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Approximation of circular arcs by parametric polynomial curves

Abstract In this paper the approximation of circular arcs by parametric polynomial curves is studied. If the angular length of the circular arc is h, a parametric polynomial curve of arbitrary degree $n \in \mathbb{N}$, which interpolates given arc at a particular point, can be constructed with radial distance bounded by h^{2n}. This is a generalization of the result obtained by Lyche and Mørken for odd n.

1 Introduction

The approximation of circular arcs is an important task in Computer Aided Geometric Design (CAGD), Computer Aided Design (CAD) and Computer Aided Manufacturing (CAM). Though a circle arc can be exactly represented by a rational quadratic Bézier curve (or, generally, by rational parametric curve of low

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degree, see [1], e.g.), some CAD/CAM systems require a polynomial representation of circular segments. Also, some important algorithms, such as lofting and blending can not be directly applied to rational curves.
On the other hand, circular arcs can not be represented by polynomials exactly, thus interpolation or approximation has to be used to represent them accurately.

Among others, Lyche and Mørken have studied the problem of approximation of circular segments by polynomial parametric curves (see [3]). They have found an excellent explicit approximation by odd degree parametric polynomial curves, but conjectured that the same problem with even degree could be a tough task. Their method is based on Taylor-type approximation and explicitly provides parametric polynomials of odd degree n with high asymptotic approximation order, i.e., $2n$.

In this paper the general case for any $n \in \mathbb{N}$ is solved. First, the approximation problem will be stated. Let

$$\boldsymbol{f}(\varphi) := \begin{pmatrix} \sin \varphi \\ \cos \varphi \end{pmatrix}, \quad 0 \le \varphi \le \alpha < 2\pi, \tag{1.1}$$

be a particular parametrization of a circular arc of angular length α. It is enough to consider arcs of the unit circle only, since any other arc of the same angular length can be obtained by affine transformations. Our goal is to find a parametric polynomial curve

$$\boldsymbol{p}_n := \boldsymbol{p} := \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{1.2}$$

of degree $\le n$ with nonconstant scalar polynomials $x_n, y_n \in \mathbb{R}[t]$,

$$x_n(t) := \sum_{j=0}^{n} a_j t^j, \quad y_n(t) := \sum_{j=0}^{n} b_j t^j, \tag{1.3}$$

which provides "the best approximation" of (1.1). The only prescribed interpolation point is $\boldsymbol{f}(0) := (x_n(0), y_n(0))^T := (0,1)^T$, thus $a_0 := 0$ and $b_0 := 1$ in (1.3). In CAGD, we are interested mainly in geometric properties of objects. A particular parametrization is just a representation of an object in a desired form. Therefore the approximation error will be considered as a distance between set of points on given curves, i.e., a circular arc and a parametric polynomial in this case. It seems natural to choose a *"radial distance"* here (see Figure 1), i.e.,

$$d(\boldsymbol{f}, \boldsymbol{p}) := \max_{t \in I} \left\{ \left| \sqrt{x_n^2(t) + y_n^2(t)} - 1 \right| \right\}, \tag{1.4}$$

where I is some interval of observation.

If $\boldsymbol{p}$ is a good approximation of $\boldsymbol{f}$ on I then (1.4) is small and $\sqrt{x_n^2(t) + y_n^2(t)} \approx 1$, thus

$$\left| \sqrt{x_n^2(t) + y_n^2(t)} - 1 \right| = \frac{\left| x_n^2(t) + y_n^2(t) - 1 \right|}{\sqrt{x_n^2(t) + y_n^2(t)} + 1} \approx \frac{1}{2} \left| x_n^2(t) + y_n^2(t) - 1 \right|,$$

and, for computational purposes, it is enough to consider only the "error"

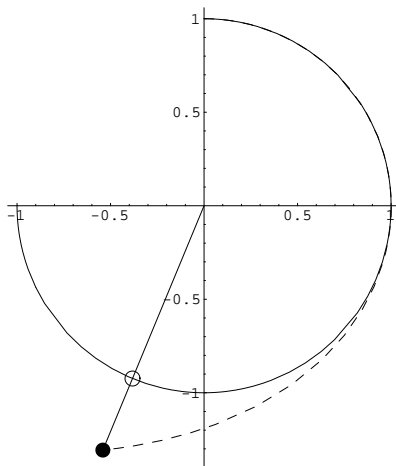$$e(t) := \left| x_n^2(t) + y_n^2(t) - 1 \right|. \tag{1.5}$$

Fig. 1 Radial distance between circular arc (solid) and a parametric curve approximation (dashed)

Ideally, e would be zero if a polynomial parametrization of a circular arc would exist. But if at least one of x_n or y_n is of degree n, then (1.3) implies

$$x_n^2(t) + y_n^2(t) = (a_n^2 + b_n^2)t^{2n} + \cdots \neq 1. \qquad (1.6)$$

Now it follows from (1.5) that e will be small (at least for small t), if coefficients at the lower degree terms in (1.6) will vanish. This implies that e will be as small as possible if

$$x_n^2(t) + y_n^2(t) = 1 + \text{const} \cdot t^{2n}. \qquad (1.7)$$

A proper reparametrization

$$t \rightarrow \frac{t}{\sqrt[2n]{a_n^2 + b_n^2}}$$

transforms (1.7) to

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n}, \qquad (1.8)$$

which gives $2n$ nonlinear equations for $2n$ unknown coefficients $(a_j)_{j=1}^n$ and $(b_j)_{j=1}^n$.

The paper is organized as follows. In Section 2 the system of nonlinear equations will be studied and a general closed form solution will be derived. In Section 3 the optimal asymptotic approximation order will be confirmed and in the last section some concluding remarks regarding the optimal approximation of circular arcs will be given.

2 Solution of the problem

Although the solutions of the system of nonlinear equations given by (1.8) can be obtained numerically for particular values of n, finding a closed form solution is a much more complicated problem. In [3], authors proposed a very nice approach

to solve this problem. They have used a particular rational parametrization of the unit circle to obtain the coefficients of the polynomials x_n and y_n. Indeed, if

$$x_0(t) := \frac{2t}{1+t^2}, \quad y_0(t) := \frac{1-t^2}{1+t^2}, \quad t \in (-\infty, \infty), \tag{2.1}$$

is a parametrization of a unit circle, then the functions

$$x_n(t) := x_0(t) - (-1)^{(n-1)/2} t^n y_0(t),$$
$$y_n(t) := y_0(t) + (-1)^{(n-1)/2} t^n x_0(t),$$

are actually polynomials of degree $\leq n$ for which (1.8) holds. It is also easy to find their explicit form, but unfortunately if n is even, their coefficients are no more real numbers. However, this idea can be applied for even n too, but slightly different rational parametrization of the unit circle has to be considered. Namely, let

$$n = 2^k (2r-1), \quad k \in \mathbb{N}_0, \ r \in \mathbb{N}, \tag{2.2}$$

and let x_0, y_0 be redefined as

$$x_0(t) := \frac{2\sqrt{1-c^2}\, t\, (1-ct)}{1 - 2ct + t^2}, \quad y_0(t) := \frac{1 - 2ct + (2c^2 - 1)t^2}{1 - 2ct + t^2}, \tag{2.3}$$

where $c \in [0,1)$. It is straightforward to see that $x_0^2(t) + y_0^2(t) = 1$. Note that (2.1) is a particular case of (2.3) where $c = 0$. The following theorem, which has already been considered in [2] in a different context, gives one of the solutions of the nonlinear system (1.8) for any n in a closed form.

Theorem 2.1 *Suppose that n, k, and r satisfy (2.2), and let the constants c_k, s_k be given as $c_k := \cos \psi_k$ and $s_k := \sin \psi_k$, where $\psi_k := \pi/2^{k+1}$. Further, suppose that x_0 and y_0 are defined by (2.3) with $c := c_k$. Then the functions x_n and y_n, defined by*

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} := \begin{pmatrix} 1 & (-1)^r t^n \\ (-1)^{r+1} t^n & 1 \end{pmatrix} \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix}, \tag{2.4}$$

are polynomials of degree $\leq n$ that satisfy (1.8). Furthermore, their coefficients are given as

$$a_j = 2s_k \cos((j-1)\psi_k) = 2s_k T_{j-1}(c_k), \quad j = 1, 2, \ldots, n-1, \tag{2.5}$$
$$a_n = 2s_k \cos((n-1)\psi_k) + (-1)^r = 2s_k T_{n-1}(c_k) + (-1)^r, \tag{2.6}$$

and

$$b_0 = 1, \ b_1 = 0, \tag{2.7}$$
$$b_j = -2s_k \sin((j-1)\psi_k) = -2s_k^2 U_{j-2}(c_k), \quad j = 2, 3, \ldots, n, \tag{2.8}$$

where T_j and U_j are Chebyshev polynomials of the first and the second kind.

Proof The proof given here is an extended version of the proof in [2].
The equation (2.4) yields

$$x_n(t) = \frac{2\sqrt{1-c_k^2}\, t\,(1-c_k t) + (-1)^r t^n \left(1 - 2c_k t + (2c_k^2 - 1)t^2\right)}{1 - 2c_k t + t^2},$$

$$y_n(t) = \frac{(-1)^{r+1} t^n \left(2\sqrt{1-c_k^2}\, t\,(1-c_k t)\right) + 1 - 2c_k t + (2c_k^2 - 1)t^2}{1 - 2c_k t + t^2}.$$

To verify that the function x_n is a polynomial of the form (1.3), it is enough to see that

$$(1 - 2c_k t + t^2) \sum_{j=0}^{n} a_j t^j = a_1 t + (a_2 - 2c_k a_1)t^2$$

$$+ \sum_{j=3}^{n} (a_j - 2c_k a_{j-1} + a_{j-2})t^j + (-2c_k a_n + a_{n-1})t^{n+1} + a_n t^{n+2}$$

$$= 2\sqrt{1-c_k^2}\, t\,(1-c_k t) + (-1)^r t^n \left(1 - 2c_k t + (2c_k^2 - 1)t^2\right).$$

A comparison of the coefficients implies the linear recurrence

$$a_1 = 2s_k,\ a_2 = c_k a_1,\ a_j - 2c_k a_{j-1} + a_{j-2} = 0, \quad j = 3, 4, \ldots, n-1, \qquad (2.9)$$

with additional conditions

$$\begin{aligned}
a_n - 2c_k a_{n-1} + a_{n-2} &= (-1)^r, \\
2c_k a_n - a_{n-1} &= (-1)^r 2c_k, \\
a_n &= (-1)^r (2c_k^2 - 1).
\end{aligned} \qquad (2.10)$$

Similarly, for y_n it is enough to see that

$$(1 - 2c_k t + t^2) \sum_{j=0}^{n} b_j t^j = b_0 + (b_1 - 2c_k b_0)t + \sum_{j=2}^{n} (b_j - 2c_k b_{j-1} + b_{j-2})t^j$$

$$+ (-2c_k b_n + b_{n-1})t^{n+1} + b_n t^{n+2}$$

$$= (-1)^{r+1} t^n \left(2\sqrt{1-c_k^2}\, t\,(1-c_k t)\right) + 1 - 2c_k t + (2c_k^2 - 1)t^2.$$

Here, the conditions on b_j are

$$b_1 = 0,\ b_2 = -2s_k^2,\ b_j - 2c_k b_{j-1} + b_{j-2} = 0, \quad j = 3, 4, \ldots, n, \qquad (2.11)$$

and

$$\begin{aligned}
-2c_k b_n + b_{n-1} &= (-1)^{r+1} 2s_k, \\
b_n &= (-1)^r 2s_k c_k.
\end{aligned} \qquad (2.12)$$

The theory of linear recurrence equations gives the general form of the solution of (2.9), namely

$$a_j = A e^{i\psi_k j} + B e^{-i\psi_k j},$$

where $i^2 := -1$. From the initial conditions

$$a_1 = 2 s_k = A e^{i \psi_k} + B e^{-i \psi_k},$$
$$a_2 = 2 c_k s_k = A e^{2 i \psi_k} + B e^{-2 i \psi_k},$$

and the relation $e^{i \psi_k} = c_k + i s_k$, it is easy to obtain $A = s_k e^{-i \psi_k}$ and $B = s_k e^{i \psi_k}$. Therefore,

$$a_j = 2 s_k \cos((j-1) \psi_k) = 2 s_k T_{j-1}(c_k), \quad j = 1, 2, \ldots, n-1.$$

Additional conditions (2.10) imply

$$a_n = 2 \sqrt{1 - c_k^2} \cos((n-1) \psi_k) + (-1)^r = 2 s_k T_{n-1}(c_k) + (-1)^r.$$

Since the general recurrence relation (2.11) is the same as in (2.9), its solution reads

$$b_j = C e^{i \psi_k j} + D e^{-i \psi_k j}.$$

The initial conditions

$$b_1 = 0 = C e^{i \psi_k} + D e^{-i \psi_k},$$
$$b_2 = -2 s_k^2 = C e^{2 i \psi_k} + D e^{-2 i \psi_k},$$

imply $C = i s_k e^{-i \psi_k}$, $D = -i s_k e^{i \psi_k}$. Hence

$$b_j = i s_k \left(e^{i \psi_k (j-1)} - e^{-i \psi_k (j-1)} \right) = -2 s_k \sin((j-1) \psi_k) = -2 s_k^2 U_{j-2}(c_k),$$

for $j = 1, 2, \ldots, n$. Additional conditions (2.12) are satisfied and the proof is complete. $\qquad \square$

3 Approximation order

The study of the approximation order in parametric case is not a trivial task. The main problem is how the distance between parametric objects is measured. Since objects are usually considered as sets of points, the distance between sets is naturally involved. This leads to a very well known Hausdorff distance d_H, which is difficult to compute in practice. As its upper bound the so called *parametric distance* d_P has been proposed by Lyche and Mørken in [3].

Definition 3.1 Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be two parametric curves defined on the intervals I_1 and I_2. The parametric distance between $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ is defined by

$$d_P(\boldsymbol{f}_1, \boldsymbol{f}_2) = \inf_{\phi} \max_{t \in I_2} \| \boldsymbol{f}_1(\phi(t)) - \boldsymbol{f}_2(t) \|,$$

where $\phi : I_2 \to I_1$ is a regular reparametrization, i.e., $\phi' \neq 0$ on I_2.

Their result will be used here to prove the following lemma.

Lemma 3.1 *Let a circular arc $\boldsymbol{f}$ be defined by (1.1) and its parametric approximation $\boldsymbol{p}$ by (1.2). Let the coefficients of x_n and y_n be given by (2.5)–(2.8). If $\boldsymbol{p} : [0,h] \to \mathbb{R}^2$, where h is sufficiently small, then*

$$d_H(\boldsymbol{f},\boldsymbol{p}) \le d_P(\boldsymbol{f},\boldsymbol{p}) \le d(\boldsymbol{f},\boldsymbol{p}) \le h^{2n},$$

where d is defined by (1.4).

Proof By [3], d_P is a metric on a set of parametric curves on $[0,h]$. Obviously, for a particular ϕ, which is a regular reparametrization of $\boldsymbol{f}$ on $[0,h]$,

$$d_P(\boldsymbol{f},\boldsymbol{p}) \le \max_{t \in [0,h]} ||\boldsymbol{f}(\phi(t)) - \boldsymbol{p}(t)||_2.$$

Thus, it is enough to find a regular reparametrization ϕ of $\boldsymbol{f}$ for which

$$\max_{t \in [0,h]} ||\boldsymbol{f}(\phi(t)) - \boldsymbol{p}(t)||_2 \le h^{2n}.$$

Let $\phi : [0,h] \to I$ be defined as

$$\phi(t) := \arctan\left(\frac{x_n(t)}{y_n(t)}\right). \tag{3.1}$$

Since $x_n(0) = 0$, $y_n(0) = 1$ and by (2.5) $x'_n(0) = 2s_k$,

$$\phi'(0) = \frac{x'_n(0) y_n(0) - x_n(0) y'_n(0)}{x_n^2(0) + y_n^2(0)} = 2s_k > 0,$$

and there exists $h_0 > 0$, such that ϕ is a regular reparametrization on $[0,h]$ for $0 < h < h_0$. But a point $(\boldsymbol{f} \circ \phi)(t)$ lies on the circular arc defined by $\boldsymbol{f}$ and on the ray from the origin to $\boldsymbol{p}(t)$. This implies

$$||(\boldsymbol{f} \circ \phi)(t) - \boldsymbol{p}(t)||_2 = |\sqrt{x_n^2(t) + y_n^2(t)} - 1| \le |x_n^2(t) + y_n^2(t) - 1| = t^{2n},$$

where the last equality follows from (1.8). Finally,

$$\max_{t \in [0,h]} ||(\boldsymbol{f} \circ \phi)(t) - \boldsymbol{p}(t)||_2 \le h^{2n}$$

and the proof of the lemma is complete. □

An interesting question is, how large can be the angular length of the circular arc, which can be approximated by the previous method. First of all, the regularity of ϕ has to be assured, i.e., h has to be small enough. Then the angular length of the reparametrized circular arc $\boldsymbol{f} \circ \phi$ can be derived at least asymptotically.

Lemma 3.2 *If ϕ is a regular reparametrization on $[0,h]$ defined by (3.1), then the length of the circular arc $\boldsymbol{f} \circ \phi : [0,h] \to \mathbb{R}^2$ is $2s_k h + \mathcal{O}(h^2)$.*

Proof The proof is straightforward. The regularity of ϕ, (1.8), (2.5)–(2.8) and the fact that $(1+t^{2n})^{-1} = 1 + \mathcal{O}(t^{2n})$, simplify the arc-length to

$$s = \int_0^h \left\| (\boldsymbol{f} \circ \phi)'(t) \right\|_2 dt = \int_0^h \frac{|x_n'(t)\, y_n(t) - x_n(t)\, y_n'(t)|}{x_n^2(t) + y_n^2(t)}\, dt =$$

$$\int_0^h \frac{x_n'(t)\, y_n(t) - x_n(t)\, y_n'(t)}{1 + t^{2n}}\, dt =$$

$$\int_0^h (x_n'(t)\, y_n(t) - x_n(t)\, y_n'(t))(1 + \mathcal{O}(t^{2n}))\, dt = 2\, s_k\, h + \mathcal{O}(h^2).$$

$\square$

4 Concluding remarks

Since we know that the best local approximation at a particular point in the functional case is the Taylor expansion, the natural question arises how good the approximation can be, if x_n and y_n are taken as Taylor polynomials for sine and cosine at $t = 0$. The result is summarized in the following lemma.

Lemma 4.1 *Let x_n and y_n be the degree n Taylor polynomials of sine and cosine, respectively. Then*

$$x_n^2(t) + y_n^2(t) = 1 + \frac{1}{w_n} t^m + \mathcal{O}(t^{m+1}),$$

where

$$m := \begin{cases} n+1, & n \text{ is odd,} \\ n+2, & n \text{ is even,} \end{cases}$$

and

$$w_n = \begin{cases} \frac{m}{2} n! & \text{if } n \bmod 4 = 1, 2, \\ -\frac{m}{2} n! & \text{otherwise.} \end{cases}$$

The proof is straightforward and will be omitted. The last lemma confirms that the Taylor polynomials are not an optimal choice if the radial distance is used as a measure of the approximation order.

The construction proposed by Theorem 2.1 has a very small local approximation error, but only interpolates one point on a circular arc. More general results concerning the Lagrange interpolation of $2n$ points on a circle-like curve by a parametric polynomial curve of degree n and the same approximation error, i.e., $2n$, are in [2].

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