Geometric Lagrange Interpolation by Planar Cubic Pythagorean-hodograph Curves

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Abstract

In this paper, the geometric Lagrange interpolation of four points by planar cubic Pythagorean-hodograph (PH) curves is studied. It is shown that such an interpolatory curve exists provided that the data polygon, formed by the interpolation points, is convex, and satisfies an additional restriction on its angles. The approximation order is 4. This gives rise to a conjecture that a PH curve of degree n can, under some natural restrictions on data points, interpolate up to $n+1$ points.

Key words: planar curve, PH curve, geometric interpolation, Lagrange interpolation

1 Introduction

Pythagorean-hodograph planar curves (PH curves) were introduced in [1]. They form an important class of planar parametric polynomial curves for which the arc-length can be computed exactly and their offsets are rational curves. This makes them very useful in many practical applications, e.g. in CAD/CAM systems, robotics, animation, NC machining, etc. The PH curves have attracted a lot of attention of researchers in the last two decades. Many results on the Hermite type interpolation by PH curves have been obtained (see [2], [3], [4], [5], [6], e.g.), but it seems that there are no results on the Lagrange type interpolation. Hermite type interpolation methods are very

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useful in computer aided geometric design (CAGD) since polynomial pieces can be easily smoothly joined. On the other hand, in practical applications it is often difficult to obtain information about derivatives. If a PH curve has to be evaluated online, where all the data are not available in advance, one has to be able to compute the values on the curve efficiently using only information on the position of already known interpolated points. In this case a Lagrange type interpolation by PH curves similar to Aitken interpolation is needed. Of course there is a serious drawback that usually one can not put piecewise polynomial Lagrange interpolants together to form a smooth spline curve.

Formally, a PH curve is defined as follows. Suppose that $\boldsymbol{p} : [a, b] \to \mathbb{R}^2$, $\boldsymbol{p}(t) := (x(t), y(t))^T$, where x and y are polynomials of degree $\leq n$, is a planar polynomial curve. Then $\boldsymbol{p}$ is said to have a *Pythagorean hodograph* if and only if

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2, \quad \forall t \in [a, b],$$

for some polynomial σ. It is well known ([1]) that a PH curve of degree n has $n + 3$ degrees of freedom, i.e., $n - 1$ less than a general planar parametric polynomial curve of the same degree. Thus it is expected that it can interpolate at most $\lfloor (n+3)/2 \rfloor$ points in the plane. But this is true only if the interpolation parameters are prescribed in advance. If one considers so called *geometric interpolation* (see [7], [8], [9], e.g.), where the interpolation parameters are supposed to be unknown, a larger number of points might be interpolated by a PH curve of the same degree n.

Suppose that we want to interpolate k points $\boldsymbol{T}_j \in \mathbb{R}^2$, $j = 0, 1, \ldots, k-1$, by a PH geometric interpolant $\boldsymbol{p}$ of degree n. Since a linear reparameterization does not affect the degree of $\boldsymbol{p}$ and preserves the PH property, we can assume that $t_0 := 0 < t_1 < t_2 < \cdots < t_{k-2} < t_{k-1} := 1$. A PH curve $\boldsymbol{p}$ has $n + 3$ degrees of freedom and $k - 2$ new ones are provided by unknown interpolation parameters t_j. Since $2\,k$ interpolation conditions $\boldsymbol{p}(t_j) = \boldsymbol{T}_j$, $j = 0, 1, \ldots, k-1$, have to be fulfilled, the following conjecture is right at hand.

Conjecture 1 *A planar PH curve of degree n can interpolate up to $n + 1$ points.*

A similar conjecture for general planar parametric curves has been stated in [10], namely, that a planar polynomial parametric curve of degree n can interpolate $2n$ data - much more than $n + 1$ as in the standard case. The conjecture has not been confirmed for general degree n yet, since the problem turned out to be very hard. But it holds true for curves of small degrees $n \leq 5$ (see [9] and the references therein).

For the geometric Lagrange interpolation by PH curves it is expected that some reasonable conditions on the geometry of data points have to be added (similarly as in the conjecture for general curves) and it is also quite clear that the conjecture on PH curves might be even harder to prove than the one for general curves. Thus it is reasonable to study some particular cases first. In this paper, we will consider the Lagrange interpolation by cubic PH curves, the case quite frequently encountered in practical applications. We will show, that a planar cubic PH curve can interpolate 4 data points under some natural restrictions.

The paper is organized as follows. In Section 2 a detailed explanation of the interpolation problem is given together with the derivation of nonlinear equations that have to be studied. Section 3 provides the main result of the paper. Since the proofs of the main theorems require several steps, they are given as a separate section. In the last section, some numerical examples are outlined, which confirm the results of the paper.

2 Interpolation problem

Let us start with the notation first. Throughout the paper $\boldsymbol{u} \cdot \boldsymbol{v}$ denotes the standard scalar product, $\boldsymbol{u} \times \boldsymbol{v}$ is the standard planar vector product, and $\|\cdot\|$ is the Euclidean norm. Further, let $\angle(\boldsymbol{u}, \boldsymbol{v})$ denote the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\Delta(\cdot)_i := (\cdot)_{i+1} - (\cdot)_i$, and

$$Q(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

be the rotation matrix.

Let us introduce the setup. Suppose that an ordered set of points $\boldsymbol{T}_j$, $j = 0, 1, 2, 3$, in the plane is given, where $\boldsymbol{T}_j \neq \boldsymbol{T}_{j+1}$, $j = 0, 1, 2$. A cubic PH curve $\boldsymbol{p}$, which interpolates the given data, needs to be found. Since the number of degrees of freedom of a PH cubic is known to be 6 ([1]), the points $\boldsymbol{T}_j$ in general can not be interpolated by a PH cubic at *the prescribed* values of interpolation parameters. Thus one has to interpolate them in a geometric sense. Let

$$t_0 := 0 < t_1 < t_2 < t_3 := 1 \tag{1}$$

be a sequence of parameters, where t_1 and t_2 are unknown. The PH curve $\boldsymbol{p}$ has to satisfy the interpolation conditions

$$\boldsymbol{p}(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, 2, 3. \tag{2}$$

It turns out that it is prosperous to consider the interpolant $\boldsymbol{p}$ in the Bézier form,

$$\boldsymbol{p}(t) = \sum_{i=0}^{3} \boldsymbol{b}_i B_i^3(t), \tag{3}$$

where $\boldsymbol{b}_i$ are the Bézier control points of the curve and

$$B_i^3(t) = \binom{3}{i} t^i (1-t)^{3-i}, \quad i = 0,1,2,3,$$

are the Bernstein polynomials of degree 3. Due to the boundary control points interpolation property, $\boldsymbol{b}_0 = \boldsymbol{T}_0$ and $\boldsymbol{b}_3 = \boldsymbol{T}_3$. In order to determine the interpolant, the unknown control points $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ as well as the parameters t_1 and t_2, have to be found.

The interpolation conditions (2) for $j = 1,2$, imply the equations

$$\boldsymbol{b}_1 B_1^3(t_j) + \boldsymbol{b}_2 B_2^3(t_j) = \boldsymbol{T}_j - \boldsymbol{b}_0 B_0^3(t_j) - \boldsymbol{b}_3 B_3^3(t_j) =: \boldsymbol{c}_j, \quad j = 1,2,$$

which can be written as a linear system for $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$,

$$B\,\boldsymbol{b} = \boldsymbol{c}, \tag{4}$$

where

$$B = \begin{pmatrix} B_1^3(t_1) & B_2^3(t_1) \\ B_1^3(t_2) & B_2^3(t_2) \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_1^T \\ \boldsymbol{b}_2^T \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} \boldsymbol{c}_1^T \\ \boldsymbol{c}_2^T \end{pmatrix}.$$

Since

$$\det B = 9\, t_1\, t_2\, (1-t_1)(1-t_2)(t_2-t_1),$$

the linear system (4) has a unique solution for any set of parameters (1). A straightforward computation reveals that $\Delta\boldsymbol{b}_i := \Delta\boldsymbol{b}_i(t_1,t_2)$, $i = 0,1,2$, can be written as

$$\begin{aligned}
\Delta\boldsymbol{b}_0 &:= \frac{t_1+t_2+t_1t_2}{3t_1t_2}\,\Delta\boldsymbol{T}_0 - \frac{t_1\,(t_2^2+(1-t_1)\,(1+t_2))}{3\,(1-t_1)\,(t_2-t_1)\,t_2}\,\Delta\boldsymbol{T}_1 \\
&\quad + \frac{t_1t_2}{3\,(1-t_1)\,(1-t_2)}\,\Delta\boldsymbol{T}_2, \\
\Delta\boldsymbol{b}_1 &:= -\frac{1-t_1t_2}{3t_1t_2}\,\Delta\boldsymbol{T}_0 + \frac{t_2t_1^2+(t_2^2+t_2+1)\,(1-t_1)}{3\,(1-t_1)\,(t_2-t_1)\,t_2}\,\Delta\boldsymbol{T}_1 \\
&\quad - \frac{t_1\,(1-t_2)+t_2}{3\,(1-t_1)\,(1-t_2)}\,\Delta\boldsymbol{T}_2, \\
\Delta\boldsymbol{b}_2 &:= \frac{(1-t_1)\,(1-t_2)}{3t_1t_2}\,\Delta\boldsymbol{T}_0 - \frac{(1-t_2)\left((1-t_1)^2+(2-t_1)\,t_2\right)}{3\,(1-t_1)\,(t_2-t_1)\,t_2}\,\Delta\boldsymbol{T}_1 \\
&\quad + \frac{t_1\,(t_2-2)-2t_2+3}{3\,(1-t_1)\,(1-t_2)}\,\Delta\boldsymbol{T}_2.
\end{aligned} \tag{5}$$

But (2) are not the only equations which have to be considered. One has to assure that $\boldsymbol{p}$ is also a PH curve. There are several equivalent characterizations of cubic PH curves, but usually the complex representation is the most suitable one. Thus if control points $\boldsymbol{b}_i$ are considered as complex numbers (using the standard representation of planar vectors as complex numbers) then by [11], a cubic parametric curve is a PH curve if and only if

$$(\Delta \boldsymbol{b}_1)^2 - \Delta \boldsymbol{b}_0 \, \Delta \boldsymbol{b}_2 = 0. \tag{6}$$

This is a complex equation for two scalar unknowns t_1 and t_2. Note that (6) is equivalent to well-known conditions on Bézier control polygon of a PH curve, namely $\|\Delta \boldsymbol{b}_1\| = \sqrt{\|\Delta \boldsymbol{b}_0\| \, \|\Delta \boldsymbol{b}_2\|}$ and $\theta_1 = \theta_2$ (see Fig. 1). Note also that the

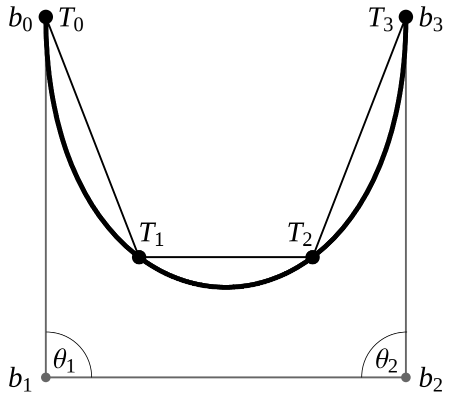

Fig. 1. A cubic PH Lagrange interpolant together with its control polygon and the data polygon.

use of the complex-valued approach does not overcome the problem of solving a nonlinear system of equations for t_1 and t_2. This makes geometric Lagrange type interpolation methods much more complicated than Hermite type ones.

The equation (6) is rational in t_1 and t_2. Taking its real and imaginary part leads to a system of two real rational equations

$$\boldsymbol{e}(t_1, t_2) := \boldsymbol{e}\,(t_1, t_2; T) = (e_i(t_1, t_2; T))_{i=1}^2 = \mathbf{0}, \tag{7}$$

where

$$\begin{aligned} e_1(t_1, t_2; T) &:= \Delta b_{1,x}^2 - \Delta b_{1,y}^2 - \Delta b_{0,x} \Delta b_{2,x} + \Delta b_{0,y} \Delta b_{2,y}, \\ e_2(t_1, t_2; T) &:= 2\Delta b_{1,x} \Delta b_{1,y} - \Delta b_{0,x} \Delta b_{2,y} - \Delta b_{0,y} \Delta b_{2,x}, \end{aligned}$$

and $\Delta \boldsymbol{b}_i := (\Delta b_{i,x}, \Delta b_{i,y})^T$. Note that $T := (\boldsymbol{T}_j)_{j=0}^3$ refers to the fact that the system depends on data points $\boldsymbol{T}_j$, $j = 0, 1, 2, 3$. It is easy to see that the denominators in (7) vanish only for $t_1 = 0, 1$, $t_2 = 0, 1$ and $t_1 = t_2$, thus the system can be transformed to the system of two polynomial equations, each of them of total degree 8. Applying the straightforward approach using the Gröbner basis or resultants on the obtained polynomial system seems hopeless. Thus an in-depth analysis is needed.

3 Cubic PH Lagrange interpolation

Some quick numerical experiments show that more than one solution of the interpolation problem might exist. This is a common fact observed already in the Hermite case. Since some of the solutions are clearly not acceptable in CAGD (having loops, e.g.), some kind of shape preserving will be required. The following definition gives a description of an appropriate interpolating cubic PH curve.

Definition 2 *A cubic Bézier PH curve* (3) *which interpolates data points* $\boldsymbol{T}_j$*,* $j = 0, 1, 2, 3$*, in a geometric sense is admissible curve if*

$$(\Delta\boldsymbol{b}_i \times \Delta\boldsymbol{b}_{i+1})\,(\Delta\boldsymbol{T}_i \times \Delta\boldsymbol{T}_{i+1}) > 0, \quad i = 0, 1.$$

Accordingly, the solution of the nonlinear system (7) *is an admissible solution if the unknowns lie in*

$$\mathcal{D} := \big\{(t_1, t_2) \in \mathbb{R}^2 \,|\, 0 < t_1 < t_2 < 1\big\},$$

and the resulting curve is admissible.

Observe that the above definition excludes those configurations of data points for which $\Delta\boldsymbol{T}_i \times \Delta\boldsymbol{T}_{i+1} = 0$, $i = 0$ or $i = 1$. But in this case at least three of them must lie on a straight line. If all of them are collinear, the interpolating PH curve must be a straight line. If three consecutive data points are collinear, then either the resulting cubic PH interpolant forms a loop since cubic PH curves do not have any inflection points (see [1], e.g.), or the data polygon has a fold. Both cases should be excluded in practical applications. Thus we can assume that three consecutive data points are not collinear.

Note further that the non-convex data points $\boldsymbol{T}_j$, $j = 0, 1, 2, 3$,

$$(\Delta\boldsymbol{T}_0 \times \Delta\boldsymbol{T}_1)\,(\Delta\boldsymbol{T}_1 \times \Delta\boldsymbol{T}_2) < 0,$$

can not be interpolated by an admissible cubic PH curve. As a consequence we shall concentrate only on those configurations for which

$$(\Delta\boldsymbol{T}_0 \times \Delta\boldsymbol{T}_1)\,(\Delta\boldsymbol{T}_1 \times \Delta\boldsymbol{T}_2) > 0.$$

It can still happen that an admissible curve forms a loop, e.g., if the data polygon has a self intersection (see Fig 4).

We are now ready to state the main results of the paper.

Theorem 3 *Suppose that the data points* $\boldsymbol{T}_j$*,* $j = 0, 1, 2, 3$*, satisfy*

$$(\Delta\boldsymbol{T}_0 \times \Delta\boldsymbol{T}_1)\ (\Delta\boldsymbol{T}_1 \times \Delta\boldsymbol{T}_2) > 0 \quad \textit{and} \quad \gamma_1(T) + \gamma_2(T) < 4\pi/3,$$

where

$$\gamma_1(T) := \angle(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_1), \quad \gamma_2(T) := \angle(\Delta\boldsymbol{T}_1, \Delta\boldsymbol{T}_2).$$

Then an admissible cubic PH curve $\boldsymbol{p}$*, which satisfies* (2)*, exists.*

The following theorem extends the asymptotic approximation order obtained in [4] to the Lagrange case. In comparison to [7] it is lower by 2 as expected since two degrees of freedom are used by the PH condition. However, the same order as in the function case is still achieved.

Theorem 4 *Let the data points be sampled from a smooth regular convex curve*

$$\boldsymbol{f} : [-h, h] \to \mathbb{R}^2.$$

One can find $h_0 > 0$ *such that for all* $h \le h_0$ *there exists an admissible cubic PH curve* $\boldsymbol{p}$ *which satisfies* (2)*. The approximation order is four.*

Theorem 3 provides us with a sufficient condition on the existence of admissible cubic PH curves. If the angle restriction is violated, the number of admissible solutions is even, in most cases zero. Consider the following example. Let the data points be chosen as

$$\begin{aligned} &\boldsymbol{T}_0 = (0,0)^T, \quad \boldsymbol{T}_1 = \left(-1, \frac{1}{4}\right)^T, \quad \boldsymbol{T}_2 = \left(-\frac{1}{2}, -1\right)^T, \\ &\boldsymbol{T}_3 = \boldsymbol{T}_2 + 10\, Q\,(\xi_0 + \xi)\, \frac{\Delta\boldsymbol{T}_1}{\|\Delta\boldsymbol{T}_1\|}, \quad \xi_0 := \frac{4\pi}{3} - \gamma_1(T), \end{aligned} \tag{8}$$

where ξ is a free parameter. Now, the assumptions of Theorem 3 hold for every $-\xi_0 < \xi < 0$, and as one can check, the interpolation problem has a unique admissible solution. But for $\xi > 0$ the angle restriction is violated. Two admissible solutions exist for $0 < \xi < \tilde{\xi}$, $\tilde{\xi} := 0.0220188\,\pi$, that turn into one at $\xi = \tilde{\xi}$. But, for $\xi > \tilde{\xi}$ no (admissible) solutions were found. Fig. 2 shows the admissible PH curves for $\xi = -0.02\pi$, 0.02π, 0.022π, 0.022018π. In the first case there is a unique interpolant, and there are two of them for the other cases.

Since the proofs of Theorem 3 and Theorem 4 take several steps, they will be given as a separate section.

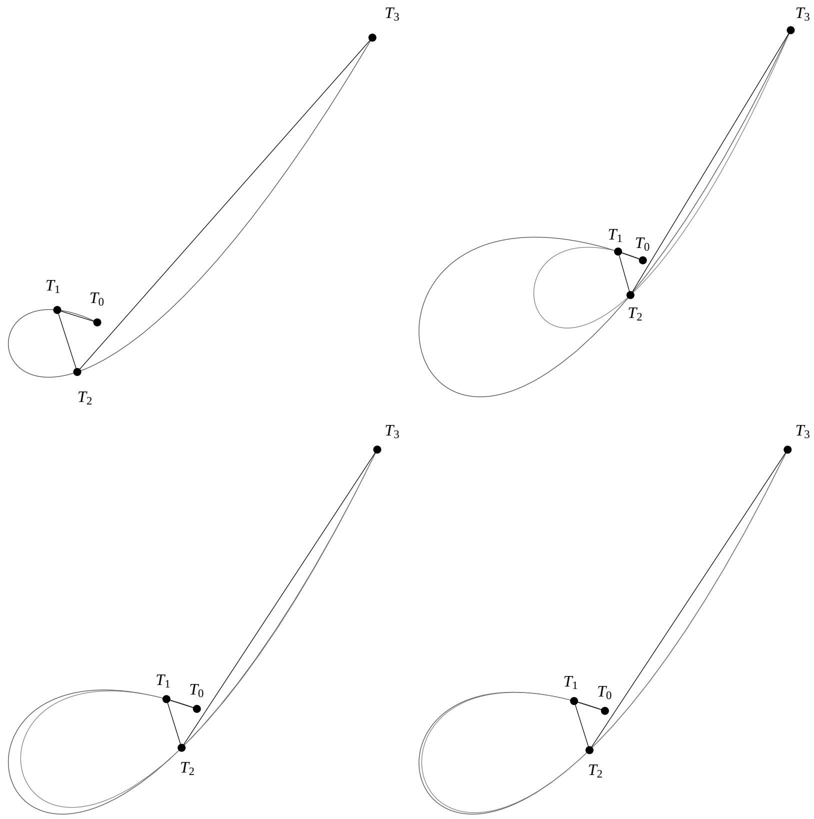

Fig. 2. Admissible PH curves for data (8) with $\xi = -0.02\pi$ and $\xi = 0.02\pi$ (top left and right), $\xi = 0.022\pi$ and $\xi = 0.022018\pi$ (bottom left and right).

4 Proofs of theorems

Without loosing generality we may assume throughout the paper that $\boldsymbol{T}_0$ is at the origin, and

$$\Delta \boldsymbol{T}_i \times \Delta \boldsymbol{T}_{i+1} > 0, \quad i = 0, 1.$$

If this is not the case, a simple transformation of data points leads to a desired configuration. To prove that an interpolating cubic PH curve exists is clearly equivalent to show that a nonlinear system of equations (7) has an admissible solution. In order to prove this, consider the following particular problem first. Let the data points

$$\boldsymbol{U}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{U}_1 = \begin{pmatrix} \frac{7}{27} \\ -\frac{2}{3} \end{pmatrix}, \boldsymbol{U}_2 = \begin{pmatrix} \frac{20}{27} \\ -\frac{2}{3} \end{pmatrix}, \boldsymbol{U}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

be given (see Fig. 1). Consider the system of nonlinear equations $\boldsymbol{e}(t_1, t_2; U) = \boldsymbol{0}$, given by (7), where $U := (\boldsymbol{U}_j)_{j=0}^3$. Since coefficients are rational numbers now, one is able to compute the Gröbner basis of the corresponding equivalent polynomial system (obtained from (7) by simply multiplying each equation by their common denominator) exactly. The first Gröbner basis polynomial, obtained by an elimination of t_2, has a precisely one solution in $[0,1]$, i.e., $t_1 = \frac{1}{3}$. Since a similar result can be obtained for t_2, the pair $\left\{\frac{1}{3}, \frac{2}{3}\right\}$ is the only solution satisfying $0 < t_1 < t_2 < 1$. It is easy to verify that in this case the control points of the resulting interpolating Bézier curve satisfy the admissibility condition. Thus this is the only admissible solution (see Fig. 1).

The fact that a particular set of data points U admits an odd number (in this case precisely one) of admissible solutions will now be carried over to the general case of data points T by a homotopy. Since a rotation of data does not affect the solution we can assume that

$$\frac{\Delta \boldsymbol{T}_0}{\|\Delta \boldsymbol{T}_0\|} = \frac{\Delta \boldsymbol{U}_0}{\|\Delta \boldsymbol{U}_0\|}.$$

However, the angle restriction in Theorem 3 requires that a homotopy is chosen carefully. Namely, let $\boldsymbol{H} : \mathcal{D} \times [0,1] \to \mathbb{R}^2$ be defined by

$$\boldsymbol{H}(t_1, t_2, \lambda) := \boldsymbol{e}(t_1, t_2; W(\lambda)),$$

where data $W(\lambda) := (\boldsymbol{W}_j(\lambda))_{j=0}^3$ are determined by angles $\angle(\boldsymbol{W}_{i-1}(\lambda), \boldsymbol{W}_i(\lambda))$ and norms $\|\Delta \boldsymbol{W}_i(\lambda)\|$. More precisely, with

$$\gamma_i(\lambda) := \gamma_i(W(\lambda)) = (1-\lambda)\,\gamma_i(T) + \lambda\,\gamma_i(U), \quad i = 1, 2,$$

and

$$L_i(\lambda) := (1-\lambda)\|\Delta \boldsymbol{T}_i\| + \lambda\|\Delta \boldsymbol{U}_i\|, \quad i = 0, 1, 2,$$

the data $\boldsymbol{W}_i(\lambda)$ reads

$$\begin{aligned}
\boldsymbol{W}_0(\lambda) &:= \boldsymbol{T}_0, \\
\boldsymbol{W}_1(\lambda) &:= \boldsymbol{T}_0 + L_0(\lambda)\frac{\Delta \boldsymbol{T}_0}{\|\Delta \boldsymbol{T}_0\|}, \\
\boldsymbol{W}_2(\lambda) &:= \boldsymbol{W}_1(\lambda) + L_1(\lambda) Q(\gamma_1(\lambda))\frac{\Delta \boldsymbol{T}_0}{\|\Delta \boldsymbol{T}_0\|}, \\
\boldsymbol{W}_3(\lambda) &:= \boldsymbol{W}_2(\lambda) + L_2(\lambda) Q(\gamma_2(\lambda))\frac{\Delta \boldsymbol{W}_1(\lambda)}{\|\Delta \boldsymbol{W}_1(\lambda)\|}.
\end{aligned}$$

This homotopy clearly transforms the general data T to the particular data U preserving the original upper bound on the angles, $4\pi/3$. Fig. 3 demonstrates how general data are connected to the particular data by the described homotopy for two examples of data points.

The following lemma reveals that there are no admissible solutions near the boundary.

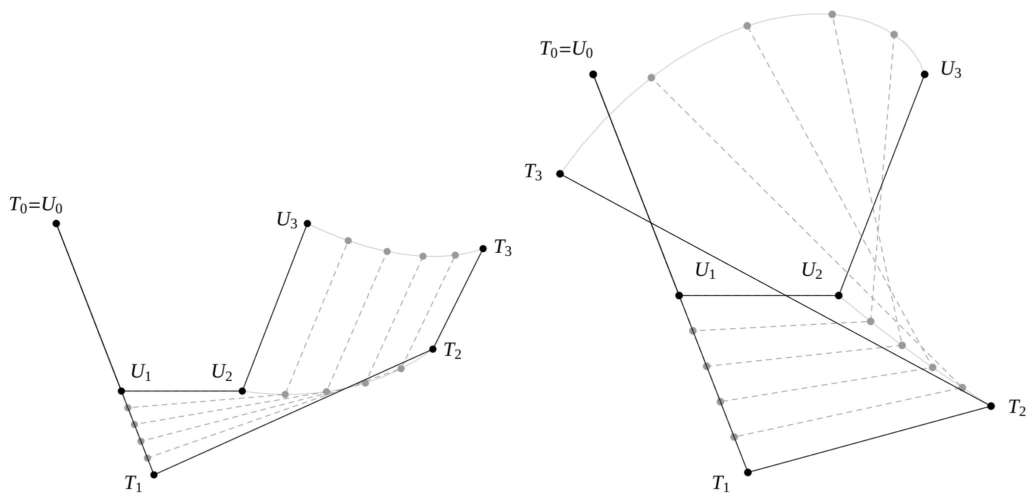

Fig. 3. Homotopy connecting general and particular data.

Lemma 5 *Suppose that the assumptions of Theorem 3 hold. Then the system* (7) *can not have an admissible solution close to the boundary* $\partial\mathcal{D}$.

PROOF. Since the data points $\boldsymbol{W}_i(\lambda)$ by the construction of the homotopy for any $\lambda \in [0,1]$ satisfy the assumption of the theorem, we need to prove the assertion for the points $\boldsymbol{T}_i$ only. The boundary $\partial\mathcal{D}$ contains $t_1 = 0$, $t_1 = t_2$ and $t_2 = 1$, thus we have to consider a couple of particular cases.

There are six possible approaches to the boundary $\partial\mathcal{D}$ to be excluded as a solution of the system (7). From

$$\begin{aligned} \|\boldsymbol{e}\,(\varepsilon, t_2)\,\|^2 &= \frac{(t_2^2 - t_2 + 1)^{\,2}}{81 t_2^4}\,\|\Delta\boldsymbol{T}_0\|^4\,\frac{1}{\varepsilon^4} + \mathcal{O}\left(\frac{1}{\varepsilon^3}\right), \\ \|\boldsymbol{e}\,(t_1, 1-\varepsilon)\,\|^2 &= \frac{(t_1^2 - t_1 + 1)^{\,2}}{81\,(1-t_1)^4}\,\|\Delta\boldsymbol{T}_2\|^4\,\frac{1}{\varepsilon^4} + \mathcal{O}\left(\frac{1}{\varepsilon^3}\right), \\ \|\boldsymbol{e}\,(t_1, t_1+\varepsilon)\,\|^2 &= \frac{(t_1^2 - t_1 + 1)^{\,2}}{81(1-t_1)^4 t_1^4}\,\|\Delta\boldsymbol{T}_1\|^4\,\frac{1}{\varepsilon^4} + \mathcal{O}\left(\frac{1}{\varepsilon^3}\right), \end{aligned} \tag{9}$$

it follows that a single t_i can not approach the boundary since $\Delta\boldsymbol{T}_i$ should not vanish. However, (9) eliminates also the possibility that both unknowns tend to the boundary, but not at the same rate. So, with some constant $\tau > 0$, we only have to consider expansions

$$\begin{aligned} \|\boldsymbol{e}\,(\varepsilon, \varepsilon(1+\tau))\,\|^2 &= \frac{1}{81\tau^4(\tau+1)^4}\,\|-\tau\Delta\boldsymbol{T}_0 + \Delta\boldsymbol{T}_1\|^4\,\frac{1}{\varepsilon^8} + \mathcal{O}\left(\frac{1}{\varepsilon^7}\right), \\ \|\boldsymbol{e}\,(1-\varepsilon(1+\tau), 1-\varepsilon)\,\|^2 &= \frac{1}{81\tau^4(\tau+1)^4}\,\|\Delta\boldsymbol{T}_1 - \tau\Delta\boldsymbol{T}_2\|^4\,\frac{1}{\varepsilon^8} + \mathcal{O}\left(\frac{1}{\varepsilon^7}\right). \end{aligned}$$

Quite clearly, the leading terms can not vanish since $\Delta\boldsymbol{T}_i$ and $\Delta\boldsymbol{T}_{i+1}$ are not collinear. As to the last possibility, the system (7) expands as

$$\boldsymbol{e}\,(\varepsilon, 1-\varepsilon\,\tau) = \frac{1}{9\tau^2}\,\tilde{\boldsymbol{e}}\,(\tau)\;\frac{1}{\varepsilon^2} + \mathcal{O}\left(\frac{1}{\varepsilon}\right).$$

The resultant $\mathcal{R}\,(\,\tilde{\boldsymbol{e}}, \tau)$ of polynomials $\tilde{\boldsymbol{e}}$ simplifies to

$$\mathcal{R}\,(\,\tilde{\boldsymbol{e}}, \tau) = (\Delta\boldsymbol{T}_0 \times \Delta\boldsymbol{T}_2)^2 \left(4\,(\Delta\boldsymbol{T}_0 \cdot \Delta\boldsymbol{T}_2)^2 - \|\Delta\boldsymbol{T}_0\|^2\|\Delta\boldsymbol{T}_2\|^2\right), \tag{10}$$

and it remains to verify that this term can't vanish. If $\Delta\boldsymbol{T}_0 \times \Delta\boldsymbol{T}_2 = 0$, then $\Delta\boldsymbol{T}_0$ and $\Delta\boldsymbol{T}_2$ must be collinear. Further, if the second factor in (10) is equal to 0, then $\cos^2(\angle(\Delta\boldsymbol{T}_0, \Delta\boldsymbol{T}_2)) = \frac{1}{4}$. This gives the range of angles $\gamma_1(T)+\gamma_2(T) \in \left\{0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{5}{3}\pi\right\}$ that has to be considered. The assumptions of lemma shrink the set of angles to $\gamma_1(T) + \gamma_2(T) \in \left\{\frac{1}{3}\pi, \frac{2}{3}\pi, \pi\right\}$. Suppose that φ is one of these angles, and let $Q(\varphi)$ be the rotation matrix that brings $\Delta\boldsymbol{T}_0$ to the direction of $\Delta\boldsymbol{T}_2$,

$$Q(\varphi)\Delta\boldsymbol{T}_0 = \omega\Delta\boldsymbol{T}_2, \quad \omega := \frac{\|\Delta\boldsymbol{T}_0\|}{\|\Delta\boldsymbol{T}_2\|} > 0.$$

Then

$$\|\boldsymbol{e}\,(\varepsilon, 1-\varepsilon\,\tau)\,\|^2 = \frac{1}{81\tau^4}\,g(\varphi)\,\|\Delta\boldsymbol{T}_2\|^4\,\frac{1}{\varepsilon^4} + \mathcal{O}\left(\frac{1}{\varepsilon^3}\right), \tag{11}$$

with

$$g(\varphi) := \left(\tau^4\omega^4 + \tau^2\omega^2 + 2\tau\omega\left(\tau^2\omega^2\cos\varphi + \tau\omega\cos 2\varphi + \cos\varphi\right) + 1\right).$$

However, the expansion (11) can not vanish at $\frac{1}{3}\pi$ and π since

$$g\left(\frac{1}{3}\pi\right) = (\tau\omega+1)^2(\tau^2\omega^2 - \tau\omega + 1) > 0, \quad g\,(\pi) = (\tau^2\omega^2 - \tau\omega + 1)^2 > 0.$$

Finally, if $\gamma_1(T) + \gamma_2(T) = \frac{2}{3}\pi$, the leading term in (11) may vanish, but the solution that crosses the boundary $\partial\mathcal{D}$ is not the admissible one since the vector product

$$\Delta\boldsymbol{b}_0 \times \Delta\boldsymbol{b}_1 = -\frac{\omega}{6\sqrt{3}\,\tau}\,\|\Delta\boldsymbol{T}_2\|^2\frac{1}{\varepsilon^2} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

is supposed to be positive. The proof, that there is no admissible solution close to the boundary $\partial\mathcal{D}$, is complete.

A very well known fact about homotopy invariants (see [12], e.g.) states that the Brouwer's degree of $\boldsymbol{H}$ is invariant as soon as the homotopy is nonzero at the boundary $\partial\mathcal{D}$ for any $\lambda \in [0,1]$. The map $\boldsymbol{H}$ is formally not defined on $\partial\mathcal{D}$, but by Lemma 5, $\boldsymbol{H}$ is nonzero close to $\partial\mathcal{D}$, so it is nonzero on the boundary of some compact set $\mathcal{E} \subset \mathcal{D}$ and the same conclusion follows. Now, since the particular system of nonlinear equations $\boldsymbol{e}(t_1, t_2; U) = \mathbf{0}$ has a precisely one simple solution, the Brouwer's degree of $\boldsymbol{H}(t_1, t_2, 1)$ is ± 1. Thus the Brouwer's degree of $\boldsymbol{H}(t_1, t_2, 0) = \boldsymbol{e}(t_1, t_2; T)$ must be odd too. Then the system $\boldsymbol{e}(t_1, t_2; T)$ has at least one admissible solution and the proof of Theorem 3 is concluded.

It is important to note that the previous analysis reveals the possibility of an another solution of the problem, which is not admissible by definition. This is not surprising, since we know that extraneous PH curves, which interpolate the same data set, exist also in the Hermite type interpolation and various approaches are known to avoid them (see [13], e.g.).

Let us conclude this section by the proof of Theorem 4. Suppose that the curve $f\colon [-h, h] \to \mathbb{R}^2$ is smooth regular and convex as required. Without loosing generality we may assume that it is parameterized by the first component, and $\boldsymbol{f}(0) = (0, 0)^T$, $\boldsymbol{f}'(0) = (1, 0)^T$. Thus $\boldsymbol{f}$ expands at 0 as

$$\boldsymbol{f}(s) = \begin{pmatrix} s \\ \frac{c_2}{2}\, s^2 + \frac{c_3}{6}\, s^3 + \frac{c_4}{24}\, s^4 + \mathcal{O}\left(s^5\right) \end{pmatrix},$$

where $c_2 > 0$ since $\boldsymbol{f}$ is assumed to be convex. Further, let the data points be sampled as

$$\boldsymbol{T}_i = \boldsymbol{f}\left(\, (2\eta_i - 1)\, h\,\right), \quad i = 0, 1, 2, 3, \tag{12}$$

where

$$0 = \eta_0 < \eta_1 < \eta_2 < \eta_3 = 1.$$

Let us introduce new unknowns z_i by a guess

$$t_i = \eta_i + (1 - \eta_i)\, \eta_i\, z_i\, h, \quad i = 1, 2\,.$$

The equations (7) expand as

$$\begin{aligned} &\frac{4}{3\,(\eta_2 - \eta_1)}\, h^3\, (z_1 - z_2) + \mathcal{O}\left(h^4\right) = 0, \\ &\frac{8c_2}{9\,(\eta_2 - \eta_1)}\, ((3\eta_1 + \eta_2 - 2)\, z_1 - (\eta_1 + 3\eta_2 - 2)\, z_2)\, h^4 \\ &\qquad\qquad - \frac{8}{9} h^4 c_3 + \mathcal{O}\left(h^5\right) = 0. \end{aligned}$$

This reveals the unknowns z_i as

$$z_1 = -\frac{c_3}{2c_2} + \mathcal{O}\,(h)\,, \quad z_2 = -\frac{c_3}{2c_2} + \mathcal{O}\,(h)\,,$$

which shows that the system (7) has an asymptotic solution

$$t_i = \eta_i - \frac{c_3}{2c_2}\, (1 - \eta_i)\, \eta_i\, h + \mathcal{O}\left(h^2\right), \quad i = 1, 2.$$

Let us insert this solution and the data points (12) in (5). The expansions simplify to

$$\Delta \boldsymbol{b}_i = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} h + \mathcal{O}\left(h^2\right), \ i = 0, 1, 2,$$

and higher order terms give

$$\Delta^2 \boldsymbol{b}_i = \begin{pmatrix} -\frac{c_3}{3c_2} \\ \frac{2}{3} c_2 \end{pmatrix} h^2 + \mathcal{O}\left(h^3\right), \ i = 0, 1, \quad \Delta^3 \boldsymbol{b}_0 = \begin{pmatrix} \frac{c_3^2}{c_2^2} \\ -\frac{2}{3} c_3 \end{pmatrix} h^3 + \mathcal{O}\left(h^4\right).$$

So the convex hull property implies that the interpolating Bézier curve satisfies $\boldsymbol{p}' = (2\,h, 0)^T + \mathcal{O}(h^2)$, $\boldsymbol{p}^{(r)} = \mathcal{O}\left(h^r\right)$, $r = 2, 3$, for all h small enough. But then $\boldsymbol{p}$ could be reparameterized by the first component $t = (\boldsymbol{p})_1^{-1}(s)$ on $s \in [-h, h]$, and the derivatives of the reparameterized curve stay bounded for all h small enough as in [7]. Since then $\boldsymbol{f}$ and $\boldsymbol{p}$ agree at η_i, $i = 1, 2, 3, 4$, the approximation order $\mathcal{O}\left(h^4\right)$ follows.

5 Numerical examples

In this section some numerical examples will be given which confirm the obtained results. Take the data points

$$\boldsymbol{T}_0 = (0, 0)^T, \quad \boldsymbol{T}_1 = \left(0, -\frac{1}{3}\right)^T, \quad \boldsymbol{T}_2 = \left(\xi, -\frac{1}{20}\xi - \frac{1}{3}\right)^T, \quad \boldsymbol{T}_3 = (1, 0)^T, \tag{13}$$

where ξ is a free parameter (Fig. 4). Consider the following six choices of the parameter ξ,

$$\xi = -\frac{1}{7}, -\frac{1}{8}, \frac{1}{10}, \frac{2}{3}, 1, \frac{7}{4}. \tag{14}$$

The system of nonlinear equations (7) will be solved by the continuation method. For the second case in (14), the system of nonlinear equations (7) has two admissible solutions. According to Theorem 3, $\gamma_1(T) + \gamma_2(T) > 4\pi/3$. For $\xi = 1/10$, the system (7) has two solutions, but one of them has an undesirable loop (the solution is not admissible). Here $\gamma_1(T) + \gamma_2(T) < 2\pi/3$. If we choose the parameter ξ such that $\gamma_1(T) + \gamma_2(T) \in (2\pi/3, 4\pi/3)$ (next two cases in (14)), one of the solutions disappears. This is to be expected according to the proof of Theorem 3. For the first and the last case in (14), $\gamma_1(T) + \gamma_2(T) > 4\pi/3$, but here no cubic PH interpolant exists. This fact can also be confirmed by computing the Gröbner basis of the equivalent polynomial system obtained from (7).

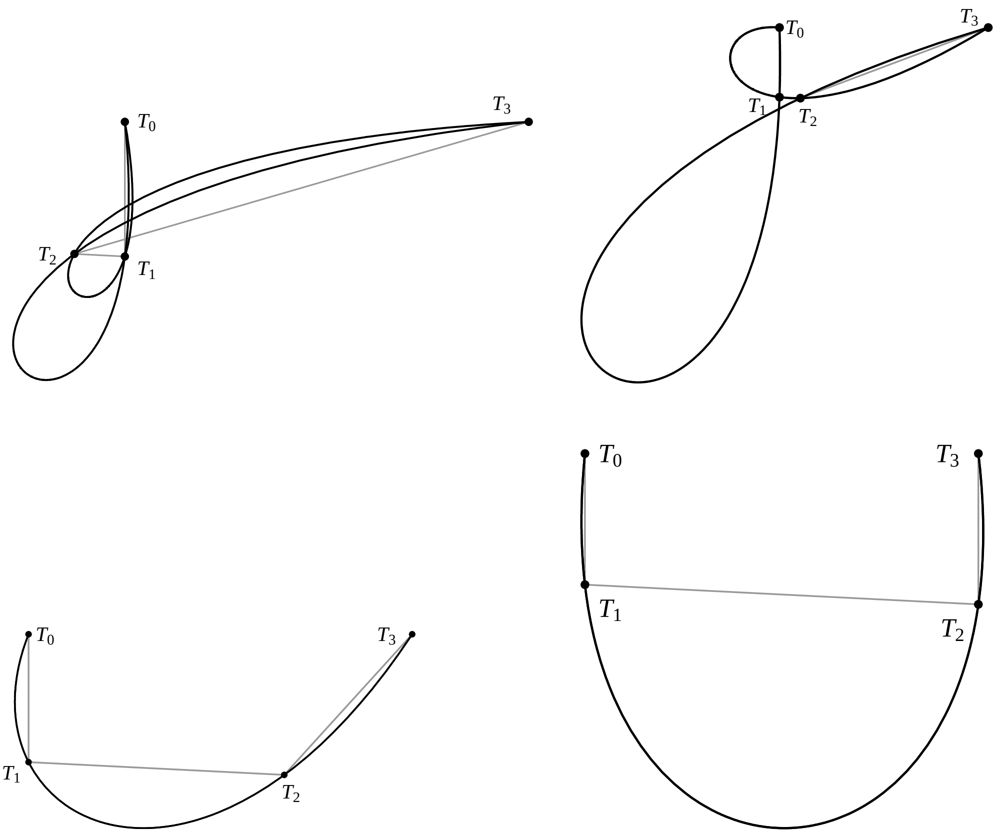

Fig. 4. Different choices of the parameter ξ imply different number of cubic PH interpolants for the data (13) and (14).

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