CAYLEY COMPOSITIONS, PARTITIONS, POLYTOPES, AND GEOMETRIC BIJECTIONS

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Abstract. In 1857, Cayley showed that certain sequences, now called *Cayley compositions*, are equinumerous with certain partitions into powers of 2. In this paper we give a simple bijective proof of this result and a geometric generalization to equality of Ehrhart polynomials between two convex polytopes. We then apply our results to give a new proof of *Braun's conjecture* proved recently by the authors [KP2].

Introduction and main results

Partition Theory is a classical field with a number of advanced modern results and applications. Its long and tumultuous history left behind a number of beautiful results which are occasionally brought to light to wide acclaim. The story of the so called *Cayley compositions* is a prime example of this. Introduced and studied by Cayley in 1857 [Cay], they were rediscovered by Minc [Minc], and remained largely forgotten until Andrews, Paule, Riese and Strehl [APRS] resurrected and christened them in 2001. This is when things became really interesting.

Theorem 1 (Cayley, 1857). *The number of integer sequences* $(a_1, \ldots, a_n)$ *such that* $1 \le a_1 \le 2$, *and* $1 \le a_{i+1} \le 2a_i$ *for* $1 \le i < n$, *is equal to the total number of partitions of integers* $N \in \{0, 1, \ldots, 2^n - 1\}$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

Our first result is a long elusive bijective proof of Cayley's theorem, and its several extensions. Our bijection construction is geometric, based on our approach in [P1].

Denote by $\mathcal{A}_n$ the set of sequences $(a_1, \ldots, a_n)$ satisfying the conditions of the theorem, which are called *Cayley compositions*. Denote by $\mathcal{B}_n$ the set of partitions into powers of 2 as in the theorem, which we call *Cayley partitions*. Now Theorem 1 states that $|\mathcal{A}_n| = |\mathcal{B}_n|$. For example,

$$\mathcal{A}_2 = \big\{ (1,1),\, (1,2),\, (2,1),\, (2,2),\, (2,3),\, (2,4) \big\}, \quad \mathcal{B}_2 = \big\{ 21,\, 2,\, 1^3,\, 1^2,\, 1,\, \varnothing \big\},$$

so $|\mathcal{A}_2| = |\mathcal{B}_2| = 6$. Following [BBL], define the *Cayley polytope* $\boldsymbol{A}_n$ to be the convex hull of all Cayley compositions $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

The main result of this paper is the following geometric extension of Theorem 1. Recall that the *Ehrhart polynomial* $\mathcal{E}_P(t)$ of a lattice polytope $P \subset \mathbb{R}^n$ is defined by

$$\mathcal{E}_P(k) \,=\, \#\{k\, P \cap \mathbb{Z}^n\}\,,$$

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where kP denotes the k-fold dilation of P, $k \in \mathbb{N}$ (see e.g. [Bar]).

Theorem 2. *Let $\mathcal{B}_n$ be the set of Cayley partitions, where a partition of the form $(2^{n-1})^{m_1}(2^{n-2})^{m_2}\ldots 1^{m_n}$ is identified with an integer point $(m_1, m_2, \ldots, m_n) \in \mathbb{R}^n$. Now let $\boldsymbol{B}_n = \operatorname{conv}\mathcal{B}_n$. Then $\mathcal{E}_{\boldsymbol{A}_n}(t) = \mathcal{E}_{\boldsymbol{B}_n}(t)$.*

In particular, when $t = 1$, we obtain Cayley's theorem. Our proof is based on an explicit volume-preserving map $\varphi : \boldsymbol{B}_n \to \boldsymbol{A}_n$, which satisfies a number of interesting properties. In particular, when restricted to integer points, this map gives the bijection $\varphi : \mathcal{B}_n \to \mathcal{A}_n$ mentioned above (see Proposition 6).

In [BBL], Ben Braun made an interesting conjecture about the volume of $\boldsymbol{A}_n$, which was recently proved by the authors [KP2]. Denote by $\mathcal{C}_n$ the set of connected graphs on n nodes, and let $C_n = |\mathcal{C}_n|$.

Theorem 3 ([KP2], formerly Braun's conjecture)**.** *Let $\mathcal{A}_n \subset \mathbb{R}^n$ be the set of Cayley compositions, and let $\boldsymbol{A}_n = \operatorname{conv}\mathcal{A}_n$ be the* Cayley polytope*. Then* $\operatorname{vol}\boldsymbol{A}_n = C_{n+1}/n!$.

Combined with Theorem 2, we immediately have $\operatorname{vol}\boldsymbol{B}_n = \operatorname{vol}\boldsymbol{A}_n$, and conclude:

Corollary 4. *Let $\mathcal{B}_n$ be the polytope defined above. Then* $\operatorname{vol}\boldsymbol{B}_n = C_{n+1}/n!$.

Curiously, one can also use $\operatorname{vol}\boldsymbol{B}_n = \operatorname{vol}\boldsymbol{A}_n$ in reverse, and derive Theorem 3 from Theorem 2 and known results on *Stanley-Pitman polytopes* (see below).

The rest of this paper is structured as follows. In Section 1 we prove Theorems 1 and 2 using an explicit bijection φ. Some applications are given in Section 2, followed by a new proof of Theorem 3 in Section 3. We finish with final remarks in Section 4.

1. Bijection construction

Recall from [BBL, KP2] (or observe directly from the definition) that Cayley polytope $\boldsymbol{A}_n \subset \mathbb{R}^n$ is defined by the following inequalities:

$$1 \le x_1 \le 2, \quad 1 \le x_2 \le 2x_1, \ \ldots \ , 1 \le x_n \le 2x_{n-1}.$$

Consider a basis

$$\boldsymbol{e}_1 = (1, 2, 4, \ldots, 2^{n-1}), \ \boldsymbol{e}_2 = (0, 1, 2, \ldots, 2^{n-2}), \ \ldots \ , \boldsymbol{e}_n = (0, 0, \ldots, 1),$$

and a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$\varphi(b_1, b_2, \ldots, b_n) \ = \ (2, 4, \ldots, 2^n) \ - \ \sum_{i=1}^{n} b_i \boldsymbol{e}_i\,.$$

Now observe that $\varphi^{-1}(\boldsymbol{A}_n)$ is a polytope defined by the following inequalities:

$$y_1 \le 1,\ 2y_1 + y_2 \le 3,\ 4y_1 + 2y_2 + y_3 \le 7,\ \ldots,\ 2^{n-1}y_1 + \ldots + 2y_{n-1} + y_n \le 2^n - 1$$
$$\text{and} \quad y_1, y_2, \ldots, y_n \ge 0.$$

Denote this polytope $\boldsymbol{Y}_n$, and by $\mathcal{Y}_n$ the set of integer points in $\boldsymbol{Y}_n$.

Lemma 5. *Polytope $\boldsymbol{B}_n = \operatorname{conv}\mathcal{B}_n$ coincides with $\boldsymbol{Y}_n = \varphi^{-1}(\boldsymbol{A}_n)$.*

First proof. Observe that $\mathcal{B}_n$ contains $\mathcal{Y}_n$, by construction of integer points in $\mathbb{R}^n$ corresponding to partitions into powers of 2 as in Theorem 2, and the last (long) inequality defining $\varphi^{-1}(\mathcal{A}_n)$. On the other hand, observe that the first $(n-1)$ inequalities on $\boldsymbol{Y}_n$ hold for Cayley partitions $\mathcal{B}_n$ by integrality. On the other hand, since φ is an affine lattice-preserving linear transformation, all integer points in $\mathcal{A}_n$ are mapped into integer points in $\boldsymbol{Y}_n$. Thus all vertices of $\boldsymbol{Y}_n$ are integer points. This immediately implies that $\boldsymbol{Y}_n = \mathrm{conv}\mathcal{B}_n$. $\square$

Second proof. As in the previous proof, vertices of $\boldsymbol{Y}_n$ must be integral and thus lie in $\mathcal{B}_n$ from the long inequality. It now follows from Theorem 1 that $\boldsymbol{A}_n$ (and thus $\boldsymbol{Y}_n$) has the same number of points as $\boldsymbol{B}_n$. Therefore, $\boldsymbol{Y}_n$ has no integer points other than those in $\mathcal{B}_n$, which implies that $\boldsymbol{Y}_n$ is a convex hull of the whole $\mathcal{B}_n$. $\square$

The second proof is shorter as it allows one to avoid checking the first $n-1$ inequalities, but it relies on Cayley's theorem. Of course, to obtain a new bijective proof of Theorem 1 we would need to go with the first proof.

Proposition 6. *Map $\varphi : \boldsymbol{B}_n \to \boldsymbol{A}_n$ is an affine volume-preserving map. Furthermore, when restricted to integer points, $\varphi : \mathcal{B}_n \to \mathcal{A}_n$ is a bijection. More generally, when restricted to lattice $\mathbb{Z}^n/k$ and then dilated by k, the map $k\varphi$ is a bijection between $\{k\boldsymbol{B}_n \cap \mathbb{Z}^n\}$ and $\{k\boldsymbol{A}_n \cap \mathbb{Z}^n\}$, for all $k \in \mathbb{N}$.*

The proposition immediately implies both Theorems 1 and 2. For example, bijection $\varphi : \mathcal{B}_2 \to \mathcal{A}_2$ is given as follows:

$$\mathbf{21} = (1,1) \to (1,1),\ \mathbf{2} = (1,0) \to (1,2),\ \mathbf{1^3} = (0,3) \to (2,1),$$
$$\mathbf{1^2} = (0,2) \to (2,2),\ \mathbf{1} = (0,1) \to (2,3),\ \varnothing = (0,0) \to (2,4).$$

Proof of Proposition 6. The affine map φ is well defined by Lemma 5. After a shift, map φ is a *unimodular* with integer coefficients. This implies that it is volume-preserving, and maps lattice $\mathbb{Z}^n/k$ into itself, which implies the result. $\square$

2. Three quick applications

Here are some interesting consequences of the bijection φ defined above.

Proposition 7. *Bijection φ maps Cayley partitions in $\mathcal{B}_n$ with one part of size 2^{n-1} into Cayley compositions $(a_1, \ldots, a_n) \in \mathcal{A}_n$, such that $a_1 = 1$.*

The proof is trivial, and the numerical result implied by the proposition is simply Theorem 1 for $n-1$.

Corollary 8. *The number of Cayley partitions of m in $\mathcal{B}_n$ is equal to the number of Cayley compositions $(a_1, \ldots, a_n) \in \mathcal{A}_n$, such that $a_n = 2^n - m$.*

Proof. First, observe by induction, that we can write $\varphi^{-1} : \boldsymbol{A}_n \to \boldsymbol{B}_n$ as

$$\varphi^{-1} : (a_1, a_2, a_3, \ldots) \to (2 - a_1, 2a_1 - a_2, 2a_2 - a_3, \ldots).$$

Now observe that the size of the partition in this notation is given by

$$m = 2^{n-1}b_1 + 2^{n-2}b_2 + \ldots + b_n = 2^{n-1}(2-a_1) + 2^{n-2}(2a_1 - a_2) + 2^{n-3}(2a_2 - a_3) + \ldots + (2a_{n-1} - a_n) = 2^n - a_n,$$

as desired. ☐

Corollary 9. *For any integer k, $1 \le k \le n$, the number of Cayley partitions in $\mathcal{B}_n$ with no part of size 2^k is equal to the number of Cayley compositions $(a_1, \ldots, a_n) \in \mathcal{A}_n$, such that $a_k = 2a_{k-1}$ for $k \ge 2$, and $a_1 = 2$ for $k = 1$.*

The corollary follows immediately from the explicit formula for map φ^{-1} given in the proof above.

3. Stanley-Pitman polytopes

In this section we give a new proof of Theorem 3, via the polytopes defined by Pitman and Stanley [SP] as follows.

Fix $c_1, \ldots, c_n > 0$. Define $\mathbf{\Pi}_n(c_1, \ldots, c_n) \subset \mathbb{R}^n$ by the following inequalities:

$$x_i \ge 0, \quad x_1 + \ldots + x_i \le c_1 + \ldots + c_i, \qquad \text{for all } i = 1..n.$$

Theorem 10 ([SP])**.** *We have:*

$$n! \operatorname{vol} \mathbf{\Pi}_n\big(1, q \ldots, q^{n-1}\big) = q^{\binom{n}{2}} \operatorname{Inv}_{n+1}\left(\frac{1}{q}\right),$$

where $\operatorname{Inv}_n(t)$ *is the* inversion polynomial (*see e.g.* [GJ]).

The proof of the theorem given in [SP] is highly non-trivial, and is based on triangulations of certain cones, and the properties of *parking functions*. However, we can now use Theorem 10 and Lemma 5 to obtain a new proof of Theorem 3, completely circumventing the original proof given in [KP2].

Proof of Theorem 3. Take $q = \frac{1}{2}$ in Theorem 10, and recall that $\operatorname{Inv}_n(2) = C_n$, see [GJ, MR]. Now, in the definition above, take $x_i = y_i/2^{i-1}$, and check that the inequalities defining $\mathbf{\Pi}_n$ in this case coincide with those defining $\boldsymbol{Y}_n = \boldsymbol{B}_n$ (see above). We conclude:

$$n! \operatorname{vol} \boldsymbol{B}_n = \big(1 \cdot 2 \cdot 2^2 \cdots 2^{n-1}\big) \cdot n! \operatorname{vol} \mathbf{\Pi}_n \left(1, \frac{1}{2}, \ldots, \frac{1}{2^{n-1}}\right) = 2^{\binom{n}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{n}{2}} \operatorname{Inv}_{n+1}(2) = C_{n+1},$$

as desired. ☐

4. Final remarks and open problems

4.1. Cayley's original statement in [Cay] (see also [APRS]) is somewhat different from Theorem 1, but easily equivalent (simply remove all parts $1'$). It also has the second part which follows along similar lines. Let us mention that Cayley's original proof uses only basic generating functions and is relatively short.

4.2. After reading the elementary bijection φ construction above, the reader may conclude that Cayley's theorem (Theorem 1) is completely straightforward. This is perhaps in sharp contrast with the first impression of Cayley's theorem, which (at least to us) appears very surprising. The explanation is as much mathematical as it is semantic. The apparently difficult structure of Cayley compositions is immediately clear from the definition: they are integer points in a difficult to describe polytope $\boldsymbol{A}_n$, combinatorially (but not metrically!) equivalent to a cube [KP2]. On the other hand, the elementary description of Cayley partitions, defined as partitions into certain parts, evokes the image of a simplex $\boldsymbol{Q}_n$, defined by the inequalities

$$2^{n-1}z_1 + 2^{n-2}z_2 + \ldots + 2z_{n-1} + z_n \le 2^n - 1, \text{ and } z_1, \ldots, z_n \ge 0.$$

(cf. the definition of $\boldsymbol{Y}_n$ in Section 1). The problem, however, is that integer points in $\boldsymbol{Q}_n$ are exactly those in $\boldsymbol{B}_n$, while polytopes $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ are equally complicated. The moral of the story can be summarized as follows: the inherent complexity of integer points in polyhedra can obscure natural bijections between such sets. We refer to [Bar] for an introduction to integer points in polytopes, and further references.

4.3. In the past decade, fueled by several beautiful applications such as *lecture hall partition identities* [BE1, BE2, Yee] and Cayley compositions, there has been a number of studies of partitions and compositions defined by inequalities (see e.g. [And, CS, CSW, P1]). Along the way a number of interesting proofs and extensions of Theorem 1 were also established [APRS, BBL, CLS2, CLS1]. Although many of these results follow directly from the structure of bijection φ, we decided not to pursue them. Let us single out [CLS1], where the authors obtain a special case of Corollary 8 using different tools.

A referee pointed out a mysterious connection between the Cayley polytopes and the $(2^n, 2^{n-1}, \ldots, 2)$-lecture hall polytope whose volume is $2^{\binom{n+2}{2}}/n!$, the total number of graphs on $n+1$ vertices. It would be interesting if φ applies in this case.

Note that the geometric approach to the construction of combinatorial bijections via integer points in polytopes was previously explored in [P1, PV]. This approach was also used by the authors in [KP1] to analyze the complexity of a bijection whose original definition was non-geometric. We refer to [P2] for a broad survey of partitions bijections and further references.

4.4. The sequence for the number C_n of connected labeled graphs on n nodes, $n = 1, 2, \ldots$ is A001187 in the Encyclopedia of Integer Sequences [Slo]. It begins

$$1, 1, 4, 38, 728, 26704, 1866256, 251548592, 66296291072, 34496488594816, \ldots$$

and is well studied in the enumerative combinatorics literature. For example, as mentioned in the proof of Theorem 3 (see Section 3), we have $C_n = \mathrm{Inv}_n(2)$. Similarly, $C_n = T_{K_n}(0, 2)$, where $T_G(x, y)$ is the *Tutte polynomial* of a graph G. We refer to [Ges, GJ, PPR, MR, Tut] for an explicit form (exponential) generating functions for numbers C_n, polynomials $\mathrm{Inv}_n(t)$ and $T_n(q, t)$.

4.5. Let us mention that Theorem 3 is one of the many results in [SP] on the Stanley–Pitman polytopes. In a different direction, it was generalized by the authors to what we call *Tutte polytopes* and general values of the Tutte polynomials of complete graphs [KP2]. It would be interesting to see how far the results of this paper can be extended in this direction.

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