Barycentric coordinates for Lagrange interpolation over lattices on a simplex

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Abstract. In this paper, a $(d+1)$-pencil lattice on a simplex in $\mathbb{R}^d$ is studied. The lattice points are explicitly given in barycentric coordinates. This enables the construction and the efficient evaluation of the Lagrange interpolating polynomial over a lattice on a simplex. Also, the barycentric representation, based on shape parameters, turns out to be appropriate for the lattice extension from a simplex to a simplicial partition.

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1. Introduction

The approximation of multivariate functions by the polynomial interpolation is one of the fundamental approaches in multivariate approximation theory. Among crucial steps in this process, one is particularly important, namely a selection of interpolation points. It is well known that the existence and the uniqueness of the Lagrange interpolant in Π_n^d $(d > 1)$, the space of polynomials in d variables of total degree $\leq n$, heavily depends on the geometry of the interpolation points. This fact makes interpolation in several variables much more complicated than the univariate one. Although a simple algebraic characterization states that a set of interpolation points is correct in Π_n^d if and only if they do not lie on an algebraic hypersurface of degree $\leq n$, it is useless in practical computations, since in general it can not be checked in the floating point arithmetic. Thus a considerable research has been focused on finding configurations of points which guarantee the existence and the uniqueness of the Lagrange interpolant.

Perhaps the most frequently used configurations of this type are lattices which satisfy the well-known GC (geometric characterization) condition (see [2]). They have a nice geometric background, since they are constructed as intersections of hyperplanes. Furthermore, they provide an easy way to construct the Lagrange basis polynomials as products of linear factors. Among several known classes of lattices ([1], [3]), principal lattices ([2]), and their generalization, $(d+1)$-pencil lattices ([6]) are the most important.

The first step in multivariate Lagrange interpolation in $\mathbb{R}^d$ is usually a construction of interpolation points on a simplex, a corner stone of the d dimensional Euclidean space. This leads to a natural generalization, lattices over triangulations, or more generally, over simplicial partitions. Two main questions arise: what are explicit coordinates of lattice points and how should such lattices be put together to ensure at least continuity over common faces of simplices.

In this paper we use a straightforward approach to answer the first question. We provide a closed form formula for lattice points on a simplex in barycentric coordinates. The novel representation, based on control points, provides shape parameters of the lattice with a clear geometric interpretation. Furthermore, the corresponding Lagrange interpolating polynomial is derived. Since the Lagrange basis polynomials are products of linear factors, some simplifications are done in order to decrease the amount of work needed.

This representation of lattices is useful in many practical applications, such as an explicit interpolation of multivariate functions, in particular, approximation of functionals defined on a simplex (numerical methods for multidimensional integrals, e.g.), finite elements methods in solving partial differential equations .... On the other hand, we provide fundamental tools for the construction of continuous splines over simplicial partitions, the problem which has already been observed in [4] but only for triangulations.

The paper is organized as follows. In the next section a novel definition of a lattice, based on control points, is given and the barycentric coordinates of lattice points are derived. In Section 3 the Lagrange interpolation polynomial is presented, and in the last section an example is given to conclude the paper.

2. Barycentric form of a $(d+1)$-pencil lattice in $\mathbb{R}^d$

In [6] an explicit representation of a $(d+1)$-pencil lattice of order n on a d-simplex was provided. This approach heavily depends on homogeneous coordinates, and a nice illustrative explanation can be found in [7], where the cases perhaps most often met in practice, i.e., $d=2$ and $d=3$, are outlined. Here our goal is an explicit representation in barycentric coordinates using a novel approach, since this enables a natural extension from a simplex to a simplicial partition (see [4] for the case $d = 2$).

A simplex in $\mathbb{R}^d$ is a convex hull of $d+1$ vertices $\boldsymbol{T}_i$, $i = 0, 1, \ldots, d$. Since for our purposes the ordering of the vertices of the simplex will be important, the notation

$$\triangle := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle,$$

which defines a simplex with a prescribed order of the vertices $\boldsymbol{T}_i$, will be used. The standard simplex of vertices

$$\boldsymbol{T}_i = (\delta_{i,j})_{j=1}^d, \quad i = 0, 1, \ldots, d, \qquad \delta_{i,j} := \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

will be denoted by $\triangle_d$.

A $(d+1)$-pencil lattice of order n on $\triangle$ is a set of $\binom{n+d}{d}$ points, generated by particular $d+1$ pencils of $n+1$ hyperplanes, such that each lattice point is an intersection of $d+1$ hyperplanes, one from each pencil. Furthermore, each pencil intersects at a center

$$\boldsymbol{C}_i \subset \mathbb{R}^d, \; i = 0, 1, \ldots, d,$$

a plane of codimension two. The lattice is actually based upon affinely

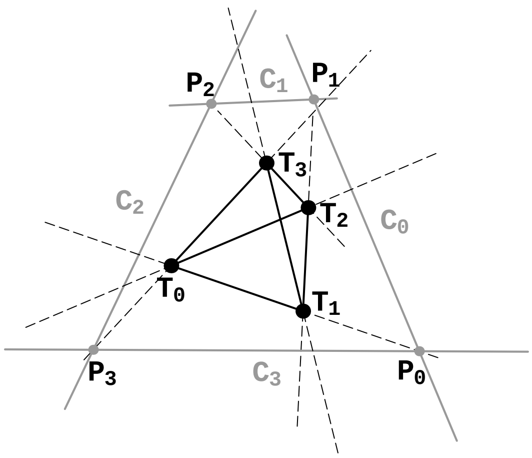

Figure 1. A 3-simplex $\langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{T}_3 \rangle$ in $\mathbb{R}^3$, lattice control points $\boldsymbol{P}_i$ and centers $\boldsymbol{C}_i$.

independent control points

$$\boldsymbol{P}_0, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_d, \quad \boldsymbol{P}_i \in \mathbb{R}^d,$$

where $\boldsymbol{P}_i$ lies on the line through $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$ outside of the segment $\boldsymbol{T}_i\boldsymbol{T}_{i+1}$ (Fig. 1). The center $\boldsymbol{C}_i$ is then uniquely determined by a sequence of control points

$$\boldsymbol{P}_i, \boldsymbol{P}_{i+1}, \ldots, \boldsymbol{P}_{i+d-2}, \tag{1}$$

where

$$\{\boldsymbol{P}_{i+1}, \boldsymbol{P}_{i+2}, \ldots, \boldsymbol{P}_{i+d-2}\} \subseteq \boldsymbol{C}_i \cap \boldsymbol{C}_{i+1}.$$

Here and throughout the paper, indices of points, centers, lattice parameters, etc., are assumed to be taken modulo $d+1$.

Thus with the given control points, the lattice on a simplex is determined. Quite clearly, the geometric construction of the lattice assures $\boldsymbol{C}_i \cap \triangle = \emptyset$, $i = 0, \ldots, d$, and also that each $\boldsymbol{C}_i$ is lying in a supporting hyperplane of a facet $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1}, \ldots, \boldsymbol{T}_{i+d-1} \rangle$ of $\triangle$ (Fig. 2).

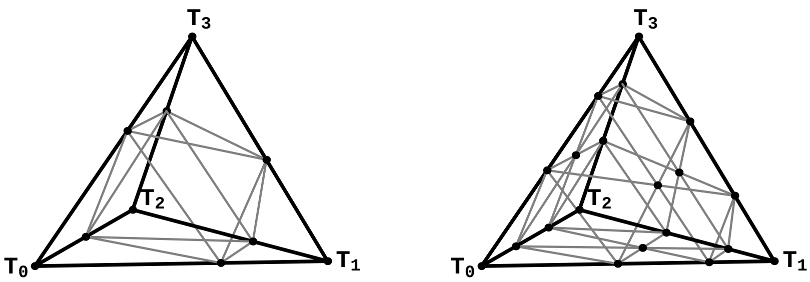

Figure 2. The 4-pencil lattices of order $n = 2, 3$ on a simplex $\triangle$ and the intersections of hyperplanes through the centers of the lattice with facets of $\triangle$.

Let $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$, $\gamma_i \in \mathbb{N}_0$, denote an index vector and let

$$|\boldsymbol{\gamma}| := \sum_{i=1}^{d} \gamma_i, \quad [j]_\alpha := \begin{cases} \dfrac{1-\alpha^j}{1-\alpha}, & \alpha \neq 1, \\ j, & \alpha = 1, \end{cases} \quad j \in \mathbb{N}_0.$$

A $(d+1)$-pencil lattice of order n on the standard simplex $\triangle_d$, as introduced in [6], is given by free parameters

$$\alpha > 0 \quad \text{and} \quad \boldsymbol{\beta} := (\beta_0, \beta_1, \ldots, \beta_d)\,,\ \beta_i > 0,\ i = 0, \ldots, d.$$

Control points $\boldsymbol{P}_i = \boldsymbol{P}_i\,(\alpha, \boldsymbol{\beta})$ of the lattice on $\triangle_d$ are determined as

$$\begin{aligned} \boldsymbol{P}_0 &= \left(\frac{\beta_1}{\beta_1 - \beta_0}, \underbrace{0, 0, \ldots, 0}_{d-1} \right), \\ \boldsymbol{P}_i &= \left(\underbrace{0, 0, \ldots, 0}_{i-1}, \frac{\beta_i}{\beta_i - \beta_{i+1}}, \frac{\beta_{i+1}}{\beta_{i+1} - \beta_i}, \underbrace{0, 0, \ldots, 0}_{d-1-i} \right), i = 1, \ldots, d-1, \end{aligned}$$

$$\boldsymbol{P}_d = \left(\underbrace{0, 0, \ldots, 0}_{d-1}, \frac{\beta_d}{\beta_d - \alpha^n \beta_0} \right). \tag{2}$$

If $\beta_{i+1} = \beta_i$ for some $0 \le i \le d$ (and $\alpha = 1$ if $i = d$), then the control point $\boldsymbol{P}_i$ is at the ideal line and the hyperplanes in the corresponding pencil are parallel. Further, lattice points are given as

$$\left(\boldsymbol{Q}_{\boldsymbol{\gamma}}\right)_{|\boldsymbol{\gamma}| \le n} := \left(\boldsymbol{Q}_{\boldsymbol{\gamma}}\left(\alpha, \boldsymbol{\beta}\right)\right)_{|\boldsymbol{\gamma}| \le n},$$

where

$$\begin{aligned} Q_{\boldsymbol{\gamma}}\left(\alpha, \boldsymbol{\beta}\right) = \frac{1}{D} \Big(& \beta_1 \alpha^{|\boldsymbol{\gamma}| - \gamma_1} \left[\gamma_1\right]_\alpha, \beta_2 \alpha^{|\boldsymbol{\gamma}| - \gamma_1 - \gamma_2} \left[\gamma_2\right]_\alpha, \\ & \beta_3 \alpha^{|\boldsymbol{\gamma}| - \gamma_1 - \gamma_2 - \gamma_3} \left[\gamma_3\right]_\alpha, \ldots, \beta_d \alpha^0 \left[\gamma_d\right]_\alpha \Big), \end{aligned} \tag{3}$$

and

$$D = \beta_0 \alpha^{|\boldsymbol{\gamma}|} \left[n - |\boldsymbol{\gamma}|\right]_\alpha + \beta_1 \alpha^{|\boldsymbol{\gamma}| - \gamma_1} \left[\gamma_1\right]_\alpha + \cdots + \beta_d \alpha^0 \left[\gamma_d\right]_\alpha.$$

Since the points $\boldsymbol{P}_i$, $\boldsymbol{T}_i$ and $\boldsymbol{T}_{i+1}$ are collinear, the barycentric coordinates $\boldsymbol{P}_{i,\triangle}$ of $\boldsymbol{P}_i$ w.r.t. $\triangle$ are particularly simple,

$$\begin{aligned} \boldsymbol{P}_{i,\triangle} &= \left(\underbrace{0, 0, \ldots, 0}_{i}, \frac{1}{1 - \xi_i}, -\frac{\xi_i}{1 - \xi_i}, \underbrace{0, 0, \ldots, 0}_{d-1-i} \right), \ i = 0, 1, \ldots, d-1, \\ \boldsymbol{P}_{d,\triangle} &= \left(-\frac{\xi_d}{1 - \xi_d}, \underbrace{0, 0, \ldots, 0}_{d-1}, \frac{1}{1 - \xi_d} \right), \end{aligned} \tag{4}$$

where

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)$$

are new free parameters of the lattice. Note that $\xi_i > 0$, since $\boldsymbol{P}_i$ is not on the line segment $\boldsymbol{T}_i \boldsymbol{T}_{i+1}$, and that a special form of barycentric coordinates is used in order to cover also the cases of parallel hyperplanes ($\xi_i = 1$). We are now able to give the relations between parameters $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$.

Theorem 1. Let $\triangle \subset \mathbb{R}^d$ be a d-simplex, and let the barycentric representation $\boldsymbol{P}_{i,\triangle}, i = 0, 1, \ldots, d$, of control points $\boldsymbol{P}_i$ of a $(d+1)$-pencil lattice on $\triangle$ be prescribed by $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)$ as in (4). Then the lattice is determined by parameters α and $\boldsymbol{\beta}$ that satisfy

$$\alpha = \sqrt[n]{\prod_{i=0}^{d} \xi_i}, \quad \frac{\beta_i}{\beta_0} = \prod_{j=0}^{i-1} \xi_j, \quad i = 1, 2, \ldots, d. \tag{5}$$

Proof. An affine map $\mathcal{A}$ carries $\triangle$ to the standard simplex $\triangle_d$, where the lattice is given by (3) with the control points (2). The i-th barycentric coordinate of a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ w.r.t. $\triangle_d = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$, $\boldsymbol{T}_i = (\delta_{i,j})_{j=1}^d$, is obtained as

$$\frac{\operatorname{vol}(\langle \overbrace{\boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_{i-2}}^{i-1}, \boldsymbol{x}, \boldsymbol{T}_i, \ldots, \boldsymbol{T}_d \rangle)}{\operatorname{vol}(\triangle_d)} = \begin{cases} 1 - \sum_{j=1}^{d} x_j, & i = 1, \\ x_{i-1}, & i \geq 2, \end{cases} \tag{6}$$

where vol is a signed volume in $\mathbb{R}^d$. So it is straightforward to compute the barycentric coordinates of (2) w.r.t. $\triangle_d$. The inverse map $\mathcal{A}^{-1}$ brings the control points (2) as well as the lattice from $\triangle_d$ back to $\triangle$. But barycentric coordinates are affinely invariant, so the barycentric coordinates of transformed control points w.r.t. $\triangle$ do not change and are given by (4). Therefore

$$\begin{aligned} \frac{\beta_1}{\beta_1 - \beta_0} &= -\frac{\xi_0}{1 - \xi_0}, \\ \frac{\beta_i}{\beta_i - \beta_{i+1}} &= \frac{1}{1 - \xi_i}, \quad i = 1, 2, \ldots, d-1, \\ \frac{\beta_d}{\beta_d - \alpha^n \beta_0} &= \frac{1}{1 - \xi_d}. \end{aligned}$$

This describes the system of $d+1$ equations for $d+1$ unknowns

$$\alpha, \quad \frac{\beta_i}{\beta_0}, \ i = 1, 2, \ldots, d.$$

Since the solution is given by (5), the proof is completed. □

Note that, in contrast to parameters $\boldsymbol{\beta}$, the introduced parameters $\boldsymbol{\xi}$ have a clear geometric meaning, and can be used as shape parameters of the lattice (see [4] and Fig. 3, e.g.).

Let us shorten the notation with the following operator

$$W : \mathbb{R}_+^k \to \mathbb{R}_+^k, \quad W\boldsymbol{x} := \frac{1}{\boldsymbol{1}^T \boldsymbol{x}} \boldsymbol{x}, \quad W\boldsymbol{0} := \boldsymbol{0}.$$

Corollary 1. The barycentric coordinates of a $(d+1)$-pencil lattice of order n on $\triangle = \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \ldots, \boldsymbol{T}_d \rangle$ w.r.t. $\triangle$ are determined by $\boldsymbol{\xi}$ as

$$B\boldsymbol{\gamma} = W\left(\alpha^{n-\gamma_0} \left[\gamma_0\right]_\alpha, \xi_0 \alpha^{n-\gamma_0-\gamma_1} \left[\gamma_1\right]_\alpha, \ldots, \xi_0 \cdots \xi_{d-1} \alpha^0 \left[\gamma_d\right]_\alpha\right), \tag{7}$$

where $\boldsymbol{\gamma} \in \mathbb{N}_0^{d+1}$, $|\boldsymbol{\gamma}| = n$, and $\alpha^n = \prod_{i=0}^d \xi_i$.

Proof. By applying (6) and (5) to (3), one obtains

$$B_{\boldsymbol{\gamma}} = W\left(\alpha^{|\boldsymbol{\gamma}|}\left[n-|\boldsymbol{\gamma}|\right]_\alpha, \xi_0\alpha^{|\boldsymbol{\gamma}|-\gamma_1}\left[\gamma_1\right]_\alpha, \ldots, \xi_0\cdots\xi_{d-1}\alpha^0\left[\gamma_d\right]_\alpha\right),$$

where $\boldsymbol{\gamma} \in \mathbb{N}_0^d$, $|\boldsymbol{\gamma}| \leq n$. To make the formula more symmetric, we can, without loss of generality, replace the index vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$ by the index vector $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)$, $\gamma_0 := n - |\boldsymbol{\gamma}|$, and (7) follows. □

Note that ξ_d appears in (7) implicitly, since $\alpha^n = \prod_{i=0}^d \xi_i$.

3. Lagrange interpolation

One of the main advantages of lattices is that they provide an explicit construction of Lagrange basis polynomials as products of linear factors. Therefore some simplifications can be done in order to decrease the amount of work needed. Details on how this can be done in barycentric coordinates are summarized in the following theorem.

Theorem 2. Let a $(d+1)$-pencil lattice of order n on $\triangle$ be given in the barycentric form by parameters $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_d)$ as in Corollary 1 and let data

$$f_{\boldsymbol{\gamma}} \in \mathbb{R},\ \boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^{d+1},\ |\boldsymbol{\gamma}| = n,$$

be prescribed. The polynomial $p_n \in \Pi_n^d$ that interpolates the data $\left(f_{\boldsymbol{\gamma}}\right)_{|\boldsymbol{\gamma}|=n}$ at the points $\left(B_{\boldsymbol{\gamma}}\right)_{|\boldsymbol{\gamma}|=n}$ is given as

$$p_n(\boldsymbol{x}) = \sum_{|\boldsymbol{\gamma}|=n} f_{\boldsymbol{\gamma}}\,\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^{d+1}, \sum_{i=0}^d x_i = 1. \tag{8}$$

The Lagrange basis polynomial $\mathcal{L}_{\boldsymbol{\gamma}}$ is a product of hyperplanes, i.e.,

$$\mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}) = \prod_{i=0}^{d}\prod_{j=0}^{\gamma_i-1} h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x}), \tag{9}$$

where

$$h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x}) := \frac{c_{i,\boldsymbol{\gamma}}}{1-\dfrac{[n-\gamma_i]_\alpha}{[n-j]_\alpha}}\left(x_i + \left(1-\frac{[n]_\alpha}{[n-j]_\alpha}\right)q_i(\boldsymbol{x})\right), \tag{10}$$

and

$$q_i(\boldsymbol{x}) := \sum_{t=i+1}^{i+d}\frac{1}{\prod\limits_{k=i}^{t-1}\xi_k}x_t, \quad c_{i,\boldsymbol{\gamma}} := \left(1-\frac{[n-\gamma_i]_\alpha}{[n]_\alpha}\right)\frac{1}{\left(B_{\boldsymbol{\gamma}}\right)_i}.$$

Proof. Let $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_d)$, $|\boldsymbol{\gamma}| = n$, be a given index vector. Let us construct the Lagrange basis polynomial $\mathcal{L}_{\boldsymbol{\gamma}}$ that satisfies

$$\mathcal{L}_{\boldsymbol{\gamma}}\left(B_{\boldsymbol{\gamma}'}\right) = \begin{cases} 1, & \boldsymbol{\gamma}' = \boldsymbol{\gamma}, \\ 0, & \boldsymbol{\gamma}' \neq \boldsymbol{\gamma}. \end{cases}$$

Based upon the GC approach, this polynomial can be found as a product of hyperplanes $H_{i,j,\boldsymbol{\gamma}}$ with the equations $h_{i,j,\boldsymbol{\gamma}} = 0$, $j = 0, 1, \ldots, \gamma_i - 1$, $i = 0, 1, \ldots, d$, where $H_{i,j,\boldsymbol{\gamma}}$ contains the lattice points indexed by

$$(\eta_0, \eta_1, \ldots, \eta_{i-1}, j, \eta_{i+1}, \ldots, \eta_d), \quad \sum_{\substack{k=0 \\ k \neq i}}^{d} \eta_k = n - j.$$

Quite clearly, the total degree of such a polynomial is bounded above by $\sum_{i=0}^{d} \sum_{j=0}^{\gamma_i - 1} 1 = n$. But, for fixed i and $j, 0 \leq j \leq \gamma_i - 1$, a hyperplane $H_{i,j,\boldsymbol{\gamma}}$ is determined by the center $\boldsymbol{C}_{i+1}$ and the point $\boldsymbol{U}$ at the edge $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1} \rangle$ with the barycentric coordinates w.r.t. $\langle \boldsymbol{T}_i, \boldsymbol{T}_{i+1} \rangle$ equal to

$$\left(\frac{[n]_\alpha - [n-j]_\alpha}{[n]_\alpha - [n-j]_\alpha + [n-j]_\alpha \, \xi_i}, \frac{[n-j]_\alpha \, \xi_i}{[n]_\alpha - [n-j]_\alpha + [n-j]_\alpha \, \xi_i} \right).$$

The equation $h_{i,j,\boldsymbol{\gamma}} = 0$ is by (1) given as

$$\det\left(\boldsymbol{x}, \boldsymbol{U}, \boldsymbol{P}_{i+1}, \boldsymbol{P}_{i+2}, \ldots, \boldsymbol{P}_{i+d-1}\right) = 0. \tag{11}$$

Let us recall the barycentric representation (4) of $\boldsymbol{P}_i$. A multiplication of the matrix in (11) by a nonsingular diagonal matrix

$$\operatorname{diag}\left(1, [n]_\alpha - [n-j]_\alpha + [n-j]_\alpha \, \xi_i, 1 - \xi_{i+1}, 1 - \xi_{i+2}, \ldots, 1 - \xi_{i+d-1}\right),$$

and a circular shift of columns simplifies the equation (11) to

$$\det \begin{pmatrix} x_i & x_{i+1} & \cdots & \cdots & & x_{i+d} \\ [n]_\alpha - [n-j]_\alpha & [n-j]_\alpha \, \xi_i & & & & \\ & 1 & -\xi_{i+1} & & & \\ & & 1 & -\xi_{i+2} & & \\ & & & \ddots & \ddots & \\ & & & & 1 & -\xi_{i+d-1} \end{pmatrix} = 0.$$

A straightforward evaluation of the determinant yields

$$[n-j]_\alpha \left(\prod_{k=i}^{i+d-1} \xi_k \right) x_i - \left([n]_\alpha - [n-j]_\alpha\right) \sum_{t=i+1}^{i+d} x_t \left(\prod_{k=t}^{i+d-1} \xi_k \right) = 0,$$

and further

$$\begin{aligned}[n-j]_\alpha x_i - \left([n]_\alpha - [n-j]_\alpha\right) \sum_{t=i+1}^{i+d} \frac{1}{\prod_{k=i}^{t-1} \xi_k} x_t &= \\ = [n-j]_\alpha x_i - \left([n]_\alpha - [n-j]_\alpha\right) q_i(\boldsymbol{x}) &= 0.\end{aligned}$$

If $j > 0$, this gives also a relation

$$q_i(\boldsymbol{x}) = \frac{[n-j]_\alpha}{[n]_\alpha - [n-j]_\alpha} x_i$$

for a particular $\boldsymbol{x}$ that satisfies (11). But $0 \le j < \gamma_i \le n$, so

$$q_i(B\boldsymbol{\gamma}) = \frac{[n-\gamma_i]_\alpha}{[n]_\alpha - [n-\gamma_i]_\alpha} (B\boldsymbol{\gamma})_i$$

simplifies the equation of the hyperplane

$$h_{i,j,\boldsymbol{\gamma}}(\boldsymbol{x}) = \frac{[n-j]_\alpha x_i - \left([n]_\alpha - [n-j]_\alpha\right) q_i(\boldsymbol{x})}{[n-j]_\alpha (B\boldsymbol{\gamma})_i - \left([n]_\alpha - [n-j]_\alpha\right) q_i(B\boldsymbol{\gamma})} \tag{12}$$

to the assertion (10). Consider now an index vector $\boldsymbol{\gamma}' \neq \boldsymbol{\gamma}$. Since $|\boldsymbol{\gamma}'| = |\boldsymbol{\gamma}| = n$, there must exist an index $i, 0 \le i \le d$, such that $\gamma_i' < \gamma_i$. So γ_i' appears as one of the indices j in the product (9). Thus $\mathcal{L}_{\boldsymbol{\gamma}}\left(B_{\boldsymbol{\gamma}'}\right) = 0$. But from (12) one deduces $\mathcal{L}_{\boldsymbol{\gamma}}(B\boldsymbol{\gamma}) = 1$, and the proof is completed. $\square$

Note that $h_{i,j,\boldsymbol{\gamma}}$ in (10) depends only on the i-th component of the corresponding point $B\boldsymbol{\gamma}$. This is not obvious from the classical representation of Lagrange basis polynomial, and is vital for an efficient computation.

Let us now give some remarks on how to organize the computations. Let $\triangle$ be a simplex in $\mathbb{R}^d$ with vertices $\boldsymbol{V}_i$, $i = 0, \ldots, d$, given in Cartesian coordinates. The Cartesian coordinates of the lattice points are

$$(Q\boldsymbol{\gamma})_i = \sum_{j=0}^{d} (B\boldsymbol{\gamma})_{j+1} (V_j)_i, \qquad i = 1, 2, \ldots, d.$$

If one is looking for an explicit expression of the Lagrange interpolating polynomial $p_n(\boldsymbol{u})$, $\boldsymbol{u} = (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d$, over the given $\triangle$, the symbolic system

$$\sum_{j=0}^{d} (\boldsymbol{V}_j)_i x_j = u_i, \quad i = 1, 2, \ldots, d, \quad \sum_{j=0}^{d} x_j = 1,$$

has to be solved. This leads to the solution of the form

$$x_j = g_j(\boldsymbol{u}), \quad j = 0, 1, \ldots, d. \tag{13}$$

After inserting (13) into (8), one obtains the interpolating polynomial $p_n(\boldsymbol{u})$ over the lattice on $\triangle$.

Let now $\boldsymbol{U}$ be an arbitrary point in $\triangle$. We would like to efficiently evaluate the interpolating polynomial p_n at the point $\boldsymbol{U}$. The previous observation gives one of the possible ways, but one can use a more efficient method by computing the barycentric coordinates $\widetilde{\boldsymbol{U}}$ of the point $\boldsymbol{U}$ w.r.t. $\triangle$. They can be obtained by solving a linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{V}_0 & \boldsymbol{V}_1 & \cdots & \boldsymbol{V}_d \end{pmatrix} \widetilde{\boldsymbol{U}} = \begin{pmatrix} 1 \\ \boldsymbol{U} \end{pmatrix}.$$

By inserting $\widetilde{\boldsymbol{U}}$ into (8), the desired value $p_n(\boldsymbol{U})$ is obtained.

It can be shown that the computational cost needed for the derivation of the lattice points on an arbitrary simplex $\triangle$ and for the evaluation of the Lagrange polynomial over $\triangle$ are roughly equivalent to the one presented in [6].

4. Example

In this section the results of the paper will be illustrated by an example for the planar case, i.e., $d = 2$, where a 2-simplex is a triangle $\triangle := \langle \boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2 \rangle$. Let a 3-pencil lattice of order 3 on $\triangle$ be given by control points $\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2 \in \mathbb{R}^2$ and let the data $f_{\boldsymbol{\gamma}} \in \mathbb{R}$, $|\boldsymbol{\gamma}| = 3$, at lattice points be prescribed. Barycentric coordinates w.r.t. $\triangle$ of the lattice on $\triangle$ are given by free parameters ξ_0, ξ_1 and ξ_2 as

$$B_{\boldsymbol{\gamma}} = W\left(\alpha^{\gamma_1+\gamma_2} \left[\gamma_0\right]_\alpha, \xi_0 \alpha^{\gamma_2} \left[\gamma_1\right]_\alpha, \xi_0 \xi_1 \left[\gamma_2\right]_\alpha\right),$$

$\alpha = \sqrt[3]{\xi_0 \xi_1 \xi_2}$, $|\boldsymbol{\gamma}| = 3$. Let us now compute the Lagrange interpolating polynomial $p_3(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, $\sum_{i=0}^{2} x_i = 1$, over the lattice that interpolates data $f_{\boldsymbol{\gamma}}$ sampled from the surface given by

$$f(\boldsymbol{\zeta}) = 3e^{-(\zeta_1^2+\zeta_2^2)} + e^{-2(\zeta_1^2+\zeta_2^2)} + 3. \tag{14}$$

By Theorem 2,

$$p_3(\boldsymbol{x}) = \sum_{|\boldsymbol{\gamma}|=3} f_{\boldsymbol{\gamma}} \mathcal{L}_{\boldsymbol{\gamma}}(\boldsymbol{x}),$$

where $\mathcal{L}_{\boldsymbol{\gamma}}$, $|\boldsymbol{\gamma}| = 3$, is equal to one of the following cases

$$\begin{array}{ll}
\prod_{j=0}^{2} h_{i,j}, & \exists\, i,\ \gamma_i = 3, \\
\left(\frac{\left(\alpha^2+\alpha+\prod_{k=i}^{j-1}\xi_k\right)^3}{\alpha^2(1+\alpha+\alpha^2)\left(\prod_{k=i}^{j-1}\xi_k\right)}\right)\left(h_{j,0}\prod_{k=0}^{1} h_{i,k}\right), & \exists\, i<j,\ \gamma_i = 2,\ \gamma_j = 1, \\
\left(\frac{\left(\alpha^2+(\alpha+1)\prod_{k=j}^{i-1}\xi_k\right)^3}{\alpha^2(1+\alpha+\alpha^2)\left(\prod_{k=j}^{i-1}\xi_k\right)^2}\right)\left(h_{j,0}\prod_{k=0}^{1} h_{i,k}\right), & \exists\, i>j,\ \gamma_i = 2,\ \gamma_j = 1, \\
\left(\frac{\left(\alpha^2+\xi_0(\alpha+\xi_1)\right)^3}{\alpha^3\xi_0^2\xi_1}\right)\left(\prod_{i=0}^{2} h_{i,0}\right), & \boldsymbol{\gamma} = (1,1,1),
\end{array}$$

with

$$h_{i,j}(\boldsymbol{x}) = x_i + \frac{[n-j]_\alpha - [n]_\alpha}{[n-j]_\alpha\ \xi_i}\, x_{i+1} + \frac{[n-j]_\alpha - [n]_\alpha}{[n-j]_\alpha\ \xi_i\,\xi_{i+1}}\, x_{i+2}.$$

Combining these with the results in [4], where 3-pencil lattices on triangulations have been studied, a continuous piecewise polynomial interpolant over the triangulation can be obtained (Fig. 3).

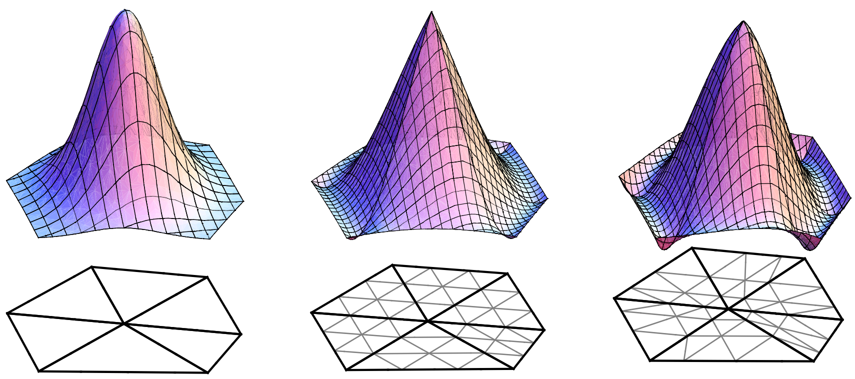

Figure 3. The surface (14) over a star ([5]) with two different continuous piecewise polynomial interpolants.

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