $w_3$ . Since  $e$  and  $f$  are not in relation  $\Theta$ ,  $W_3(m, n), n \geq 2, m = 1$ , is not a partial cube.

It is straightforward to check that  $W_3(1, 1)$  is a partial cube.

In the case where  $n = 0$  we see that  $W_k(1, 0)$  are median graphs which makes them partial cubes. Finally, let  $m \geq 3$  be odd. We claim that in this case

$W_k(m, 0)$  is a partial cube. Since it is bipartite, by Theorem 1.1 it is enough to show that the sets  $W_{ab}$  are convex. First note that  $W_{w_1u}$  is a path of length  $m + 1$  and clearly convex. Likewise the set  $W_{uw_1}$  is easily seen to be convex. Consider now an arbitrary edge  $ab$  of the outer cycle and assume without loss of generality that  $d(a, u) < d(b, u)$ . But then we infer that  $W_{ba}$  induces a path, and we easily conclude that again  $W_{ba}$  and  $W_{ab}$  are convex.  $\square$

Combining Theorems 2.2 and 3.3 we can now answer a question of Chepoi and Tardif as follows.

**Corollary 3.4.** *Let  $n \geq 2$ , and let  $m$  be an odd integer,  $m \leq 2n$ . Then  $W_3(m, n)$  is a bipartite, interval monotone graph, which does not admit an isometric embedding into a hypercube.*

We note that also nonsymmetric subdivided  $K_4$ 's can be interval monotone, bipartite and not partial cubes. For example, consider the class of graphs

$$W(2k + 1, 2k, 2k; 2k + 1, 2k + 2, 2k + 2)$$

for all  $k \geq 1$ . We believe that there are more such cases. However, any interval monotone, bipartite graph, that is not a partial cube, contains a subdivision of  $K_4$ , as our final result claims.

**Theorem 3.5.** *Let  $G$  be a bipartite interval monotone graph. Then either  $G$  is a partial cube or it contains a subdivision of  $K_4$ .*

PROOF: Let  $G$  be a bipartite graph in which all intervals are convex, and suppose that  $G$  is not a partial cube. Then by Theorem 1.3 there exist edges  $ab, xy \in E(G)$  which are in relation  $\Theta$  such that  $W_{ab} \neq W_{xy}$ . Hence, since  $G$  is bipartite, there exists a vertex  $w \in W_{xy}$ , such that also  $w \in W_{ba}$ . We select edges  $ab$  and  $xy$  so that  $d(a, x)$  is as small as possible, and among such pairs let  $ab$  and  $xy$  be chosen in such a way that for some  $w \in W_{xy} \cap W_{ba}$  the sum  $d(x, w) + d(w, b)$  is as small as possible. Note that under these conditions vertex  $w$  can still be chosen in such a way that its neighbor on a shortest  $w, x$ -path is in  $W_{ab}$ .

Let  $x' \in I(x, a) \cap I(x, w)$  be such that its neighbor  $x''$  on a shortest  $x, w$ -path is not in  $I(x, a)$  (clearly, such a neighbor  $x''$  of  $x'$  exists, since  $w$  cannot be in  $I(x, a)$ ). Then, it is easy to see that the remainder of the shortest path from  $x''$  to  $w$  is disjoint with  $I(x, a)$ . Let  $b'$  be the first vertex on a shortest path from  $w$  to  $b$  which is in  $I(y, b)$ . Then obviously  $I(b', b) \subseteq I(y, b)$ . Let  $P$  be a path from  $x'$  to  $b'$  which is a concatenation of a shortest  $x', w$ -path, and a shortest  $w, b'$ -path. Since  $w \notin I(y, a)$  and  $I(y, a)$  is convex, it follows that  $P$  cannot be a shortest  $x', b'$ -path. We shall now prove that  $a, b \in I(x', b')$  or  $x, y \in I(x', b')$ .

Suppose there is a shortest  $x', b'$ -path  $P'$  that avoids all four vertices  $x, y, a$  and  $b$ . Let  $v$  be the first vertex on  $P'$  which is in  $W_{ba}$  (such a vertex exists since  $b' \in W_{ba}$ ). Hence its preceding neighbor  $u$  on  $P'$  is in  $W_{ab}$ , thus  $ab\Theta uv$ . We distinguish two cases.

**Case 1:**  $xy\Theta uv$ .

Note that in this case  $v$  cannot be closer to  $x$  than to  $y$ , because then we would derive that  $v \in W_{ba} \cap W_{xy}$ , and by the choice of  $P'$  we would have  $d(x, v) + d(v, b) < d(x, w) + d(w, b)$ . Hence  $v \in W_{yx}$  and  $u \in W_{xy}$ . Now we have two possibilities: if  $w \in W_{uv}$  then  $w$  is a vertex in  $W_{uv} \cap W_{ba}$ , where  $d(a, x) > d(a, u)$ , a contradiction to the choice of  $x$  and  $a$  being the vertices with the smallest distance such that  $W_{ab} \neq W_{xy}$ . On the other hand, if  $w \in W_{vu}$  then  $w$  is a vertex in  $W_{vu} \cap W_{xy}$ , where  $d(a, x) > d(u, x)$ , again the same contradiction.

**Case 2:**  $\neg(xy\Theta uv)$ .

In this case both  $x$  and  $y$  are either in  $W_{uv}$  or in  $W_{vu}$ . If  $x, y \in W_{uv}$  then  $y \in W_{uv} \cap W_{ba}$ , where  $d(u, a) < d(x, a)$ , again a contradiction with the choice of  $x$  and  $a$ . If  $x, y \in W_{vu}$  then  $x \in W_{vu} \cap W_{ab}$ , where  $d(u, a) < d(x, a)$ , the same contradiction.

Hence  $I(x', b')$  includes at least one pair of vertices  $x, y$  or  $a, b$ . Without loss of generality assume that  $x, y \in I(x', b')$ . We have noted in the beginning of the proof that  $w$  can be chosen in such a way that its neighbor  $w'$  on  $P$  is in  $W_{ab}$ . Then obviously  $w' \in I(w, a)$  and  $b' \in I(w, a)$ . Now, if  $x'$  would also be in  $I(w, a)$ , then since  $G$  is interval monotone and  $y \in I(x', b')$  that would imply  $y \in I(w, a)$ . This is possible only if  $b' = y$  which leads straightforward to a contradiction with the choice of  $w$  as a vertex in  $W_{xy} \cap W_{ba}$ . Thereby  $x' \notin I(w, a)$ , and let  $a'$  be a nearest vertex to  $w$  in  $I(w, a) \cap I(a, x)$ , and  $w''$  a vertex in  $I(w, a') \cap P$  at the largest distance from  $w$ . We have thus obtained a subdivided  $K_4$  in  $G$  with vertices  $w'', a', x'$  and  $b'$  and the proof is complete.  $\square$

It would be interesting to see whether one can strengthen Theorem 3.5 to derive the existence of an isometric subdivided  $K_4$  in  $G$ . Moreover, a characterization of partial cubes as bipartite interval monotone graphs with some (nice) additional condition(s) seems to be a challenging task.

## REFERENCES

- [1] Aurenhammer F., Formann M., Idury R., Schäffer A., Wagner F., *Faster isometric embeddings in products of complete graphs*, Discrete Appl. Math. **52** (1994), 17–28.
- [2] Aurenhammer F., Hagauer J., *Recognizing binary Hamming graphs in  $O(n^2 \log n)$  time*, Math. Systems Theory **28** (1995), 387–395.
- [3] Avis D., *Hypermetric spaces and the Hamming cone*, Canad. J. Math. **33** (1981), 795–802.
- [4] Chepoi V.,  *$d$ -convexity and isometric subgraphs of Hamming graphs*, Cybernetics **1** (1988), 6–9.
- [5] Chepoi V., *On distances in benzenoid graphs*, J. Chem. Inform. Comput. Sci. **36** (1996), 1169–1172.
- [6] Chepoi V., Klavžar S., *The Wiener index and the Szeged index of benzenoid systems in linear time*, J. Chem. Inform. Comput. Sci. **37** (1997), 752–755.
- [7] Chepoi V., Tardif C., *personal communication*, 1994.
- [8] Djoković D., *Distance preserving subgraphs of hypercubes*, J. Combin. Theory Ser. B **14** (1973), 263–267.



- [9] Fukuda K., Handa K., *Antipodal graphs and oriented matroids*, Discrete Math. **111** (1993), 245–256.
- [10] Graham R.L., Pollak H., *On addressing problem for loop switching*, Bell System Tech. J. **50** (1971), 2495–2519.
- [11] Imrich W., Klavžar S., *On the complexity of recognizing Hamming graphs and related classes of graphs*, European J. Combin. **17** (1996), 209–221.
- [12] Imrich W., Klavžar S., *A convexity lemma and expansion procedures for bipartite graphs*, European J. Combin. **19** (1998), 677–685.
- [13] Imrich W., Klavžar S., *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.
- [14] Klavžar S., Gutman I., *Wiener number of vertex-weighted graphs and a chemical application*, Discrete Appl. Math. **80** (1997), 73–81.
- [15] Lawrence J., *Lopsided sets and orthant-intersection by convex sets*, Pacific J. Math. **104** (1983), 155–173.
- [16] Mollard M., *Interval-regularity does not lead to interval monotonicity*, Discrete Math. **118** (1993), 233–237.
- [17] Mulder H.M., *The Interval Function of a Graph*, Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.
- [18] Wilkeit E., *Isometric embeddings in Hamming graphs*, J. Combin. Theory Ser. B **50** (1990), 179–197.
- [19] Winkler P., *Isometric embeddings in products of complete graphs*, Discrete Appl. Math. **7** (1984), 221–225.

UNIVERSITY OF MARIBOR, FERI, SMETANOVA 17, 2000 MARIBOR, SLOVENIA

*E-mail:* bostjan.bresar@uni-mb.si

DEPARTMENT OF MATHEMATICS, PEF, UNIVERSITY OF MARIBOR, KOROŠKA 160,  
2000 MARIBOR, SLOVENIA

*E-mail:* sandi.klavzar@uni-lj.si

(Received July 3, 2001)