On geometric Lagrange interpolation by quadratic parametric patches

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Abstract

In the paper, the geometric Lagrange interpolation by quadratic parametric patches is considered. The freedom of parameterization is used to raise the number of interpolated points from the usual 6 up to 10, i.e., the number of points commonly interpolated by a cubic patch. At least asymptotically, the existence of a quadratic geometric interpolant is confirmed for data taken on a parametric surface with locally nonzero Gaussian curvature and interpolation points based upon a three-pencil lattice. Also, the asymptotic approximation order 4 is established.

Key words: Interpolation, Approximation, Parametric surface.

1 Introduction

Interpolation by polynomial parametric patches is one of the fundamental problems in CAGD. For a given set of three-dimensional data points one has to find a parametric polynomial patch passing through them. Parameterization of the patch is important, since it is well known that the choice of parameters affects not only the shape of the interpolating patch but also its approximating properties. Moreover, in the multivariate case it is essential that a chosen parameter set is unisolvent, since this leads to a correct interpolation problem, which can be solved componentwise by any standard interpolation technique, such as Newton, Lagrange, ...

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But one can also leave the interpolating parameters unknown, which leads to so called geometric (or parametric) interpolation. This is a well known topic in polynomial curve interpolation, but only a few results have been obtained for surfaces. Recently, Mørken ([1]) carried over the idea of the geometric interpolation from the curve to the surface case by considering geometric interpolation schemes of the Taylor type, and, in detail, the parametric quadratic interpolation at a point. His approach goes back to the earlier joint work with Scherer ([2]) on the curve interpolation. It is based upon the freedom of choosing the reparameterization of the interpolated curve or surface. This helps to reduce the degree of the Taylor interpolating parametric surface, and to achieve higher asymptotic approximation order, i.e., four in the quadratic case.

However, a higher approximation order is not the only advantage of the geometric interpolation. As observed in the curve case (see [3], e.g.), a geometric polynomial interpolant could do much better as far as the shape is concerned than its ordinary higher degree counterpart, constructed by the componentwise interpolation. So it is worthwhile to extend the results, obtained in [1], to Lagrange interpolation, the case perhaps most often encountered in practical computations.

In this paper a special case will be studied. Suppose that data points $\boldsymbol{T}_{\boldsymbol{\alpha}} \in \mathbb{R}^3$ are interpolated by a quadratic parametric patch $\boldsymbol{p}_2$, where, more generally, a patch $\boldsymbol{p}_n$ of total degree $\leq\ n$ is given as

$$\boldsymbol{p}_n : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad \boldsymbol{u} \mapsto \boldsymbol{p}_n(\boldsymbol{u}).$$

The components of $\boldsymbol{p}_n$ belong to Π_n^2, the space of bivariate polynomials of total degree $\leq\ n$. The domain Ω contains the usual triangular domain $\Delta \subset \Omega$ of a Bézier triangular patch, but is assumed to be as large as needed.

The power of the geometric interpolation is implied by the fact that a patch is allowed to pass through points $\boldsymbol{T}_{\boldsymbol{\alpha}}$ at the parameter values it chooses. Since a linear transformation of the domain Ω preserves the degree of the patch, at least three points $\boldsymbol{T}_{\boldsymbol{\alpha}}$ should be interpolated at prescribed parameter values. For simplicity we shall assume that these three points will be interpolated at the vertices of $\Delta \subset \Omega$. For a correct interpolation problem it is well known (see [4], e.g.) that a parametric polynomial patch $\boldsymbol{p}_n$ can, in general, interpolate at most

$$\binom{n+2}{2} \;=\; \dim \Pi_n^2 \tag{1}$$

points $\boldsymbol{T}_{\boldsymbol{\alpha}}$ at given parameter values. In fact, this case makes the interpolation problem linear. But encouraged by the curve case, one is tempted to keep n as low as possible and increase the number of interpolated points as much as possible, with interpolating parameters not known in advance. In this way a lower degree patch would replace the higher degree one, but with the same approximation order. Of course, the price paid is nonlinearity.

Suppose that the number of points, which will be interpolated at known parameter values, is $k+3$ (3 points are always interpolated at prescribed parameter values) and let m denote the total number of interpolated points. By (1) it is clear that

$$k \leq \binom{n+2}{2} - 3.$$

A simple counting reveals that the number of nonlinear equations, arising from interpolation, is $3\,m$, and the number of unknowns involved is

$$3\binom{n+2}{2} + 2\,(m - (k+3)).$$

The first term in the above expression counts unknown coefficients of the components of the patch, and the second one the number of unknown parameter values at which data points are interpolated. Note that the interpolation at a point with free parameter values imposes only one scalar constraint. Quite clearly, the only hope that the solution might exist in general is that the number of equations is smaller or equal to the number of unknowns. Thus the following conjecture can be stated.

Conjecture 1 *A polynomial parametric patch of degree n can, in general, interpolate at most*

$$3\binom{n+2}{2} - 2\,k - 6, \tag{2}$$

points, where

$$k+3 \leq \binom{n+2}{2}$$

is the number of points interpolated at the prescribed parameter values.

Table 1 gives the optimal number (i.e., the number arising from the conjecture where the upper bound (2) is achieved) of points interpolated by a quadratic and a cubic patch for all possible values of k.

Table 1
The optimal number of points (m) possibly interpolated by a polynomial patch of degree n, where $k+3$ points are interpolated at known parameter values.

n	2				3							
k	0	1	2	3	0	1	2	3	4	5	6	7
m	12	10	8	6	24	22	20	18	16	14	12	10

A similar conjecture has been proposed for interpolation by parametric polynomial curves (see [5]). As it has turned out, the only promising way of proving it is the asymptotic analysis which requires some additional information on interpolated points (convexity of sampling function e.g.). Even in this case only partial results have been obtained. Since it might be even worse for the surfaces, the asymptotic analysis for the first nontrivial case, i.e., quadratic interpolation, will be considered here.

The existence and the approximation order will be studied by the asymptotic approach. However, assumptions not only on the smoothness of the underlying surface but also on the parameter positions at which data are sampled must be made. Three-pencil lattices ([8]) will be used since they allow some flexibility in positions of the data points and determine the interpolating points by only a few degrees of freedom. A particular subset of these configurations are the principal lattices, most often used in practical computations. Thus the interpolating parameters will represent a unisolvent set for the space Π_3^2 independently of the domain magnitude.

The outline of the paper is as follows. In the next section the main result of the paper is presented. In Section 3 the equations that determine the interpolating patch are derived. Section 4 introduces three-pencil lattices and provides their algebraic definition. In Section 5 the asymptotic analysis is carried out and Section 6 completes the proof of the main theorem. The last section illustrates the results of the paper with numerical examples.

2 The main result

Unfortunately, as already observed in [1], the number of free parameters introduced by the interpolation at unknown parameter values and $\dim \Pi_n^2$ do not match so nicely as in the curve case. As an example, take the quadratic case, $n = 2$. Table 1 shows that a quadratic polynomial patch might interpolate 12 points. But

$$\dim \Pi_3^2 = 10 < 12 < \dim \Pi_4^2 = 15,$$

and the interpolant $\boldsymbol{p}_2$ at 12 points will obviously replace a polynomial interpolant of minimal degree ([6],[7]), lying in a space between Π_3^2 and Π_4^2. Such an implicit joining of two additional Newton basic polynomials to Π_3^2 will increase the precision in particular directions only, and will not raise the asymptotic approximation order in general. If the data are not of a particular nature that requires such an approach, it seems better to stick to Π_3^2 only, and to interpolate some additional points at prescribed parameter values $\boldsymbol{u} \in \Delta$. The only useful choice is given by the second column of Table 1 for $n = 2$, namely not only the points, corresponding to the vertices of the triangle Δ, but also one additional point is interpolated at the prescribed parameter values.

The main result of the paper can be summarized in the following theorem.

Theorem 2 *Suppose that S is a smooth surface having a nonzero Gaussian curvature in a vicinity of a point $\boldsymbol{T} \in S$, with an (injective) regular parameterization $\boldsymbol{s} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^3$, defined on a triangular domain Δ. Let $0 < h \leq 1$, and*

$$\Delta_h := \left\{ h \left(\boldsymbol{x} - \boldsymbol{s}^{-1}(\boldsymbol{T}) \right) + \boldsymbol{s}^{-1}(\boldsymbol{T}), \ \boldsymbol{x} \in \Delta \right\}.$$

Then there is $h_0 > 0$ such that for all h, $0 < h \leq h_0$, the quadratic patch that interpolates $\boldsymbol{s}$ at data points given by a three-pencil lattice on Δ_h, exists and depends on the parameters determining the lattice. The asymptotic approximation order is optimal, i.e., 4.

3 Nonlinear equations in the quadratic case

The choice of the second column for $n = 2$ in Table 1 as a basis for the interpolation problem will now be studied in detail. In order to shorten the notation, the standard multiindex notation will be used. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ is a multiindex and $\boldsymbol{x} = (x_1, x_2)^T \in \mathbb{R}^2$, then

$$|\boldsymbol{\alpha}| := \sum_{i=1}^{2} \alpha_i, \quad \boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2}, \quad D^{\boldsymbol{\alpha}} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}.$$

Let

$$\mathcal{I}_n := \{(0,0), (1,0), (0,1), \ldots, (1, n-1), (0, n)\} \subset \mathbb{N}_0^2$$

denote the ordered set of multiindices of degree $\leq n$ in $\mathbb{N}_0^2$ in a grevlex (i.e., the graded reverse lexicographical term) order. Further, let

$$\left\{ \boldsymbol{T}_{\boldsymbol{\alpha}} \in \mathbb{R}^3, \quad \boldsymbol{\alpha} \in \mathcal{I}_3 \right\} \tag{3}$$

be a given set of distinct points and $\boldsymbol{p}_2 : \Omega \to \mathbb{R}^3$ a quadratic parametric polynomial patch that should interpolate data points (3) at some values $\boldsymbol{u}_{\boldsymbol{\alpha}} \in \Omega$, i.e.,

$$\boldsymbol{p}_2(\boldsymbol{u}_{\boldsymbol{\alpha}}) = \boldsymbol{T}_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathcal{I}_3. \tag{4}$$

Since four interpolation parameters are supposed to be prescribed in advance, we may choose them as $\boldsymbol{u}_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{I}_3$, $|\boldsymbol{\alpha}| = 3$. Note that (4) is a nonlinear system of 30 equations for 30 unknowns, the coefficients of $\boldsymbol{p}_2$ and the unknown parameters $\boldsymbol{u}_{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| \leq 2$.

The first step is to reduce equations (4) to a nonlinear system for the unknown parameters $\boldsymbol{u}_{\boldsymbol{\alpha}}$ only. Once they are determined, the coefficients of the parametric polynomial patch $\boldsymbol{p}_2$ are derived as a solution of the system of linear equations

$$\boldsymbol{p}_2(\boldsymbol{u}_{\boldsymbol{\alpha}}) = \boldsymbol{T}_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathcal{J} \subset \mathcal{I}_3, \quad |\mathcal{J}| = 6,$$

with $U_2 := \{\boldsymbol{u}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{J}\}$ an arbitrary unisolvent subset of $\{\boldsymbol{u}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{I}_3\}$. Thus

$$\boldsymbol{p}_2(\boldsymbol{u}) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \ell_{\boldsymbol{\alpha},2}(\boldsymbol{u}; U_2)\, \boldsymbol{T}_{\boldsymbol{\alpha}}, \tag{5}$$

where $\ell_{\boldsymbol{\alpha},2}(\cdot; U_2)$ are the Lagrange fundamental polynomials of total degree ≤ 2 on the set of points U_2. In order to do the reduction, take a cubic parametric polynomial patch $\boldsymbol{p}_3$,

$$\boldsymbol{p}_3(\boldsymbol{u}) = \sum_{\boldsymbol{\alpha} \in \mathcal{I}_3} \boldsymbol{a}_{\boldsymbol{\alpha}}\, \boldsymbol{u}^{\boldsymbol{\alpha}}.$$

Suppose that $U_3 := \{\boldsymbol{u}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{I}_3\}$ is a unisolvent subset of $\mathbb{R}^2$. Then the interpolation problem

$$\boldsymbol{p}_3(\boldsymbol{u}_{\boldsymbol{\alpha}}) = \boldsymbol{T}_{\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \mathcal{I}_3, \tag{6}$$

has a unique solution. If the coefficients $\boldsymbol{a}_{\boldsymbol{\alpha}}$ satisfy

$$\boldsymbol{a}_{\boldsymbol{\alpha}} = \boldsymbol{0}, \quad |\boldsymbol{\alpha}| = 3, \tag{7}$$

$\boldsymbol{p}_3$ is reduced to a quadratic patch $\boldsymbol{p}_2$. So (7) are the equations that will determine the unknown $\boldsymbol{u}_{\boldsymbol{\alpha}}$. Let $A := [\boldsymbol{a}_{\boldsymbol{\alpha}}]_{\boldsymbol{\alpha} \in \mathcal{I}_3} \in \mathbb{R}^{3\times 10}$ and $T := [\boldsymbol{T}_{\boldsymbol{\alpha}}]_{\boldsymbol{\alpha} \in \mathcal{I}_3} \in \mathbb{R}^{3\times 10}$. The equations (6) can be written as

$$M(U_3)\, A^T = T^T,$$

where

$$M(U_3) := \left[\boldsymbol{u}_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}\right]_{\boldsymbol{\alpha},\, \boldsymbol{\beta} \in \mathcal{I}_3}$$

is a Vandermonde matrix. Since U_3 is unisolvent, $\det M(U_3) \neq 0$, and by the Cramer's rule the equations (7) can be rewritten as

$$\sum_{\boldsymbol{\beta} \in \mathcal{I}_3} (-1)^{\mathrm{pos}(\boldsymbol{\alpha}) + \mathrm{pos}(\boldsymbol{\beta})}\, M_{\boldsymbol{\beta},\boldsymbol{\alpha}}(U_3)\, \boldsymbol{T}_{\boldsymbol{\beta}} = \boldsymbol{0}, \quad |\boldsymbol{\alpha}| = 3, \tag{8}$$

where $M_{\boldsymbol{\beta},\boldsymbol{\alpha}}(U_3)$ denotes the minor of $M(U_3)$ with the row $\mathrm{pos}(\boldsymbol{\beta})$ and the column $\mathrm{pos}(\boldsymbol{\alpha})$ omitted. Here and through the rest of the paper $\mathrm{pos}(\boldsymbol{\alpha})$ denotes the position of $\boldsymbol{\alpha}$ in the ordered set $\mathcal{I}_3$. The system (8) for the unknown parameters $\boldsymbol{u}_{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| \leq 2$, is the required reduction of the original nonlinear one, given by (4).

Although a significant reduction has been made, the system is still nonlinear and difficult to analyse in general. The asymptotic analysis will be used instead.

4 Three-pencil lattices

Three-pencil lattices (Fig. 1) are often encountered geometric configurations that provide unisolvent sets of interpolation parameters ([8]). They represent a particular subset of generalized principal lattices ([9]). *Pencil* is a set of

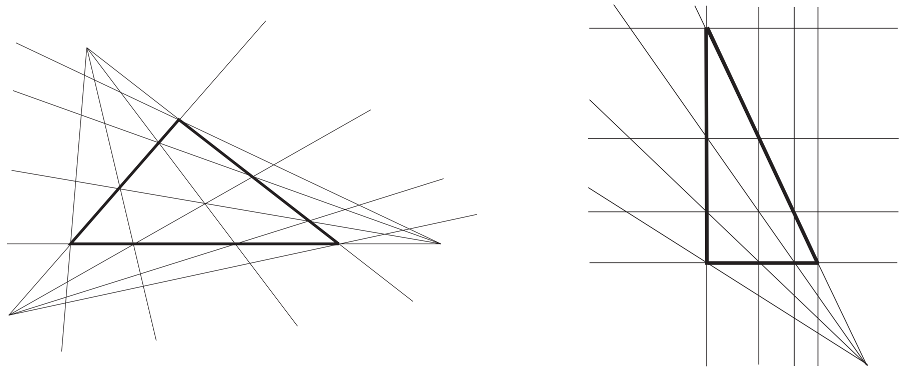

Fig. 1. A three-pencil lattice with three finite centers (left) and with 2 centers at the ideal line (right).

lines intersecting at one point (center of the pencil) or a set of parallel lines (center is at the ideal line). The latter case defines *principal lattices*. Here three-pencil lattices will be three-pencil lattices of order 3, defined as a set of 10 points, generated by 3 pencils of 4 lines each. Every point of the lattice is an intersection of three lines, one from each pencil (Fig. 1). Three-pencil lattices are uniquely determined by 3 lines and 3 center points ([8]).

The asymptotic analysis will require an explicit representation of lattice points, i.e., domain parameters of the interpolation data. For this purpose, let us use barycentric coordinates w.r.t. the vertices of a triangle in $\mathbb{R}^2$ (Fig. 2). Then $\boldsymbol{P}_{(3,0)} = (1,0,0)^T$, $\boldsymbol{P}_{(2,1)} = (0,1,0)^T$, $\boldsymbol{P}_{(1,2)} = (0,0,1)^T$. It is obvious that all

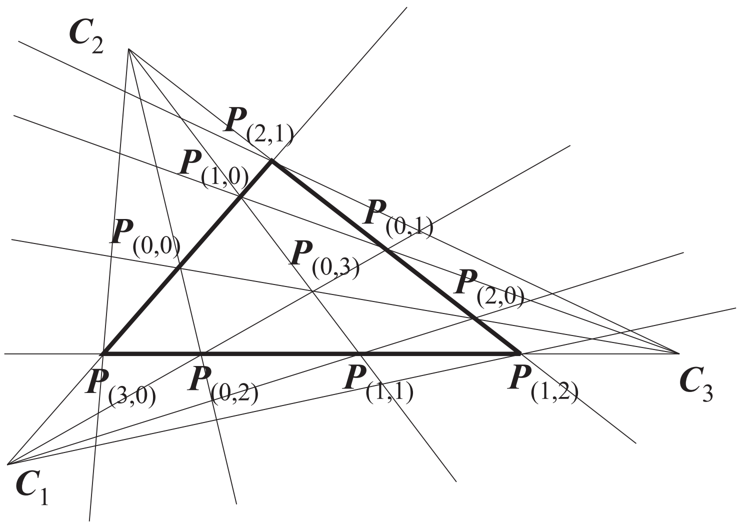

Fig. 2. A three-pencil lattice with barycentric coordinates $\{\boldsymbol{P}_{\boldsymbol{\alpha}},\ |\boldsymbol{\alpha}| \leq 3\}$.

the corner points of Δ must belong to the lattice. Let

$$\boldsymbol{q}_{\mathrm{pos}(\boldsymbol{\alpha}),\mathrm{pos}(\boldsymbol{\beta})}(\lambda) := (1-\lambda)\,\boldsymbol{P}_{\boldsymbol{\alpha}} + \lambda\,\boldsymbol{P}_{\boldsymbol{\beta}}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{I}_3,$$

denote a line through the vertices $\boldsymbol{P}_{\boldsymbol{\alpha}}$ and $\boldsymbol{P}_{\boldsymbol{\beta}}$. The edges of the given triangle lie on the lines $\boldsymbol{q}_{8,7}$, $\boldsymbol{q}_{9,8}$ and $\boldsymbol{q}_{7,9}$, respectively. Therefore the vertices on those lines can be written as

$$\begin{aligned}
\boldsymbol{P}_{(0,0)} &= \boldsymbol{q}_{8,7}(\tau_1), \quad \boldsymbol{P}_{(1,0)} = \boldsymbol{q}_{8,7}(\tau_2), \quad \boldsymbol{P}_{(0,1)} = \boldsymbol{q}_{9,8}(\tau_3),\\
\boldsymbol{P}_{(2,0)} &= \boldsymbol{q}_{9,8}(\tau_4), \quad \boldsymbol{P}_{(1,1)} = \boldsymbol{q}_{7,9}(\tau_5), \quad \boldsymbol{P}_{(0,2)} = \boldsymbol{q}_{7,9}(\tau_6),
\end{aligned}$$

where

$$0 < \tau_2 < \tau_1 < 1, \quad 0 < \tau_4 < \tau_3 < 1, \quad 0 < \tau_6 < \tau_5 < 1 \tag{9}$$

are unknown parameters. Let the remaining point $\boldsymbol{P}_{(0,3)}$ be

$$\boldsymbol{P}_{(0,3)} = (\xi_1, \xi_2, \xi_3)^T, \quad \xi_i \in (0,1), \quad \xi_1 + \xi_2 + \xi_3 = 1.$$

If the parameters η_1 and η_2 are defined as

$$\eta_1 := \frac{\xi_1}{\xi_2}, \quad \eta_2 := \frac{\xi_2}{\xi_3}, \tag{10}$$

then

$$\boldsymbol{P}_{(0,3)} = \left(\frac{\eta_1\eta_2}{1+(1+\eta_1)\eta_2}, \frac{\eta_2}{1+(1+\eta_1)\eta_2}, \frac{1}{1+(1+\eta_1)\eta_2}\right)^T.$$

Centers

$$\boldsymbol{C}_1 = \boldsymbol{q}_{8,7}\left(\frac{\mu_{8,7}}{1-\tau_5-\tau_4}\right), \ \boldsymbol{C}_2 = \boldsymbol{q}_{9,8}\left(\frac{\mu_{9,8}}{1-\tau_1-\tau_6}\right), \ \boldsymbol{C}_3 = \boldsymbol{q}_{7,9}\left(\frac{\mu_{7,9}}{1-\tau_3-\tau_2}\right),$$

where $\mu_{8,7}$, $\mu_{9,8}$, $\mu_{7,9}$ are unknown parameters too, are also on the lines on which the edges of the triangle lie (Fig. 2). Observe that

$$\boldsymbol{q}_{5,4} \cap \boldsymbol{q}_{6,3} = \{\boldsymbol{C}_1\}, \quad \boldsymbol{q}_{2,5} \cap \boldsymbol{q}_{1,6} = \{\boldsymbol{C}_2\}, \quad \boldsymbol{q}_{3,2} \cap \boldsymbol{q}_{4,1} = \{\boldsymbol{C}_3\}$$

determine 6 scalar equations. Furthermore, the lines $\boldsymbol{q}_{4,1}$, $\boldsymbol{q}_{2,5}$ and $\boldsymbol{q}_{6,3}$ intersect at the vertex $\boldsymbol{P}_{(0,3)}$, which gives additional 4 scalar equations. This describes the system of 10 equations for 11 unknowns τ_i, $i = 1, 2, \ldots, 6$, $\mu_{8,7}, \mu_{9,8}, \mu_{7,9}$ and η_1, η_2, which determines the three-pencil lattice. Let $\omega := \tau_6$. By a proper sequence of elementary linear eliminations the solution can be expressed by three independent parameters ω, η_1, η_2 as

$$\begin{aligned}
\tau_1 &= \frac{(\omega-1)\eta_1}{\omega-1+\eta_1(\omega-1+\omega\eta_2)}, \quad \tau_2 = \frac{\omega\eta_1^2\eta_2}{1-\omega+\omega\eta_1^2\eta_2},\\
\tau_3 &= \frac{(\omega-1)\eta_2}{\omega-1+\eta_2(\omega-1+\omega\eta_1)}, \quad \tau_4 = \frac{\omega\eta_1\eta_2^2}{1-\omega+\omega\eta_1\eta_2^2},\\
\tau_5 &= \frac{\omega-1}{\omega-1+\eta_1\eta_2(\omega-1+\omega\eta_1\eta_2)}.
\end{aligned} \tag{11}$$

From (9) and (10), the parameters ω, η_1, and η_2 must satisfy $0 < \omega < 1$, $\eta_1 > 0$, and $\eta_2 > 0$. But the inequality $\tau_6 < \tau_5 < 1$ and (11) imply that

$\omega < \frac{1}{1+\eta_1\eta_2}$ must be satisfied too. So the conditions

$$\eta_1 > 0, \quad \eta_2 > 0, \quad 0 < \omega < \frac{1}{1+\eta_1\eta_2} \tag{12}$$

are necessary. On the other hand, it is straightforward to verify that the relations (12) together with (11) imply all the inequalities (9). Let us summarize the results in the following lemma.

Lemma 3 *Let Δ be a given triangle. A three-pencil lattice of ten distinct points in Δ is uniquely determined by three parameters ω, η_1, and η_2 iff they satisfy* (12).

As an immediate consequence of Lemma 3 one can give the barycentric coordinates of a three-pencil lattice in a matrix $P := [\boldsymbol{P}_{\boldsymbol{\alpha}}]_{\boldsymbol{\alpha}\in\mathcal{I}_3} \in \mathbb{R}^{3\times 10}$ as

$$P = \begin{bmatrix} \tau_1 & \tau_2 & 0 & 0 & 1-\tau_5 & 1-\tau_6 & 1 & 0 & 0 & \xi_1 \\ 1-\tau_1 & 1-\tau_2 & \tau_3 & \tau_4 & 0 & 0 & 0 & 1 & 0 & \xi_2 \\ 0 & 0 & 1-\tau_3 & 1-\tau_4 & \tau_5 & \tau_6 & 0 & 0 & 1 & \xi_3 \end{bmatrix}. \tag{13}$$

Note that the centers $\{\boldsymbol{C}_i,\ i = 1, 2, 3\}$, and the inner vertices on the triangle edges $\{\boldsymbol{P}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathcal{I}_2\}$, together with the adjoined lines form the Pappus' configuration. The construction of three-pencil lattices ([8]) depends heavily on the Pappus' theorem ([10]).

From now on we will consider unisolvent interpolation parameters, generated by a three-pencil lattice on a domain triangle. Since affine transformations preserve barycentric coordinates, the analysis carried through depends only on the domain triangle vertices. The interpolation parameters depend on ω, η_1 and η_2. Note that similar analysis could be carried out for a three-pencil lattice on $\binom{n+2}{2}$ points. From the construction of three-pencil lattices ([8]) it follows that, in general, a three-pencil lattice on a given triangle depends on 3 parameters only.

5 Asymptotic analysis of the quadratic case

Suppose that a surface S, a point $\boldsymbol{T} \in S$, and a parameterization $\boldsymbol{s}$ are as required in Theorem 2. Without loss of generality we can assume that Δ is a triangle defined by the vertices $(0,0)^T$, $(1,0)^T$, $(0,1)^T$, and $\boldsymbol{s}^{-1}(\boldsymbol{T}) = (0,0)^T$. Therefore $\Delta_h = h\,\Delta$. It is well known ([10]) that for h small enough S can be locally parameterized on Δ_h by a particular regular parameterization

$$\boldsymbol{s} : \Delta_h \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad \boldsymbol{x} \mapsto \boldsymbol{s}(\boldsymbol{x}) = (\boldsymbol{x}, f(\boldsymbol{x}))^T, \quad \boldsymbol{s}(\boldsymbol{0}) = \boldsymbol{0}, \tag{14}$$

such that

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T K\, \boldsymbol{x} + \mathcal{O}(\boldsymbol{x}^{\boldsymbol{\alpha}}), \quad |\boldsymbol{\alpha}| = 3. \tag{15}$$

Here $K := \operatorname{diag}(\kappa_1, \kappa_2)$, where κ_1 and κ_2 are nonzero principal curvatures of S at $\boldsymbol{T}$. Now, let the data points be sampled as

$$\boldsymbol{T}_{\boldsymbol{\alpha}} = \boldsymbol{s}(h\, \boldsymbol{x}_{\boldsymbol{\alpha}}), \quad \boldsymbol{\alpha} \in \mathcal{I}_3, \tag{16}$$

where $X_3 := \{\boldsymbol{x}_{\boldsymbol{\alpha}},\, \boldsymbol{\alpha} \in \mathcal{I}_3\}$ is a unisolvent subset of $\mathbb{R}^2$ given by the three-pencil lattice on Δ. Since four parameters $\{\boldsymbol{x}_{\boldsymbol{\alpha}},\ |\boldsymbol{\alpha}| = 3\}$ need to be fixed, let us choose them as the vertices of Δ and the interior vertex (Fig. 3). The trian-

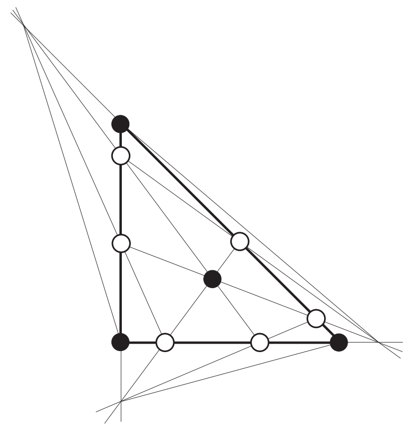

Fig. 3. A set of interpolating parameters in a triangular domain Δ. Black circles indicate the prescribed parameter values.

gle Δ is based upon the vertices $(0,0)^T$, $(1,0)^T$, $(0,1)^T$, thus the interpolating parameters $\boldsymbol{x}_{\boldsymbol{\alpha}}$ are determined as

$$[\boldsymbol{x}_{\boldsymbol{\alpha}}]_{\boldsymbol{\alpha} \in \mathcal{I}_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} P, \tag{17}$$

with P given by (13). From (15) and (16) one obtains

$$\boldsymbol{T}_{\boldsymbol{\alpha}} = \boldsymbol{s}(h\, \boldsymbol{x}_{\boldsymbol{\alpha}}) = D_h \begin{pmatrix} \boldsymbol{x}_{\boldsymbol{\alpha}} \\ \frac{1}{2}\boldsymbol{x}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{x}_{\boldsymbol{\alpha}} + \mathcal{O}(h) \end{pmatrix}, \quad D_h := \operatorname{diag}\left(h, h, h^2\right).$$

Since the parameters $\boldsymbol{u}_{\boldsymbol{\alpha}} = \boldsymbol{x}_{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}| = 3$, are prescribed, one has to show that the system (8) has a real solution $\{\boldsymbol{u}_{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| \le 2\}$ for h small enough. In order to make the nonlinear system (8) well defined as $h \to 0$, equations (8) need to be scaled by D_h^{-1}. The system becomes $\boldsymbol{F}(U_3, X_3, h, \boldsymbol{\alpha}) = \boldsymbol{0}$, $|\boldsymbol{\alpha}| = 3$, where

$$\boldsymbol{F}(U_3, X_3, h, \boldsymbol{\alpha}) := D_h^{-1} \sum_{\boldsymbol{\beta} \in \mathcal{I}_3} (-1)^{\operatorname{pos}(\boldsymbol{\alpha})+\operatorname{pos}(\boldsymbol{\beta})} M_{\boldsymbol{\beta},\boldsymbol{\alpha}}(U_3)\, \boldsymbol{T}_{\boldsymbol{\beta}}.$$

The limit solution at $h = 0$ is

$$\boldsymbol{u}^*_{\boldsymbol{\alpha}} = \boldsymbol{x}_{\boldsymbol{\alpha}}, \quad |\boldsymbol{\alpha}| \leq 2. \tag{18}$$

Namely,

$$\lim_{h\to 0} \boldsymbol{F}(U_3, X_3, h, \boldsymbol{\alpha}) = \sum_{\boldsymbol{\beta} \in \mathcal{I}_3} (-1)^{\operatorname{pos}(\boldsymbol{\alpha})+\operatorname{pos}(\boldsymbol{\beta})} M_{\boldsymbol{\beta},\boldsymbol{\alpha}}(X_3) \left(\boldsymbol{x}_{\boldsymbol{\beta}}, \frac{1}{2}\boldsymbol{x}_{\boldsymbol{\beta}}^T K\, \boldsymbol{x}_{\boldsymbol{\beta}}\right)^T = \boldsymbol{0}, \quad |\boldsymbol{\alpha}| = 3.$$

The last equality holds since the points $\left\{\left(\boldsymbol{x}_{\boldsymbol{\beta}}, \frac{1}{2}\boldsymbol{x}_{\boldsymbol{\beta}}^T K\, \boldsymbol{x}_{\boldsymbol{\beta}}\right),\ \boldsymbol{\beta} \in \mathcal{I}_3\right\}$ are taken from a quadratic patch and the cubic terms must be zero. In order to apply the Implicit Function theorem, one has to study the Jacobian. This turns out to be not so straightforward. Since by (18) only the unknown differences $\boldsymbol{u}_{\boldsymbol{\alpha}} - \boldsymbol{x}_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{I}_2$, as functions of h need to be studied, it will be simpler to exchange the role of the parameters and the unknowns, i.e., let $\boldsymbol{u}_{\boldsymbol{\alpha}}$ be given parameters and $\boldsymbol{x}_{\boldsymbol{\alpha}}$ the unknowns. Let the unknowns be ordered as $\{\boldsymbol{x}_{\boldsymbol{\alpha}}^{(1,0)}, \boldsymbol{\alpha} \in \mathcal{I}_2\}$, followed by $\{\boldsymbol{x}_{\boldsymbol{\alpha}}^{(0,1)}, \boldsymbol{\alpha} \in \mathcal{I}_2\}$, and J be the Jacobian of $\boldsymbol{F}(U_3, X_3, h, \boldsymbol{\alpha})$, $|\boldsymbol{\alpha}| = 3$, at the limit solution (18). It is straightforward to verify that

$$J = \begin{bmatrix} C & 0 & C_1 \\ 0 & C & C_2 \end{bmatrix}^T, \quad C := \left[(-1)^{\operatorname{pos}(\boldsymbol{\alpha})+\operatorname{pos}(\boldsymbol{\beta})} M_{\boldsymbol{\beta},\boldsymbol{\alpha}}(X_3)\right]_{\boldsymbol{\alpha}\in\mathcal{I}_2,\, \boldsymbol{\beta}\in\mathcal{I}_3,\, |\boldsymbol{\beta}|=3},$$

and

$$C_1 := \kappa_1\, C \operatorname{diag}\left(\boldsymbol{x}_{\boldsymbol{\alpha}}^{(1,0)}\right)_{\boldsymbol{\alpha}\in\mathcal{I}_2}, \quad C_2 := \kappa_2\, C \operatorname{diag}\left(\boldsymbol{x}_{\boldsymbol{\alpha}}^{(0,1)}\right)_{\boldsymbol{\alpha}\in\mathcal{I}_2}.$$

Let us recall (17). The structure of J leads to

$$\det J = \kappa_1\, \kappa_2\, r(\omega, \eta_1, \eta_2, \kappa_1, \kappa_2),$$

where r is a rational function of its variables. One can extract common rational factors in the rows and the columns of J to diagonal matrices D_1, D_2, and determine the determinants of the factors $J = D_1 \left(D_1^{-1} J D_2^{-1}\right) D_2$ separately. It is straightforward to compute

$$\begin{aligned}\det\left(D_1^{-1} J D_2^{-1}\right) = {}& \kappa_1\, \kappa_2\, \eta_1^3 \eta_2^2 \left(\eta_1\eta_2 + \eta_2 + 1\right)^3 \\ & \times \left(\omega\eta_1\eta_2^2 - \omega + 1\right)^2 \left(\omega\eta_1\eta_2 + \omega\eta_2 + \omega - \eta_2 - 1\right)^2 \\ & \times \left((\omega-1)^2 + (\omega-1)\omega\eta_1\eta_2 + \omega^2\eta_1^2\eta_2^2\right)^6\end{aligned}$$

by using a Computer Algebra system's symbolic facilities. This gives the function r as

$$r(\omega, \eta_1, \eta_2, \kappa_1, \kappa_2) = 16\, \kappa_1\, \kappa_2\, \frac{Q_1}{Q_2},$$

where

$$\begin{aligned} Q_1 = {} & \eta_1^{71}\, \eta_2^{82}\, \omega^{30}\, (\omega - 1)^{60}\, \left(\omega + \omega\eta_1\eta_2 - 1\right)^{30} \\ & \times \left((\omega-1)^2 + (\omega-1)\omega\eta_1\eta_2 + \omega^2\eta_1^2\eta_2^2\right)^{30}, \\ Q_2 = {} & \left((1-\omega+\omega\eta_1^2\eta_2)(1-\omega+\omega\eta_1\eta_2^2)\right)^{30} \\ & \times \left(\omega - 1 + \eta_1(-1+\omega+\omega\eta_2)\right)^{30} \left(\omega - 1 + \eta_2(-1+\omega+\omega\eta_1)\right)^{30} \\ & \times \left(1 + (1+\eta_1)\eta_2\right)^{33} \left(\omega - 1 + \eta_1\eta_2(\omega - 1 + \omega\eta_1\eta_2)\right)^{30}. \end{aligned}$$

Note that the terms in Q_2 are the powers of denominators of the parameters (17). Moreover, for real η_1, η_2, ω, the expression Q_1 is equal to zero iff

$$\eta_1 = 0 \quad \text{or} \quad \eta_2 = 0 \quad \text{or} \quad \omega = 0 \quad \text{or} \quad \omega = 1 \quad \text{or} \quad \omega = \frac{1}{1+\eta_1\eta_2},$$

which, by Lemma 3, are not allowed for three-pencil lattices. Therefore

$$\det J \neq 0, \tag{19}$$

and the first part of Theorem 2 follows. The next section will conclude the proof of Theorem 2 by confirming the optimal approximation order.

6 Approximation order

Methods of geometric interpolation are of particular interest since they provide interpolants with high approximation order. The main problem with parametric objects is, of course, how to measure the distance in a proper way. Here the fact that a parametric surface $\boldsymbol{s}$ given by (14) and (15) is actually a function on Δ_h, will be used. Thus the estimate known for the functional case can be applied ([11],[12]). Let $q \in \Pi_3^2(\Delta_h)$ denote the polynomial interpolant of total degree ≤ 3, that interpolates $\boldsymbol{s}$ at 10 points, i.e.,

$$q(h\,\boldsymbol{x}_{\boldsymbol{\alpha}}) = f(h\,\boldsymbol{x}_{\boldsymbol{\alpha}}), \quad \boldsymbol{\alpha} \in \mathcal{I}_3.$$

Recall $X_3 = \{\boldsymbol{x}_{\boldsymbol{\alpha}},\, \boldsymbol{\alpha} \in \mathcal{I}_3\}$ and let $X_3^h := h\, X_3$. Let $\ell_{\boldsymbol{\alpha},3}\left(\cdot; X_3^h\right) \in \Pi_3^2(\Delta_h)$, $\boldsymbol{\alpha} \in \mathcal{I}_3$, be the Lagrange fundamental polynomials of total degree ≤ 3 on X_3^h, $\boldsymbol{k}_{\boldsymbol{\alpha}}(\boldsymbol{x}) := \boldsymbol{x} - h\,\boldsymbol{x}_{\boldsymbol{\alpha}}$ and $\boldsymbol{D} := (D^{(1,0)}, D^{(0,1)})^T$. Then the Ciarlet's error formula

([11],[12]) for $f \in \mathcal{C}^4(\Delta_h)$ reads

$$f - q = \frac{1}{4!} \sum_{\boldsymbol{\alpha} \in \mathcal{I}_3} \ell_{\boldsymbol{\alpha},3}\left(\cdot; X_3^h\right) \left(\left(\boldsymbol{k}_{\boldsymbol{\alpha}}^T(\cdot)\, \boldsymbol{D}\right)^4 f\right)(\cdot - c_{\boldsymbol{\alpha}}\, \boldsymbol{k}_{\boldsymbol{\alpha}}(\cdot)), \tag{20}$$

where $c_{\boldsymbol{\alpha}} \in (0,1)$. Note that $\boldsymbol{x} - c_{\boldsymbol{\alpha}}\, \boldsymbol{k}_{\boldsymbol{\alpha}}(\boldsymbol{x}) \in \Delta_h$ for any $\boldsymbol{x} \in \Delta_h$, $\boldsymbol{\alpha} \in \mathcal{I}_3$. Since $\Delta_h = h\,\Delta$, $\|\boldsymbol{k}_{\boldsymbol{\alpha}}\| = \mathcal{O}(h)$. The Lagrange fundamental polynomials are given by

$$\ell_{\boldsymbol{\alpha},3}\left(\boldsymbol{x}; X_3^h\right) = \frac{\det M_{\boldsymbol{\alpha}}\left([\boldsymbol{x}^{\boldsymbol{\beta}}]_{\boldsymbol{\beta}\in\mathcal{I}_3}; X_3^h\right)}{\det M\left(X_3^h\right)}, \tag{21}$$

where $M_{\boldsymbol{\alpha}}\left([\boldsymbol{x}^{\boldsymbol{\beta}}]_{\boldsymbol{\beta}\in\mathcal{I}_3}; X_3^h\right)$ denotes the matrix $M\left(X_3^h\right)$ with its row corresponding to $\boldsymbol{\alpha}$ replaced by $[\boldsymbol{x}^{\boldsymbol{\beta}}]_{\boldsymbol{\beta}\in\mathcal{I}_3}$. Since $\ell_{\boldsymbol{\alpha},3}\left(\cdot; X_3^h\right) = \ell_{\boldsymbol{\alpha},3}\left(\frac{1}{h}\cdot; \frac{1}{h}X_3^h\right)$, the Lagrange polynomials are obviously bounded on Δ_h as $h \to 0$. Now, to show that the error formula (20) implies the approximation order 4, only the boundedness of the derivatives $D^{\boldsymbol{\alpha}} f$, $|\boldsymbol{\alpha}| \le 4$, has to be proved.

Unfortunately the formula (20) cannot be directly applied to our interpolating polynomial $\boldsymbol{p}_2$ described by (4), since $\boldsymbol{p}_2$ is not defined on Δ_h. A regular reparameterization $\varphi : \Delta_h \to \Omega$ which parameterizes $\boldsymbol{p}_2 = (p_{21}, p_{22}, p_{23})^T$ to $\boldsymbol{p}_2 \circ \varphi : \Delta_h \to \mathbb{R}^3$, such that $\boldsymbol{p}_2(\varphi(\boldsymbol{x})) = (\boldsymbol{x}, p_{23}(\varphi(\boldsymbol{x})))^T$, must first be found. Then

$$\|f - q + q - p_{23} \circ \varphi\| \le \|f - q\| + \|p_{23} \circ \varphi - q\|,$$

and the Ciarlet's formula can be used for both terms. Since f is smooth, $\|f - q\| = \mathcal{O}(h^4)$. To bound the second term, the boundedness of the derivatives $D^{\boldsymbol{\alpha}} p_{23}(\varphi(\boldsymbol{x}))$, $|\boldsymbol{\alpha}| \le 4$, has to be proved.

By (18) and (19) the interpolating abscissae $h\,\boldsymbol{x}_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{I}_3$, are necessarily of the form

$$h\,\boldsymbol{x}_{\boldsymbol{\alpha}} = h\,\boldsymbol{u}_{\boldsymbol{\alpha}} + \mathcal{O}(h^2). \tag{22}$$

Some basic properties of Lagrange polynomials, (5), (15) and (16) imply

$$\begin{aligned}\boldsymbol{p}_2(\boldsymbol{u}) &= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}(\boldsymbol{u}; U_2)\, \boldsymbol{T}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}(\boldsymbol{u}; U_2)\, (h\,\boldsymbol{x}_{\boldsymbol{\alpha}}, f(h\,\boldsymbol{x}_{\boldsymbol{\alpha}}))^T \\ &= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}(\boldsymbol{u}; U_2)\, (h\,\boldsymbol{u}_{\boldsymbol{\alpha}} + \mathcal{O}(h^2), \mathcal{O}(h^2))^T = (h\,\boldsymbol{u} + \mathcal{O}(h^2), \mathcal{O}(h^2))^T.\end{aligned}$$

Let us choose now the reparameterization φ as $\varphi := \left((p_{21}, p_{22})^T\right)^{-1}$. Then by the Implicit Function theorem

$$\varphi : \Delta_h \to \Omega, \quad \boldsymbol{x} \mapsto \varphi(\boldsymbol{x}) = \frac{1}{h}\boldsymbol{x} + h \sum_{|\boldsymbol{\alpha}|\ge 2} \boldsymbol{d}_{\boldsymbol{\alpha}}(h)\, \boldsymbol{x}^{\boldsymbol{\alpha}}, \quad \boldsymbol{d}_{\boldsymbol{\alpha}}(h) = \mathcal{O}(1), \tag{23}$$

and

$$\boldsymbol{p}_2(\varphi(\boldsymbol{x})) = (\boldsymbol{x}, p_{23}(\varphi(\boldsymbol{x})))^T.$$

Moreover

$$\det\left(\boldsymbol{D}\varphi\right)(\boldsymbol{x}) = \frac{1}{h^2} + \mathcal{O}(1) > 0,$$

thus φ is a regular reparameterization. The following lemma concludes the proof of Theorem 2.

Lemma 4 *Partial derivatives* $D^{\boldsymbol{\alpha}}\left(p_{23}\circ\varphi\right)$, $|\boldsymbol{\alpha}| \le 4$, *are bounded on* Δ_h *as* $h \to 0$.

PROOF. Take any $\boldsymbol{x} \in \Delta_h$. By (5) and (14)–(16)

$$\begin{aligned} p_{23}(\varphi(\boldsymbol{x})) &= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}(\varphi(\boldsymbol{x}); U_2)\, f(h\, \boldsymbol{x}_{\boldsymbol{\alpha}}) \\ &= \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\varphi(\boldsymbol{x}); U_2\right)\, \left(h^2\, \boldsymbol{x}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{x}_{\boldsymbol{\alpha}} + \mathcal{O}(h^3)\right). \end{aligned}$$

But (22) implies

$$h^2\, \boldsymbol{x}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{x}_{\boldsymbol{\alpha}} + \mathcal{O}(h^3) = h^2\, \boldsymbol{u}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{u}_{\boldsymbol{\alpha}} + \mathcal{O}(h^3),$$

and further, since $\varphi(\boldsymbol{x}) = \boldsymbol{u}$,

$$\begin{aligned} &\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\varphi(\boldsymbol{x}); U_2\right)\, \left(h^2\, \boldsymbol{x}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{x}_{\boldsymbol{\alpha}} + \mathcal{O}(h^3)\right) = \\ &\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\boldsymbol{u}; U_2\right)\, \left(h^2\, \boldsymbol{u}_{\boldsymbol{\alpha}}^T K\, \boldsymbol{u}_{\boldsymbol{\alpha}} + \mathcal{O}(h^3)\right) = \\ &h^2\, \boldsymbol{u}^T K \boldsymbol{u} + \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\varphi(\boldsymbol{x}); U_2\right) \mathcal{O}(h^3) = \\ &\boldsymbol{x}^T K \boldsymbol{x} + \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\varphi(\boldsymbol{x}); U_2\right) \mathcal{O}(h^3) + \mathcal{O}(h^3). \end{aligned}$$

Now, by (23),

$$\sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\varphi(\boldsymbol{x}); U_2\right) \mathcal{O}(h^3) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \ell_{\boldsymbol{\alpha},2}\left(\frac{1}{h}\,\boldsymbol{x} + \mathcal{O}(h); U_2\right) \mathcal{O}(h^3).$$

It remains to prove that the derivatives

$$D^{\boldsymbol{\alpha}}\left(\ell_{\boldsymbol{\alpha},2}\left(\frac{1}{h}\,\boldsymbol{x} + \mathcal{O}(h); U_2\right)\; \mathcal{O}(h^3)\right), \quad |\boldsymbol{\alpha}| \le 4,$$

are bounded. Since $\ell_{\boldsymbol{\alpha},2}(\cdot; U_2)$ can be defined similarly as $\ell_{\boldsymbol{\alpha},3}\left(\cdot; X_3^h\right)$ in (21), it is easy to see that the denominators of the terms in $\ell_{\boldsymbol{\alpha},2}(\frac{1}{h}\,\boldsymbol{x} + \mathcal{O}(h); U_2)$ are $\mathcal{O}(h^2)$. This implies

$$\ell_{\boldsymbol{\alpha},2}\left(\frac{1}{h}\,\boldsymbol{x} + \mathcal{O}(h); U_2\right)\; \mathcal{O}(h^3) = h\, g(\boldsymbol{x}) + \mathcal{O}(h^2),$$

where g is a smooth function and the proof of the lemma is complete.

7 Examples

Let us illustrate the results by two numerical examples. First, let us consider

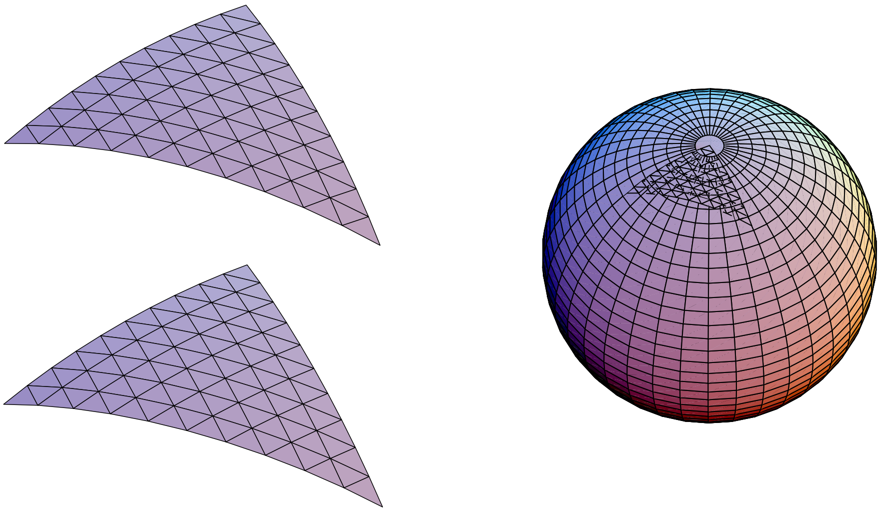

Fig. 4. The quadratic geometric interpolant and the approximated part of the sphere (left) and the interpolant on the sphere (right).

an approximation of a part of the unit sphere over the domain triangle based upon the points

$$(0,0)^T,\ \left(\frac{1}{2},0\right)^T,\ \left(0,\frac{1}{2}\right)^T.$$

The interpolation points $\left(u,v,\sqrt{1-u^2-v^2}\right)^T$ are sampled from the sphere at parameter values

$$\begin{gathered}(0,0)^T,\ \left(\frac{1}{6},0\right)^T,\ \left(\frac{1}{3},0\right)^T,\ \left(\frac{1}{2},0\right)^T,\ \left(0,\frac{1}{6}\right)^T,\\ \left(\frac{1}{6},\frac{1}{6}\right)^T,\ \left(\frac{1}{3},\frac{1}{6}\right)^T,\ \left(0,\frac{1}{3}\right)^T,\ \left(\frac{1}{6},\frac{1}{3}\right)^T,\ \left(0,\frac{1}{2}\right)^T,\end{gathered}$$

determined by a principal lattice. In Table 2 the approximation error, measured as a radial distance, and numerical approximation order are presented as the domain triangle, defined by

$$(0,0)^T,\ \left(\frac{1}{2}h,0\right)^T,\ \left(0,\frac{1}{2}h\right)^T,$$

shrinks. The quadratic geometric interpolant and the sphere for $h=1$ are shown in Fig. 4. The differences between the interpolant (top) and the approximated sphere patch (bottom) are almost indistinguishable (Fig. 4, left, and Table 2).

As the second example, consider an approximation of the surface

$$(u, v, \exp(u) + \exp(v) - u - v - 1)^T$$

over the same domain triangle and at the same prescribed parameter values as in the previous example. The error (Table 2) is given as a radial distance, i.e., the distance between surfaces on the rays from the origin $(0, 0, 0)^T$.

Table 2
The error in geometric interpolation.

h	Approximation error		Decay exponent	
	Sph.	Exp.	Sph.	Exp.
1	5.10989×10^{-4}	1.64540×10^{-4}	—	—
$\frac{9}{10}$	3.16648×10^{-4}	1.02523×10^{-4}	-4.54	-4.49
$\frac{8}{10}$	1.88103×10^{-4}	6.07809×10^{-5}	-4.42	-4.44
$\frac{7}{10}$	1.05643×10^{-4}	3.38710×10^{-5}	-4.32	-4.38
$\frac{6}{10}$	5.49874×10^{-5}	1.74568×10^{-5}	-4.24	-4.30
$\frac{5}{10}$	2.57269×10^{-5}	8.06412×10^{-6}	-4.17	-4.24
$\frac{4}{10}$	1.02831×10^{-5}	3.17273×10^{-6}	-4.11	-4.18
$\frac{3}{10}$	3.19296×10^{-6}	9.67434×10^{-7}	-4.07	-4.13
$\frac{2}{10}$	6.22342×10^{-7}	1.84838×10^{-7}	-4.03	-4.08
$\frac{1}{10}$	3.85875×10^{-8}	1.12178×10^{-8}	-4.01	-4.04

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