GENERALIZED RANK FUNCTIONS AND QUILTS OF ALTERNATING SIGN MATRICES

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Abstract. In this paper, we present new objects, *quilts of alternating sign matrices* with respect to two given posets. Quilts generalize several commonly used concepts in mathematics. For example, the rank function on submatrices of a matrix gives rise to a quilt with respect to two Boolean lattices. When the two posets are chains, a quilt is equivalent to an alternating sign matrix and its corresponding corner sum matrix. Quilts also generalize the monotone Boolean functions counted by the Dedekind numbers. Quilts form a distributive lattice with many beautiful properties and contain many classical and well-known sublattices, such as the lattice of matroids of a given rank and ground set. While enumerating quilts is hard in general, we prove two major enumerative results, when one of the posets is an antichain and when one of them is a chain. We also give some bounds for the number of quilts when one poset is the Boolean lattice.

1. Introduction

The *rank* of a matrix is fundamental in mathematics, science, and engineering. The notion of rank can also be associated to graphs, matroids, and partial orders. One can refine the rank function to submatrices, subgraphs, etc. as well to get a family of ranks to associate to each object. We observe that such families always follow certain Boolean growth rules leading to the concept of a generalized rank function, which we call a quilt. The goal of this paper is to consider families of generalized rank functions and their connection with the well-studied alternating sign matrices (ASMs). We present some applications and some related enumeration questions.

One motivating example of generalized rank functions comes from the Bruhat decomposition of the general linear group, permutations, and the geometry of flag manifolds. Given a matrix $M \in GL_n$, let $\mathrm{rank}_M(i,j)$ be the rank of the submatrix of M on rows $[i,n] = \{i, i+1, \ldots, n\} \subseteq [n]$ and columns $[j] = \{1, 2, \ldots, j\} \subseteq [n]$. We call the $n \times n$ matrix of values $\mathrm{rank}_M(i,j)$ for $i, j \in [n]$ the *southwest rank table* of M. The southwest rank tables of all $n \times n$ invertible matrices can be classified by permutations in the symmetric group S_n and their associated permutation matrices. To find the permutation matrix to associate to M with the same southwest rank table, simply apply all possible elementary column reduction moves from left to right and elementary row reductions from bottom to top. These moves preserve the southwest rank of the matrix and, since M is invertible, will eventually terminate with a single nonzero entry in each row and column. Finally, rescale each entry to be 1 to obtain the associated permutation matrix.

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Bruhat order on the symmetric group S_n is the partial order given by $v \leq w$ if and only if $\mathrm{rank}_{M(v)}(i,j) \leq \mathrm{rank}_{M(w)}(i,j)$ for all $1 \leq i,j \leq n$, where $M(v), M(w)$ are the permutation matrices corresponding to v, w. Bruhat order also determines the partial order on permutations given by containment of Schubert varieties in the flag manifold [Ful97, Ch. 9]. One observes from the Hasse diagram of S_3 that Bruhat order is not a lattice, as a partial order. In general, the *Dedekind–MacNeille* completion of a finite poset is a lattice which contains the poset and is isomorphic to some subposet of any other lattice containing it. Lascoux and Schützenberger proved that the Dedekind–MacNeille completion of Bruhat order to a lattice is given by a natural partial order on square alternating sign matrices (ASMs) [LS96, Las08]. The (square) alternating sign matrices were defined by Mills, Robbins and Rumsey to be the $n \times n$ matrices with entries from the set $\{-1, 0, 1\}$ such that the nonzero entries in each row and each column alternate between 1 and -1 and must both start and end with 1 [MRR83, RR86]. They arise in the process of computing a determinant using Dodgson condensation. By definition, permutation matrices are examples of ASMs.

The partial order on $n \times n$ ASMs generalizing Bruhat order is given by $A = (a_{ij}) \leq B = (b_{ij})$ if and only if the corresponding entries in the *southwest corner sum matrices* (CSMs) are increasing,

$$\sum_{i \leq i' \leq n, 1 \leq j' \leq j} a_{i'j'} \leq \sum_{i \leq i' \leq n, 1 \leq j' \leq j} b_{i'j'} \quad \text{for all } 1 \leq i,j \leq n. \tag{1.1}$$

One can show that the ASM poset is a lattice with meet and join given by taking the entry-wise min and max in the corresponding corner sum matrices, and every such matrix does indeed correspond to an ASM. See Section 2 for more details. Since the corner sum matrix of a permutation matrix is exactly its southwest rank table, Bruhat order on S_n embeds into the ASM lattice.

There are exactly $\prod_{j=0}^{n-1}(3j+1)!/(n+j)!$ alternating sign matrices of size $n \times n$. This result was conjectured by Mills, Robbins and Rumsey and proved first by Zeilberger [Zei96], and further established independently by Kuperberg and Fischer [Kup96, Fis07]. It was notoriously difficult to prove the enumeration formula for square ASMs, and no simple formula for the number of rectangular ASMs is known.

In this paper, we define new objects, *quilts of alternating sign matrices*, corresponding to two ranked partially ordered sets with greatest and least elements. They are a generalization of rectangular alternating sign matrices and their corner sum matrices. For example, the southwest rank table of a matrix in GL_n corresponds to a quilt on two copies of the chain C_n. Also, if B_n is the Boolean lattice of subsets of $[n]$ ordered by inclusion and M is a $k \times n$ matrix of full rank (i.e., the rank is $\min\{k,n\}$), the function $f_M : B_k \times B_n \longrightarrow \mathbb{N}$ given by setting $f_M(I,J)$ to be the rank of the submatrix of M in rows I and columns J is a quilt of type (B_k, B_n). See Example 3.11.

Given the complexity of enumerating rectangular ASMs, it is surprising that we are still able to say quite a bit about the enumeration of quilts, especially when one of the two posets is an antichain (see Theorem 6.1) or a chain. One of our main results is the following theorem, more precisely stated as Theorem 7.1. Here $S(P)$ is the set of standard quilts, defined in Section 7.

Theorem 1.1. *The number of quilts of type (P, C_n) for $n \geq \mathrm{rank}\, P$ is a polynomial of degree $b(P) = \sum_{x \in P} \mathrm{rank}\, x$. Furthermore, the leading coefficient is $\frac{|S(P)|}{b(P)!}$.*

The following is an easy consequence of the theorem and the hook-length formula for shifted standard tableaux. To the best of our knowledge, this is a new observation for rectangular ASMs.

Corollary 1.2. *The number of rectangular ASMs of size $k \times n$, for $n \geq k$, is a polynomial in n of degree $\binom{k+1}{2}$ with leading coefficient $\prod_{i=0}^{k-1} \frac{(2i)!}{(k+i)!}$.*

In general, it is rare to find exact formulas for quilt enumeration. We show the problem of counting quilts on two arbitrary posets is #P-complete by a reduction to the enumeration of antichains, see Theorem 3.18. The quilts form a distributive lattice with a number of beautiful properties, see Sections 3 and 5. They generalize the notions of matroids and flag matroids, as we will show in Section 5. As a precursor to defining quilts, we also define Dedekind maps of posets generalizing the monotone Boolean functions on B_n counted by the Dedekind numbers. These numbers also count the number of antichains in B_n.

Our main application of quilts is to an embedding of a partial order on Fubini words (equivalently Cayley permutations or ordered set partitions) into the lattice of quilts of type (C_k, B_n). This partial order on Fubini words arose in the context of a generalization of southwest rank tables for decomposing rectangular matrices based on the geometry of spanning line configurations due to Pawlowski and Rhoades [PR19].

The literature on alternating sign matrices and matroids is vast, and we are certain that quilts hide many riches way beyond the scope of this paper. The paper is structured as follows. In Section 2, we give some standard definitions about partially ordered sets, alternating sign matrices, and matroids. In Section 3, we define our main objects, *Dedekind maps* and *quilts* of alternating sign matrices of type (P, Q) for P and Q finite ranked posets with least and greatest elements, and we develop some of their basic properties. When Q is a chain, we can view a quilt as a filling of the poset P with interlacing sets generalizing the notion of a monotone triangle, see Proposition 3.17. In Section 4, we explain how our definition of quilts was motivated via the medium roast order on Fubini words. In Section 5 we prove some interesting properties of the quilt lattice. For example, there is a natural bijection between quilts of type (P, Q) and (Q, P). In Sections 6 and 7, we present our main enumerative results. We show that the number of quilts when Q is an antichain with j elements (and added least and greatest elements) is exponential in j, and that it is a polynomial in n when Q is a chain of rank n. The results also give asymptotic formulas in both cases. In Section 8, we give some bounds for the number of quilts when Q is a Boolean lattice. In Section 9, we point out some possible future research directions. In Appendix A, we give some of the more unwieldy and computationally intensive results. In Appendix B, we give some terms of the newly identified integer sequences related to quilts.

2. Background

In this section, we lay out some of the standard notation for permutations, posets, and alternating sign matrices. See [Sta12] for more information.

2.1. **Posets and lattices.** Let us write $\mathbb{N}$ for the set of non-negative integers. For $n \in \mathbb{N}$, let $[n]$ be the set $\{1, \ldots, n\}$; in particular, $[0] = \emptyset$. Recall that a partially ordered set (poset) is a set P with a reflexive, antisymmetric and transitive relation $\leq$. We denote by $[x, y]$ the *interval* between x and y for $x \leq y$. We say that y *covers* x, denoted $x \lessdot y$, if $x < y$ and there is no z satisfying $x < z < y$. A *chain* is a set of comparable elements in P, and an *antichain* is a set of incomparable elements. A chain is *maximal* if it is not contained in a larger chain. A poset is *ranked* if all maximal chains have the same size. A ranked poset P with least element $\hat{0}_P$ and greatest element $\hat{1}_P$ has a *rank function*, which is a map $\operatorname{rank}\colon P \to \mathbb{N}$ satisfying $\operatorname{rank}\hat{0}_P = 0$ and $x \lessdot y \Rightarrow \operatorname{rank} y = \operatorname{rank} x + 1$. We define $\operatorname{rank} P = \operatorname{rank}\hat{1}_P$, and we write $b(P) = \sum_{x\in P} \operatorname{rank} x$ for the *sum of ranks* of P. We omit the subscript in $\hat{0}_P$ and $\hat{1}_P$ if the poset is clear from the context.

Every poset in this paper is finite, ranked, and has the least element $\hat{0}$ and the greatest element $\hat{1}$. The elements covering $\hat{0}$ are called *atoms*, and the elements covered by $\hat{1}$ are *coatoms*. We assume that P comes with a fixed total ordering of the elements that respects rank: the element $\hat{0} = \hat{0}_P$ comes first, then the atoms in some order, then the elements of rank 2, etc.

A poset is a *lattice* if every two elements x, y have a unique greatest common lower bound (meet) $x \wedge y$ and a unique least common upper bound (join) $x \vee y$. A lattice is *distributive* if $(x\vee y)\wedge z = (x\wedge z)\vee(y\wedge z)$ and $(x\wedge y)\vee z = (x\vee z)\wedge(y\vee z)$ for all x, y, z. Some important examples of distributive lattices that we will consider are the following:

- C_n for $n \geq 0$ is the poset $\{0, 1, \ldots, n\}$ with the usual order $\leq$ (the *chain* of rank n);
- $A_2(j)$ for $j \geq 1$ is the poset with $\hat{0}$, $\hat{1}$, and j other elements that are incomparable with each other (the *antichain* of j elements, with $\hat{0}$ and $\hat{1}$ added);
- B_n for $n \geq 1$ is the poset of subsets of $[n]$ ordered by inclusion (the *Boolean lattice* of rank n).

See Figure 1 for examples. Note that the subscript always denotes the rank of the poset.

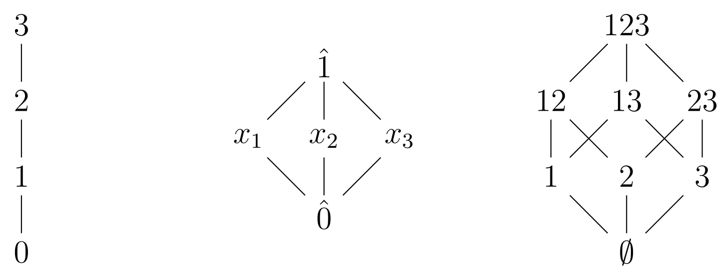


FIGURE 1. Hasse diagrams for C_3, $A_2(3)$ and B_3.

For posets P and Q, the *Cartesian product* $P \times Q$ has cover relations $(x, y) \lessdot (x', y)$ for $x \lessdot x'$ and $(x, y) \lessdot (x, y')$ for $y \lessdot y'$. For example, B_2 is isomorphic to $C_1 \times C_1$, and B_n to C_1^n. See [Sta12, §3] for more details on products and sums of posets.

2.2. **Alternating sign matrices and Fubini words.** An *alternating sign matrix* (or ASM for short) is a matrix of size $k \times n$ with entries in $\{-1, 0, 1\}$ such that:

- in each row and each column the non-zero entries alternate,
- the leftmost non-zero entry in every row and the bottommost non-zero entry in every column is 1,
- if $k \leq n$, the rightmost non-zero entry in every row is 1, and
- if $k \geq n$, the topmost non-zero entry in every column is 1.

In particular, if $n = k$, the non-zero entries of every row and every column alternate and begin and end with 1. Note that what we call an ASM is typically called a *rectangular* or *truncated* ASM in the literature; we will instead emphasize that we have a *square* ASM when

$n = k$. Denote the set of all ASMs of size $k \times n$ by $\mathrm{ASM}_{k,n}$. As mentioned in the introduction, a very famous result tells us that

$$|\,\mathrm{ASM}_{n,n}\,| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

See [BP99] for more background.

For example, every permutation matrix is a square ASM. To spell out some notation, a permutation in S_n is a bijection $w : [n] \longrightarrow [n]$, which can be denoted simply by its one-line notation $w = w(1)w(2)\cdots w(n)$. The permutation matrix corresponding with w has a 1 in position (w_j, j) for each $j \in [n]$ and 0's elsewhere. More generally, a *Fubini word* w is a surjective map $w : [n] \longrightarrow [k]$, denoted by its one-line notation $w = w(1)w(2)\cdots w(n)$. Let $W_{n,k}$ denote all such Fubini words for a fixed pair $1 \leq k \leq n$. The Fubini words in $W_{n,k}$ are in natural bijection with ordered set partitions on $[n]$ into k nonempty parts given by $w^{-1}(1)|w^{-1}(2)|\ldots|w^{-1}(k)$, but in this context it is helpful to think of them as a $k \times n$ generalization of a permutation matrix where again the matrix for w has a 1 in position (w_j, j) for each $j \in [n]$ and 0's elsewhere. Such matrices are *not* examples of rectangular ASMs of size $k \times n$ when $k < n$ because some row must have two 1's with no -1 between them.

Remark 2.1. We note that the nomenclature "Fubini words" comes from [OEI24, A000670]. Others refer to the same words as "Cayley permutations", "packed words", "surjective words", "normal patterns", and "initial words." We thank Anders Cleasson for this list. Some history of this terminology is given in [CCEG24].

The following is an example of an ASM of size 5×6:

$$\begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

All ASMs of size 3×3 are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and we can get all ASMs of size 3×2 (resp., 2×3) by deleting the rightmost column (resp., the topmost row).

Note that reflecting across the vertical axis gives an involution on the set of ASMs of size $k \times n$ for $k \leq n$ (but not for $k > n$). When $n = k$, we have involutions coming from reflections across the horizontal and vertical axes and from the two diagonal transpositions of a matrix, as well as rotations by 90°, 180° and 270°. This gives a faithful action of the dihedral group D_4 on the set of ASMs of size $n \times n$ for $n \geq 2$.

Given $A \in \mathrm{ASM}_{k,n}$, we can define a new matrix $C(A)$, called its *corner sum matrix*, or CSM for short, of size $(k+1) \times (n+1)$ by adding a row and column of 0's below and to the left of the matrix, and taking the sum of all entries weakly to the left and weakly below a given entry. For example, the 5×6 ASM in (2.1) gives rise to the 6×7 CSM

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

In the resulting matrix $C(A)$, the entries change by 0 or 1 when moving to the right or up, and the bottom row and the leftmost column always consist of 0's. Furthermore, if $k \geq n$, the top row consists of $0, 1, \ldots, n$, and if $k \leq n$, the rightmost column consists of $0, 1, \ldots, k$. Conversely, given a matrix B, with rows numbered $0, 1, \ldots, k$ and starting at the bottom, and columns numbered $0, 1, \ldots, n$ and starting on the left, satisfying these properties, the $k \times n$ matrix $A = (a_{ij})$ given by

$$a_{i,j} = b_{i,j} - b_{i-1,j} - b_{i,j-1} + b_{i-1,j-1}$$

for all $i \in [k]$ and $j \in [n]$ is an ASM. Therefore, we can equivalently define CSMs directly as follows.

Definition 2.2. A *corner sum matrix (CSM)* of size $(k+1) \times (n+1)$ is a map

$$f \colon C_k \times C_n \longrightarrow \mathbb{N}$$

satisfying:

- $f(i,j) = 0$ whenever $i = 0$ or $j = 0$,
- $f(k,n) = \min\{k, n\}$, and
- if $(i,j) \lessdot (i',j')$ in $C_k \times C_n$, then $f(i',j') \in \{f(i,j), f(i,j)+1\}$.

Let $\mathrm{CSM}_{k,n}$ denote the set of all CSMs of size $(k+1)\times(n+1)$. We refer to the third condition in the definition of a CSM as a *Boolean growth rule*, which is a central concept in this paper.

One more way to think of ASMs/CSMs is as monotone triangles (MTs). Given a CSM $f \colon C_k \times C_n \longrightarrow \mathbb{N}$, record the positions of *jumps* in each row, denoted by

$$J_f(i) = \{j \in [n] \colon f(i,j) = f(i,j-1)+1\}. \tag{2.3}$$

Then, the *monotone triangle* corresponding to f is the "triangular array" of jump sequences with rows $J_f(k), \ldots, J_f(2), J_f(1)$ (from top to bottom). The rows of the monotone triangle are *interlacing* in the sense that if $J_f(i) = \{s_1 < s_2 < \cdots < s_p\}$ and $J_f(i+1) = \{t_1 < t_2 < \cdots < t_q\}$ then either $p = q-1$ and

$$t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq s_{q-1} \leq t_q, \tag{2.4}$$

or $p = q$ and

$$t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_q \leq s_q. \tag{2.5}$$

Clearly, the original CSM can be recovered from its interlacing jump sets so there are easy bijections between the ASMs, CSMs, and MTs for given k, n.

For the CSM in (2.2), we get jump sequences $J_f(0) = \emptyset$, $J_f(1) = \{3\}$, $J_f(2) = \{3,4\}$, $J_f(3) = \{2,3,5\}$, $J_f(4) = \{1,3,4,6\}$, $J_f(5) = \{1,2,3,5,6\}$. This can be presented in the monotone triangle (MT) form as

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 5 & & 6 \\ & 1 & & 3 & & 4 & & 6 & \\ & & 2 & & 3 & & 5 & & \\ & & & 3 & & 4 & & & \\ & & & & 3. & & & & \end{array}$$

Let g be the CSM given by the transpose of (2.2). The jump sets are $J_g(0) = \emptyset$, $J_g(1) = \{4\}$, $J_g(2) = \{3, 5\}$, $J_g(3) = \{1, 3, 5\}$, $J_g(4) = \{1, 2, 4\}$, $J_g(5) = \{1, 2, 3, 5\}$, $J_g(6) = \{1, 2, 3, 4, 5\}$, and the MT form is

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 1 & & 2 & & 3 & & 5 & \\ & & 1 & & 2 & & 4 & & \\ & & 1 & & 3 & & 5 & & \\ & & & 3 & & 5 & & & \\ & & & & 4. & & & & \end{array}$$

Observe that $J_g(3)$ and $J_g(4)$ have the same size, so the MT form is not a classical triangle of numbers.

Remark 2.3. Terwilliger introduced a poset Φ_n on the subsets of $[n]$ with covering relations given by $S \lessdot T$ whenever S and T are interlacing in the sense of (2.4). The poset Φ_n contains the Boolean lattice B_n as a subposet. He showed maximal chains in Φ_n are in bijection with $\mathrm{ASM}_{n,n}$, just as the maximal chains of B_n are in bijection with S_n [Ter18, Thm. 3.4]. Building on this work, Hamaker and Reiner [HR20] showed that Φ_n is a shellable poset, introduced a notion of descents for monotone triangles, and connected them to a generalization of the Malvenuto–Reutenauer Hopf algebra of permutations. See Section 9 for some follow up questions in this direction.

2.3. **Matroids and flag matroids.** A *matroid* M on ground set $[n]$ is determined by a rank function $r : 2^{[n]} \longrightarrow \mathbb{N}$ such that the following three conditions hold:

(1) (bounded by size) $0 \leq r(X) \leq |X|$ for all $X \subseteq [n]$,
(2) (monotonicity) $r(X) \leq r(Y)$ for all $X \subseteq Y$,
(3) (submodularity) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

The *rank of* M is $r(M) = r([n])$. For example, given a $k \times n$ matrix with real entries, the rank function on the subsets of columns of the matrix satisfies the three conditions above.

Remark 2.4. Observe that if $i \in [n] \setminus X$, then $0 \leq r(X \cup \{i\}) - r(X)$ by monotonicity. By the bounded size property, $r(\emptyset) = 0$ and $r(\{i\}) \leq 1$. Hence, $r(X \cup \{i\}) - r(X) \leq r(\{i\}) \leq 1$ by submodularity. Hence, the rank function of a matroid is surjective on $[0, r(M)]$ and also respects the Boolean growth property: $r(X \cup \{i\}) \in \{r(X), r(X) + 1\}$.

Let M, N be matroids on $[n]$. We say N is a *quotient* of M provided

$$r_M(Y) - r_M(X) \geq r_N(Y) - r_N(X) \tag{2.6}$$

for all $X \subseteq Y \subseteq [n]$. In particular, $r_M(Y) \geq r_N(Y)$ since $r_M(\emptyset) = r_N(\emptyset) = 0$.

A *flag matroid* of type $(1 \leq k_1 < \cdots < k_s \leq n)$ on $[n]$ is a collection of matroids $\mathcal{M} = (M_1, M_2, \ldots, M_s)$ on the ground set $[n]$ where the rank of M_j is k_j and M_j is a quotient of M_{j+1} for each $1 \leq j < s$. For example, given a $k \times n$ matrix with complex entries, one can construct a flag matroid $\mathcal{M} = (M_1, M_2, \dots, M_s)$ where M_i for $1 \le i \le k$ is the matroid with rank function defined by the submatrix using only the top i rows.

For more information on matroids and flag matroids, the standard reference is the book by Oxley [Oxl11]. A nice survey can be found in [CDMS22].

3. Dedekind maps and quilts

In this section, we introduce a generalization of the Dedekind numbers, which count the number of monotone increasing Boolean functions [OEI24, A000372]. Such functions are closely related to the CSMs defined in Definition 2.2 and are natural precursors to the notion of a quilt defined later in this section.

Definition 3.1. A *Dedekind map of rank* k on P is a surjective map $f\colon P \to C_k$ satisfying $x \lessdot y \Rightarrow f(y) \in \{f(x), f(x)+1\}$. The set of all Dedekind maps of rank k on P is denoted by $D_k(P)$, their union by $D(P)$, and we write $d_k(P) = |D_k(P)|$ and $d(P) = |D(P)| = \sum_k d_k(P)$ for the *kth Dedekind number of* P and *Dedekind number of* P, respectively.

Example 3.2. By Remark 2.4, the rank function of a matroid on ground set $[n]$ of rank k is a Dedekind map of rank k on the Boolean lattice B_n. However, not every Dedekind map of rank k is the rank function of a matroid of rank k. In fact, any Dedekind map on B_n with $f(\{i\}) = 0$ for all $i \in [n]$ could not be the rank function for a rank $k > 0$ matroid on $[n]$ because submodularity would be violated. In general, there are far more Dedekind maps than matroids. For example, there are only 7 matroids on [3] of rank 2, but there are 18 Dedekind maps on B_3 of rank 2.

Observe that for a Dedekind map of rank k, $f\colon P \to C_k = [0,k]$, the following three conditions are satisfied:

- $f(\hat{0}) = 0$,
- $f(\hat{1}) = k$, and
- if $x \lessdot y$, then $f(y) \in \{f(x), f(x)+1\}$ (Boolean growth).

Consequently, $f(x) \le \operatorname{rank} x$ for all $x \in P$. The k-Dedekind number of a chain is $d_k(C_n) = \binom{n}{k}$. For the antichain poset $A_2(j)$, we have

$$d_k(A_2(j)) = \begin{cases} 1 & : \ k = 0,2 \\ 2^j & : \ k = 1 \\ 0 & : \ k \ge 3. \end{cases}$$

Some values of $d_k(B_n)$ are given in the following table:

$n\backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	4	1	0	0	0
3	1	18	18	1	0	0
4	1	166	656	166	1	0
5	1	7579	189967	189967	7579	1.

The first column determined by $d_1(B_n)$ for $n \ge 0$ is given by [OEI24, A007253].

Remark 3.3. Given $f \in D_1(P)$, the set of minimal elements satisfying $f(x) = 1$ is a non-empty antichain in $P \setminus \{\hat{0}\}$; so $d_1(P)$ counts the number of antichains in P (except for $\emptyset$ and

$\{\hat{0}\}$), which is a #P-complete problem [PB83]. See also [Sap91] on antichain enumeration in ranked posets.

In particular, $d_1(B_n) + 2$ is the classical Dedekind number and is notoriously difficult to compute. The exact value for $n = 9$ was first computed in 2023, thirty years after the value for $n = 8$ [Jäk23]. These numbers grow very quickly, $d_1(B_9) \approx 2.86 \cdot 10^{41}$. See also [OEI24, A000372] and [HCG+23] for more history and an independent computation of the 9th Dedekind number.

Lemma 3.4. *For any poset P and $k \geq 1$, we have $d_k(P) \leq d_1(P)^k$.*

Proof. For $f \in d_k(P)$ and $1 \leq i \leq k$, take A_i to be the set of minimal elements of the set $\{x \in P\colon f(x) = i\}$. Clearly, A_i is an antichain, $A_i \neq \emptyset$, $A_i \neq \{\hat{0}\}$; there are $d_1(P)$ such antichains. The map $f \mapsto (A_1, \ldots, A_k)$ is an injection, which proves the statement. □

Every column of a CSM of size $(k+1) \times (n+1)$ can be seen as a Dedekind map on C_k and every row as a Dedekind map on C_n. As one reads left to right in columns or bottom to top in rows, another Boolean growth rule must hold. This second type of Boolean growth rule gives rise to the following graphs.

Definition 3.5. Let $G_D(P)$ denote the *Dedekind graph of P*, defined as the directed graph with vertex set given by the Dedekind maps in $D(P)$ and an edge from f to g if $g(x) \in \{f(x), f(x)+1\}$ for all $x \in P$. The *restricted Dedekind graph of P*, $G'_D(P)$, is the directed graph with vertex set $D(P)$ and an edge from f to g if $g(\hat{1}_P) = f(\hat{1}_P) + 1$ and $g(x) \in \{f(x), f(x)+1\}$ for all $x \in P$.

If we order the vertices $D(P) = \cup_{k=0}^{\operatorname{rank} P} D_k(P)$ by rank k, and the vertices within $D_k(P)$ lexicographically (according to our fixed linear order on P), the adjacency matrix $A_D(P)$ of $G_D(P)$ is upper triangular with 1's on the diagonal. The adjacency matrix $A'_D(P)$ of $G'_D(P)$ is strictly upper triangular. In particular, both $G_D(P)$ and $G'_D(P)$ are acyclic and have a unique source and sink. One can naturally identify the walks in the Dedekind graph of a chain with CSMs, which leads to the next proposition. Furthermore, we will use the (restricted) Dedekind graph of a poset to prove the enumerative results in Theorem 7.5.

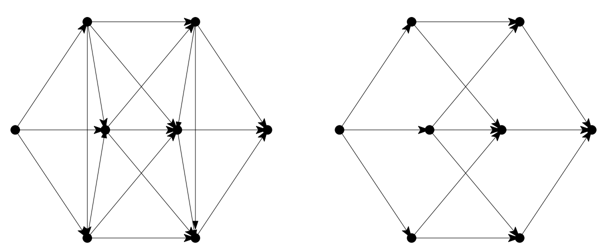


FIGURE 2. The Dedekind graph (the loops are not shown) and the restricted Dedekind graph of C_3.

Proposition 3.6. *For any $1 \leq k \leq n$, the map between the set $CSM_{k,n}$ and the set of walks in the Dedekind graph $G_D(C_n)$ from its unique sink to a vertex in $D_k(C_n)$ determined by the consecutive list of columns is a bijection.*

Proof. The proof follows by comparing Definition 2.2 and Definition 3.1. □

Recall the definition of interlacing sets and monotone triangles from Section 2. The Dedekind graph of a chain C_n also respects the interlacing conditions. The next statement follows from Proposition 3.6. See Proposition 3.17 for more connections with the interlacing conditions.

Corollary 3.7. *The restricted Dedekind graph $G'_D(C_n)$ is isomorphic to the directed graph on B_n, with an edge from S to T if $|S| = |T| - 1$ and the sets S and T are interlacing: $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \ldots \leq s_{|T|-1} \leq t_{|T|}$. Similarly, the Dedekind graph of C_n is isomorphic to the directed graph on B_n, edges as above plus an edge from S to T whenever $|S| = |T|$ and $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \ldots \leq t_{|T|} \leq s_{|T|}$.*

Remark 3.8. Corollary 3.7 implies that the Dedekind graph $G_D(C_n)$ is the Hasse diagram of the Terwilliger poset Φ_n mentioned in Remark 2.3. The restricted Dedekind graph $G'_D(C_n)$ is the Hasse diagram of the interlacing poset on the subsets of $[n]$ with covering relations given by both types of interlacing conditions.

The following is the main definition of this paper. It generalizes the definition of a CSM in Definition 2.2.

Definition 3.9. Let P and Q be finite ranked posets with least and greatest elements. A *quilt of alternating sign matrices of type* (P, Q) is a map

$$f : P \times Q \longrightarrow \mathbb{N}$$

satisfying:

- $f(x, y) = 0$ whenever $x = \hat{0}_P$ or $y = \hat{0}_Q$,
- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P, \ \operatorname{rank} Q\}$, and
- if $(x, y) \lessdot (x', y')$ in $P \times Q$, then $f(x', y') \in \{f(x, y), f(x, y) + 1\}$ (Boolean growth).

We will also call such a map an *ASM quilt* or just a *quilt* for short. The set of all quilts of type (P, Q) will be denoted by $\mathrm{Quilts}(P, Q)$.

Remark 3.10. A quilt of type (C_k, C_n) is a CSM of size $(k + 1) \times (n + 1)$, so there is also a corresponding ASM and MT. Similarly, for any $f \in \mathrm{Quilts}(P, Q)$ and any pair of maximal chains

$$\hat{0}_P = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{k-1} \lessdot x_k = \hat{1}_P, \quad \hat{0}_Q = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{n-1} \lessdot y_n = \hat{1}_Q$$

in P and Q, the map $(i, j) \mapsto f(x_i, y_j)$ is a CSM of size $(k + 1) \times (n + 1)$, which again has a corresponding ASM and MT. So we can think of quilts as encoding collections of alternating sign matrices, one for each pair of maximal chains in the two posets, appropriately "pieced" together like the fabric of a quilt.

Example 3.11. Let M be a $k \times n$ matrix of full rank. Take the function $f_M : B_k \times B_n \longrightarrow \mathbb{N}$ given by setting $f_M(I, J)$ to be the rank of the submatrix of M in rows I and columns J; here $f_M(I, J) = 0$ if either I or J is the empty set. Since M is full rank, we know $f_M([k], [n]) = \min\{k, n\}$. Since the rank of any matrix increases by at most 1 when we add in one more row or column, it is clear that the Boolean growth rule in Definition 3.9 is satisfied as well. Hence, $f_M : B_k \times B_n \longrightarrow \mathbb{N}$ is a quilt of type (B_k, B_n). This example also proves rank functions of graphs and their subgraphs are encoded by quilts since the rank of a graph can be defined as the rank of its adjacency matrix.

Example 3.12. Take posets P and Q with $\operatorname{rank} P \leq \operatorname{rank} Q$, and an l-Dedekind map g on Q, $l \geq \operatorname{rank} P$. Then the map

$$f\colon P \times Q \longrightarrow \mathbb{N}, \qquad f(x,y) = \min\{\operatorname{rank} x,\ g(y)\}$$

is a quilt of type (P,Q). In particular, if g_1 and g_2 map to the same quilt f, then $g_1(y) = f(\hat{1}_P, y) = g_2(y)$ for all $y \in Q$, so $D_{\operatorname{rank} P}(Q) \longrightarrow \operatorname{Quilts}(P,Q)$ is injective.

Lemma 3.13. *Let $f \in \operatorname{Quilts}(P,Q)$. If $\operatorname{rank} P \geq \operatorname{rank} Q$, then $f(\hat{1}_P, y) = \operatorname{rank}_Q y$ for all $y \in Q$. If $\operatorname{rank} P \leq \operatorname{rank} Q$, then $f(x, \hat{1}_Q) = \operatorname{rank}_P x$ for all $x \in P$.*

Proof. Assume $k = \operatorname{rank} P \geq n = \operatorname{rank} Q$. Fix any maximal chain $\hat{0}_Q = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_{n-1} \lessdot y_n = \hat{1}_Q$ in Q. Then by definition of a quilt, we have $0 = f(\hat{1}_P, y_0) \leq f(\hat{1}_P, y_1) \leq \cdots \leq f(\hat{1}_P, y_n) = \min\{k,n\} = n$ and $f(\hat{1}_P, y_i) - f(\hat{1}_P, y_{i-1}) \in \{0,1\}$ for $i = 1, \ldots, n$, so the first claim follows. The second statement follows similarly. $\square$

The lemma also shows that the entire rank function of the smaller ranked poset is encoded in each quilt. This justifies our claim that quilts generalize rank functions of posets.

There is a natural partial order on $\operatorname{Quilts}(P,Q)$: we say that $f \leq g$ if $f(x,y) \leq g(x,y)$ for all $x \in P$, $y \in Q$. For $P = C_k$, $Q = C_n$, this is the well-known partial order on the set of CSMs or ASMs. We will call $\operatorname{Quilts}(P,Q)$ the *quilt lattice*, as justified by the following.

Theorem 3.14. *Let P, Q be finite ranked posets with least and greatest elements. The poset $\operatorname{Quilts}(P,Q)$ is a distributive lattice ranked by*

$$\operatorname{quiltrank} f = \sum_{x \in P,\, y \in Q} f(x,y) - \sum_{x \in P,\, y \in Q} f_{\hat{0}}(x,y),$$

where $f_{\hat{0}}(x,y) = \max\{0, \operatorname{rank} x + \operatorname{rank} y - \max\{n,k\}\}$ is the least element of $\operatorname{Quilts}(P,Q)$. The greatest element of $\operatorname{Quilts}(P,Q)$ is $f_{\hat{1}}(x,y) = \min\{\operatorname{rank} x,\ \operatorname{rank} y\}$.

Before we prove the theorem, let us introduce some notation. Given $f, g \in \operatorname{Quilts}(P,Q)$ such that $f \leq g$, define the *difference set*

$$\Delta(f,g) = \{(x,y) \in P \times Q \colon f(x,y) < g(x,y)\}. \tag{3.1}$$

Given $(x,y) \in \Delta(f,g)$ and $(x',y') \in P \times Q$, write $(x,y) \to (x',y')$ if either $(x,y) \lessdot (x',y')$ and $f(x,y) = f(x',y')$ or $(x',y') \lessdot (x,y)$ and $f(x',y') = f(x,y) - 1$. In the first case, $f(x',y') = f(x,y) < g(x,y) \leq g(x',y')$, which means that $(x',y') \in \Delta(f,g)$. In the second case, $f(x',y') = f(x,y) - 1 < g(x,y) - 1 \leq g(x',y')$, which also implies $(x',y') \in \Delta(f,g)$. In other words, we have constructed a directed graph $G_\Delta(f,g)$ on the difference set $\Delta(f,g)$. An edge from (x,y) to (x',y') means that the value of f on (x',y') prevents us from increasing the value of f in (x,y) to get another valid quilt of type (P,Q).

Proof of Theorem 3.14. To prove that $\operatorname{Quilts}(P,Q)$ is a lattice, observe that for every $f, g \in \operatorname{Quilts}(P,Q)$, the pointwise minimum and maximum of f and g is again in $\operatorname{Quilts}(P,Q)$ by definition. These quilts are $f \wedge g$ and $f \vee g$ respectively, hence $\operatorname{Quilts}(P,Q)$ is a lattice. Distributivity follows from the fact that

$$\max\{a, \min\{b,c\}\} = \min\{\max\{a,b\}, \max\{a,c\}\}$$

and

$$\min\{a, \max\{b,c\}\} = \max\{\min\{a,b\}, \min\{a,c\}\}$$

hold for $a, b, c \in \mathbb{N}$.

It is easy to check that $f_{\hat{0}}$ and $f_{\hat{1}}$ are in $\mathrm{Quilts}(P, Q)$ by checking the properties of the definition. To see $f_{\hat{1}}$ is the unique maximal element in $\mathrm{Quilts}(P, Q)$, consider any $(x, y) \in P \times Q$ with $\operatorname{rank} x = r$, $\operatorname{rank} y = s$. Since P, Q are ranked posets, there exist maximal chains

$$\hat{0}_P = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r = x \lessdot \cdots \lessdot x_k = \hat{1}_P$$

and

$$\hat{0}_Q = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_s = y \lessdot \cdots \lessdot y_n = \hat{1}_Q.$$

Then, for any $f \in \mathrm{Quilts}(P, Q)$, we know $f(x, y) \leq f(x_{r-1}, y) + 1 \leq \ldots \leq f(\hat{0}_P, y) + r = r$ and $f(x, y) \leq f(x, y_{s-1}) + 1 \leq \ldots \leq f(x, \hat{0}_Q) + s = s$, so $f(x, y) \leq \min\{r, s\} = f_{\hat{1}}(x, y)$. Hence, $f_{\hat{1}}$ is maximal. On the other hand, $f(x, y) \geq f(x, y_{s+1}) - 1 \geq \cdots \geq f(x, \hat{1}_Q) - (n - s) \geq f(x_{r+1}, \hat{1}_Q) - 1 - (n - s) \geq \cdots \geq f(\hat{1}_P, \hat{1}_Q) - (k - r) - (n - s) = \min\{k, n\} - k - n + r + s = r + s - \max\{k, n\}$ and so $f(x, y) \geq f_{\hat{0}}(x, y)$. Therefore, $f_{\hat{0}}$ is minimal in $\mathrm{Quilts}(P, Q)$.

To see $\mathrm{Quilts}(P, Q)$ is ranked by the given function, choose $f, g \in \mathrm{Quilts}(P, Q)$ with $f < g$. Our goal is to prove that $f \lessdot g$ if and only if f and g differ in exactly one $(x, y) \in P \times Q$, and $g(x, y) = f(x, y) + 1$. The condition is clearly sufficient, let us prove that it is also necessary.

Since $f < g$, the directed graph $G_\Delta(f, g)$ constructed above is non-empty. We claim that it has no directed cycles, and that it therefore has at least one sink. To observe the claim, note that if $(x, y) \to (x', y')$, then $f(x, y) \geq f(x', y')$, and if $f(x, y) = f(x', y')$, then $\operatorname{rank}(x, y) < \operatorname{rank}(x', y')$ in $P \times Q$. If $(x, y) \to (x', y') \to (x'', y'') \to \cdots \to (x, y)$, then the value of f must stay constant (if it strictly decreases, it can never increase to $f(x, y)$ again), and that implies that $\operatorname{rank}(x, y) < \operatorname{rank}(x', y') < \operatorname{rank}(x'', y'') < \cdots < \operatorname{rank}(x, y)$, which is a contradiction.

Take (x, y) to be an arbitrary sink in $G_\Delta(f, g)$. Observe by choice of (x, y) that if $(x, y) \lessdot (x', y')$, then $f(x', y') = f(x, y) + 1$, and if $(x', y') \lessdot (x, y)$, then $f(x', y') = f(x, y)$. Since $f(\hat{0}_P, \hat{0}_Q) = g(\hat{0}_P, \hat{0}_Q) = 0$ and $f(\hat{1}_P, \hat{1}_Q) = g(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$ by definition of a quilt, (x, y) is not $(\hat{0}_P, \hat{0}_Q)$ or $(\hat{1}_P, \hat{1}_Q)$. Therefore, the function $f' \colon P \times Q \longrightarrow \mathbb{N}$, defined to be equal to f everywhere except $f'(x, y) = f(x, y) + 1$, is also a quilt of type (P, Q), hence $f \lessdot f'$. Furthermore, since (x, y) is a vertex in $G_\Delta(f, g)$, we know $f'(x, y) = f(x, y) + 1 \leq g(x, y)$, so $f' \leq g$ as well. Therefore, we conclude that $f \lessdot g$ if and only if $f' = g$. $\square$

From the proof of the theorem, we learned that the sinks in the difference graph $G_\Delta(f, g)$ determine all of the quilts covering f in the interval $[f, g]$. This includes the case $g = f_{\hat{1}}$ so all covering relations in the quilt lattice are relatively easy to identify.

Corollary 3.15. *Given $f, g \in \mathrm{Quilts}(P, Q)$ such that $f < g$, the atoms of the interval $[f, g]$ are in a one-to-one correspondence with the sinks of the graph $G_\Delta(f, g)$. Similarly, the co-atoms of $[f, g]$ are in a one-to-one correspondence with the sources of $G_\Delta(f, g)$.* $\square$

We will call a quilt in $\mathrm{Quilts}(P, C_n)$ or $\mathrm{Quilts}(C_n, P)$ a *chain quilt*, a quilt in $\mathrm{Quilts}(P, A_2(j))$ or $\mathrm{Quilts}(A_2(j), P)$ an *antichain quilt*, a quilt in $\mathrm{Quilts}(P, B_n)$ or $\mathrm{Quilts}(B_n, P)$ a *Boolean quilt* etc. As we will explain in Section 4, our most important and motivating example will be when one of the posets is the Boolean lattice and the other one is a chain.

There are three important ways to think of a chain quilt $f \in \mathrm{Quilts}(P, C_n)$. One is to see it as a sequence of Dedekind maps in $D(P)$ that correspond with a walk in the Dedekind graph $G_D(P)$, generalizing Proposition 3.6.

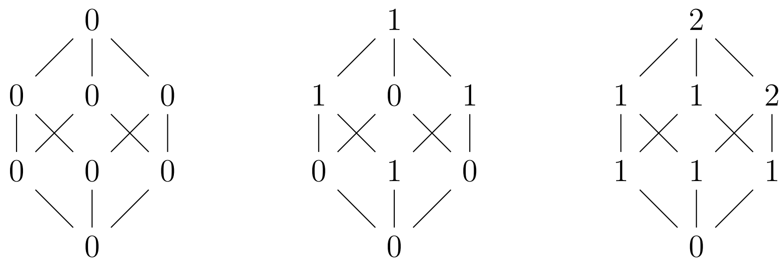


FIGURE 3. A visual representation of an element in Quilts(B_3, C_2)

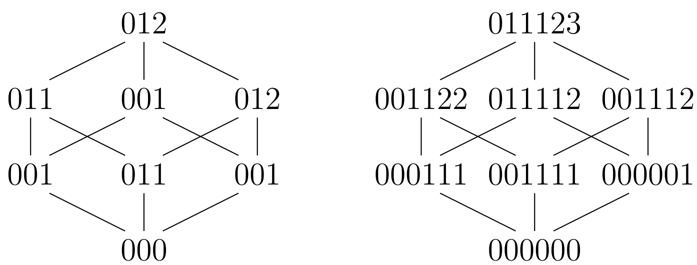


FIGURE 4. Another visualization of quilts of types (B_3, C_2) and (B_3, C_5).

Example 3.16. Figure 3 shows a visual representation of the quilt $f \in \text{Quilts}(B_3, C_2)$ given by

$$f(\{2\},1) = f(\{1,2\},1) = f(\{2,3\},1) = f(\{1,2,3\},1) = 1$$

and $f(T,1) = 0$ for all other subsets T, while $f(\{2,3\},2) = f(\{1,2,3\},2) = 2$ and $f(T,2) = 1$ for all other nonempty subsets T. From Remark 3.10, we know that for every maximal chain in B_3 (and the unique maximal chain in C_2), there is a corresponding 3×2 ASM encoded by the quilt f. Here are the six ASMs paired with maximal chains encoded by f,

$$\begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ for }, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ for }, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ for }, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ for }, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ for }, \text{ and } \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for }.$$

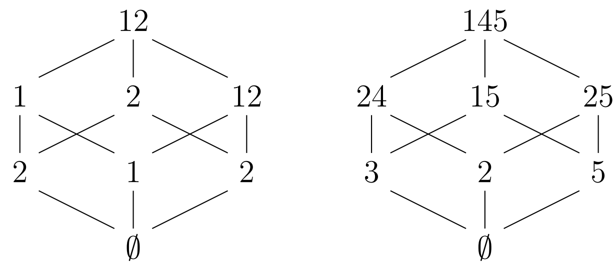
The second way to represent a chain quilt is to observe that f maps an arbitrary $x \in P$ to the sequence $(f(x,0), f(x,1), \ldots, f(x,n))$ of length $n+1$. This sequence has the property that every two consecutive elements are either equal or they differ by one. We also have $f(y,i) \in \{f(x,i), f(x,i)+1\}$ when $x \lessdot y$. The element $\hat{0}_P$ is mapped to the zero sequence, and the sequence corresponding to $\hat{1}_P$ ends with $\min\{\operatorname{rank} P,\ n\}$. The pictures in Figure 4 represent chain quilts with $P = B_3$ for $n = 2$ and $n = 5$, respectively mapping $x \in B_3$ to the sequence $(f(x,0), f(x,1), \ldots, f(x,n))$. Note how the top element on the left is $01\ldots n$, and the rightmost element of every sequence on the right is equal to its rank, as stated in Lemma 3.13. The picture on the right in Figure 4 represents the same quilt at shown in Figure 3.

Another equivalent, and probably even more intuitive, way to represent a chain quilt $f : P \times C_n \longrightarrow \mathbb{N}$ is to say that it is a map that sends $x \in P$ to the set of *jumps of f at x*,

$$J_f(x) = \{i \in [n] \colon f(x,i) = f(x,i-1)+1\}. \tag{3.2}$$

We will call this the *monotone triangle (MT) form* of the quilt f. It is easy to go back from the MT form of a quilt: given $J : P \longrightarrow B_n$, then $f(x, i) = |J(x) \cap [i]|$ defines $f\colon P \times C_n \longrightarrow \mathbb{N}$. The Boolean growth condition translates into adjacent sets interlacing for quilts in MT form, see Proposition 3.17.

The following shows the two quilts from Figure 4 in MT form, where we omit braces and commas for sets since all integers are below 10.



Proposition 3.17. *Take $f \in \mathrm{Quilts}(P, C_n)$. For all $x, y \in P$ with $x \lessdot y$, the sets $S = J_f(x)$ and $T = J_f(y)$ are interlacing. When $n \leq \operatorname{rank} P$, $J_f(\hat{1}_P) = [n]$. When $n \geq \operatorname{rank} P$, we have $|J_f(x)| = \operatorname{rank} x$ for all $x \in P$.*

Proof. Assume $f \in \mathrm{Quilts}(P, C_n)$. By construction, we have $|S| = f(x, n)$ and $|T| = f(y, n)$. Since $x \lessdot y$ and f is a quilt, $|T| = |S|$ or $|T| = |S| + 1$. Furthermore, the i-th jump in $J_f(y)$ cannot come after the i-th jump in $J_f(x)$. That implies that $t_i \leq s_i$. Also, the i-th jump in $J_f(x)$ cannot come after the $(i+1)$-st jump in $J_f(y)$, as that would violate the rule that $f(x, j)$ and $f(y, j)$ can differ by at most 1. Therefore $s_i \leq t_{i+1}$. The last two statements follow from Lemma 3.13. □

The general problem of enumerating quilts is hard, as the following theorem shows. Recall that a counting problem is in #P if we can represent it as counting the number of accepting paths of a polynomial-time non-deterministic Turing machine, and it is #P-complete if every problem in #P has a polynomial-time counting reduction to it. To prove a problem is $\#P$-complete, it suffices to show it is in $\#P$ and is as hard as some $\#P$-complete problem.

Theorem 3.18. *Computing $|\mathrm{Quilts}(P, Q)|$ for general P and Q is a #P-complete problem.*

Proof. One can check if a given map $P \times Q \to \mathbb{N}$ satisfies the properties of a quilt in polynomial time in terms of the sizes of P and Q. It follows that the problem of computing $|\mathrm{Quilts}(P, Q)|$ is in #P. To prove #P-completeness, note that mapping $f \in \mathrm{Quilts}(P, C_1)$ to the set of minimal elements of $\{x \in P\colon f(x, \hat{1}) = 1\}$ gives a bijection between $\mathrm{Quilts}(P, C_1)$ and the set $\{A \subseteq P, A \text{ antichain}\} \setminus \{\emptyset, \{\hat{0}_P\}\}$. The fact that counting antichains in finite posets is #P-complete is Part 3 of the main theorem proved by Provan and Ball in [PB83, p. 783]. One can easily adapt their proof to ranked posets with $\hat{0}$ and $\hat{1}$: the poset constructed in the proof is already ranked, and adding $\hat{0}$ and $\hat{1}$ just adds two extra antichains. □

Even though computing $|\mathrm{Quilts}(P, Q)|$ for general P and Q is out of reach, we do have a simple upper bound. Recall that $b(P) = \sum_{x \in P} \operatorname{rank} x$.

Theorem 3.19. *If $\operatorname{rank} P \leq \operatorname{rank} Q$, then $|\mathrm{Quilts}(P, Q)| \leq d_1(Q)^{b(P)}$.*

Proof. Consider a quilt $f \in \mathrm{Quilts}(P, Q)$. For $x \in P$, we have $f(x, \hat{1}_Q) = \operatorname{rank} x$, and the map $f^x\colon y \mapsto f(x, y)$ is in $D_{\operatorname{rank} x}(Q)$. The map $f \mapsto (f^x)_{x \in P}$ is an injection, so $|\mathrm{Quilts}(P, Q)| \leq$

$\prod_{x\in P} d_{\operatorname{rank} x}(Q)$. By Lemma 3.4,

$$|\operatorname{Quilts}(P,Q)| \leq \prod_{x\in P} d_{\operatorname{rank} x}(Q) \leq \prod_{x\in P} d_1(Q)^{\operatorname{rank} x} = d_1(Q)^{b(P)}.$$

□

4. Motivation from the space of spanning line configurations

An interesting analog of the flag manifold and Schubert varieties was given by Pawlowski and Rhoades in [PR19]. For any $k \leq n$, they define a *spanning line configuration* $l_\bullet = (l_1, \ldots, l_n)$ to be an ordered n-tuple in the product of complex projective spaces $(\mathbb{P}^{k-1})^n$ whose vector space sum is $\mathbb{C}^k$. Each such configuration can be identified by a $k \times n$ full rank matrix over $\mathbb{C}$ with no zero columns. Note, multiplying a $k \times n$ matrix on the right by an invertible $n \times n$ diagonal matrix determines the same spanning line configuration. Therefore, the space of all such configurations can be identified with the orbits

$$X_{n,k} = \mathcal{F}_{k\times n}(\mathbb{C})/T = \{l_\bullet = (l_1, \ldots, l_n) \in (\mathbb{P}^{k-1})^n \,:\, l_1 + \cdots + l_n = \mathbb{C}^k\} \tag{4.1}$$

where $\mathcal{F}_{k\times n}(\mathbb{C})$ is the set of full rank $k \times n$ matrices with no zero columns and T is the set of diagonal matrices in $GL_n(\mathbb{C})$.

Pawlowski and Rhoades give a cell decomposition of $X_{n,k} = \bigcup C_w$ indexed by Fubini words $w \in W_{n,k}$, which were defined in Section 2.2. This cell decomposition plays an important role in the geometry and topology of the space of spanning line configurations. The closure of the cell C_w in $X_{n,k}$ is determined by certain rank conditions on the matrices representing points in C_w. The Pawlowski-Rhoades varieties, or *PR varieties* for short, are the cell closures $\overline{C}_w$.

The required rank conditions for PR varieties can be described in terms of quilts as follows. Every $k\times n$ matrix M determines a Boolean-chain quilt $f_M : C_k \times B_n \longrightarrow \mathbb{N}$ given by sending (h, J) to the rank of the submatrix of M on rows $[h] = \{1, 2, \ldots, h\}$ and columns in J, denoted $f_M(h, J) = \operatorname{rank}(M[[h], J])$. Here we define the boundary cases by $f_M(0, J) = f_M(h, \emptyset) = 0$ for all $1 \leq h \leq k$ and $J \subseteq [n]$. The quilt f_M also encodes the rank functions of the flag matroid corresponding to M as mentioned in Section 2.3. Furthermore, if $D \in T$ is an invertible diagonal matrix then $f_M = f_{MD}$, so such collections of rank functions are constant on the T-orbits in $\mathcal{F}_{k\times n}(\mathbb{C})$. Hence, each spanning line configuration gives rise to a well-defined Boolean-chain quilt. It was shown in [Sta22, Lemma 4.1.2] that f_M determines which cell $C_w \subset X_{n,k}$ contains the spanning line configuration determined by the columns of M for each $M \in \mathcal{F}_{k\times n}(\mathbb{C})$. On the other hand, if we define a function $C_k \times B_n \longrightarrow \mathbb{N}$ for each Fubini word $w \in W_{n,k}$ by

$$f_w(h, J) = \max\{f_M(h, J) \,|\, M \in C_w\} \tag{4.2}$$

for each $1 \leq h \leq k$ and $J \subseteq [n]$, then we have the following observations.

Lemma 4.1. *For $w \in W_{n,k}$, the map $f_w \in \operatorname{Quilts}(C_k, B_n)$. Furthermore, the spanning line configuration defined by $M \in \mathcal{F}_{k\times n}(\mathbb{C})$ is in $\overline{C}_w$ if and only if $f_M \leq f_w$ in the quilt lattice of type (C_k, B_n).*

Proof. The fact that f_w is a quilt of type (C_k, B_n) follows from the fact that $f_w = f_M$ for any generically chosen $M \in C_w$. By (4.2), if $M \in C_w$, then $f_M \leq f_w$ in the quilt lattice. This claim extends any $M \in \overline{C}_w$ since the closure is defined by such rank conditions. Conversely, if $M \in \mathcal{F}_{k\times n}(\mathbb{C})$ and $f_M \leq f_w$ in the quilt lattice, then M satisfies the defining rank conditions for the PR variety $\overline{C}_w$. □

There is a natural analog of Bruhat order for Fubini words in $W_{n,k}$ given by the (reverse) containment order on the PR varieties, so $v \leq w$ if and only if $\overline{C}_w \subseteq \overline{C}_v$. This partial order, originally studied by Pawlowski–Rhoades, is called the *medium roast Fubini–Bruhat order* on $W_{n,k}$ following terminology in [BR24, Sta22]. Billey–Ryan observed that the medium roast order can also be determined in terms of certain vanishing flag minors. An algorithm to determine $f_w : C_k \times B_n \longrightarrow C_k$ directly from w is given by [Sta22, Lemma 5.2.8].

Corollary 4.2. *The medium roast order on $W_{n,k}$ is the subposet of* $\mathrm{Quilts}(C_k, B_n)$ *defined by*

$$v \leq w \iff f_v \geq f_w.$$

Corollary 4.3. *The interval $[v, w]$ in medium roast Fubini order on $W_{n,k}$ can be determined from the interval $[f_v, f_w]$ in* $\mathrm{Quilts}(C_k, B_n)$ *by identifying all of the quilts in $[f_v, f_w]$ which correspond to some Fubini word. In particular, testing if v is covered by w reduces to checking that the open interval (f_v, f_w) contains no f_u for $u \in W_{n,k}$.*

Remark 4.4. Finding a concise description of all covering relations in the medium roast order on $W_{n,k}$ is still an open problem as of the writing of this paper. Can the embedding into the quilt lattice $\mathrm{Quilts}(C_k, B_n)$ be used to find such a characterization?

5. Properties of the quilt lattice

In this section, we prove some further properties of the lattice $\mathrm{Quilts}(P, Q)$ using natural involutions and embeddings. We begin with some examples of how the CSM/ASM poset, the medium roast poset, and a natural poset on matroids all embed into quilt lattices.

Observe from the definitions that for all posets P, Q with $\operatorname{rank} P = k$ and $\operatorname{rank} Q = n$, the map

$$\iota\colon \mathrm{CSM}_{k,n} \longrightarrow \mathrm{Quilts}(P, Q), \qquad \iota(f)(x, y) = f(\operatorname{rank} x,\ \operatorname{rank} y) \tag{5.1}$$

is a lattice embedding. This embedding is not necessarily surjective; in fact, it can be quite sparse. For example, Figure 5 shows the lattice $\mathrm{Quilts}(C_2, B_3)$, which has 199 elements. For $k \leq n$, we illustrate that $\mathrm{Quilts}(C_k, B_n)$ contains (among others) the following three overlapping subposets.

- ASMs/CSMs of size $k \times n$, embedded via ι from (5.1) (the seven ASMs of size 2×3 are marked with red and purple);
- The Fubini words $W_{n,k}$ with the medium roast order on Fubini words are embedded into $\mathrm{Quilts}(C_k, B_n)$ by Lemma 4.1. The six Fubini words of length 3 with letters 1 and 2 are marked with blue and purple.
- Matroids on ground set $[n]$ with rank k are embedded into $\mathrm{Quilts}(C_k, B_n)$ via the map that sends a matroid on $[n]$ to the quilt $f : C_k \times B_n \longrightarrow \mathbb{N}$ with $f(i, T) = \min\{i,\ \operatorname{rank} T\}$, where $\operatorname{rank} T$ is the cardinality of the largest independent set contained in T. The seven matroids on [3] with rank 2 are marked as squares.

The rank functions of flag matroids $\mathcal{M}$ on the ground set $[n]$ of type $(k_1 < \cdots < k_s)$ and rank functions $\mathrm{rank}_1, \ldots, \mathrm{rank}_s$ can also be encoded as a Boolean-chain quilt. Specifically, the embedding into $\mathrm{Quilts}(C_{k_s}, B_n)$ is the map that sends $\mathcal{M}$ to $f_{\mathcal{M}}$, where

$$f_{\mathcal{M}}(i, X) = \begin{cases} \min\{i, \mathrm{rank}_1 X\} & 1 \leq i \leq k_1 \\ \min\{\mathrm{rank}_j X + i - k_j, \mathrm{rank}_{j+1} X\} & k_j < i \leq k_{j+1}, 1 \leq j \leq s-1. \end{cases}$$

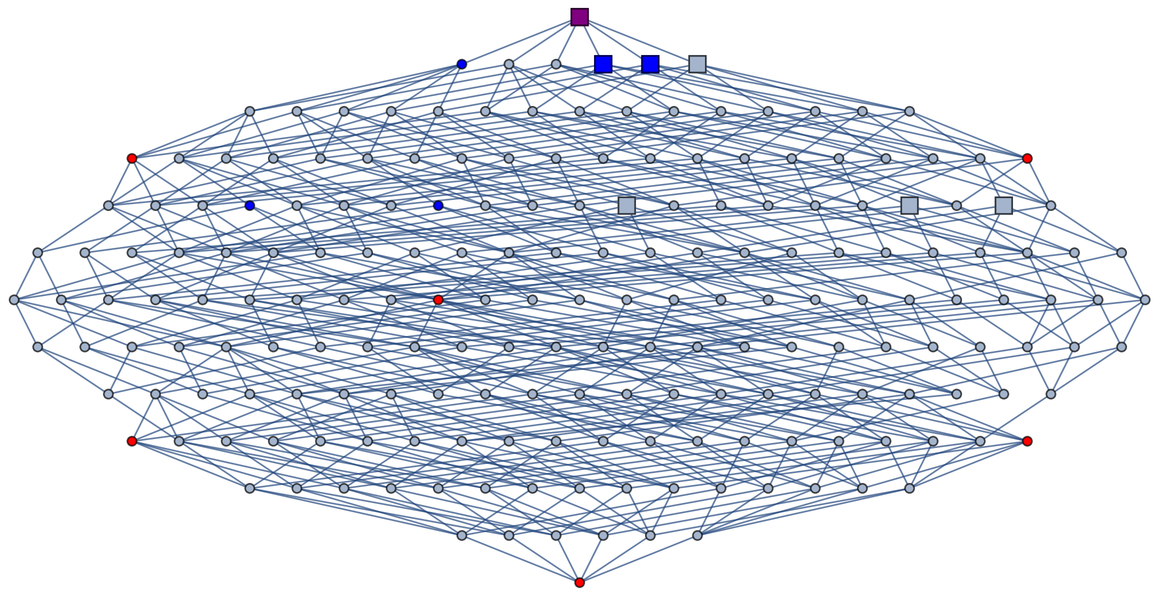


FIGURE 5. The Hasse diagram of $\mathrm{Quilts}(C_2, B_3)$.

One can verify $f_{\mathcal{M}}$ satisfies the properties of a quilt from the definitions in Section 2.3 as follows. By definition, $f_{\mathcal{M}}(0, X) = f_{\mathcal{M}}(i, \emptyset) = \mathrm{rank}_j(\emptyset) = 0$ for all i, j, X and $f_{\mathcal{M}}(k_s, [n]) = \mathrm{rank}_s[n] = k_s$. Matroid rank functions satisfy Boolean growth, hence $f_{\mathcal{M}}(i, X \cup \{y\}) - f_{\mathcal{M}}(i, X) \in \{0, 1\}$ for all $y \in [n] \setminus X$. Finally, observe that $\min\{\mathrm{rank}_j X + i - k_j,\ \mathrm{rank}_{j+1} X\}$ exhibits Boolean growth as a function of i since $\mathrm{rank}_j X \le \mathrm{rank}_{j+1} X$ and because M_j being a quotient of M_{j+1} implies $\mathrm{rank}_j X + k_{j+1} - k_j \ge \mathrm{rank}_{j+1} X$ by (2.6).

Under certain circumstances, (anti)automorphisms of the posets P and Q give us (anti)automorphisms of the quilt lattice. Here an antiautomorphism of a ranked poset P is an isomorphism between $(P, \le)$ and its dual poset $(P, \ge)$. If there exists an involutive antiautomorphism on P, there is a dihedral group D_4 action on $\mathrm{Quilts}(P, P)$, like with square ASMs.

Theorem 5.1. *Let P and Q be finite ranked posets with least and greatest elements.*

(1) *The switch map*

$$\Sigma\colon\ \mathrm{Quilts}(P, Q) \to \mathrm{Quilts}(Q, P), \quad \text{where } \Sigma(f)(x, y) = f(y, x)$$

is an involutive lattice isomorphism.

(2) *If γ is an automorphism of P, then*

$$\Gamma\colon\ \mathrm{Quilts}(P, Q) \to \mathrm{Quilts}(P, Q), \quad \text{where } \Gamma(f)(x, y) = f(\gamma(x), y)$$

is an automorphism of the lattice $\mathrm{Quilts}(P, Q)$.

(3) *If φ is an (involutive) antiautomorphism of P and* $\mathrm{rank}\, P \ge \mathrm{rank}\, Q$, *then*

$$\Phi\colon\ \mathrm{Quilts}(P, Q) \to \mathrm{Quilts}(P, Q), \quad \text{where } \Phi f(x, y) = \mathrm{rank}\, y - f(\varphi(x), y)$$

is an (involutive) antiautomorphism of the lattice $\mathrm{Quilts}(P, Q)$.

(4) *Given an involutive antiautomorphism $\varphi\colon P \to P$, there is an action of the dihedral group D_4 acting on* $\mathrm{Quilts}(P, P)$ *that sends the horizontal reflection of the square to Φ and the diagonal reflection of the square to Σ. If* $\mathrm{rank}\, P \ge 2$, *the action is faithful.*

Proof. Parts (1) and (2) are straightforward to verify from Definition 3.9. For (3) one can verify the conditions of Definition 3.9 as follows. Note that we have $\Phi f(x, \hat{0}_Q) = \operatorname{rank} \hat{0}_Q - f(\varphi(x), \hat{0}_Q) = 0$, $\Phi f(\hat{0}_P, y) = \operatorname{rank} y - f(\varphi(\hat{0}_P), y) = \operatorname{rank} y - f(\hat{1}_P, y) = \operatorname{rank} y - \operatorname{rank} y = 0$ and $\Phi f(\hat{1}_P, \hat{1}_Q) = \operatorname{rank} \hat{1}_Q - f(\varphi(\hat{1}_P), \hat{1}_Q) = \operatorname{rank} Q - f(\hat{0}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$. If $x \lessdot x'$, then $\varphi(x') \lessdot \varphi(x)$, so $\Phi f(x', y) - \Phi f(x, y) = f(\varphi(x), y) - f(\varphi(x'), y) \in \{0, 1\}$. If $y \lessdot y'$, then $1 = \operatorname{rank} y' - \operatorname{rank} y$, so

$$\Phi f(x, y') - \Phi f(x, y) = 1 - (f(\varphi(x), y') - f(\varphi(x), y)) \in \{0, 1\}.$$

That means that $\Phi f \in \operatorname{Quilts}(P, Q)$. If $f \leq g$, then $\Phi f(x, y) = \operatorname{rank} y - f(\varphi(x), y) \geq \operatorname{rank} y - g(\varphi(x), y) = \Phi g(x, y)$. If λ is the inverse of φ, then $\Lambda\colon \operatorname{Quilts}(P, Q) \to \operatorname{Quilts}(P, Q)$, where $(\Lambda f)(x, y) = \operatorname{rank} y - f(\lambda(x), y))$, is easily seen to be the inverse of Φ. In particular, if φ is an involution, so is Φ.

To prove (4), compute

$$(\Sigma \circ \Phi) f(x, y) = \Phi f(y, x) = \operatorname{rank} x - f(\varphi(y), x),$$

$$(\Sigma \circ \Phi)^2 f(x, y) = \operatorname{rank} x - (\Sigma \circ \Phi) f(\varphi(y), x) = \operatorname{rank} x - \operatorname{rank} \varphi(y) + f(\varphi(x), \varphi(y)),$$

$$(\Sigma \circ \Phi)^3 f(x, y) = \operatorname{rank} x - (\Sigma \circ \Phi)^2 f(\varphi(y), x) = \operatorname{rank} x - \operatorname{rank} \varphi(y) + \operatorname{rank} \varphi(x) - f(y, \varphi(x)),$$

$$(\Sigma \circ \Phi)^4 f(x, y) = \operatorname{rank} x - (\Sigma \circ \Phi)^3 f(\varphi(y), x) = \operatorname{rank} x - \operatorname{rank} \varphi(y) + \operatorname{rank} \varphi(x) - \operatorname{rank} y + f(x, y),$$

and because $\operatorname{rank} x + \operatorname{rank} \varphi(x) = \operatorname{rank} y + \operatorname{rank} \varphi(y) = \operatorname{rank} P$, we know $(\Sigma \circ \Phi)^4 = \mathrm{id}$. Therefore D_4 acts on $\operatorname{Quilts}(P, P)$.

Finally, $\operatorname{Quilts}(P, P)$ contains a representative of all square CSMs of size $\operatorname{rank} P \times \operatorname{rank} P$ under the map ι defined in (5.1), and the symmetries of the square act faithfully on this subset if $\operatorname{rank} P \geq 2$. Therefore, they also act faithfully on $\operatorname{Quilts}(P, P)$. ☐

Example 5.2. Any permutation of the labels gives an automorphism of B_k, which gives many automorphisms of $\operatorname{Quilts}(B_k, Q)$ for any Q. The maps $i \mapsto k - i$ and $T \mapsto [k] \setminus T$ are involutive antiautomorphisms on C_k and B_k, which gives involutive automorphisms of $\operatorname{Quilts}(C_k, Q)$ and $\operatorname{Quilts}(B_k, Q)$ for $\operatorname{rank} Q \leq k$ and a faithful D_4 action on $\operatorname{Quilts}(B_n, B_n)$ for $n \geq 2$.

Take posets P_1 and P_2 with the same rank. The *disjoint union* $P_1 + P_2$ is the poset we get my "merging" $\hat{0}_{P_1}$ with $\hat{0}_{P_2}$ and $\hat{1}_{P_1}$ with $\hat{1}_{P_2}$, and adding the other elements of P_1 and P_2 without any new relations. For example, $A_2(j_1) + A_2(j_2)$ is isomorphic to $A_2(j_1 + j_2)$. Write jP for the disjoint union of j copies of P. For example, $A_2(j) = jC_2$.

Proposition 5.3. *Assume that* $\operatorname{rank} P_1 = \operatorname{rank} P_2 \geq \operatorname{rank} Q$. *Then the map*

$$\Theta\colon \operatorname{Quilts}(P_1 + P_2, Q) \longrightarrow \operatorname{Quilts}(P_1, Q) \times \operatorname{Quilts}(P_2, Q)$$

defined by

$$f \mapsto (f_1, f_2), \qquad f_i(x_i, y) = f(x_i, y) \text{ for } x_i \in P_i, y \in Q,$$

is an isomorphism of lattices.

Proof. The only non-trivial part to prove is that the map is surjective. Say that we are given $f_1 \in \mathrm{Quilts}(P_1, Q)$ and $f_2 \in \mathrm{Quilts}(P_2, Q)$. Because $\operatorname{rank} P_i \geq \operatorname{rank} Q$, $f_i(\hat{1}_{P_i}, y) = \operatorname{rank} y$ for all $y \in Q$. That means that f_1 and f_2 are compatible in the only two "common" elements of P_1 and P_2 in $P_1 + P_2$, and the map $f : (P_1 + P_2) \times Q \longrightarrow \mathbb{N}$ given by

$$f(x, y) = \begin{cases} f_1(x, y) & : \quad x \in P_1 \\ f_2(x, y) & : \quad x \in P_2 \end{cases}$$

is a well-defined quilt and maps to (f_1, f_2). □

Remark 5.4. Without the assumption that $\operatorname{rank} P_1 = \operatorname{rank} P_2 \geq \operatorname{rank} Q$, the map Θ is still a well-defined injective homomorphisms of lattices, but is not necessarily a surjection. Therefore, we have $|\,\mathrm{Quilts}(P_1 + P_2, Q)| \leq |\,\mathrm{Quilts}(P_1, Q)| \cdot |\,\mathrm{Quilts}(P_2, Q)|$.

Corollary 5.5. *For $k \geq n$ and arbitrary positive integer j, $|\,\mathrm{Quilts}(jC_k, C_n)| = |\,\mathrm{ASM}_{k\times n}|^j$. For any n, i, j, we have $|\,\mathrm{Quilts}(iC_n, jC_n)| = |\,\mathrm{ASM}_{n\times n}|^{ij}$.*

Proposition 5.6. *Assume that a map $\psi\colon Q' \to Q$ has the following properties:*

- *ψ is surjective,*
- *$x \lessdot y \Rightarrow \psi(x) = \psi(y)$ or $\psi(x) \lessdot \psi(y)$.*

For P with $\operatorname{rank} P \leq \operatorname{rank} Q$, the map

$\Psi\colon\ \mathrm{Quilts}(P, Q) \longrightarrow \mathrm{Quilts}(P, Q'),\qquad$ *where $\Psi f(x, y') = f(x, \psi(y'))$ for all $f \in \mathrm{Quilts}(P, Q)$*

is an injective lattice homomorphism.

Proof. Let us first check that Ψf is indeed a quilt. It follows from the assumptions that $\psi(\hat{0}_{Q'}) = \hat{0}_Q$ and $\psi(\hat{1}_{Q'}) = \hat{1}_Q$, so $\operatorname{rank} Q \leq \operatorname{rank} Q'$. We have $\Psi f(\hat{0}_P, y) = f(\hat{0}_P, \psi(y)) = 0$, $\Psi f(x, \hat{0}_{Q'}) = f(x, \psi(\hat{0}_{Q'})) = f(x, \hat{0}_Q) = 0$ and

$$\Psi f(\hat{1}_P, \hat{1}_{Q'}) = f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\} = \operatorname{rank} P = \min\{\operatorname{rank} P,\ \operatorname{rank} Q'\}.$$

If $x_1 \lessdot x_2$ in P, then $\Psi f(x_2, y') = f(x_2, \psi(y'))$ is either $f(x_1, \psi(y'))$ or $f(x_1, \psi(y')) + 1$. On the other hand, if $y'_1 \lessdot y'_2$ in Q', then $\psi(y'_1) = \psi(y'_2)$ or $\psi(y'_1) \lessdot \psi(y'_2)$. In both cases, either $\Psi f(x, y'_2) = \Psi f(x, y'_1)$ or $\Psi f(x, y'_2) = \Psi f(x, y'_1) + 1$. That completes the verification that Ψf is a quilt.

If $f \leq g$ in $\mathrm{Quilts}(P, Q)$, then for $x \in P$, $y' \in Q'$, $\Psi f(x, y') = f(x, \psi(y')) \leq g(x, \psi(y')) = \Psi g(x, y')$, so $\Psi f \leq \Psi g$. If $\Psi f = \Psi g$ and $x \in P$, $y \in Q$ are arbitrary, then $y = \psi(y')$ for some $y' \in Q'$ by the surjectivity of ψ, and $f(x, y) = \Psi f(x, y') = \Psi g(x, y') = g(x, y)$. That means that Ψ is an injective homomorphism on two quilt lattices. □

Example 5.7. Take $m \leq n$. The following maps have the required properties:

- $\psi_1\colon C_n \to C_m$, $\psi(i) = \min\{i, m\}$,
- $\psi_2\colon B_n \to B_m$, $\psi(T) = T \cap [m]$

We therefore have injective lattice homomorphisms

$$\Psi_1\colon\ \mathrm{Quilts}(P, C_m) \longrightarrow \mathrm{Quilts}(P, C_n) \text{ and } \Psi_2\colon\ \mathrm{Quilts}(P, B_m) \longrightarrow \mathrm{Quilts}(P, B_n).$$

for $\operatorname{rank} P \leq m$. Since ψ_1 has the added property that $\psi_1(i) = \psi_1(j) \Rightarrow i = j$ or $\psi_1(i) = \psi_1(j) = m$, Ψ_1 preserves cover relations.

Thus, the embedding of $\mathrm{Quilts}(P, C_m)$ into $\mathrm{Quilts}(P, C_n)$ is *isometric*, meaning $\operatorname{rank} g - \operatorname{rank} f = \operatorname{rank} \Psi g - \operatorname{rank} \Psi f$ for all f, g. For $P = C_k$, $k \leq m \leq n$, this is equivalent to adding $n - m$ zero columns to a $k \times m$ ASM to get a $k \times n$ ASM. The map Ψ_2 does not preserve covering relations: take quilts $f, g \in \operatorname{Quilts}(C_2, B_2)$ satisfying $f(1,\{1\}) = 0$, $g(1,\{1\}) = 1$ and $f \lessdot g$; then $\Psi_2 f(1,\{1\}) = \Psi_2 f(1,\{1,3\}) = 0$, $\Psi_2 g(1,\{1\}) = \Psi_2 g(1,\{1,3\}) = 1$, so $\Psi_2 f \not\lessdot \Psi_2 g$.

6. Enumeration of antichain quilts

In this section, we consider the case of counting the number of quilts of type (P, Q) when Q is an antichain poset. The enumeration is in terms of the number of antichains in convex cut sets of P. We begin by defining the necessary vocabulary and notation. Several specific examples are included following the corollaries.

We say that a subset S of a poset P is *convex* if $x, y \in S$ implies $[x, y] \subseteq S$. We say that S is a *cut set* if it intersects every maximal chain in P. If you have a convex cut set C, it makes sense to say that an element $x \in P \setminus C$ is *above* C or *below* C: x lies on a maximal chain, the maximal chain intersects C in some element x', and x is above C if $x > x'$ and below C if $x < x'$. This is well defined, as $x' < x < x''$ for $x', x'' \in C$ would imply $x \in C$. For example, if $\operatorname{rank} P \geq 2$, then $C = P \setminus \{\hat{0}_P, \hat{1}_P\}$ is a convex cut set, $\hat{0}_P$ is below C, and $\hat{1}_P$ is above C.

Recall from Section 3 that $d_1(P)$ counts the number of nonempty antichains in P other than $\{\hat{0}\}$. Such antichains are in bijection with antichains in $P \setminus \{\hat{0}_P, \hat{1}_P\}$. For any $S \subseteq P$, denote by $\alpha_P(S)$ the number of antichains in S. Then, we have $\alpha_P(P \setminus \{\hat{0}_P, \hat{1}_P\}) = d_1(P)$. Given two infinite sequences (a_n) and (b_n), we write $a_n \sim b_n$ to mean $a_n/b_n \to 1$ as n goes to infinity.

Theorem 6.1. *Take a ranked poset P with least and greatest elements,* $\operatorname{rank} P \geq 2$, *and $j \geq 1$. We have*

$$|\operatorname{Quilts}(P, A_2(j))| = \sum_C \alpha_P(C)^j, \tag{6.1}$$

where the sum is over all subsets C of $P \setminus \{\hat{0}_P, \hat{1}_P\}$ that are convex cut sets of P. In particular, as j goes to infinity, we have

$$|\operatorname{Quilts}(P, A_2(j))| \sim d_1(P)^j. \tag{6.2}$$

Proof. For $f \in \operatorname{Quilts}(P, A_2(j))$, we have $f(x,y) \leq \min\{\operatorname{rank} P, \operatorname{rank} A_2(j)\} = 2$ for all $x \in P$ and $y \in A_2(j)$. Take $x \in P$. If $f(x, \hat{1}) = 0$, then $f(x,y) = 0$ for all $y \in A_2(j)$. If $f(x,\hat{1}) = 2$, then $f(x, \hat{0}) = 0$ and $f(x,y) = 1$ for $\operatorname{rank} y = 1$. If, however, $f(x,\hat{1}) = 1$, then $f(x,y)$ can be either 0 or 1 for $\operatorname{rank} y = 1$. If $x \leq x''$ and $f(x,\hat{1}) = f(x'',\hat{1}) = 1$, then $f(x',\hat{1}) = 1$ for every $x' \in [x, x'']$ since $f \in \operatorname{Quilts}(P, A_2(j))$. Furthermore, for a maximal chain $\hat{0}_P = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = \hat{1}_P$, we have $f(x_0, \hat{1}) = 0$, $f(x_k, \hat{1}) = 2$, and $f(x_i, \hat{1}) - f(x_{i-1}, \hat{1}) \in \{0, 1\}$, so $f(x_i, \hat{1})$ must be 1 for some i. Therefore, the set $C_f = \{x \in P \colon f(x, \hat{1}) = 1\}$ is a convex cut set contained in $P \setminus \{\hat{0}_P, \hat{1}_P\}$, and for every rank 1 element y in $A_2(j)$, the set $F_y = \{x \in C_f \colon f(x,y) = 1\}$ is an order filter in C_f: if $x \in F_y$ and $x' \in C_f$, $x' \geq x$, then $x' \in F_y$.

It remains to enumerate quilts f in $\operatorname{Quilts}(P, A_2(j))$ for which $\{x \in P \colon f(x, \hat{1}) = 1\}$ is equal to a given convex cut subset C of $P \setminus \{\hat{0}_P, \hat{1}_P\}$. Given any choice of an order filter F_y in C for each y independently, one can define a corresponding quilt $f \in \operatorname{Quilts}(P, A_2(j))$ as follows. The value $f(x, \hat{1})$ is 1 if $x \in C$, 0 if x is below C, and 2 if x is above C. If $f(x, \hat{1})$ is 0 or 2, the other $f(x,y)$ are uniquely determined, and if $f(x, \hat{1}) = 1$, the other $f(x,y)$ are determined by F_y. Since the order filters of C are in a natural bijection with the antichains in C, this completes the proof of Equation (6.1).

It is clear that if C is a proper subset of $P \setminus \{\hat{0}_P, \hat{1}_P\}$, it contains strictly fewer antichains than $P \setminus \{\hat{0}_P, \hat{1}_P\}$, so the term $d_1(P)^j$ coming from $C = P \setminus \{\hat{0}_P, \hat{1}_P\}$ dominates. This proves (6.2). ☐

Example 6.2. As a simple example, take $P = A_2(i)$ for $i \geq 1$. There is only one (convex) cut set in $P \setminus \{\hat{0}_P, \hat{1}_P\}$, namely $P \setminus \{\hat{0}_P, \hat{1}_P\}$ itself. Every subset is an antichain, so $\alpha_P(P \setminus \{\hat{0}_P, \hat{1}_P\}) = 2^i$. Therefore $|\operatorname{Quilts}(A_2(i), A_2(j))| = 2^{ij}$. This is consistent with Corollary 5.5 for $n = 2$. In fact, it is easy to see that $\operatorname{Quilts}(A_2(i), A_2(j)) \cong B_{ij}$ as lattices.

As a more involved application, let us show some enumerative results about quilts when P is a chain or a product of chains, and Q is an antichain. The b_n's that appear in the proposition are the Bernoulli numbers $1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots$

Corollary 6.3. *For arbitrary integers $j \geq 1$ and $k \geq 2$, we have*

$$|\operatorname{Quilts}(C_k, A_2(j))| = \sum_{i=2}^{k} (k+1-i) i^j. \tag{6.3}$$

Therefore, $|\operatorname{Quilts}(C_k, A_2(j))|$ as a function of k is given by the polynomial

$$\frac{1}{(j+1)(j+2)} \left(k^{j+2} + \sum_{l=1}^{j} \binom{j+2}{l} \left(l\, b_{l-1} - (l-1) b_l\right) k^{j+2-l} \right) + (b_j - b_{j+1} - 1)k. \tag{6.4}$$

For arbitrary k we have

$$|\operatorname{Quilts}(C_k \times C_1, A_2(j))| = \sum_{\substack{0 \leq d \leq b,\, c \leq a \leq k \\ 1 \leq b \leq c+1 \leq k}} \left((a-c)(c-d+2) + \tbinom{c-d+3}{2} - \tbinom{b-d+1}{2}\right)^j.$$

For arbitrary k_1, k_2, we have

$$|\operatorname{Quilts}(C_{k_1} \times C_{k_2}, A_2(j))| \sim \left(\binom{k_1+k_2+2}{k_1+1} - 2 \right)^j.$$

Proof. A non-empty convex set in $C_k \setminus \{0, k\}$ is an interval $[i, j]$ for $1 \leq i \leq j \leq k-1$, and every such interval is a cut set. Furthermore, $[i, j]$ contains $|[i, j]| + 1 = j - i + 2$ antichains—the empty set and all singletons. There are $k + 1 - i$ intervals of size $i - 1$, $i \geq 2$, so Theorem 6.1 gives $|\operatorname{Quilts}(C_k, A_2(j))| = \sum_{i=2}^{k} (k+1-i) i^j = (k+1) \sum_{i=2}^{k} i^j - \sum_{i=2}^{k} i^{j+1}$. The famous Faulhaber's formula gives (6.4).

A subset C of $C_k \times C_1 \setminus \{(0,0), (k,1)\}$ is a convex cut set if and only if it there exist a, b, c, d so that $C = \{(i, 0), b \leq i \leq a\} \cup \{(i, 1), d \leq i \leq c\}$, where

- $b \geq 1$, otherwise C contains $(0, 0)$;
- $c \leq k - 1$, otherwise C contains $(k, 1)$;
- $d \leq b$, otherwise the point $(b, 1)$ is between $(b, 0)$ and $(d, 1)$ but not in C;
- $c \leq a$ for a similar reason; and
- $b \leq c + 1$, otherwise there is a maximal chain that avoids C.

It is not hard to see that the number of antichains in such a set is $(a-c)(c-d+2) + \binom{c-d+3}{2} - \binom{b-d+1}{2}$. Now use Theorem 6.1.

We can interpret an antichain in $C_{k_1} \times C_{k_2}$ as (the southwest corners of) a lattice path between $(k_1, 0)$ and $(0, k_2)$, and there are $\binom{k_1+k_2+2}{k_1+1}$ of those. We subtract 2 because we do not count the antichains $\{(0,0)\}$ and $\{(k_1, k_2)\}$. Again, use Theorem 6.1. □

Example 6.4. The following illustrates the various statements of Corollary 6.3. We have

$$|\operatorname{Quilts}(C_4, A_2(j))| = 3 \cdot 2^j + 2 \cdot 3^j + 4^j$$

for $j \geq 1$ and

$$|\operatorname{Quilts}(C_k, A_2(3))| = \frac{k^5}{20} + \frac{k^4}{4} + \frac{5k^3}{12} + \frac{k^2}{4} - \frac{29k}{30}$$

for $k \geq 2$. The next statement gives

$$|\operatorname{Quilts}(C_2 \times C_1, A_2(j))| = 3^j + 2 \cdot 4^j + 2 \cdot 5^j + 2 \cdot 6^j + 8^j,$$

and

$$|\operatorname{Quilts}(C_3 \times C_1, A_2(j))| = 2 \cdot 3^j + 3 \cdot 4^j + 4 \cdot 5^j + 5 \cdot 6^j + 2 \cdot 7^j + 4 \cdot 8^j + 3 \cdot 9^j + 2 \cdot 11^j + 13^j.$$

Example 6.5. It follows from Theorem 6.1 that $|\operatorname{Quilts}(B_n, A_2(j))| \sim d_1(B_n)^j$. It does not seem likely that a simple exact formula for $|\operatorname{Quilts}(B_n, A_2(j))|$ exists. Since $B_2 = A_2(2)$, we have $|\operatorname{Quilts}(B_2, A_2(j))| = 4^j$, and some (computer) time is needed to find

$$|\operatorname{Quilts}(B_3, A_2(j))| = 2 \cdot 8^j + 3 \cdot 9^j + 6 \cdot 10^j + 6 \cdot 13^j + 18^j.$$

The exact formula for $|\operatorname{Quilts}(B_4, A_2(j))|$ can be found in Appendix A.

7. Enumeration of chain quilts

Given a finite poset P, we will consider the enumeration of chain quilts $\operatorname{Quilts}(P, C_n)$ in this section. The formulas are in terms of *fundamental* and *standard* quilts for P. We begin with some additional notation and vocabulary.

Recall that we defined the sum of ranks $b(P) = \sum_{x \in P} \operatorname{rank} x$. If $f \in \operatorname{Quilts}(P, C_{b(P)})$, we say that $i \in [b(P)]$ is a *jump for* f if there exists $x \in P$ so that $f(x, i) = f(x, i-1) + 1$. If the set of jumps of f is equal to $[m]$, we say that f is *m-fundamental for P*. A *standard* quilt is one that is $b(P)$-fundamental. Denote by $F_m(P)$ the set of all m-fundamental quilts for P, and write $S(P) = F_{b(P)}$ and $F(P) = \bigcup_m F_m(P)$.

A chain quilt is m-fundamental if and only if its MT form contains precisely the elements $1, \ldots, m$. In particular, it is standard if and only if its MT form contains exactly one of each of $1, \ldots, b(P)$. For example, consider $P = B_2$. We have $b(P) = 4$, and there are four 2-fundamental, five 3-fundamental, and two 4-fundamental (standard) quilts, presented in Figure 6 in MT form.

```
  12      12      12      12      13      13      13      13      13      14      14
 /  \    /  \    /  \    /  \    /  \    /  \    /  \    /  \    /  \    /  \    /  \
1    1  1    2  2    1  2    2  1    2  2    1  2    2  2    3  3    2  2    3  3    2
 \  /    \  /    \  /    \  /    \  /    \  /    \  /    \  /    \  /    \  /    \  /
  ∅       ∅       ∅       ∅       ∅       ∅       ∅       ∅       ∅       ∅       ∅
```

Figure 6. All fundamental quilts for B_2.

Observe that $\operatorname{rank} P \leq b(P)$ and a quilt can be m-fundamental only for $\operatorname{rank} P \leq m \leq b(P)$. Note too that $S(P)$ is not empty. Indeed, to find a standard chain quilt for P, consider the map that sends each $x \in P$ of rank i to the subset $\{1, 2, \ldots, i\}$, and then "standardize": change all the 1's in the reverse of the chosen total order on P to $1, 2, \ldots, j_1$, then change all of the original 2's to $j_1 + 1, j_1 + 2, \ldots, j_1 + j_2$, then all original 3's to $j_1 + j_2 + 1, j_1 + j_2 + 2, \ldots, j_1 + j_2 + j_3$ etc. Figure 7 shows the standard quilt we get for B_3. There are 1344 standard chain quilts for B_3 in total.

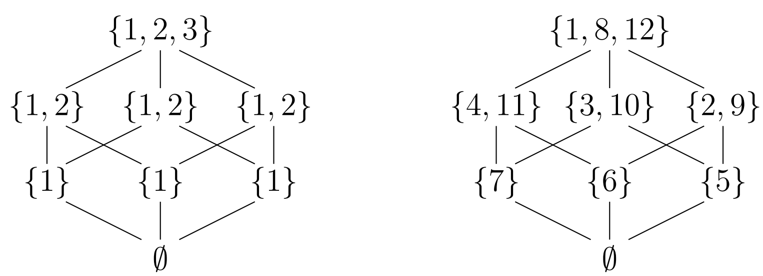


FIGURE 7. A 3-fundamental quilt for B_3 along with its standardization.

Theorem 7.1. *For a fixed poset P of rank k with least and greatest elements and any integer $n \geq k$, the number of chain quilts of type (P, C_n) is given by a polynomial in n, namely*

$$|\operatorname{Quilts}(P, C_n)| = \sum_{m=k}^{b(P)} |F_m(P)| \binom{n}{m}. \tag{7.1}$$

In particular,

$$|\operatorname{Quilts}(P, C_n)| \sim \frac{|S(P)|}{b(P)!} \cdot n^{b(P)}. \tag{7.2}$$

Proof. We will abuse notation and consider a quilt in $f \in \operatorname{Quilts}(P, C_n)$, with $n \geq k$, to be synonymous with its jump set map $f \colon P \to B_n$ for which $|f(x)| = \operatorname{rank} x$ for all $x \in P$ and for which $f(x)$ and $f(y)$ interlace when $x \lessdot y$, see Proposition 3.17. Say that $\bigcup_{x \in P} f(x) = \{i_1, \ldots, i_m\} \subseteq [n]$. Replace all instances of i_j with j for all $j \in [m]$. This gives us an m-fundamental quilt, and for every m-fundamental quilt, there are $\binom{n}{m}$ ways to choose the map $j \mapsto i_j$. This proves Equation (7.1). The highest degree terms are clearly the ones with $m = b(P)$, which implies Equation (7.2). □

To illustrate the procedure employed in the proof, take the chain quilt on the left of Figure 8. The union of all the jump sets is $\{2, 4, 6, 7, 9\}$, so we replace 2, 4, 6, 7, 9 by 1, 2, 3, 4, 5, respectively. We get the 5-fundamental quilt on the right.

Example 7.2. From the previous example it follows that

$$|\operatorname{Quilts}(B_2, C_n)| = 4\binom{n}{2} + 5\binom{n}{3} + 2\binom{n}{4} = \frac{n^4}{12} + \frac{n^3}{3} + \frac{5n^2}{12} - \frac{5n}{6} \sim 2 \cdot \frac{n^4}{4!}$$

for $n \geq 2$. This agrees with Corollary 6.3 (note that $B_2 = A_2(2) = C_1 \times C_1$). With more effort, we can compute

$$|\operatorname{Quilts}(B_3, C_n)| \sim \frac{1344}{12!} \cdot n^{12} \quad \text{and} \quad \operatorname{Quilts}(B_4, C_n)| \sim \frac{10651644896477184}{32!} \cdot n^{32},$$

see Appendix A.

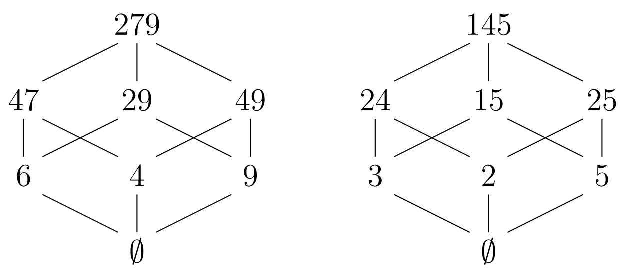


FIGURE 8. A chain quilt and a 5-fundamental quilt for B_3.

Example 7.3. We can think of a standard quilt for $P = C_k$ as a monotone triangle (in the classical sense) in which all numbers $1, \dots, \binom{k+1}{2}$ appear. After an up-down reflection and a 45° rotation, we get a shifted standard Young tableau of staircase shape $(k, k-1, \dots, 1)$. For example, for $k = 3$, we get monotone triangles

$$\begin{array}{ccccc} 1 & & 3 & & 6 \\ & 2 & & 5 & \\ & & 4 & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & & 4 & & 6 \\ & 2 & & 5 & \\ & & 3 & & \end{array}$$

and shifted standard Young tableaux

$$\begin{array}{ccc} 1 & 2 & 4 \\ & 3 & 5 \\ & & 6 \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{array}.$$

The hook-length formula for shifted standard Young tableaux [Thr52] gives (for fixed k and $n \to \infty$)

$$|\operatorname{ASM}_{k,n}| \sim \frac{\prod_{i=0}^{k-1}(2i)!}{\prod_{i=0}^{k-1}(k+i)!} \cdot n^{\binom{k+1}{2}}.$$

An m-fundamental quilt in this case can be interpreted as a shifted tableau of staircase shape $(k, k-1, \dots, 1)$ with weakly increasing rows and columns, strictly increasing diagonals, and entries in $[m]$, with each number in $[m]$ appearing at least once.

Remark 7.4. Every m-fundamental quilt can be obtained from a standard quilt by replacing $1, \dots, b(P)$ by a sequence of the type $1, \dots, 1, 2, \dots, 2, \dots, m, \dots, m$, with each number appearing at least once. The only requirement when choosing the replacing sequence is that if i, j appear in the same set of the standard quilt, i and j cannot be replaced by the same number. However, one m-fundamental quilt can be obtained from several standard ones. For example, for $P = B_2$, the replacing sequence 1223 on both fundamental quilts (the last two in Figure 6) gives the same 3-fundamental quilt (the seventh quilt in Figure 6). That means that we can rephrase Equation (7.1) as

$$|\operatorname{Quilts}(P, C_n)| = \sum_{f \in S(T)} \sum_{u} \binom{n}{\max u},$$

where u runs over integer sequences that are "compatible" with f, where the definition of compatibility ensures that there are no repetitions of fundamental quilts. We omit the details.

There is in fact one more way to compute $|\operatorname{Quilts}(P, C_n)|$ and prove the polynomiality property via the transfer-matrix method [Sta12, Thm. 4.7.2] using the adjacency matrix $A_D(P)$ of the Dedekind graph of P defined in Definition 3.5. While it does not imply Equation (7.2), it is the authors' experience that it is in practice easier to compute the inverse of the (upper-triangular) matrix $I - xA_D(P)$ than the cardinalities of $F_m(P)$ for $m = k, \ldots, b(P)$. Furthermore, this method also gives us a way to compute $|\operatorname{Quilts}(P, C_1)|, \ldots, |\operatorname{Quilts}(P, C_{k-1})|$.

Theorem 7.5. *For a finite poset P of rank $k \geq 1$ with least and greatest elements, we have*

$$\sum_{n=k}^{\infty} |\operatorname{Quilts}(P, C_n)| x^n = (I - xA_D(P))^{-1}_{1,d(P)} = \frac{(-1)^{d(P)-1}}{(1-x)^{d(P)}} \det T(P), \tag{7.3}$$

where $T(P)$ is the transfer-matrix $I - xA_D(P)$ with the first column and last row removed. In particular, the sequence $0, 0, \ldots, |\operatorname{Quilts}(P, C_k)|, |\operatorname{Quilts}(P, C_{k+1})|, \ldots$ is given by a polynomial of degree $< d(P)$. Furthermore,

$$\sum_{n=0}^{k-1} |\operatorname{Quilts}(P, C_n)| x^n = \sum_{i=1}^{d(P)-1} (I - xA'_D(P))^{-1}_{1,i} = \sum_{i=1}^{d(P)-1} (-1)^{i-1} \det T'(P)_i, \tag{7.4}$$

where $T'(P)_i$ is the matrix $I - xA'_D(P)$ with the first column and i-th row removed.

Proof. A chain quilt $f \in \operatorname{Quilts}(P, C_n)$ can be viewed as a sequence of Dedekind maps, with the i-th one, $0 \leq i \leq n$, sending x to $f(x, i)$. The map $x \mapsto f(x, 0)$ is always the zero map, which is the only element in $D_0(P)$. If $n \geq \operatorname{rank} P = k$, $f(x, n) = \operatorname{rank} x$ is the only element in $D_k(P)$. Furthermore, there is an edge from $x \mapsto f(x, i-1)$ to $x \mapsto f(x, i)$ in the Dedekind graph $G_D(P)$. In other words, we can interpret a chain quilt as a walk on the graph $G_D(P)$ starting in the first vertex (the all zero map) and ending in the last vertex (the only element in $D_k(P)$). The transfer-matrix method [Sta12, Thm. 4.7.2] tells us that the generating function for such walks is the $(1, d(P))$ entry of the matrix $(I - xA_D(P))^{-1}$, or, equivalently, is given by the corresponding determinantal expression, which proves (7.3). By definition, $\det T(P)$ is a polynomial of degree $< d(P)$, which implies $(-1)^{d(P)-1} \det T(P)(1-x)^{-d(P)}$ is a rational function of x of degree < 0. Hence, its coefficients as a power series, namely $|\operatorname{Quilts}(P, C_n)|$, are given by a polynomial function of n.

If $n < \operatorname{rank} P$, we have $f(\hat{1}_P, i) = i$ for every $i \in [n]$. In other words, given $f \in \operatorname{Quilts}(P, C_n)$ and $i \in [n]$ there is an edge from $x \mapsto f(x, i-1)$ to $x \mapsto f(x, i)$ in the restricted Dedekind graph $G'_D(P)$, and we are looking at walks on the graph $D'(P)$ starting in the first vertex and ending anywhere except in the last vertex. This proves (7.4). $\square$

Recall from Section 2.2 that there is an easy bijection between $k \times n$ ASMs and monotone triangles with all possible length k top row sequences. Such a top row sequence will be denoted by $(a_1, \ldots, a_k)$ with $1 \leq a_1 < a_2 < \cdots < a_k \leq n$. Fischer proved that the cardinality of $\operatorname{MT}(a_1, \ldots, a_k)$, the set of monotone triangles with top row $(a_1, \ldots, a_k)$ is a polynomial in variables $a_1, \ldots, a_k$, and she also found an explicit (operator) formula for $|\operatorname{MT}(a_1, \ldots, a_k)|$, see [Fis06]. The definition can be extended to arbitrary chain quilts: given a poset P of rank k and $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, define $\operatorname{MT}_P(a_1, \ldots, a_k)$ as the set of quilts $f \in \operatorname{Quilts}(P, C_n)$ for which $J_f(\hat{1}_P) = \{a_1, \ldots, a_k\}$. Here, we equate the quilt f with its jump set map $f : P \longrightarrow B_n$ as in the proof of Theorem 7.1. We call $J_f(\hat{1}_P)$ the *top set of the quilt* f. Note that strictly speaking, $\operatorname{MT}(a_1, \ldots, a_k)$ depends on n, but there is a natural bijection between $\operatorname{MT}(a_1, \ldots, a_k) \subseteq \operatorname{Quilts}(P, C_n)$ and $\operatorname{MT}(a_1, \ldots, a_k) \subseteq \operatorname{Quilts}(P, C_m)$ whenever $m, n \geq a_k$ so we can ignore this. Let $J_f(\hat{1}_P)_i$ denote the i-th largest element of the set $J_f(\hat{1}_P)$.

Theorem 7.6. *For a finite poset P of rank k with least and greatest elements, we have*

$$|\operatorname{MT}_P(a_1,\dots,a_k)| = \sum_{f\in F(P)}\prod_{i=2}^{k}\binom{a_i-a_{i-1}-1}{J_f(\hat{1}_P)_i - J_f(\hat{1}_P)_{i-1}-1}. \tag{7.5}$$

Proof. This is similar to the proof of Theorem 7.1. Given a chain quilt $g \in \operatorname{Quilts}(P, C_n)$ with top set $\{a_1,\dots,a_k\}$, let $\{i_1 < \cdots < i_m\} = \bigcup_{x\in P} g(x) \subseteq [n]$. Replace all instances of i_j with j to get an m-fundamental quilt $f_g \in F(P)$. Each quilt in the inverse image under the replacement map of an m-fundamental quilt $f \in F(P)$ with $J_f(\hat{1}_P) = \{j_1,\dots,j_k\}$ is determined by replacing j_i by a_i everywhere in the MT form of f for each $1 \le i \le k$, making a choice of $j_2 - j_1 - 1$ elements among $a_1+1,\dots,a_2-1$ to replace $j_1+1,\dots,j_2-1$, making a choice of $j_3 - j_2 - 1$ elements among $a_2+1,\dots,a_3-1$ etc., which proves Equation (7.5). □

To illustrate the proof, say that we want to enumerate quilts in $\operatorname{Quilts}(B_3, C_{20})$ with top set $(2, 10, 16)$. We can get all such quilts from m-fundamental quilts for $3 \le m \le 12$. For example, we can take the 5-fundamental quilt on the right in Figure 8 with top set $\{1, 4, 5\}$, and replace 1 by 2, 4 by 10 and 5 by 16 to get the correct top set. We have choices for what we replace 2 and 3 by: we can select any 2 of the elements between 3 and 9 for that, and there are $\binom{7}{2}$ ways to do that. Since $4, 5$ are adjacent values there are no further choices to make in this case. On the other hand, if we take a, say, standard quilt with top set $(1, 8, 12)$, like the one in Figure 7 we replace 1 by 2, 8 by 10 and 12 by 16, and we select any 6 elements between 3 and 9 to replace $2,\dots,7$ by, and any 3 elements between 11 and 15 to replace $9, 10, 11$ by. Therefore we have $\binom{7}{6}\cdot\binom{5}{3}$ choices. Using Theorem 7.6, one can compute $|\operatorname{MT}_{B_3}(2, 10, 16)| = 52202240$.

Example 7.7. From Figure 6, we get that

$$|\operatorname{MT}_{B_2}(a_1,a_2)| = 4 + 5\binom{a_2-a_1-1}{3-1-1} + 2\binom{a_2-a_1-1}{4-1-1} = (a_2-a_1+1)^2.$$

By Proposition 3.17, one can observe that the following general formula holds:

$$|\operatorname{MT}_{A_2(j)}(a_1,a_2)| = (a_2-a_1+1)^j.$$

See Appendix A for the much less obvious expression for $|\operatorname{MT}_{B_3}(a_1,a_2,a_3)|$.

We conclude the section with the following observation about the k-fundamental quilts for a poset of rank k. These quilts are the most compressed fundamental quilts for such a poset.

Corollary 7.8. *Assume P has rank k. Then we have*

$$|F_k(P)| = |\operatorname{MT}_P(1,\dots,k)| = |\operatorname{Quilts}(P, C_k)| = |\operatorname{Quilts}(P, C_{k-1})|. \tag{7.6}$$

Proof. The first equality holds by Theorem 7.6. Take a quilt $f \in \operatorname{Quilts}(P, C_k)$. Since $\operatorname{rank} P = \operatorname{rank} C_k$, Lemma 3.13 gives $f(x, k) = \operatorname{rank} x$ for all $x \in P$. This means that the map $\operatorname{Quilts}(P, C_k) \longrightarrow \operatorname{Quilts}(P, C_{k-1})$ defined by $f \mapsto f|_{P\times C_{k-1}}$ is an isomorphism of lattices proving the last equality. Lemma 3.13 also says that the top set $J_f(\hat{1}_P)$ is $[k]$, which proves $|\operatorname{Quilts}(P, C_k)| = |\operatorname{MT}_P(1,\dots,k)|$. □

8. Enumeration of Boolean quilts

Exact enumeration of Dedekind maps for B_n and Boolean quilts is at least as difficult as finding a formula for the Dedekind numbers. However, some bounds can be given. For example, we can construct $2^{\binom{n}{\lfloor n/2\rfloor}}$ 1-Dedekind maps on B_n by taking $f(T) = 0$ for $|T| < \lfloor n/2\rfloor$, $f(T) = 1$ for $|T| > \lfloor n/2\rfloor$, $f(T) \in \{0,1\}$ for $|T| = \lfloor n/2\rfloor$. It follows that $d_1(B_n) \geq 2^{\binom{n}{\lfloor n/2\rfloor}}$. In 1966, Hansel proved that $d_1(B_n) \leq 3^{\binom{n}{\lfloor n/2\rfloor}}$ [Han66]. Kleitman [Kle69] improved that to

$$d_1(B_n) \leq 2^{(1+c\ln n/\sqrt{n})\binom{n}{\lfloor n/2\rfloor}} \tag{8.1}$$

for some constant c. We will use that result in the following.

Lemma 8.1. *There exists a constant $c > 0$ so that for all $1 \leq k \leq n$,*

$$d_k(B_n) \leq 2^{k(1+c\ln n/\sqrt{n})\binom{n}{\lfloor n/2\rfloor}}.$$

Furthermore, for every $\varepsilon > 0$,

$$d_k(B_n) \geq 2^{(k-\varepsilon)\binom{n}{\lfloor n/2\rfloor}}$$

for large enough n.

Proof. The upper bound follows from Lemma 3.4 and (8.1). To prove the lower bound, assume that $n \geq 2k$. Consider the set of maps on B_n defined by the following criteria:

- $f(T) = 0$ if $|T| \leq \lfloor n/2\rfloor - 2\lfloor k/2\rfloor$;
- $f(T) \in \{0,1\}$ if $|T| = \lfloor n/2\rfloor - 2\lfloor k/2\rfloor + 1$;
- $f(T) = 1$ if $|T| = \lfloor n/2\rfloor - 2\lfloor k/2\rfloor + 2$;
- $f(T) \in \{1,2\}$ if $|T| = \lfloor n/2\rfloor - 2\lfloor k/2\rfloor + 3$;

$\vdots$

- $f(T) \in \{\lfloor k/2\rfloor - 1, \lfloor k/2\rfloor\}$ if $|T| = \lfloor n/2\rfloor - 1$;
- $f(T) = \lfloor k/2\rfloor$ if $|T| = \lfloor n/2\rfloor$;
- $f(T) \in \{\lfloor k/2\rfloor, \lfloor k/2\rfloor + 1\}$ if $|T| = \lfloor n/2\rfloor + 1$;

$\vdots$

- $f(T) \in \{k-1, k\}$ if $|T| = \lfloor n/2\rfloor + 2\lfloor k/2\rfloor - 1$;
- $f(T) = k$ if $|T| \geq \lfloor n/2\rfloor + 2\lfloor k/2\rfloor$.

Every such map is a k-Dedekind map. It follows that

$$d_k(B_n) \geq 2^{\sum_{i=-\lfloor k/2\rfloor+1}^{\lceil k/2\rceil}\binom{n}{\lfloor n/2\rfloor+2i-1}}.$$

For a fixed k and for $-k \leq i \leq k$,

$$\lim_{n\to\infty}\frac{\binom{n}{\lfloor n/2\rfloor+i}}{\binom{n}{\lfloor n/2\rfloor}} = 1.$$

That means that for a chosen $\varepsilon > 0$, we have $\binom{n}{\lfloor n/2\rfloor+2i-1} \geq (1-\varepsilon/k)\binom{n}{\lfloor n/2\rfloor}$ for $i = -\lfloor k/2\rfloor + 1, \ldots, \lceil k/2\rceil$ for n large enough. The second statement of the lemma now follows. $\square$

Theorem 8.2. *Let P be a finite ranked poset with least and greatest elements. There exists a constant $c > 0$ so that if $n \geq \operatorname{rank} P$, then*

$$2^{\binom{n}{\lfloor n/2\rfloor}} \leq |\operatorname{Quilts}(P, B_n)| \leq 2^{b(P)(1+c\ln n/\sqrt{n})\binom{n}{\lfloor n/2\rfloor}}.$$

If $n \geq 2\operatorname{rank} P$, we have the improved lower bound

$$|\operatorname{Quilts}(P, B_n)| \geq d_1(P)^{\binom{n}{\lfloor n/2 \rfloor}}.$$

In particular,

$$2^{\binom{k}{\lfloor k/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(B_k, B_n)| \leq 2^{k2^{k-1}(1+c\ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$$

for $n \geq 2k$.

Proof. By Theorem 3.19, we have

$$|\operatorname{Quilts}(P, B_n)| \leq d_1(B_n)^{b(P)},$$

and the upper bound for $|\operatorname{Quilts}(P, B_n)|$ now follows from (8.1).

The $2^{\binom{n}{\lfloor n/2 \rfloor}}$ quilts of type (P, B_n) defined by

$$f(x,T) = \begin{cases} \min\{\operatorname{rank} x,\ |T|\} & : \ |T| < \lfloor k/2 \rfloor \\ \min\{\operatorname{rank} x,\ \lfloor k/2 \rfloor\} & : \ \lfloor k/2 \rfloor \leq |T| < \lfloor n/2 \rfloor \\ \min\{\operatorname{rank} x,\ \lfloor k/2 \rfloor + \epsilon_T\} & : \ |T| = \lfloor n/2 \rfloor \\ \min\{\operatorname{rank} x,\ \lfloor k/2 \rfloor + 1\} & : \ \lfloor n/2 \rfloor < |T| \leq n - \lceil k/2 \rceil \\ \min\{\operatorname{rank} x,\ |T| - n + k\} & : \ n - \lceil k/2 \rceil < |T| \end{cases},$$

where $\epsilon_T \in \{0,1\}$, prove the first lower bound.

For the second lower bound, assume $n \geq 2k$. For each $T \subseteq [n]$ with $|T| = \lfloor n/2 \rfloor$, choose a 1-Dedekind map $g_T \in D_1(P)$. Then, each such collection of choices determines a distinct quilt of type (P, B_n) given by

$$f(x,T) = \begin{cases} 0 & |T| < \lfloor n/2 \rfloor \\ g_T(x) & |T| = \lfloor n/2 \rfloor \\ \min\{\operatorname{rank} x,\ |T| - \lfloor n/2 \rfloor, k\} & |T| > \lfloor n/2 \rfloor. \end{cases}$$

It follows that $|\operatorname{Quilts}(P, B_n)| \geq d_1(P)^{\binom{n}{\lfloor n/2 \rfloor}}$. The last inequality follows from $d_1(B_k) \geq 2^{\binom{k}{\lfloor k/2 \rfloor}}$ from the beginning of this section. ☐

Remark 8.3. Theorem 8.2 guarantees that for a poset P, there are positive numbers A_P and B_P such that

$$\frac{\ln|\operatorname{Quilts}(P, B_n)|}{\binom{n}{\lfloor n/2 \rfloor}} \in [A_P, B_P]$$

for $n \geq \operatorname{rank} P$. It is natural to ask if the limit

$$L(P) = \lim_{n\to\infty} \frac{\ln|\operatorname{Quilts}(P, B_n)|}{\binom{n}{\lfloor n/2 \rfloor}}$$

exists. By the last part of the theorem, if $L(B_k)$ exists, it must be in the interval

$$\left[\binom{k}{\lfloor k/2 \rfloor}\ln 2, k2^{k-1}\ln 2\right].$$

We do not have enough data to state an explicit conjecture, but we believe that the limit does indeed exist; if we had to venture a guess as to what this number would be, we would say $L(P) = b(P)\ln 2$. In other words, we believe that $2^{b(P)\binom{n}{\lfloor n/2 \rfloor}}$ is the best estimate for $|\operatorname{Quilts}(P, B_n)|$ among functions of the form $C^{\binom{n}{\lfloor n/2 \rfloor}}$.

9. Final remarks

Representability. Call a quilt $f \in \mathrm{Quilts}(B_k, B_n)$ *representable* if there exists a matrix $A \in \mathbb{R}^{k\times n}$, $\operatorname{rank} A = \min\{k, n\}$, so that $f(I, J)$ is equal to the rank of the matrix obtained by taking rows in I and columns in J in the matrix A. For $n = k = 2$, there are 7 representable quilts $f_1, \ldots, f_7$ coming from, say, matrices $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}1&1\\-1&1\end{smallmatrix}\right]$. The lattice $\mathrm{Quilts}(B_2, B_2) \cong B_4$, on the other hand, contains 16 elements, so they are not all representable. It would be interesting to understand representable quilts better.

Open problem 9.1. *Characterize the representable quilts of type* (B_k, B_n)*.*

Open problem 9.2. *Characterize the representable chain quilts of types* (C_k, B_n) *and those which correspond to* f_w *for some* $w \in W_{n,k}$*.*

Dedekind–MacNeille completion. It is well known that the lattice of alternating sign matrices is the Dedekind–MacNeille completion of (i.e., the smallest lattice containing) the strong Bruhat order on S_n. A natural question is whether the lattice $\mathrm{Quilts}(B_k, B_n)$ is the Dedekind–MacNeille completion of the poset of representable quilts. The answer, however, is no. The poset and the completion are shown in Figure 9. The following problem inspired the exploration of quilts as a generalization of ASMs. However, it remains open.

Open problem 9.3. *(Posed by Stark Ryan and independently by Jessica Striker) What is the Dedekind–MacNeille completion of the medium roast partial order on Fubini words in* $W_{n,k}$ *defined in Section 4? By Corollary 4.2, this complete lattice must be isomorphic to a sublattice of* $\mathrm{Quilts}(C_k, B_n)$ *and contain the quilts of the form* f_w *for* $w \in W_{n,k}$*.*

Open problem 9.4. *Find the Dedekind–MacNeille completion of the poset of representable quilts of type* (B_k, B_n)*.*

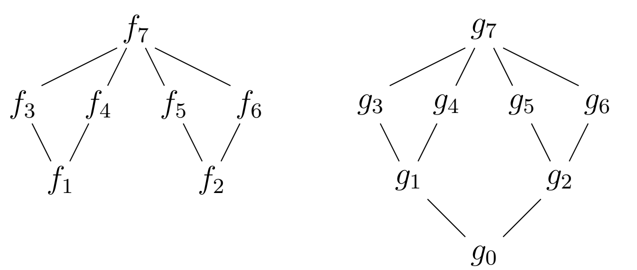


Figure 9. Induced poset of the 7 representable quilts of type (B_2, B_2) and its Dedekind–MacNeille completion. Note, $|\mathrm{Quilts}(B_2, B_2)| = 16$.

Quilt polytopes. There are beautiful results about the polytopes of alternating sign matrices, matroids, and flag matroids, see e.g. [Str09] and [CDMS22]. In 2018, Sanyal-Stump [SS18] defined the *Lipschitz polytope* of a poset P, denoted $\mathcal{L}(P)$, as the set of functions $f \in \mathbb{R}^P$ such that $0 \le f(a) \le 1$ for all minimal elements $a \in P$ and $0 \le f(y) - f(x) \le 1$ for all $x \lessdot y$ in P. Therefore, the vertices of the Lipschitz polytopes are closely related to the Dedekind maps on P. This variation on Boolean growth leads us to define $\mathcal{L}(P, Q)$ for a pair of finite ranked posets P, Q with least and greatest elements as the set of functions $f \in \mathbb{R}^{P\times Q}$ such that

- $f(x, y) = 0$ whenever $x = \hat{0}_P$ or $y = \hat{0}_Q$,

- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and
- if $(x, y) \lessdot (x', y')$ in $P \times Q$, then $0 \le f(x', y') - f(x, y) \le 1$ (bounded growth).

Thus, integer lattice points of $\mathcal{L}(P, Q)$ are exactly the quilts of type (P, Q). What is the Ehrhart polynomial for these generalized Lipschitz polytopes? What more can be said about these polytopes?

Enumeration. As we stated in the introduction, one of the most fascinating facts in the area is that there is a product formula for the number of square ASMs. Corollary 5.5 gives a simple generalization of this statement. Is there a simple formula for $|\operatorname{Quilts}(P, P)|$ for some family of posets $P \neq jC_n$? Can we at least find asymptotic formulas for $|\operatorname{Quilts}(P_n, P_n)|$ for some nice families of posets P_n, or upper and lower estimates? Can we improve the bounds for $|\operatorname{Quilts}(P, B_n)|$ beyond Theorem 8.2?

Alternative definitions. We can generalize the definition of a quilt slightly to account for finite ranked posets that do not necessarily have a unique minimal and maximal element. In that case, we should replace the first two conditions in Definition 3.9 by

- $f(x, y) = 0$ if x is a minimal element of P or y is a minimal element of Q,
- $f(x, y) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$ if x is a maximal element of P and y is a maximal element of Q.

Most of our results still hold, but not all; for example, there is no longer the concept of a monotone triangle with a specified top set. Another possibility is to keep the least and greatest elements and remove the condition $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$. One drawback of that is that elements of $\operatorname{Quilts}(C_k, C_n)$ are no longer (in bijection with) alternating sign matrices; instead, we get *partial alternating sign matrices*, see [HS22].

Standard quilts. The standard quilts we defined in Section 7 seem worthy of further study, as they generalize shifted standard Young tableau of shape $(k, k-1, \ldots, 1)$ and determine the asymptotics of $|\operatorname{Quilts}(P, C_n)|$. Is there a simple way to count standard quilts (generalizing the hook-length formula) or is that a #P-complete problem like computing $|\operatorname{Quilts}(P, C_1)|$? Since determining the asymptotics of antichain quilts is #P-complete by Theorem 6.1 and since antichain quilts are typically simpler than chain quilts, we would assume that enumerating standard quilts is also a #P-hard problem.

Monotone triangles. In Section 7, we proved that $m_P(a_1, \ldots, a_k) = |\operatorname{MT}_P(a_1, \ldots, a_k)|$ is a polynomial function of $a_1, \ldots, a_k$. For $P = C_k$, the crucial results are the *operator formula*, expressing the number of monotone triangles via the number of Gelfand–Tsetlin patterns, and the *rotation formula*, which states that $m_{C_k}(a_2, \ldots, a_k, a_1 - k)| = (-1)^{k-1} m_{C_k}(a_1, \ldots, a_k)$, see [Fis06, Theorem 1] and [Fis07, Lemma 5]. Is there a way to generalize these results to arbitrary (or at least some) posets?

Generalizing ASMs. The literature on permutations and alternating sign matrices provide a rich source of problems for quilts, some of which are mentioned in the introduction and Remark 2.3. Following Hamaker–Reiner [HR20], what are the descents for quilts? Is $\operatorname{Quilts}(P, Q)$ a shellable poset? Is there a Hopf algebra interpretation for quilts and an analog of the shuffle product? See also the work of Cheballah–Giraudo–Maurice, who defined a Hopf algebra with basis given by alternating sign matrices [CGM15].

We could also consider generalizing Terwilliger's extension of the Boolean lattice to include interlacing sets not just of the type in (2.4), but also to include (2.5). What can be said about the interlacing Boolean lattice with both types of interlacing conditions?

Appendix A: Computational Results

The next three equalities illustrate Theorems 6.1 and 7.1. The formula for $|\operatorname{Quilts}(B_4, C_n)|$ was actually produced using Theorem 7.5.

$$
\begin{aligned}
|\operatorname{Quilts}(B_4,A_2(j))|={}&2{\cdot}16^j+12{\cdot}20^j+6{\cdot}25^j+24{\cdot}26^j+8{\cdot}27^j+24{\cdot}34^j+8{\cdot}35^j+14{\cdot}36^j+8{\cdot}38^j+24{\cdot}39^j+24{\cdot}42^j+6{\cdot}47^j+24{\cdot}49^j\\
&+12{\cdot}50^j+24{\cdot}52^j+24{\cdot}55^j+24{\cdot}59^j+12{\cdot}61^j+49{\cdot}64^j+24{\cdot}70^j+20{\cdot}72^j+24{\cdot}77^j+12{\cdot}80^j+4{\cdot}81^j+12{\cdot}82^j+12{\cdot}83^j+24{\cdot}90^j+24{\cdot}91^j\\
&+8{\cdot}95^j+6{\cdot}100^j+24{\cdot}101^j+6{\cdot}102^j+8{\cdot}103^j+24{\cdot}104^j+2{\cdot}113^j+24{\cdot}114^j+24{\cdot}115^j+8{\cdot}122^j+12{\cdot}128^j+4{\cdot}129^j+12{\cdot}133^j+8{\cdot}147^j+166^j
\end{aligned}
$$

$$
|\operatorname{Quilts}(B_3,C_n)|=1344\binom{n}{12}+10080\binom{n}{11}+33444\binom{n}{10}+64506\binom{n}{9}+79788\binom{n}{8}+65652\binom{n}{7}+35876\binom{n}{6}+12471\binom{n}{5}+2456\binom{n}{4}+199\binom{n}{3} \quad \text{for } n\geq 3
$$

$$
\begin{aligned}
|\operatorname{Quilts}(B_4,C_n)|={}&10651644896477184\binom{n}{32}+197055430584827904\binom{n}{31}+1738665057137541120\binom{n}{30}+9735818288500039680\binom{n}{29}\\
&+38839556977856928768\binom{n}{28}+117471942156471614976\binom{n}{27}+279881902757513059200\binom{n}{26}+538793272789014417984\binom{n}{25}\\
&+852913906502788631808\binom{n}{24}+1124093660783042183328\binom{n}{23}+1244204557392229952160\binom{n}{22}+1163423387552452501296\binom{n}{21}\\
&+922421269447363713000\binom{n}{20}+621185943976110723780\binom{n}{19}+355315109292664467516\binom{n}{18}+172335637248751133958\binom{n}{17}\\
&+70636458716011510126\binom{n}{16}+24338243155860965610\binom{n}{15}+6997548154002120846\binom{n}{14}+1662187981311784640\binom{n}{13}\\
&+321944626547285880\binom{n}{12}+49970302238834940\binom{n}{11}+6073377257995792\binom{n}{10}+560131126345528\binom{n}{9}\\
&+37512372358044\binom{n}{8}+1710540931365\binom{n}{7}+48063694812\binom{n}{6}+703244285\binom{n}{5}+3813042\binom{n}{4} \quad \text{for } n\geq 4
\end{aligned}
$$

The following is a list of the numbers of fundamental quilts for B_3 with a given top set given in reverse lexicographic order:

1,2,3	199	1,2,4	1228	1,3,4	1228	1,2,5	3428	1,3,5	5615	1,4,5	3428	1,2,6	5175
1,3,6	12763	1,4,6	12763	1,5,6	5175	1,2,7	4416	1,3,7	16518	1,4,7	23784	1,5,7	16518
1,6,7	4416	1,2,8	2016	1,3,8	12501	1,4,8	25377	1,5,8	25377	1,6,8	12501	1,7,8	2016
1,2,9	384	1,3,9	5184	1,4,9	16038	1,5,9	21294	1,6,9	16038	1,7,9	5184	1,8,9	384
1,3,10	912	1,4,10	5664	1,5,10	10146	1,6,10	10146	1,7,10	5664	1,8,10	912	1,4,11	864
1,5,11	2640	1,6,11	3072	1,7,11	2640	1,8,11	864	1,5,12	288	1,6,12	384	1,7,12	384
1,8,12	288												

Note that the sum of the numbers of fundamental quilts with last element of the top set equal to m is equal to the coefficient of $\binom{n}{m}$ in the formula for $|\operatorname{Quilts}(B_3, C_n)|$, e.g. $3428 + 5615 + 3428 = 12471$. Using this table, we can compute $|\operatorname{MT}_{B_3}(a_1, a_2, a_3)|$ using Theorem 7.6:

$$
\begin{aligned}
|\operatorname{MT}_{B_3}(a_1,a_2,a_3)|&=199+1228(a_3-a_2-1)+1228(a_2-a_1-1)+\cdots+384\binom{a_2-a_1-1}{5}\binom{a_3-a_2-1}{4}+288\binom{a_2-a_1-1}{6}\binom{a_3-a_2-1}{3}\\
&=\tfrac{1}{15}a_1a_2^8-\tfrac{1}{15}a_2^8a_3-\tfrac{2a_2^8}{15}-\tfrac{4}{15}a_1^2a_2^7+\tfrac{4}{15}a_2^7a_3^2+\tfrac{8}{15}a_1a_2^7+\tfrac{8}{15}a_2^7a_3+\tfrac{7}{15}a_1^3a_2^6-\tfrac{7}{15}a_2^6a_3^3-\tfrac{14}{15}a_1^2a_2^6-\tfrac{7}{15}a_1a_2^6a_3^2
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{14}{15}a_2^6a_3^2-\tfrac{3}{40}a_1a_2^6+\tfrac{7}{15}a_1^2a_2^6a_3-\tfrac{28}{15}a_1a_2^6a_3+\tfrac{3}{40}a_2^6a_3+\frac{3a_2^6}{20}-\tfrac{7}{15}a_1^4a_2^5+\tfrac{7}{15}a_2^5a_3^4+\tfrac{14}{15}a_1^3a_2^5+\tfrac{14}{15}a_1a_2^5a_3^3+\tfrac{14}{15}a_2^5a_3^3+\tfrac{9}{40}a_1^2a_2^5\\
&+\tfrac{14}{5}a_1a_2^5a_3^2-\tfrac{9}{40}a_2^5a_3^2-\tfrac{9}{20}a_1a_2^5-\tfrac{14}{15}a_1^3a_2^5a_3+\tfrac{14}{5}a_1^2a_2^5a_3-\tfrac{9}{20}a_2^5a_3+\tfrac{4}{15}a_1^5a_2^4-\tfrac{4}{15}a_2^4a_3^5-\tfrac{1}{3}a_1^4a_2^4-a_1a_2^4a_3^4-\tfrac{1}{3}a_2^4a_3^4-\tfrac{31}{24}a_1^3a_2^4\\
&-\tfrac{1}{3}a_1^2a_2^4a_3^3-\tfrac{10}{3}a_1a_2^4a_3^3+\tfrac{31}{24}a_2^4a_3^3+\tfrac{21}{8}a_1^2a_2^4+\tfrac{1}{3}a_1^3a_2^4a_3^2-2a_1^2a_2^4a_3^2-\tfrac{11}{4}a_1a_2^4a_3^2+\tfrac{21}{8}a_2^4a_3^2-\tfrac{13}{12}a_1a_2^4+a_1^4a_2^4a_3-\tfrac{10}{3}a_1^3a_2^4a_3+\tfrac{11}{4}a_1^2a_2^4a_3\\
&-3a_1a_2^4a_3+\tfrac{13}{12}a_2^4a_3-\frac{6a_2^4}{5}-\tfrac{1}{15}a_1^6a_2^3+\tfrac{1}{15}a_2^3a_3^6-\tfrac{4}{15}a_1^5a_2^3+\tfrac{2}{3}a_1a_2^3a_3^5-\tfrac{4}{15}a_2^3a_3^5+\tfrac{53}{24}a_1^4a_2^3+\tfrac{1}{3}a_1^2a_2^3a_3^4+\tfrac{8}{3}a_1a_2^3a_3^4-\tfrac{53}{24}a_2^3a_3^4\\
&-\tfrac{9}{2}a_1^3a_2^3+\tfrac{4}{3}a_1^2a_2^3a_3^3+\tfrac{11}{3}a_1a_2^3a_3^3-\tfrac{9}{2}a_2^3a_3^3+\tfrac{13}{6}a_1^2a_2^3-\tfrac{1}{3}a_1^4a_2^3a_3^2+\tfrac{4}{3}a_1^3a_2^3a_3^2+3a_1a_2^3a_3^2-\tfrac{13}{6}a_2^3a_3^2+\tfrac{12}{5}a_1a_2^3-\tfrac{2}{3}a_1^5a_2^3a_3+\tfrac{8}{3}a_1^4a_2^3a_3-\\
&\tfrac{11}{3}a_1^3a_2^3a_3+3a_1^2a_2^3a_3+\tfrac{12}{5}a_2^3a_3+\tfrac{1}{5}a_1^6a_2^2-\tfrac{1}{5}a_1a_2^2a_3^6+\tfrac{1}{5}a_2^2a_3^6-\tfrac{13}{15}a_1^5a_2^2-\tfrac{2}{5}a_1^2a_2^2a_3^5-\tfrac{2}{5}a_1a_2^2a_3^5+\tfrac{13}{15}a_2^2a_3^5+\tfrac{17}{24}a_1^4a_2^2+\tfrac{1}{3}a_1^3a_2^2a_3^4\\
&-3a_1^2a_2^2a_3^4+\tfrac{55}{24}a_1a_2^2a_3^4+\tfrac{17}{24}a_2^2a_3^4+\tfrac{133}{40}a_1^3a_2^2-\tfrac{1}{3}a_1^4a_2^2a_3^3+\tfrac{8}{3}a_1^3a_2^2a_3^3-\tfrac{121}{12}a_1^2a_2^2a_3^3+\tfrac{32}{3}a_1a_2^2a_3^3-\tfrac{133}{40}a_2^2a_3^3-\tfrac{361}{60}a_1^2a_2^2+\tfrac{2}{5}a_1^5a_2^2a_3^2\\
&-3a_1^4a_2^2a_3^2+\tfrac{121}{12}a_1^3a_2^2a_3^2-\tfrac{41}{2}a_1^2a_2^2a_3^2+\tfrac{659}{40}a_1a_2^2a_3^2-\tfrac{361}{60}a_2^2a_3^2-\tfrac{127}{120}a_1a_2^2+\tfrac{1}{5}a_1^6a_2^2a_3-\tfrac{2}{5}a_1^5a_2^2a_3-\tfrac{55}{24}a_1^4a_2^2a_3+\tfrac{32}{3}a_1^3a_2^2a_3-\tfrac{659}{40}a_1^2a_2^2a_3\\
&+\tfrac{29}{6}a_1a_2^2a_3+\tfrac{127}{120}a_2^2a_3+\frac{11a_2^2}{60}-\tfrac{1}{5}a_1^6a_2+\tfrac{1}{5}a_1^2a_3^6a_2-\tfrac{2}{5}a_1a_2a_3^6+\tfrac{1}{5}a_2a_3^6+\tfrac{22}{15}a_1^5a_2-\tfrac{2}{15}a_1^3a_2a_3^5+\tfrac{8}{5}a_1^2a_2a_3^5-\tfrac{44}{15}a_1a_2a_3^5+\tfrac{22}{15}a_2a_3^5\\
&-\tfrac{529}{120}a_1^4a_2-\tfrac{2}{3}a_1^3a_2a_3^4+\tfrac{121}{24}a_1^2a_2a_3^4-\tfrac{35}{4}a_1a_2a_3^4+\tfrac{529}{120}a_2a_3^4+\tfrac{289}{60}a_1^3a_2+\tfrac{2}{15}a_1^5a_2a_3^3-\tfrac{2}{3}a_1^4a_2a_3^3+\tfrac{41}{6}a_1^2a_2a_3^3-\tfrac{659}{60}a_1a_2a_3^3+\tfrac{289}{60}a_2a_3^3\\
&+\tfrac{127}{120}a_1^2a_2-\tfrac{1}{5}a_1^6a_2a_3^2+\tfrac{8}{5}a_1^5a_2a_3^2-\tfrac{121}{24}a_1^4a_2a_3^2+\tfrac{41}{6}a_1^3a_2a_3^2-\tfrac{29}{12}a_1a_2a_3^2-\tfrac{127}{120}a_2a_3^2-\tfrac{11}{60}a_1a_2-\tfrac{2}{5}a_1^6a_2a_3+\tfrac{44}{15}a_1^5a_2a_3-\tfrac{35}{4}a_1^4a_2a_3\\
&+\tfrac{659}{60}a_1^3a_2a_3-\tfrac{29}{12}a_1^2a_2a_3-\tfrac{11}{60}a_2a_3+\frac{a_1^6}{15}-\tfrac{1}{15}a_1^3a_3^6+\tfrac{1}{5}a_1^2a_3^6-\tfrac{1}{5}a_1a_3^6+\frac{a_3^6}{15}-\frac{3a_1^5}{5}+\tfrac{2}{15}a_1^4a_3^5-\tfrac{14}{15}a_1^3a_3^5+\tfrac{31}{15}a_1^2a_3^5-\tfrac{28}{15}a_1a_3^5+\frac{3a_3^5}{5}\\
&+\frac{55a_1^4}{24}-\tfrac{2}{15}a_1^5a_3^4+\tfrac{4}{3}a_1^4a_3^4-\tfrac{41}{8}a_1^3a_3^4+\tfrac{217}{24}a_1^2a_3^4-\tfrac{889}{120}a_1a_3^4+\frac{55a_3^4}{24}-\frac{15a_1^3}{4}+\tfrac{1}{15}a_1^6a_3^3-\tfrac{14}{15}a_1^5a_3^3+\tfrac{41}{8}a_1^4a_3^3-\tfrac{43}{3}a_1^3a_3^3+\tfrac{2437}{120}a_1^2a_3^3\\
&-\tfrac{839}{60}a_1a_3^3+\frac{15a_3^3}{4}+\frac{137a_1^2}{120}+\tfrac{1}{5}a_1^6a_3^2-\tfrac{31}{15}a_1^5a_3^2+\tfrac{217}{24}a_1^4a_3^2-\tfrac{2437}{120}a_1^3a_3^2+\tfrac{1331}{60}a_1^2a_3^2-\tfrac{1223}{120}a_1a_3^2+\frac{137a_3^2}{120}-\frac{3a_1}{20}+\tfrac{1}{5}a_1^6a_3\\
&-\tfrac{28}{15}a_1^5a_3+\tfrac{889}{120}a_1^4a_3-\tfrac{839}{60}a_1^3a_3+\tfrac{1223}{120}a_1^2a_3-\tfrac{21}{10}a_1a_3+\frac{3a_3}{20}.
\end{aligned}
$$

10. Appendix B: Numerical Sequences

10.1. **Generalized Dedekind Numbers.** From Section 3, $d_k(B_n)$ for $0 \le k \le n \le 5$ are given by

$n\backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	0	0	0	0
2	1	4	1	0	0	0
3	1	18	18	1	0	0
4	1	166	656	166	1	0
5	1	7579	189967	189967	7579	1.

Note OEIS A007153 appears in column 1. Reading the triangle of nonzero entries by rows from the top we have the sequence 1, 1, 1, 1, 4, 1, 1, 18, 18, 1, 1, 166, 656, 166, 1, 1, 7579, 189967, 189967, 7579, 1. Note the symmetry naturally comes from complementing sets and values. This sequence is [OEI24, A374819].

10.2. **Boolean-Chain numbers.** The square table of numbers $|\operatorname{Quilts}(B_n, C_k)|$ for $1 \le n \le 4$ and $1 \le k \le 7$ plus for $n = 5, k = 1, 2$ and $n = 6, k = 1$ are given by

$n\backslash k$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	4	4	17	46	100	190	329
3	18	199	199	3252	26741	151522	671600
4	166	47000	3813042	3813042	722309495	52340356152	2061888381504
5	7579	410131245	?	?	?	?	?
6	7828352	?	?	?	?	?	?

Reading antidiagonals starting at $k = n = 1$, we have 1, 2, 4, 3, 4, 18, 4, 17, 199, 166, 5, 46, 199, 47000, 7579, 6, 100, 3252, 3813042, 410131245, 7828352. This sequence is [OEI24, A374820].

10.3. **Antichain-Boolean numbers.** The table of $|\operatorname{Quilts}(A_2(j), B_n)|$ for $1 \le n \le 4$ and $1 \le j \le 6$, plus $|\operatorname{Quilts}(A_2(1), B_5)|$ is given by

$n\backslash j$	1	2	3	4	5	6
1	2	4	8	16	32	64
2	4	16	64	256	1024	4096
3	199	2309	28225	364217	4960009	71091689
4	47000	4001278	384285926	40139162386	4455115959710	517943027803618
5	410131245	?	?	?	?	?

Reading antidiagonals starting at $k = n = 1$, we have 2, 4, 4, 8, 16, 199, 16, 64, 2309, 47000, 32, 256, 28225, 4001278, 410131245, 64, 1024, 364217, 384285926. This sequence is [OEI24, A374821].

10.4. **Antichain-Chain Quilt Numbers.** The number of quilts of type $(A_2(j), C_k)$, where j is the column index for $1 \le j \le 8$ and k is the row index for $1 \le k \le 8$ is given by the table

$k\backslash j$	1	2	3	4	5	6	7	8
1	2	4	8	16	32	64	128	256
2	2	4	8	16	32	64	128	256
3	7	17	43	113	307	857	2443	7073
4	16	46	142	466	1606	5746	21142	79426
5	30	100	366	1444	6030	26260	117966	542404
6	50	190	806	3718	18230	93430	494726	2684998
7	77	329	1589	8393	47237	278249	1695029	10592393
8	112	532	2884	17164	109012	725212	4992484	35277004.

Reading antidiagonals we have 2, 4, 2, 8, 4, 7, 16, 8, 17, 16, etc. This sequence is [OEI24, A374822].

10.5. **Chain-Chain Numbers.** The numbers of quilts of type (C_k, C_n) is also the number of rectangular alternating sign matrices in $\mathrm{ASM}_{k,n}$. This sequence is included [OEI24, A297622], where they include the cases where $k = 0$. Note, $|\operatorname{Quilts}(C_0, C_n)| = 1$ for all $n \ge 0$. The numbers $|\operatorname{Quilts}(C_k, C_n)|$ for $1 \le k \le 10$ are

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2		2	7	16	30	50	77	112	156	210
3			7	42	149	406	938	1932	3654	6468
4				42	429	2394	9698	31920	90576	229680
5					429	7436	65910	403572	1931325	7722110
6						7436	218348	3096496	29020904	205140540
7							218348	10850216	247587252	3586953760
8								10850216	911835460	33631201864.

Reading down columns for the triangle of numbers we have 1, 2, 2, 3, 7, 7, 4, 16, 42, 42, 5, 30, 149, 429, 429, 6, 50, 406, 2394, 7436, 7436, ... Note, [OEI24, A005130] is the sequence counting the number of square ASMs. It starts out 1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, as shown in the diagonal.

10.6. **Antichain-Antichain Quilt Numbers.** The quilts of type $(A_2(j), A_2(k))$ are in bijection with the $j \times k$ binary arrays, so the formula is 2^{jk} for all $j, k \geq 1$.

10.7. **Boolean-Boolean Quilt Numbers.** The triangular array of the number of ASM quilts of type (B_n, B_k) begins with

$n\backslash k$	1	2	3	4	5
1	1	4	18	166	7579
2		16	2309	4001278	?
3			2406862	?	?

Reading down columns we have 1, 4, 16, 18, 2309, 2406862, 166, 4001278. This sequence is [OEI24, A374824].

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