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Interpolation scheme for planar cubic G^2 spline curves

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Abstract In this paper a method for interpolating planar data points by cubic G^2 splines is presented. A spline is composed of polynomial segments that interpolate two data points, tangent directions and curvatures at these points. Necessary and sufficient, purely geometric conditions for the existence of such a polynomial interpolant are derived. The obtained results are extended to the case when the derivative directions and curvatures are not prescribed as data, but are obtained by some local approximation or implied by shape requirements. As a result, the G^2 spline is constructed entirely locally.

Keywords G^2 spline · polynomial curve · geometric interpolation · existence of solution · shape · algorithm

Mathematics Subject Classification (2000) 65D05 · 65D07 · 65D17

1 Introduction

One of the problems encountered often in CAGD applications is to find a smooth curve of a proper shape that interpolates given data points. It may be important that the interpolant depends on geometric quantities such as data points, tangent directions and curvatures only. In this case, the geometric interpolation schemes are considered as a proper tool to be used. The main property of such schemes is that the parameters at which the points are interpolated, the lengths of tangents, etc., are not prescribed in advance but are considered as unknowns. The additional freedom is used to interpolate more data which results in a higher approximation order and a more desirable shape obtained.

However, geometric interpolation schemes include nonlinear problems. The questions of the existence of the solution, the approximation order and an efficient implementation may turn out to be a hard nut to crack. Most of the results are thus

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obtained by the asymptotic approach ([2], [7], [8], [3], [5], [6], [19], [9], [20], [17], [21], etc.). But, in practical applications the results obtained by the asymptotic analysis are not adequate. For robust algorithms geometric conditions for the existence of the interpolant should be known in advance. Some of results of this nature can be found in [13], [14], [15], [18], [16], [11], [12], [4], [10], etc.

In this paper the interpolation of planar data points by a cubic G^2 spline curve is considered. On each polynomial segment two points, tangent directions and curvatures are interpolated. Directions and curvatures may not be explicitly prescribed as a data. If not, a way they can be chosen so that the spline exists is presented. For this reason we first revisit a well known interpolation problem that started the theory of geometric interpolation, the BHS scheme proposed by C. de Boor, K. Höllig, and M. Sabin in [2]. Here, the analysis is carried over based entirely on geometric considerations rather then by the original asymptotic approach. The existence analysis of the BHS scheme is extended with precise sufficient and necessary conditions on data points, tangent directions and curvatures. Furthermore, based on the obtained results a local interpolation scheme is derived in such a way that the resulting spline is G^2 continuous. It offers a freedom of choice between better approximation order and a local shape control based on the choice of curvatures. The numerical work is comparable to linear schemes since it requires the solution of one quartic polynomial equation per each segment.

The outline of the paper is the following. Section 2 introduces the interpolation problem. In the next section, a single segment case is considered and geometric conditions for the existence of the interpolant are derived. In Section 4 the obtained results are used to construct a cubic G^2 spline. The paper is concluded with numerical examples and approximation order considerations.

2 Interpolation problem

Suppose that a sequence of data points

$$\boldsymbol{T}_i \in \mathbb{R}^2, \quad i = 0, 1, 2, \ldots, m, \quad \boldsymbol{T}_i \neq \boldsymbol{T}_{i+1}, \tag{1}$$

is given. The task is to find a cubic G^2 spline curve

$$\boldsymbol{S} : [a, b] \to \mathbb{R}^2$$

composed of cubic polynomials between two adjacent points (breakpoints). Data points are clearly not enough to construct a spline. On each polynomial segment there are still four degrees of freedom left. If also tangent directions and curvatures at the breakpoints are prescribed, then the spline is completely determined and G^2 by a construction.

Since derivative directions and curvatures are independent of the parameterization, the spline can be given in a piecewise representation. The interpolation problem thus reduces to the following one: for two data points $\boldsymbol{T}_{\ell-1}$ and $\boldsymbol{T}_\ell$, two normalized tangent directions $\boldsymbol{d}_{\ell-1}$ and $\boldsymbol{d}_\ell$, and two values $\kappa_{\ell-1}$ and κ_ℓ find a cubic polynomial curve $\boldsymbol{p}_\ell : [0, 1] \to \mathbb{R}^2$, such that

$$\begin{aligned}
&\boldsymbol{p}_\ell(0) = \boldsymbol{T}_{\ell-1}, \quad \boldsymbol{p}_\ell(1) = \boldsymbol{T}_\ell, \\
&\boldsymbol{p}_\ell'(0) = \lambda_{\ell,0}\, \boldsymbol{d}_{\ell-1}, \quad \boldsymbol{p}_\ell'(1) = \lambda_{\ell,1}\, \boldsymbol{d}_\ell, \qquad \ell = 1, 2, \ldots, m, \\
&\kappa(0; \ell) = \kappa_{\ell-1}, \quad \kappa(1; \ell) = \kappa_\ell,
\end{aligned} \tag{2}$$

where the parameters $\lambda_{\ell,0}$, $\lambda_{\ell,1}$ are the unknown lengths of the tangents that must be positive,

$$\lambda_{\ell,0} > 0, \quad \lambda_{\ell,1} > 0, \quad \ell = 1, 2, \ldots, m, \tag{3}$$

and a function

$$\kappa(\cdot;\ell) : [0,1] \to \mathbb{R}, \quad \kappa(\cdot;\ell) = \frac{\boldsymbol{p}_\ell' \times \boldsymbol{p}_\ell''}{\|\boldsymbol{p}_\ell'\|^3},$$

is a curvature of a polynomial $\boldsymbol{p}_\ell$. A symbol $\times$ denotes a standard planar vector product, i.e., $(x_1, x_2)^T \times (y_1, y_2)^T := x_1 y_2 - x_2 y_1$, and $\|.\|$ is the Euclidean norm. The parameters that satisfy (3) are called *admissible* parameters and the solution of (2) with admissible parameters is called *admissible* solution. Note that (2) is a nonlinear system of ten equations for ten unknowns, eight unknown coefficients of $\boldsymbol{p}_\ell$ and two unknown lengths $\lambda_{\ell,0}$, $\lambda_{\ell,1}$. The system can be reduced to only two nonlinear equations for $\lambda_{\ell,0}$ and $\lambda_{\ell,1}$. Once the lengths are determined the rest of the problem is linear. The coefficients can be obtained by any standard interpolation scheme componentwise.

3 The analysis of one segment

Since the problem (2) is nonlinear the admissible solutions may not exist in general. In this section a polynomial problem (2) is analysed and simple necessary and sufficient conditions for the existence of the solution are derived. The exact number of admissible solutions is stated too.

Let us express the polynomial curve $\boldsymbol{p}_\ell$ in the Bézier form

$$\boldsymbol{p}_\ell(t) = \sum_{i=0}^{3} \boldsymbol{b}_i \; B_{3,i}(t),$$

where $\boldsymbol{b}_i \in \mathbb{R}^2$ are the unknown control points and $B_{n,i}(t) := \binom{n}{i} t^i (1-t)^{n-i}$ are standard Bernstein basis polynomials. From (2) it follows

$$\boldsymbol{b}_0 = \boldsymbol{T}_{\ell-1}, \quad \boldsymbol{b}_1 = \boldsymbol{T}_{\ell-1} + \frac{1}{3}\lambda_{\ell,0}\boldsymbol{d}_{\ell-1}, \quad \boldsymbol{b}_2 = \boldsymbol{T}_\ell - \frac{1}{3}\lambda_{\ell,1}\boldsymbol{d}_\ell, \quad \boldsymbol{b}_3 = \boldsymbol{T}_\ell,$$

and

$$\begin{aligned}
\kappa_{\ell-1} &= \frac{6}{\lambda_{\ell,0}^2 \|\boldsymbol{d}_{\ell-1}\|^3} \boldsymbol{d}_{\ell-1} \times \left(\Delta \boldsymbol{T}_{\ell-1} - \frac{\lambda_{\ell,1}}{3} \boldsymbol{d}_\ell \right), \\
\kappa_\ell &= \frac{6}{\lambda_{\ell,1}^2 \|\boldsymbol{d}_\ell\|^3} \left(\Delta \boldsymbol{T}_{\ell-1} - \frac{\lambda_{\ell,0}}{3} \boldsymbol{d}_{\ell-1} \right) \times \boldsymbol{d}_\ell,
\end{aligned}$$

where Δ denotes the forward finite difference. Since $\|\boldsymbol{d}_{\ell-1}\| = \|\boldsymbol{d}_\ell\| = 1$, the unknown tangent lengths $\lambda_{\ell,0}$ and $\lambda_{\ell,1}$ are determined by prescribed curvatures as a solution of the nonlinear system

$$\begin{aligned}
\lambda_{\ell,0}^2 &= \frac{6}{\kappa_{\ell-1}} \left(D_{\ell,0} - \frac{\lambda_{\ell,1}}{3} D_{\ell,2} \right), \\
\lambda_{\ell,1}^2 &= \frac{6}{\kappa_\ell} \left(D_{\ell,1} - \frac{\lambda_{\ell,0}}{3} D_{\ell,2} \right),
\end{aligned} \tag{4}$$

where

$$D_{\ell,0} := \boldsymbol{d}_{\ell-1} \times \Delta\boldsymbol{T}_{\ell-1}, \quad D_{\ell,1} := \Delta\boldsymbol{T}_{\ell-1} \times \boldsymbol{d}_{\ell}, \quad D_{\ell,2} := \boldsymbol{d}_{\ell-1} \times \boldsymbol{d}_{\ell}.$$

Let us assume that $D_{\ell,0}, D_{\ell,1}, D_{\ell,2} \neq 0$, i.e., tangent directions are not collinear with $\Delta\boldsymbol{T}_{\ell-1}$. Following the path applied in [2] new unknowns ρ_0 and ρ_1 are introduced by

$$\rho_0 := \frac{1}{3}\lambda_{\ell,0}\frac{D_{\ell,2}}{D_{\ell,1}}, \quad \rho_1 := \frac{1}{3}\lambda_{\ell,1}\frac{D_{\ell,2}}{D_{\ell,0}}. \tag{5}$$

Equations (4) simplify to

$$\boldsymbol{F}(\rho_0, \rho_1; R_0, R_1) := \left(\rho_0 - 1 + R_1\rho_1^2, \rho_1 - 1 + R_0\rho_0^2\right) = \boldsymbol{0}, \tag{6}$$

where

$$R_0 := \frac{3}{2}\kappa_{\ell-1}\frac{1}{D_{\ell,0}}\left(\frac{D_{\ell,1}}{D_{\ell,2}}\right)^2, \quad R_1 := \frac{3}{2}\kappa_{\ell}\frac{1}{D_{\ell,1}}\left(\frac{D_{\ell,0}}{D_{\ell,2}}\right)^2 \tag{7}$$

depend only on data. By some simple transformations of $\boldsymbol{F}$ we obtain an equivalent system

$$R_0^2R_1\rho_0^4 - 2R_0R_1\rho_0^2 + \rho_0 + R_1 - 1 = 0, \quad R_0\rho_0^2 + \rho_1 - 1 = 0, \tag{8}$$

which indicates that the numerical work required is to solve a quartic polynomial equation. In practice, one should work with (8), but for theoretical purpose system (6) will be used. Clearly, not all the solutions are admissible. From (5) it follows that the solution (ρ_0, ρ_1) is admissible iff it lies in the open set

$$\mathcal{D} := \left\{(\rho_0, \rho_1) : \quad \operatorname{sign}(\rho_0) = \operatorname{sign}\bigl(D_{\ell,1}D_{\ell,2}\bigr),\ \operatorname{sign}(\rho_1) = \operatorname{sign}\bigl(D_{\ell,0}D_{\ell,2}\bigr)\right\}.$$

Four different cases will thus be distinguished:

$$\mathcal{D}_1 := \{(\rho_0, \rho_1) : \quad \rho_0 > 0,\ \rho_1 > 0\}, \qquad \mathcal{D}_2 := \{(\rho_0, \rho_1) : \quad \rho_0 > 0,\ \rho_1 < 0\},$$
$$\mathcal{D}_3 := \{(\rho_0, \rho_1) : \quad \rho_0 < 0,\ \rho_1 > 0\}, \qquad \mathcal{D}_4 := \{(\rho_0, \rho_1) : \quad \rho_0 < 0,\ \rho_1 < 0\}.$$

The following two Lemmas reveal when the Jacobian $J_{\boldsymbol{F}}$ at the solution is singular, and when the solution approaches the boundary $\partial\mathcal{D}$.

Lemma 1 *If the determinant of the Jacobian $J_{\boldsymbol{F}}(\rho_0, \rho_1; R_0, R_1)$ vanishes at the solution then the relation*

$$r(R_0, R_1) := 256\left(R_0^2R_1^2 - R_0^2R_1 - R_0R_1^2\right) + 288\,R_0R_1 - 27 = 0$$

must hold.

Proof It is straightforward to compute the determinant of the Jacobian of $\boldsymbol{F}$:

$$\det J_{\boldsymbol{F}} = 1 - 4R_0R_1\rho_0\rho_1.$$

Let us supplement the system $\boldsymbol{F}(\rho_0,\rho_1;R_0,R_1)=\mathbf{0}$ with the equation $1-4R_0R_1\rho_0\rho_1=0$. The Gröbner basis of this new system with respect to lexicographic ordering $\rho_0>\rho_1$ yields an equivalent system

$$
\begin{aligned}
-27+288R_0R_1-256R_0^2R_1-256R_0R_1^2+256R_0^2R_1^2&=0,\\
15-16R_0-16R_1+16R_0R_1-9\rho_0+12R_0\rho_0&=0,\\
36-108R_1+64R_0R_1+64R_1^2-64R_0R_1^2-27\rho_0+36R_1\rho_0&=0,\\
288R_1-256R_0R_1-256R_1^2+256R_0R_1^2-108\rho_0+81\rho_0^2&=0,\\
-27+16R_0+16R_1-16R_0R_1+9\rho_0+9\rho_1&=0.
\end{aligned}
$$

The first element of the Gröbner basis that is independent of the unknowns gives the necessary relation between R_0 and R_1.

Lemma 2 *The solution of the system $\boldsymbol{F}(\rho_0,\rho_1;R_0,R_1)=\mathbf{0}$ lies on the boundary $\rho_0=0$ iff $R_1=1$ and $\rho_1=1$. Similarly, the solution lies on the boundary $\rho_1=0$ iff $R_0=1$ and $\rho_0=1$.*

The proof of Lemma 2 is straightforward. By Lemma 1 and Lemma 2, the (R_0,R_1)–plane is decomposed into fifteen connected regions $\mathcal{A}_i:=\mathcal{A}_i(R_0,R_1)$ as shown in Fig. 1:

$$
\begin{aligned}
\mathcal{A}_1&:=\{(R_0,R_1):\quad R_0>1,\ R_1>1\},\\
\mathcal{A}_2&:=\{(R_0,R_1):\quad 3/4<R_0<1,\ 3/4<R_1<1,\ r(R_0,R_1)>0\},\\
\mathcal{A}_3&:=\{(R_0,R_1):\quad 0<R_0<1,\ 0<R_1<1\}\setminus\mathcal{A}_2,\\
\mathcal{A}_4&:=\{(R_0,R_1):\quad R_0<0,\ R_1<0,\ r(R_0,R_1)<0\},\\
\mathcal{A}_5&:=\{(R_0,R_1):\quad R_0<0,\ R_1<0\}\setminus\mathcal{A}_4,\\
\mathcal{A}_6&:=\{(R_0,R_1):\quad 3/4<R_0<1,\ R_1>1,\ r(R_0,R_1)>0\},\\
\mathcal{A}_7&:=\{(R_0,R_1):\quad 0<R_0<1,\ R_1>1\}\setminus\mathcal{A}_6,\\
\mathcal{A}_8&:=\{(R_0,R_1):\quad R_0<0,\ R_1>1,\ r(R_0,R_1)<0\},\\
\mathcal{A}_9&:=\{(R_0,R_1):\quad R_0<0,\ 0<R_1<1\},\\
\mathcal{A}_{10}&:=\{(R_0,R_1):\quad R_0<0,\ R_1>1\}\setminus\mathcal{A}_8,\\
\mathcal{A}_{i+5}&:=\mathcal{A}_i(R_1,R_0),\quad i=6,7,8,9,10.
\end{aligned}
$$

The main existence theorem is the following.

Theorem 1 *The number of solutions of the system $\boldsymbol{F}(\rho_0,\rho_1;R_0,R_1)=\mathbf{0}$ that lie in $\mathcal{D}_k$, $k=1,2,3,4$, is constant for $(R_0,R_1)\in\mathcal{A}_i$, $i=1,2,\ldots,15$, and it is given in Table 1.*

Proof The proof is in two parts. First part: let us show that for some particularly chosen points in each region $\mathcal{A}_i$ the number of solutions in $\mathcal{D}_k$ matches the numbers in Table 1. From the symmetry in equations and the symmetry between regions it is enough to consider only $\mathcal{A}_i$, $i=1,2,\ldots,10$. Table 2 shows particularly chosen $(R_0^*,R_1^*)\in\mathcal{A}_i$. Consider for example $(R_0^*,R_1^*)=(2,2)$. The equivalent system (8) reads

$$(\rho_0+1)(2\rho_0-1)(4\rho_0^2-2\rho_0-1)=0,\quad 2\rho_0^2+\rho_1-1=0,$$

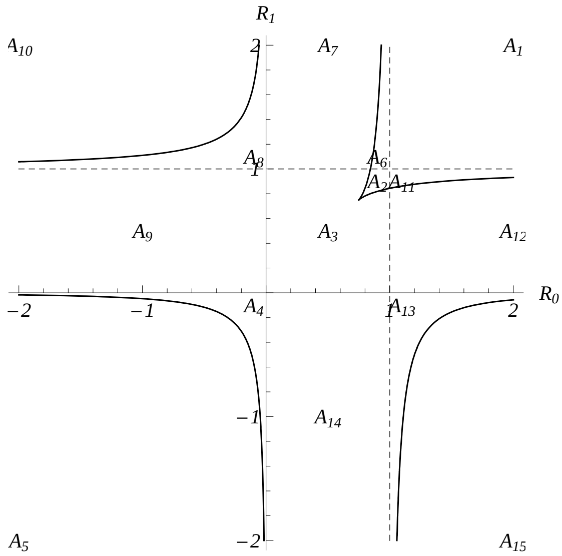

Fig. 1 The decomposition of (R_0, R_1)–plane into 15 connected regions $\mathcal{A}_i$. The number of admissible solutions is constant on each $\mathcal{A}_i$.

Table 1 The number of solutions of the system $\boldsymbol{F}(\rho_0, \rho_1; R_0, R_1) = \mathbf{0}$ in $\mathcal{D}_k$ for $(R_0, R_1) \in \mathcal{A}_i$.

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$
$\mathcal{D}_1$	1	3	1	2	0	2	0	0
$\mathcal{D}_2$	1	0	0	0	0	0	0	0
$\mathcal{D}_3$	1	0	0	0	0	1	1	2
$\mathcal{D}_4$	1	1	1	0	0	1	1	0

	$\mathcal{A}_9$	$\mathcal{A}_{10}$	$\mathcal{A}_{11}$	$\mathcal{A}_{12}$	$\mathcal{A}_{13}$	$\mathcal{A}_{14}$	$\mathcal{A}_{15}$	
$\mathcal{D}_1$	1	0	2	0	0	1	0	
$\mathcal{D}_2$	0	0	1	1	2	1	0	
$\mathcal{D}_3$	1	0	0	0	0	0	0	
$\mathcal{D}_4$	0	0	1	1	0	0	0	

and the solutions (ρ_0, ρ_1) are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \ \left(\frac{1}{4}(1 \pm \sqrt{5}), \frac{1}{4}(1 \mp \sqrt{5})\right), \ (-1, -1),$$

i.e., one solution in each $\mathcal{D}_k$. Similarly, with the use of some computer algebra facility, the solutions can analytically be computed for other (R_0^*, R_1^*) and numbers in Table 1 confirmed.

Table 2 Particularly chosen data points.

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_5$
(R_0^*, R_1^*)	$(2, 2)$	$\left(\frac{9}{10}, \frac{9}{10}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(-\frac{1}{10}, -\frac{1}{10}\right)$	$(-2, -2)$

	$\mathcal{A}_6$	$\mathcal{A}_7$	$\mathcal{A}_8$	$\mathcal{A}_9$	$\mathcal{A}_{10}$
(R_0^*, R_1^*)	$\left(\frac{9}{10}, \frac{11}{10}\right)$	$\left(\frac{1}{2}, 2\right)$	$\left(-\frac{1}{10}, \frac{11}{10}\right)$	$\left(-1, \frac{1}{2}\right)$	$(-2, 2)$

Th second part: let us connect the general $(R_0, R_1) \in \mathcal{A}_i$ with the particular $(R_0^*, R_1^*) \in \mathcal{A}_i$, defined in Table 2, by a homotopy $\boldsymbol{H}(\rho_0, \rho_1; \lambda) : \mathcal{D}_k \times [0,1] \to \mathbb{R}$, such that

$$\boldsymbol{H}(\rho_0, \rho_1; 0) := \boldsymbol{F}(\rho_0, \rho_1; R_0^*, R_1^*), \quad \boldsymbol{H}(\rho_0, \rho_1; 1) := \boldsymbol{F}(\rho_0, \rho_1; R_0, R_1).$$

There clearly exist functions $\gamma_0 : [0,1] \to \mathbb{R}$ and $\gamma_1 : [0,1] \to \mathbb{R}$ that satisfy

$$\gamma_j(0) = R_j^*, \quad \gamma_j(1) = R_j, \quad j = 0, 1, \quad \text{and} \quad \{(\gamma_0(\lambda), \gamma_1(\lambda)), \quad \lambda \in [0,1]\} \subset \mathcal{A}_i.$$

If a homotopy $\boldsymbol{H}$ is defined as

$$\boldsymbol{H}(\rho_0, \rho_1; \lambda) := \boldsymbol{F}(\rho_0, \rho_1; \gamma_0(\lambda), \gamma_1(\lambda)), \quad \lambda \in [0,1],$$

then by Lemma 1 and Lemma 2, the Jacobian of $\boldsymbol{H}(\rho_0, \rho_1; \lambda)$ is nonzero and the solutions of $\boldsymbol{H}(\rho_0, \rho_1; \lambda) = 0$ are bounded away from the boundary $\partial\mathcal{D}_k$ for every $\lambda \in [0,1]$. By the Brouwer's mapping degree ([1]) the number of solutions of $\boldsymbol{H}(\rho_0, \rho_1; \lambda) = 0$ in $\mathcal{D}_k$ is constant for $\lambda \in [0,1]$, i.e., the number of solutions in $\mathcal{D}_k$ of $\boldsymbol{F}(\rho_0, \rho_1; R_0, R_1)$ and $\boldsymbol{F}(\rho_0, \rho_1; R_0^*, R_1^*)$ is equal. This completes the proof.

Theorem 1 gives the sufficient and necessary conditions for the existence of the interpolant but in terms of R_i and ρ_i, $i = 0, 1$. Using (5) and (7) the conditions can be expressed with data constants $D_{\ell,0}, D_{\ell,1}, D_{\ell,2}$ and curvatures $\kappa_{\ell-1}, \kappa_\ell$. Let us define

$$\boldsymbol{\delta}_\ell := \left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right),$$

and the sets $\mathcal{S}_{\ell,j} \subset \mathbb{R}^3$,

$$\begin{aligned}
\mathcal{S}_{\ell,1} &:= \left\{\boldsymbol{\delta}_\ell : \quad D_{\ell,1}D_{\ell,2} > 0,\ D_{\ell,0}D_{\ell,2} > 0\right\},\\
\mathcal{S}_{\ell,2} &:= \left\{\boldsymbol{\delta}_\ell : \quad D_{\ell,1}D_{\ell,2} > 0,\ D_{\ell,0}D_{\ell,2} < 0\right\},\\
\mathcal{S}_{\ell,3} &:= \left\{\boldsymbol{\delta}_\ell : \quad D_{\ell,1}D_{\ell,2} < 0,\ D_{\ell,0}D_{\ell,2} > 0\right\},\\
\mathcal{S}_{\ell,4} &:= \left\{\boldsymbol{\delta}_\ell : \quad D_{\ell,1}D_{\ell,2} < 0,\ D_{\ell,0}D_{\ell,2} < 0\right\}.
\end{aligned}$$

The solution (ρ_0, ρ_1) is admissible for $\boldsymbol{\delta}_\ell \in \mathcal{S}_{\ell,k}$ iff it belongs to $\mathcal{D}_k$, $k = 1, 2, 3, 4$. Recall that $\operatorname{sign}(R_0) = \operatorname{sign}\left(\kappa_{\ell-1}D_{\ell,0}\right)$ and $\operatorname{sign}(R_1) = \operatorname{sign}\left(\kappa_\ell D_{\ell,1}\right)$. From Fig. 2 one can

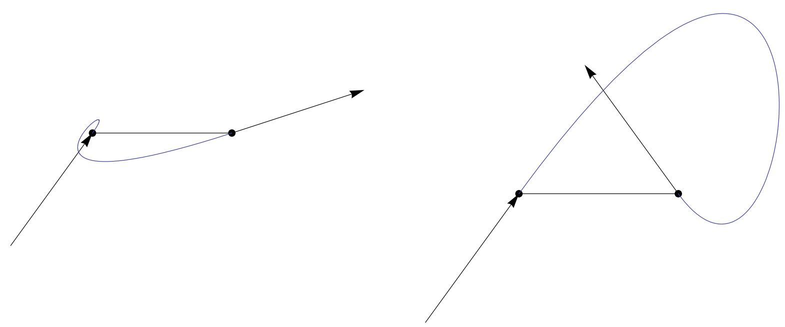

Fig. 2 The examples of interpolants with $\kappa_{\ell-1}D_{\ell,0} < 0$ (left) and $\kappa_\ell D_{\ell,1} < 0$ (right).

see that if $\kappa_{\ell-1}D_{\ell,0} < 0$ or $\kappa_\ell D_{\ell,1} < 0$ the interpolant is not shape preserving. For this reason our analysis will further be limited to the data with $\kappa_{\ell-1+i} D_{\ell,i} > 0$, $i = 0, 1$ (the first quadrant in Fig. 1). Moreover, when working with splines it is desirable to have the conditions that connect only local data. Therefore we will consider only the conditions that do not connect curvatures. By defining

$$K_{\ell,0} := \frac{2}{3} \left| D_{\ell,0} \right| \left(\frac{D_{\ell,2}}{D_{\ell,1}} \right)^2, \quad K_{\ell,1} := \frac{2}{3} \left| D_{\ell,1} \right| \left(\frac{D_{\ell,2}}{D_{\ell,0}} \right)^2, \tag{9}$$

the following sufficient conditions on curvatures for the solution to exist are obtained straightforwardly from Theorem 1.

Theorem 2 *Suppose that the curvatures satisfy*

$$\kappa_{\ell-1} D_{\ell,0} > 0 \quad \text{and} \quad \kappa_\ell D_{\ell,1} > 0.$$

If $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,1}$ *and the curvatures fulfill one of the following conditions:*

$$|\kappa_{\ell-1}| < K_{\ell,0} \quad \text{and} \quad |\kappa_\ell| < K_{\ell,1}, \qquad \text{or} \qquad |\kappa_{\ell-1}| > K_{\ell,0} \quad \text{and} \quad |\kappa_\ell| > K_{\ell,1},$$

then the system (4) *has an admissible solution. Furthermore, the system* (4) *has an unique admissible solution if* $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,2}$ *and* $|\kappa_{\ell-1}| > K_{\ell,0}$*, or if* $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,3}$ *and* $|\kappa_\ell| > K_{\ell,1}$*, or if* $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,4}$.

Proof The suppositions of the theorem imply $R_0 > 0$ and $R_1 > 0$, which means that $(R_0, R_1) \in \mathcal{A}_i$ for $i = 1, 2, 3, 6, 7, 11, 12$. Suppose first that $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,1}$. The solution of (4) is thus admissible iff the solution of (6) lies in $\mathcal{D}_1$. Table 1 implies that in $\mathcal{D}_1$ there is one solution for $(R_0, R_1) \in \mathcal{A}_1 \cup \mathcal{A}_3$, two solutions for $(R_0, R_1) \in \mathcal{A}_6 \cup \mathcal{A}_{11}$, three solutions for $(R_0, R_1) \in \mathcal{A}_2$ and no solutions for $(R_0, R_1) \in \mathcal{A}_7 \cup \mathcal{A}_{12}$. From (7) and (9) follows that $R_0 > 1$ iff $|\kappa_{\ell-1}| > K_{\ell,0}$ and similarly $R_1 > 1$ iff $|\kappa_\ell| > K_{\ell,1}$. Therefore if $|\kappa_{\ell-1+i}| > K_{\ell,i}$, $i = 0, 1$, the unique solution exists. Further if $|\kappa_{\ell-1+i}| < K_{\ell,i}$, $i = 0, 1$, there exist one or three solutions. Note that the regions $\mathcal{A}_6$ and $\mathcal{A}_{11}$ are omitted. For $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,2}$ the solution of (4) is admissible iff the solution of (6) lies in $\mathcal{D}_2$. Table 1 implies that in $\mathcal{D}_2$ there exists a unique solution for $(R_0, R_1) \in \mathcal{A}_1 \cup \mathcal{A}_{11} \cup \mathcal{A}_{12}$ and no solutions in other regions. This confirms the statement of the theorem. The case with $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,3}$ is symmetric to the previous one. For the last case $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,4}$ the solution of (4) is admissible iff the solution of (6) lies in $\mathcal{D}_4$. Table 1 shows that in $\mathcal{D}_4$ there exists a unique solution for every $R_0 > 0$ and $R_1 > 0$. The proof is completed.

Remark 1 Under the suppositions of Theorem 2 the conditions for the unique admissible solution of the system (4) to exist are sufficient and necessary for $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,k}$, $k = 2, 3, 4$. For $\left(D_{\ell,0}, D_{\ell,1}, D_{\ell,2}\right) \in \mathcal{S}_{\ell,1}$ the conditions are only sufficient since the conditions that connect the curvatures $\kappa_{\ell-1}$ and κ_ℓ are omitted. Moreover, if $|\kappa_{\ell-1}| < K_{\ell,0}$ and $|\kappa_\ell| < K_{\ell,1}$ the number of admissible solutions can be one or three.

4 The G^2 spline curve

Let us now return to the G^2 spline problem. We will show how the tangent directions and the curvatures for a general set of data points can be chosen so that the spline exists. To simplify the analysis let us assume that three consecutive data points do not lie on the same line, i.e., $\Delta \boldsymbol{T}_{\ell-1} \times \Delta \boldsymbol{T}_\ell \neq 0$. In order to get a nice shape of the interpolating spline it is natural to choose each direction $\boldsymbol{d}_\ell$ in the wedge between $\Delta\boldsymbol{T}_{\ell-1}$ and $\Delta\boldsymbol{T}_\ell$. Namely, let

$$\boldsymbol{d}_\ell = \boldsymbol{d}_\ell(\xi_\ell) := \frac{1}{\|(1-\xi_\ell)\Delta\boldsymbol{T}_{\ell-1} + \xi_\ell\Delta\boldsymbol{T}_\ell\|}\left((1-\xi_\ell)\Delta\boldsymbol{T}_{\ell-1} + \xi_\ell\Delta\boldsymbol{T}_\ell\right), \qquad (10)$$
$$\ell = 0, 1, \ldots, m,$$

where the parameters ξ_ℓ are limited to the interval $(0,1)$, and $\boldsymbol{T}_{-1}, \boldsymbol{T}_{m+1}$ are additionally chosen points that define the first and the last direction. The expressions $D_{\ell,i} = D_{\ell,i}(\xi_{\ell-1}, \xi_\ell)$, $i = 0,1,2$, now depend on $\xi_{\ell-1}$ and ξ_ℓ. Let us define $\Delta_{\ell-1,\ell} := \Delta\boldsymbol{T}_{\ell-1} \times \Delta\boldsymbol{T}_\ell$. It is straightforward to check that

$$D_{\ell,0} \cdot \Delta_{\ell-2,\ell-1} > 0 \quad \text{and} \quad D_{\ell,1} \cdot \Delta_{\ell-1,\ell} > 0$$

for any chosen $\boldsymbol{\xi} := (\xi_\ell)_{\ell=0}^m \in (0,1)^{m+1}$. Therefore we require the curvatures κ_ℓ to satisfy

$$\operatorname{sign}(\kappa_\ell) = \operatorname{sign}\bigl(\Delta_{\ell-1,\ell}\bigr), \quad \ell = 0,1,\ldots,m, \qquad (11)$$

as discussed in the previous section. For any chosen $\boldsymbol{\xi}$ the task is to find the intervals where the curvatures κ_ℓ can be taken from to have the existence of the G^2 spline guaranteed. Let us define bounds B_ℓ in the following way:

$$B_\ell := \begin{cases} \max\left(K_{\ell,1}, K_{\ell+1,0}\right), & \boldsymbol{\delta}_\ell \in \mathcal{S}_{\ell,1} \cup \mathcal{S}_{\ell,3}, \quad \boldsymbol{\delta}_{\ell+1} \in \mathcal{S}_{\ell+1,1} \cup \mathcal{S}_{\ell+1,2} \\ K_{\ell,1}, & \boldsymbol{\delta}_\ell \in \mathcal{S}_{\ell,1} \cup \mathcal{S}_{\ell,3}, \quad \boldsymbol{\delta}_{\ell+1} \in \mathcal{S}_{\ell+1,3} \cup \mathcal{S}_{\ell+1,4} \\ K_{\ell+1,0}, & \boldsymbol{\delta}_\ell \in \mathcal{S}_{\ell,2} \cup \mathcal{S}_{\ell,4}, \quad \boldsymbol{\delta}_{\ell+1} \in \mathcal{S}_{\ell+1,1} \cup \mathcal{S}_{\ell+1,2} \\ 0, & \boldsymbol{\delta}_\ell \in \mathcal{S}_{\ell,2} \cup \mathcal{S}_{\ell,4}, \quad \boldsymbol{\delta}_{\ell+1} \in \mathcal{S}_{\ell+1,3} \cup \mathcal{S}_{\ell+1,4} \end{cases},$$

$$\ell = 1, 2, \ldots, m-1, \qquad (12)$$

$$B_0 := \begin{cases} K_{1,0}, & \boldsymbol{\delta}_1 \in \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2} \\ 0, & \boldsymbol{\delta}_1 \in \mathcal{S}_{1,3} \cup \mathcal{S}_{1,4} \end{cases}, \qquad B_m := \begin{cases} K_{m,1}, & \boldsymbol{\delta}_m \in \mathcal{S}_{m,1} \cup \mathcal{S}_{m,3} \\ 0, & \boldsymbol{\delta}_m \in S_{m,2} \cup S_{m,4} \end{cases}. \qquad (13)$$

The next theorem is the main result of this section.

Theorem 3 *Suppose that the data points* (1) *are given such that three consecutive points are not collinear. Let the tangent directions $\boldsymbol{d}_\ell$ be defined by* (10) *for some chosen $\xi_\ell \in (0,1)$, $\ell = 0,1,\ldots,m$. If the curvatures κ_ℓ satisfy*

$$\operatorname{sign}(\kappa_\ell) = \operatorname{sign}\bigl(\Delta_{\ell-1,\ell}\bigr) \quad \text{and} \quad |\kappa_\ell| > B_\ell, \quad \ell = 0,1,\ldots,m,$$

then there exists a unique G^2 spline $\boldsymbol{S}$ that satisfies (2).

Remark 2 Suppose that the suppositions of Theorem 3 hold. If the curvatures κ_ℓ satisfy $|\kappa_0| > K_{1,0}$, $|\kappa_m| > K_{m,1}$, and

$$|\kappa_\ell| > \max\left(K_{\ell,1}, K_{\ell+1,0}\right), \quad \ell = 1,2,\ldots,m-1,$$

then there exists a unique G^2 spline $\boldsymbol{S}$ that satisfies (2).

Remark 3 If the data points (1) are convex, i.e., $\Delta_{\ell-1,\ell}$ are of the same sign for all ℓ, then the G^2 spline $\boldsymbol{S}$ that satisfies (2) exists also if $|\kappa_0| < K_{1,0}$, $|\kappa_m| < K_{m,1}$, and

$$|\kappa_\ell| < \min\left(K_{\ell,1}, K_{\ell+1,0}\right), \quad \ell = 1, 2, \ldots, m-1.$$

The spline may not be unique.

The proof of Theorem 3 follows straightforward from Theorem 2 and from the definition of B_ℓ.

Finally, let us suggest a simple algorithm for the selection of tangent directions and curvatures. The numerical experiments showed that the choice of tangent directions has much less influence on the shape of the spline as the choice of curvatures. A simple way of choosing the directions is the following. Compute the interpolating parabola through points $\boldsymbol{T}_{\ell-1}$, $\boldsymbol{T}_\ell$, $\boldsymbol{T}_{\ell+1}$ with respect to some chosen parameterization $\{0, u_\ell, 1\}$. Then take the direction $\boldsymbol{d}_\ell$ as the direction of the derivative at the parameter u_ℓ:

$$\overline{\boldsymbol{d}}_\ell = \frac{1-u_\ell}{u_\ell}\Delta\boldsymbol{T}_{\ell-1} + \frac{u_\ell}{1-u_\ell}\Delta\boldsymbol{T}_\ell, \quad \boldsymbol{d}_\ell = \frac{\overline{\boldsymbol{d}}_\ell}{\|\overline{\boldsymbol{d}}_\ell\|}.$$

It is easy to check that $\boldsymbol{d}_\ell$ is obtained from (10) by taking

$$\xi_\ell = \frac{u_\ell^2}{u_\ell^2 + (1-u_\ell)^2}. \tag{14}$$

A well known α–parameterization, i.e.,

$$u_\ell = \frac{\|\Delta T_{\ell-1}\|^\alpha}{\|\Delta T_{\ell-1}\|^\alpha + \|\Delta T_\ell\|^\alpha}, \tag{15}$$

is right at hand. Usually one uses $\alpha = 1$ (chord–length), $\alpha = 1/2$ (centripetal) or $\alpha = 0$ (uniform). In the last case $\xi_\ell = 1/2$. The next algorithm suggests a simple way of choosing the curvatures.

algorithm ChooseCurvatures($(\boldsymbol{T}_i)_{i=-1}^{m+1}$, $(v_\ell)_{\ell=0}^m$, α, ϵ)

1. compute $(\xi_\ell)_{\ell=0}^m$, by formulaes (14) and (15) for the given α;
2. compute $(B_\ell)_{\ell=0}^m$ by (12) and (13);
3. for $\ell = 0, 1, \ldots, m$
4.   if $v_\ell > B_\ell$
5. $\kappa_\ell := \operatorname{sign}\left(\Delta_{\ell-1,\ell}\right) \cdot v_\ell$
6.   else
7. $\kappa_\ell := \operatorname{sign}\left(\Delta_{\ell-1,\ell}\right) \cdot (B_\ell + \epsilon)$
8.   end
9. end
10. return $\left\{(B_\ell)_{\ell=0}^m, (\kappa_\ell)_{\ell=0}^m\right\}$

The algorithm returns the list $(B_\ell)_{\ell=0}^m$ of boundary values for curvatures and the list $(\kappa_\ell)_{\ell=0}^m$ of admissible curvatures. The input vector $\boldsymbol{v} = (v_\ell)_{\ell=0}^m$ tells the absolute values of curvatures that one would like to have. If the existence conditions are violated, then κ_ℓ obtains the nearest possible value up to the tolerance ϵ. By choosing very small values v_ℓ, one obtains the curvatures that equal the boundary curvatures plus-minus the tolerance ϵ. The simplest way is to choose $\boldsymbol{v}$ as a constant vector or, like for the directions, value v_ℓ can be computed as the curvature of the interpolating parabola at u_ℓ:

$$v_\ell = \frac{2\,\Delta_{\ell-1,\ell}\,u_\ell^2\,(1-u_\ell)^2}{\sqrt{\left((1-u_\ell)^4\,\|\Delta \boldsymbol{T}_{\ell-1}\| + 2(1-u_\ell)^2 u_\ell^2\,\Delta \boldsymbol{T}_{\ell-1}\cdot\Delta \boldsymbol{T}_\ell + u_\ell^4\,\|\Delta \boldsymbol{T}_\ell\|\right)^3}}. \tag{16}$$

5 Examples

Let us conclude the paper with some numerical examples. As the first one suppose that the data points are sampled from a logarithmic spiral

$$\boldsymbol{f}(t) := \log(1+t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \quad t\in[0,b], \quad b>0, \tag{17}$$

at equidistantly chosen parameters $t_i = ih$, $h=\frac{b}{m}$. Clearly, the distance between the G^2 spline approximant and the curve $\boldsymbol{f}$ depends on the choice of derivative directions and curvatures. Three different cases are examined. The first one is the BHS scheme where directions and curvatures are obtained from the curve $\boldsymbol{f}$. Secondly, directions and curvatures are computed by local parabolic interpolation (Eq. (10), (14)–(16)) based upon the centripetal parameterization, and thirdly a curvature at each points is constant (examples below use $|\kappa_\ell| = 1$). Let the splines be denoted by $\boldsymbol{S}_{bhs}, \boldsymbol{S}_p$ and $\boldsymbol{S}_c$ respectively. Table 3 shows the parametric distances ([17]) between $\boldsymbol{f}$ and the G^2

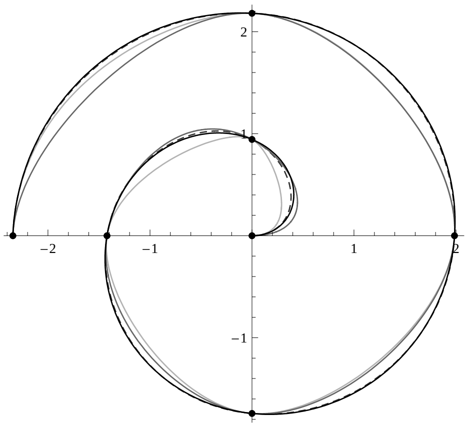

Fig. 3 A logarithmic spiral (17) (black) and the G^2 spline interpolants $\boldsymbol{S}_{bhs}$ (dashed), $\boldsymbol{S}_p$ (light gray), $\boldsymbol{S}_c$ (gray) composed of six segments.

splines composed of six segments. Moreover, the approximation order is estimated as a decay exponent between errors as h tends to zero. As proven in [2], the approximation order is six for the BHS scheme. Clearly, the order drops in the second case since the directions and curvatures are approximated only. Namely, in the second case the approximation order is four, and as expected, if no information of the curve is used for the choice of curvatures, the order drops to two. The curve $\boldsymbol{f}$ and its G^2 spline interpolants are shown in Fig. 3.

h	Approximation error			Decay exponent		
	$\boldsymbol{S}_{bhs}$	$\boldsymbol{S}_p$	$\boldsymbol{S}_c$	$\boldsymbol{S}_{bhs}$	$\boldsymbol{S}_p$	$\boldsymbol{S}_c$
$\frac{\pi}{2}$	$1.72638 \cdot 10^{-2}$	$1.36736 \cdot 10^{-1}$	$1.38954 \cdot 10^{-1}$	/	/	/
$\frac{\pi}{2^2}$	$5.02469 \cdot 10^{-3}$	$1.34298 \cdot 10^{-2}$	$1.1929 \cdot 10^{-2}$	1.78	3.35	3.54
$\frac{\pi}{2^3}$	$3.8764 \cdot 10^{-4}$	$3.1574 \cdot 10^{-3}$	$3.1574 \cdot 10^{-3}$	3.70	2.09	1.91
$\frac{\pi}{2^4}$	$7.07445 \cdot 10^{-6}$	$3.31523 \cdot 10^{-4}$	$7.31019 \cdot 10^{-4}$	5.78	3.25	2.11
$\frac{\pi}{2^5}$	$1.14998 \cdot 10^{-7}$	$2.94436 \cdot 10^{-5}$	$1.75602 \cdot 10^{-4}$	5.94	3.49	2.06
$\frac{\pi}{2^6}$	$1.65879 \cdot 10^{-9}$	$1.91446 \cdot 10^{-6}$	$4.36683 \cdot 10^{-5}$	6.12	3.94	2.01
$\frac{\pi}{2^7}$	$2.18787 \cdot 10^{-11}$	$1.05276 \cdot 10^{-7}$	$1.09395 \cdot 10^{-5}$	6.24	4.18	1.997
$\frac{\pi}{2^8}$	$2.9916 \cdot 10^{-13}$	$5.90469 \cdot 10^{-9}$	$2.74053 \cdot 10^{-6}$	6.19	4.16	1.997
$\frac{\pi}{2^9}$	$4.30257 \cdot 10^{-15}$	$3.44097 \cdot 10^{-10}$	$6.86007 \cdot 10^{-7}$	6.12	4.10	1.998

Table 3 Errors between a curve $\boldsymbol{f}$ and its G^2 spline interpolants with different tangent directions and curvatures.

Now, let us consider the examples with only the points as given data and let us observe the effect of magnitudes of curvatures at breakpoints on the shape of the interpolating spline $\boldsymbol{S}$. In all the following examples tangent directions and curvatures are computed by the algorithm *ChooseCurvatures* with $\alpha = \frac{1}{2}$ (centripetal parameterization) and $\epsilon = 10^{-3}$.

The first example of data points (see Fig. 4) shows how the shape of the interpolating spline $\boldsymbol{S}$ changes with values v_ℓ. Let us simply choose $\boldsymbol{v} = (v, v, \ldots, v)$ as a constant vector. Recall that at the breakpoint where the curvature v is not admissible, it is replaced by the nearest possible curvature up to the tolerance ϵ. Figures show that the spline approaches the data polygon by increasing $|\kappa_\ell|$. Therefore this local scheme is right at hand when one wants to have sharper edges at some breakpoints and more smooth edges at others.

In the second example (see Fig. 5) the data points are convex. Again, $\boldsymbol{v} = (v, v, \ldots, v)$ is a constant vector. Left figure shows the interpolating spline $\boldsymbol{S}$ with curvatures chosen up to the ϵ close to the boundary curvatures. By Remark 3, for the convex data the spline exists also for sufficiently small curvatures. In the right figure $|\kappa_\ell| = 0.01$ for all ℓ, but the shape of the spline is quite bad. Further examples showed that if $\kappa_\ell \to 0$ the spline is not shape preserving since it can have loops and other undesirous properties. Some more examples are shown in Fig. 6. Directions and curvatures are computed using local parabolas (Eq. (10), (14)–(16)).

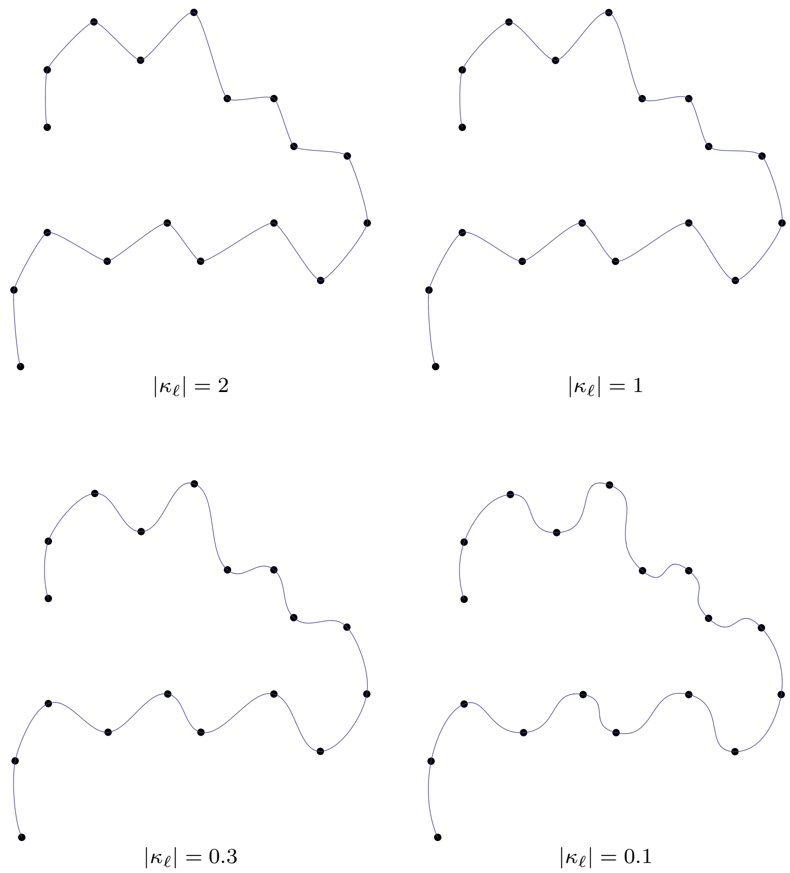

Fig. 4 The G^2 splines with different curvatures $|\kappa_\ell| = v$.

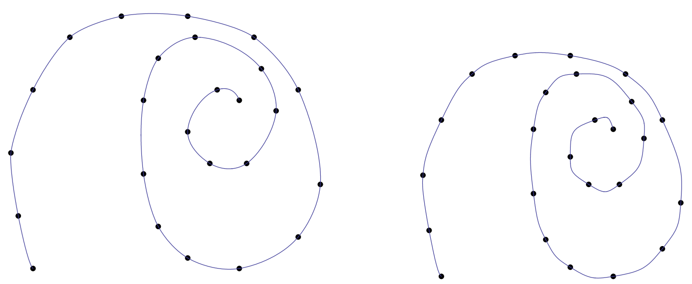

Fig. 5 The G^2 spline $\boldsymbol{S}$ with curvatures close to the boundary ones (left) and curvatures $|\kappa_\ell| = 0.01$ (right).

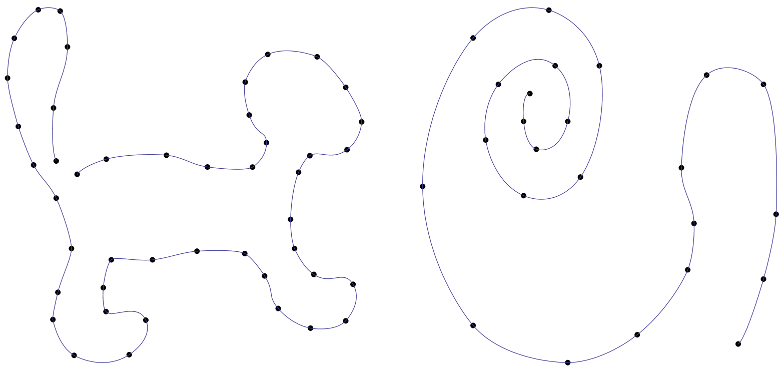

Fig. 6 The G^2 spline curves with derivative directions and curvatures computed by local parabolic interpolation.

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