

A BIJECTIVE PROOF OF THE ASM THEOREM PART I: THE OPERATOR FORMULA

ILSE FISCHER AND MATJAŽ KONVALINKA

ABSTRACT. Alternating sign matrices are known to be equinumerous with descending plane partitions, totally symmetric self-complementary plane partitions and alternating sign triangles, but no bijective proof for any of these equivalences has been found so far. In this paper we provide the first bijective proof of the operator formula for monotone triangles, which has been the main tool for several non-combinatorial proofs of such equivalences. In this proof, signed sets and sijections (signed bijections) play a fundamental role.

1. INTRODUCTION

An *alternating sign matrix* (ASM) is a square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column the non-zero entries alternate and sum up to 1. Robbins and Rumsey introduced alternating sign matrices in the 1980s [RR86] when studying their λ -determinant (a generalization of the classical determinant) and showing that the λ -determinant can be expressed as a sum over all alternating sign matrices of fixed size. The classical determinant is obtained from this by setting $\lambda = -1$, in which case the sum reduces so that it extends only over all ASMs *without* -1 's, i.e., permutation matrices, and the well-known formula of Leibniz is recovered. Numerical experiments led Robbins and Rumsey to conjecture that the number of $n \times n$ alternating sign matrices is given by the surprisingly simple product formula

$$(1) \quad \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Back then the surprise was even bigger when they learned from Stanley (see [BP99, Bre99]) that this product formula had recently also appeared in Andrews' paper [And79] on his proof of the weak Macdonald conjecture, which in turn provides a formula for the number of *cyclically symmetric plane partitions*. As a byproduct, Andrews had introduced *descending plane partitions* and had proven that the number of descending plane partitions (DPPs) with parts at most n is also equal to (1). Since then the problem of finding an explicit bijection between alternating sign matrices and descending plane partitions has attracted considerable attention from combinatorialists and to many of them it is a miracle that such a bijection has not been found so far. All the more so because Mills, Robbins and Rumsey had also introduced several "statistics" on alternating sign matrices and on descending plane partitions for which they had strong numerical evidence that the joint distributions coincide as well, see [MRR83].

Date: October 1, 2019.

The first author acknowledges the financial support from the Austrian Science Foundation FWF (SFB F50) and the second author acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0294).

There were a few further surprises yet to come. Robbins introduced a new operation on plane partitions, *complementation*, and had strong numerical evidence that totally symmetric self-complementary plane partitions (TSSCPPs) in a $2n \times 2n \times 2n$ -box are also counted by (1). Again this was further supported by statistics that have the same joint distribution as well as certain refinements, see [MRR86, Kra96, Kra16, BC16]. We still lack an explicit bijection between TSSCPPs and ASMs, as well as between TSSCPPs and DPPs.

In his collection of bijective proof problems (which is available from his webpage) Stanley says the following about the problem of finding all these bijections: “*This is one of the most intriguing open problems in the area of bijective proofs.*” In Krattenthaler’s survey on plane partitions [Kra16] he expresses his opinion by saying: “*The greatest, still unsolved, mystery concerns the question of what plane partitions have to do with alternating sign matrices.*”

Many of the above mentioned conjectures have since been proved by non-bijective means: Zeilberger [Zei96a] was the first who proved that $n \times n$ ASMs are counted by (1). Kuperberg gave another shorter proof [Kup96] based on the remarkable observation that the *six-vertex model* (which had been introduced by physicists several decades earlier) with domain wall boundary conditions is equivalent to ASMs, see [EKLP92a, EKLP92b], and he used the techniques that had been developed by physicists to study this model. Andrews enumerated TSSCPPs in [And94]. The equivalence of certain statistics for ASMs and of certain statistics for DPPs has been proved in [BDFZJ12, BDFZJ13], while for ASMs and TSSCPPs see [Zei96b, FZJ08], and note in particular that already in Zeilberger’s first ASM paper [Zei96a] he could deal with an important refinement. Further work including the study of *symmetry classes* has been accomplished; for a more detailed description of this we defer to [BFK17]. Then, in very recent work, alternating sign triangles (ASTs) were introduced in [ABF16], which establishes a fourth class of objects that are equinumerous with ASMs, and also in this case nobody has so far been able to construct a bijection.

Another aspect that should be mentioned here is Okada’s work [Oka06] (see also [Str04]), which hints at a connection between ASMs and representation theory that has not yet been well understood. He observed that a certain multivariate generating function (a specialization at a root of unity of the partition function that had been introduced by physicists in their study of the six-vertex model) can be expressed—up to a power of 3—by a single *Schur polynomial*. Since Schur polynomials are generating functions of semistandard tableaux, this establishes yet another challenging open problem for combinatorialists inclined to find bijections.

The proofs of the results briefly reviewed above contain rather long and complicated computations, and include hardly any arguments of a combinatorial flavor. In fact it seems that all ASM-related identities for which there exists a bijective proofs are trivial, with the exception of the rotational invariance of fully packed loop configurations. This was proved by Wieland [Wie00] bijectively and is also used in the celebrated proof of the Razumov-Stroganov (ex-)conjecture [CS11].

We come now to the purpose of the current paper. This is the first paper in a planned series that seeks to give the first bijective proofs of several results described so far. The seed of the idea to do so came from a brief discussion of the first author with Zeilberger on the problem of finding such bijections at the AMS-MAA Joint Mathematics Meetings 2019. Zeilberger mentioned that such bijections can be constructed from existing “computational” proofs, however, most likely these bijections are complicated. The authors of the current paper agree,

in fact the first author gave her “own” proof of the ASM theorem in [Fis06, Fis07, Fis16] and expressed some speculations in this direction in the final section of the last paper. It is also not implausible that a simple satisfactory bijective proof of the ASM theorem does not exist at all. Combinatorialists have failed to find such bijections for decades now, and we may start to ask ourselves why we are not rewarded for these efforts.

This is how the authors of the current paper decided to work on converting the proof in [Fis16] into a bijective proof. After having figured out how to actually convert computations and also having shaped certain useful fundamental concepts related to *signed sets* (see Section 2), the translation of several steps became quite straightforward; some steps were quite challenging. Then a certain type of (exciting) dynamics evolved, where the combinatorial point of view led to simplifications and other modifications, and after this process the original “computational” proof is in fact rather difficult to recognize. For several obvious reasons, we find it essential to check all our constructions with computer code; to name one it can possibly be used to identify new equivalent statistics.

After the above mentioned simplifications, it seems that *signs* seem to be unavoidable. After all, if there would be a simple bijective proof that avoided signs, would it not also be plausible that such a proof could be converted into a simple “computational” proof that avoids signs? Such a proof has also not been found so far.

In the remainder of the introduction we discuss the result that is proved bijectively in this paper, in particular we discuss why signed enumerations seem to be unavoidable from this point of view. We also sketch a few ideas informally before giving rigorous definitions and proofs later on.

The operator formula. We use the well-known correspondence between order $n \times n$ ASMs and *monotone triangles* with bottom row $1, 2, \dots, n$. A *monotone triangle* is a triangular array $(a_{i,j})_{1 \leq j \leq i \leq n}$ of integers, where the elements are usually arranged as follows

$$(2) \quad \begin{array}{cccccccc} & & & & & & & a_{1,1} \\ & & & & & & & a_{2,1} & & a_{2,2} \\ & & & & & & & \dots & & \dots & & \dots \\ & & & & & & & \dots & & \dots & & \dots & & a_{n-2,n-2} \\ & & & & & & & a_{n-2,1} & & \dots & & \dots & & a_{n-1,n-1} \\ & & & & & & & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} \\ & & & & & & & a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n} \end{array},$$

such that the integers increase weakly along \nearrow -diagonals and \searrow -diagonals, and increase strictly along rows, i.e., $a_{i,j} \leq a_{i-1,j} \leq a_{i,j+1}$ and $a_{i,j} < a_{i,j+1}$ for all i, j with $1 \leq j < i \leq n$. In order to convert an ASM into the corresponding monotone triangle, add to each entry all the entries that are in the same column above it, and record then row by row the positions of the 1’s, see Figure 1 for an example.

The following *operator formula* for the number of monotone triangles with prescribed bottom row was first proved in [Fis06] (see [Fis10, Fis16] for simplifications and generalizations). Note that we allow arbitrary strictly increasing bottom rows.

Theorem. *Let $k_1 < k_2 < \dots < k_n$ be a sequence of strictly increasing integers. The number of monotone triangles with bottom row k_1, \dots, k_n is*

$$(3) \quad \prod_{1 \leq p < q \leq n} (E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i},$$

extension of the combinatorial interpretation of (4) to any sequence k_1, \dots, k_n of integers. Such an interpretation was given in [Fis05] and is repeated below in Section 4.

Outline of the bijective proof. Given a sequence $k_1 < \dots < k_n$, it suffices to find an injective map from the set of monotone triangles with bottom row k_1, \dots, k_n to our shifted Gelfand-Tsetlin patterns associated with k_1, \dots, k_n so that the images under this map have positive signs, along with a sign-reversing involution on the set of shifted Gelfand-Tsetlin patterns that are not the image of a monotone triangle.

We will accomplish something more general, as we will also consider an extension of monotone triangles to all integer sequences k_1, \dots, k_n , see Section 5, along with a sign function on these objects, and prove that the operator formula also holds in this instance. To do that, we will construct a sign-reversing involution on a subset of monotone triangles, another sign-reversing involution on a subset of shifted Gelfand-Tsetlin patterns, and a sign-preserving bijection between the remaining monotone triangles and the remaining shifted Gelfand-Tsetlin patterns. Note that this is actually equivalent to the construction of a bijection between the (disjoint) union of the “positive” monotone triangles and the “negative” shifted Gelfand-Tsetlin patterns, and the (disjoint) union of the “negative” monotone triangles and the “positive” shifted Gelfand-Tsetlin patterns. We call such maps *sijections* for general signed sets.

The actual construction here will make use of the recursion underlying monotone triangles. For a monotone triangle with bottom row k_1, \dots, k_n , the eligible penultimate rows l_1, \dots, l_{n-1} are those with

$$k_1 \leq l_1 \leq k_2 \leq l_2 \leq \dots \leq l_{n-1} \leq k_n,$$

and $l_1 < l_2 < \dots < l_{n-1}$. This establishes a recursion that can be used to construct all monotone triangles. Phrased differently, “at” each k_i we need to sum over all l_{i-1}, l_i such that $l_{i-1} \leq k_i \leq l_i$ and $l_{i-1} < l_i$.³ However, we can split this into the following three cases:

- (1) Consider all l_{i-1}, l_i with $l_{i-1} < k_i \leq l_i$.
- (2) Consider all l_{i-1}, l_i with $l_{i-1} \leq k_i < l_i$.
- (3) Combining (1) and (2), we have done some double counting, thus we need to subtract the intersection, i.e., all l_{i-1}, l_i with $l_{i-1} < k_i < l_i$.

This can be written as a recursion. The *arrow rows* in Section 5 are used to describe this recursion: we choose \nwarrow “at” k_i if we are in Case (1), \nearrow in Case (2), and $\not\propto$ in Case (3). Our main effort will be to show “sijectively” that shifted Gelfand-Tsetlin patterns also fulfill this recursion.

Outline of the paper. The remainder of this paper is devoted to the bijective proof of Theorem 1 (or rather, the more general version with the increasing condition on k_1, \dots, k_n dropped). In Section 2 we lay the groundwork by defining concepts like signed sets and sijections, and we extend known concepts such as disjoint union, Cartesian product and composition for ordinary sets and bijections to signed sets and sijections. The composition of sijections will use a variation of the well-known Garsia-Milne involution principle [GM81, And86]. Many of the signed sets we will be considering are signed boxes (Cartesian products of signed intervals) or at least involve them, and we define some sijections on them in Section 3. These sijections will be the building blocks of our bijective proof later on. In Section 4 we introduce the extended Gelfand-Tsetlin patterns and construct some

³The degenerate cases k_1 and k_n are slightly different.

related bijections. In Section 5, we finally define the extended monotone triangles as well as the shifted Gelfand-Tsetlin patterns (i.e., the combinatorial interpretation of (3)), and use all the preparation to construct the bijection between monotone triangles and shifted Gelfand-Tsetlin patterns. In the final section, we discuss further projects.

To emphasize that we are not merely interested in the fact that two signed sets have the same size, but want to use the constructed signed bijection later on, we will be using a convention that is slightly unorthodox in our field. Instead of listing out results as lemmas and theorems with their corresponding proofs, we will be using the Problem–Construction terminology. See for instance [Voe] and [Bau].

2. SIGNED SETS AND SIJECTIONS

Signed sets. A *signed set* is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \emptyset$. Equivalently, a signed set is a finite set S together with a sign function $\text{sign}: S \rightarrow \{1, -1\}$. While we will mostly avoid the use of the sign function altogether (with the exception of monotone triangles defined in Section 5), it is useful to keep this description at the back of one's mind. Note that throughout the paper, signed sets are underlined. We will write $i \in \underline{S}$ to mean $i \in S^+ \cup S^-$.

The *size* of a signed set \underline{S} is $|\underline{S}| = |S^+| - |S^-|$. The *opposite* signed set of \underline{S} is $-\underline{S} = (S^-, S^+)$. We have $|-\underline{S}| = -|\underline{S}|$. The *Cartesian product* of signed sets \underline{S} and \underline{T} is

$$\underline{S} \times \underline{T} = (S^+ \times T^+ \cup S^- \times T^-, S^+ \times T^- \cup S^- \times T^+),$$

and we can similarly (or recursively) define the Cartesian product of a finite number of signed sets. We have

$$|\underline{S} \times \underline{T}| = |S^+| \cdot |T^+| + |S^-| \cdot |T^-| - |S^+| \cdot |T^-| - |S^-| \cdot |T^+| = |\underline{S}| \cdot |\underline{T}|.$$

The intersection of signed sets \underline{S} and \underline{T} is defined as $\underline{S} \cap \underline{T} = (S^+ \cap T^+, S^- \cap T^-)$, while the union $\underline{S} \cup \underline{T} = (S^+ \cup T^+, S^- \cup T^-)$ is only defined when $S^+ \cap T^- = S^- \cap T^+ = \emptyset$. Again, we can extend these definitions to a finite family of signed sets.

Example 1. *One of the crucial signed sets is the signed interval*

$$\underline{[a, b]} = \begin{cases} ([a, b], \emptyset) & \text{if } a \leq b \\ (\emptyset, [b+1, a-1]) & \text{if } a > b \end{cases}$$

for $a, b \in \mathbb{Z}$, where $[a, b]$ stands for an interval in \mathbb{Z} in the usual sense. We have $\underline{[b+1, a-1]} = -\underline{[a, b]}$ and $|\underline{[a, b]}| = b - a + 1$.

We will also see many signed boxes, Cartesian products of signed intervals. Note that $S^+ = \emptyset$ or $S^- = \emptyset$ for every signed box \underline{S} .

Signed subsets $\underline{T} \subseteq \underline{S}$ are defined in an obvious manner, in particular, for $s \in \underline{S}$, we have

$$\underline{\{s\}} = \begin{cases} (\{s\}, \emptyset) & \text{if } s \in S^+ \\ (\emptyset, \{s\}) & \text{if } s \in S^- \end{cases}.$$

The *disjoint union* of signed sets \underline{S} and \underline{T} is the signed set

$$\underline{S} \sqcup \underline{T} = (\underline{S} \times (\{0\}, \emptyset)) \cup (\underline{T} \times (\{1\}, \emptyset))$$

with elements $(s, 0)$ for $s \in \underline{S}$ and $(t, 1)$ for $t \in \underline{T}$. If \underline{S} and \underline{T} are signed sets with $(S^+ \cup S^-) \cap (T^+ \cup T^-) = \emptyset$, we can identify $\underline{S} \cup \underline{T}$ and $\underline{S} \sqcup \underline{T}$.

More generally, we can define the disjoint union of a family of signed sets \underline{S}_t , where the family is indexed with a signed set \underline{T} :

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigcup_{t \in \underline{T}} (\underline{S}_t \times \{t\}).$$

We get $\bigsqcup_{t \in [0,1]} \underline{S}_t = S_0 \sqcup S_1$. For $a, b \in \mathbb{Z}$, we may also write $\bigsqcup_{i=a}^b \underline{S}_i$ instead of $\bigsqcup_{i \in [a,b]} \underline{S}_i$. As for the size, we have

$$\left| \bigsqcup_{t \in \underline{T}} \underline{S}_t \right| = \sum_{t \in \underline{T}} |\underline{S}_t| \cdot |\{t\}|.$$

The usual properties such as associativity $(\underline{S} \sqcup \underline{T}) \sqcup \underline{U} = \underline{S} \sqcup (\underline{T} \sqcup \underline{U})$ and distributivity $(\underline{S} \sqcup \underline{T}) \times \underline{U} = \underline{S} \times \underline{U} \sqcup \underline{T} \times \underline{U}$ also hold. Strictly speaking, the = sign here and sometimes later on indicates that there is an obvious and natural sign-preserving bijection between the two signed sets. We summarize a few more basic properties that will be needed in the following and that are easy to prove.

(1)

$$\bigsqcup_{\mathbf{l} \in [a_1, b_1] \times \dots \times [a_n, b_n]} \underline{S}_{l_1+c_1, \dots, l_n+c_n} = \bigsqcup_{\mathbf{l} \in [a_1+c_1, b_1+c_1] \times \dots \times [a_n+c_n, b_n+c_n]} \underline{S}_{l_1, \dots, l_n}$$

(2)

$$\bigsqcup_{t \in \underline{T}} \bigsqcup_{u \in \underline{U}} \underline{S}_{t,u} = \bigsqcup_{(u,t) \in \underline{U} \times \underline{T}} \underline{S}_{t,u} = \bigsqcup_{(t,u) \in \underline{T} \times \underline{U}} \underline{S}_{t,u} = \bigsqcup_{u \in \underline{U}} \bigsqcup_{t \in \underline{T}} \underline{S}_{t,u}$$

(3)

$$\bigsqcup_{t \in \bigsqcup_{u \in \underline{U}} \underline{T}_u} \underline{S}_t = \bigsqcup_{u \in \underline{U}} \bigsqcup_{t \in \underline{T}_u} \underline{S}_t$$

(4)

$$-\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigsqcup_{t \in \underline{T}} -\underline{S}_t = \bigsqcup_{t \in -\underline{T}} \underline{S}_t.$$

Sijctions. The role of bijections for signed sets is played by “signed bijections”, which we call *sijctions*. A sijction φ from \underline{S} to \underline{T} ,

$$\varphi: \underline{S} \Rightarrow \underline{T},$$

is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with the property $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$, where \sqcup refers to the disjoint union for ordinary (“unsigned”) sets. It follows that also $\varphi(S^- \sqcup T^+) = S^+ \sqcup T^-$. There is an obvious sijction $\text{id}_{\underline{S}}: \underline{S} \Rightarrow \underline{S}$.

We can think of a sijction as a collection of a sign-reversing involution on a subset of \underline{S} , a sign-reversing involution on a subset of \underline{T} , and a sign-preserving matching between the remaining elements of \underline{S} with the remaining elements of \underline{T} . When $S^- = T^- = \emptyset$, the signed sets can be identified with ordinary sets, and a sijction in this case is simply a bijection.

A sijction is a manifestation of the fact that two signed sets have the same size. Indeed, if there exists a sijction $\varphi: \underline{S} \Rightarrow \underline{T}$, we have $|S^+| + |T^-| = |S^+ \sqcup T^-| = |S^- \sqcup T^+| = |S^-| + |T^+|$ and therefore $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$. A sijction $\varphi: \underline{S} \Rightarrow \underline{T}$ has an inverse $\varphi^{-1}: \underline{T} \Rightarrow \underline{S}$ that we obtain by identifying $(T^+ \cup T^-) \sqcup (S^+ \cup S^-)$ with $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$.

For a signed set \underline{S} , there is a natural sijction φ from $\underline{S} \sqcup (-\underline{S})$ to the empty signed set $\emptyset = (\emptyset, \emptyset)$. Indeed, the involution should be defined on $(S^+ \times \{0\} \cup S^- \times \{1\}) \cup (S^- \times \{0\} \cup S^+ \times \{1\})$ and map $S^+ \times \{0\} \cup S^- \times \{1\}$ to $S^+ \times \{1\} \cup S^- \times \{0\}$, and so we can take $\varphi((s, 0), 0) = ((s, 1), 0)$,

$\varphi((s, 1), 0) = ((s, 0), 0)$. Note that in general, a sijection from a signed set \underline{S} to $\underline{\emptyset}$ is simply a sign-reversing involution on \underline{S} , in other words, a bijection between S^+ and S^- .

If we have a sijection $\varphi: \underline{S} \Rightarrow \underline{T}$, there is a natural sijection $-\varphi: -\underline{S} \Rightarrow -\underline{T}$ (as a map, it is actually precisely the same).

If we have sijections $\varphi_i: \underline{S}_i \Rightarrow \underline{T}_i$ for $i = 0, 1$, then there is a natural sijection $\varphi: \underline{S}_0 \sqcup \underline{S}_1 \Rightarrow \underline{T}_0 \sqcup \underline{T}_1$. More interesting ways to create new sijections are described below in Proposition 2, but we will need this in our first construction for the special case $\underline{S}_0 = \underline{T}_0$ and $\varphi_0 = \text{id}_{\underline{S}_0}$.

To motivate our first result, note that if $a \leq b \leq c$ or $c < b < a$, then $\underline{[a, c]} = \underline{[a, b]} \sqcup \underline{[b+1, c]} = \underline{[a, b]} \sqcup \underline{[b+1, c]}$. Of course, this does not hold in general; for $a = 1, b = 8, c = 5$, we have $\underline{[1, 5]} = (\{1, 2, 3, 4, 5\}, \emptyset)$, $(\underline{[1, 8]} \sqcup \underline{[9, 5]})^+ = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0)\}$ and $(\underline{[1, 8]} \sqcup \underline{[9, 5]})^- = \{(6, 1), (7, 1), (8, 1)\}$. The following, however, tells us that there is in general a sijection between $\underline{[a, c]}$ and $\underline{[a, b]} \sqcup \underline{[b+1, c]}$. This map will be the crucial building block for more complicated sijections.

Problem 1. *Given $a, b, c \in \mathbb{Z}$, construct a sijection*

$$\alpha = \alpha_{a,b,c}: \underline{[a, c]} \Rightarrow \underline{[a, b]} \sqcup \underline{[b+1, c]}.$$

Construction. For $a \leq b \leq c$ and $c < b < a$, there is nothing to prove. For, say, $a \leq c < b$, we have

$$\underline{[a, b]} \sqcup \underline{[b+1, c]} = (\underline{[a, c]} \sqcup \underline{[c+1, b]}) \sqcup \underline{[b+1, c]} = \underline{[a, c]} \sqcup (\underline{[c+1, b]} \sqcup \underline{-[c+1, b]})$$

and since there is a sijection $\underline{[c+1, b]} \sqcup \underline{-[c+1, b]} \Rightarrow \underline{\emptyset}$, we get a sijection $\underline{[a, b]} \sqcup \underline{[b+1, c]} \Rightarrow \underline{[a, c]}$. The cases $b < a \leq c$, $b \leq c < a$, and $c < a \leq b$ are analogous. \square

The following proposition describes composition, Cartesian product, and disjoint union of sijections. The composition is a variant of the well-known Garsia-Milne involution principle. All the statements are easy to prove, and the proofs are left to the reader.

Proposition 2.

- (1) *(Composition) Suppose that we have sijections $\varphi: \underline{S} \Rightarrow \underline{T}$ and $\psi: \underline{T} \Rightarrow \underline{U}$. For $s \in \underline{S}$ (resp. $u \in \underline{U}$), define $\psi \circ \varphi(s)$ (resp. $\psi \circ \varphi(u)$) as the last well-defined element in the sequence $s, \varphi(s), \psi(\varphi(s)), \varphi(\psi(\varphi(s))), \dots$ (resp. $u, \psi(u), \varphi(\psi(u)), \psi(\varphi(\psi(u))), \dots$). Then $\psi \circ \varphi$ is a well-defined sijection from \underline{S} to \underline{U} .*
- (2) *(Cartesian product) Suppose we have sijections $\varphi_i: \underline{S}_i \Rightarrow \underline{T}_i$, $i = 1, \dots, k$. Then $\varphi = \varphi_1 \times \dots \times \varphi_k$, defined by*

$$\varphi(s_1, \dots, s_k) = \begin{cases} (\varphi_1(s_1), \dots, \varphi_k(s_k)) & \text{if } \varphi_i(s_i) \in \underline{T}_i \text{ for } i = 1, \dots, k \\ (s_1, \dots, s_{j-1}, \varphi_j(s_j), s_{j+1}, \dots, s_k) & \text{if } \varphi_j(s_j) \in \underline{S}_j, \varphi_i(s_i) \in \underline{T}_i \text{ for } i < j \end{cases}$$

if $(s_1, \dots, s_k) \in \underline{S}_1 \times \dots \times \underline{S}_k$ and

$$\varphi(t_1, \dots, t_k) = \begin{cases} (\varphi_1(t_1), \dots, \varphi_k(t_k)) & \text{if } \varphi_i(t_i) \in \underline{S}_i \text{ for } i = 1, \dots, k \\ (t_1, \dots, t_{j-1}, \varphi_j(t_j), t_{j+1}, \dots, t_k) & \text{if } \varphi_j(t_j) \in \underline{T}_j, \varphi_i(t_i) \in \underline{S}_i \text{ for } i < j \end{cases}$$

if $(t_1, \dots, t_k) \in \underline{T}_1 \times \dots \times \underline{T}_k$, is a well-defined sijection from $\underline{S}_1 \times \dots \times \underline{S}_k$ to $\underline{T}_1 \times \dots \times \underline{T}_k$.

(3) (*Disjoint union*) Suppose we have signed sets $\underline{T}, \underline{\tilde{T}}$ and a sijection $\psi: \underline{T} \Rightarrow \underline{\tilde{T}}$. Furthermore, suppose that for every $t \in \underline{T} \sqcup \underline{\tilde{T}}$, we have a signed set \underline{S}_t and a sijection $\varphi_t: \underline{S}_t \Rightarrow \underline{S}_{\psi(t)}$ satisfying $\varphi_{\psi(t)} = \varphi_t^{-1}$. Then $\varphi = \bigsqcup_{t \in \underline{T} \sqcup \underline{\tilde{T}}} \varphi_t$, defined by

$$\varphi(s_t, t) = \begin{cases} (\varphi_t(s_t), t) & \text{if } t \in \underline{T} \sqcup \underline{\tilde{T}}, s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_t \\ (\varphi_t(s_t), \psi(t)) & \text{if } t \in \underline{T} \sqcup \underline{\tilde{T}}, s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_{\psi(t)} \end{cases}$$

is a sijection $\bigsqcup_{t \in \underline{T}} \underline{S}_t \Rightarrow \bigsqcup_{t \in \underline{\tilde{T}}} \underline{S}_t$.

One important special case of Proposition 2 (3) is $\underline{T} = \underline{\tilde{T}}$ and $\psi = \text{id}$. We have two sets of signed sets indexed by \underline{T} , $\underline{S}_{(t,0)} =: \underline{S}_t^0$ and $\underline{S}_{(t,1)} =: \underline{S}_t^1$, and sijections $\varphi_t: \underline{S}_t^0 \Rightarrow \underline{S}_t^1$. By the proposition, these sijections have a disjoint union that is a sijection $\bigsqcup_{t \in \underline{T}} \underline{S}_t^0 \Rightarrow \bigsqcup_{t \in \underline{T}} \underline{S}_t^1$.

By the proposition, the relation

$$\underline{S} \approx \underline{T} \iff \text{there exists a sijection from } \underline{S} \text{ to } \underline{T}$$

is an equivalence relation.

Elementary signed sets and normal sijections. Often, we will be interested in disjoint unions of Cartesian products of signed intervals. An element of such a signed set is a pair, consisting of a tuple of integers and an element of the indexing signed set. Intuitively, the first one is “more important”, as the second one serves just as an index. We formalize this notion in the following definition.

Definition 3. A signed set \underline{A} is elementary of dimension n and depth 0 if its elements are in \mathbb{Z}^n . A signed set \underline{A} is elementary of dimension n and depth d , $d \geq 1$, if it is of the form

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t,$$

where \underline{T} is a signed set, and \underline{S}_t are all signed sets of dimension n and depth at most $d-1$, with the depth of at least one of them equal to $d-1$. A signed set \underline{A} is elementary of dimension n if it is an elementary signed set of dimension n and depth d for some $d \in \mathbb{N}$.

The projection map on an elementary set of dimension n is the map

$$\xi: \underline{A} \rightarrow \mathbb{Z}^n$$

defined as follows. If the depth of \underline{A} is 0, then ξ is simply the inclusion map. Once ξ is defined on elementary signed sets of depth $< d$, and the depth of \underline{A} is d , then $\underline{A} = \bigsqcup_{t \in \underline{T}} \underline{S}_t$, where ξ is defined on all \underline{S}_t . Then define $\xi(s, t) = \xi(s)$ for $(s, t) \in \underline{A}$.

A sijection $\psi: \underline{T} \Rightarrow \underline{\tilde{T}}$ between elementary signed sets \underline{T} and $\underline{\tilde{T}}$ of the same dimension is normal if $\xi(\psi(t)) = \xi(t)$ for all $t \in \underline{T} \sqcup \underline{\tilde{T}}$.

Simple examples of elementary signed sets are $[a, c]$, $[a, b] \sqcup [b+1, c]$ and $[a, c] \sqcup ([a, b] \sqcup [b+1, c])$. They are all of dimension 1 and depth 0, 1 and 2, respectively.⁴ It is easy to see that the sijection $\alpha_{a,b,c}$ from Problem 1 is normal.

⁴To avoid ambiguity, we should consider signed intervals in this case to be subsets of \mathbb{Z}^1 (1-tuples of integers), not \mathbb{Z} . Otherwise, $[0, 1] \sqcup [2, 3] = (\{(0, 0), (1, 0), (2, 1), (3, 1)\}, \emptyset)$, and this can be seen either as an elementary set of dimension 1 and depth 1, or as an elementary signed set of dimension 2 and depth 0. So the interpretation depends on the “representation” of the set as disjoint union. Instead, we should understand $[0, 1] \sqcup [2, 3]$ to mean $(\{((0), 0), ((1), 0), ((2), 1), ((3), 1)\}, \emptyset)$, with dimension 1 and depth 1. For coding, the distinction is important, but in the paper we nevertheless think of elements of signed intervals as integers.

Let us illustrate this with the example $a = 1$, $b = 5$, $c = 3$. We have $\underline{[a, c]} = (\{1, 2, 3\}, \emptyset)$ and $\underline{[a, b]} \sqcup \underline{[b+1, c]} = (\{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0)\}, \{(4, 1), (5, 1)\})$. The sijection $\alpha_{1,5,3}$ is the involution on $\underline{[1, 3]} \sqcup (\underline{[1, 5]} \sqcup \underline{[6, 3]})$ defined by

$$(1, 0) \leftrightarrow ((1, 0), 1), \quad (2, 0) \leftrightarrow ((2, 0), 1), \quad (3, 0) \leftrightarrow ((3, 0), 1), \\ ((4, 0), 1) \leftrightarrow ((4, 1), 1), \quad ((5, 0), 1) \leftrightarrow ((5, 1), 1).$$

Since $\xi(i, 0) = i$ for $i = 1, 2, 3$, $\xi((i, 0), 1) = i$ for $i = 1, 2, 3, 4, 5$ and $\xi((i, 1), 1) = i$ for $i = 4, 5$, $\alpha_{1,5,3}$ is indeed normal.

Other examples of elementary signed sets appear in the statements of Problems 2 and 3 (in both cases, they are of dimension $n - 1$).

Normality is preserved under Cartesian product, disjoint union etc. For example, the sijection

$$\underline{[a_1, c_1]} \times \underline{[a_2, c_2]} \Rightarrow \\ \underline{[a_1, b_1]} \times \underline{[a_2, b_2]} \sqcup \underline{[a_1, b_1]} \times \underline{[b_2+1, c_2]} \sqcup \underline{[b_1+1, c_1]} \times \underline{[a_2, b_2]} \sqcup \underline{[b_1+1, c_1]} \times \underline{[b_2+1, c_2]},$$

obtained by using $\alpha_{a_1, b_1, c_1} \times \alpha_{a_2, b_2, c_2}$ and distributivity on disjoint unions, is normal.

The main reason normal sijections are important is that they give a very natural special case of Proposition 2 (3). Suppose that \underline{T} and $\underline{\tilde{T}}$ are elementary signed sets of dimension n , and that $\psi: \underline{T} \Rightarrow \underline{\tilde{T}}$ is a normal sijection. Furthermore, suppose that we have a signed set $\underline{S}_{\mathbf{k}}$ for every $\mathbf{k} \in \mathbb{Z}^n$. Then we have a sijection

$$\bigsqcup_{t \in \underline{T}} \underline{S}_{\xi(t)} \Rightarrow \bigsqcup_{t \in \underline{\tilde{T}}} \underline{S}_{\xi(t)}.$$

Indeed, Proposition 2 gives us a sijection provided that we have a sijection $\varphi_t: \underline{S}_{\xi(t)} \Rightarrow \underline{S}_{\xi(\psi(t))}$ satisfying $\varphi_{\psi(t)} = \varphi_t^{-1}$ for every $t \in \underline{T} \sqcup \underline{\tilde{T}}$. But since $\xi(\psi(t)) = \xi(t)$, we can take φ_t to be the identity.

3. SOME SIJECTIONS ON SIGNED BOXES

The first sijection in this section will serve as the base of induction for Problem 5.

Example 4. For $a, b \in \mathbb{Z}$, we have a normal sijection

$$\bigsqcup_{(l_1, l_2) \in \underline{[a+1, b+1]} \times \underline{[a, b]}} \underline{[l_1, l_2]} \Rightarrow \emptyset$$

defined by $\varphi((x, (l_1, l_2)), 0) = ((x, (l_2 + 1, l_1 - 1)), 0)$. It is well defined because $(l_1, l_2) \in \underline{[a+1, b+1]} \times \underline{[a, b]}$ if and only if $(l_2 + 1, l_1 - 1) \in \underline{[a+1, b+1]} \times \underline{[a, b]}$, and because $x \in \underline{[l_1, l_2]}$ if and only if $x \in \underline{[l_2 + 1, l_1 - 1]}$.

Note that the 0 as the second coordinate in the example comes from the fact that a sijection in question is an involution on the disjoint union

$$\left(\bigsqcup_{(l_1, l_2) \in \underline{[a+1, b+1]} \times \underline{[a, b]}} \underline{[l_1, l_2]} \right) \sqcup \emptyset = \left(\bigsqcup_{(l_1, l_2) \in \underline{[a+1, b+1]} \times \underline{[a, b]}} \underline{[l_1, l_2]} \right) \times \{\underline{0}\} \cup \emptyset \times \{\underline{1}\}.$$

We could be a little less precise and write $\varphi(x, (l_1, l_2)) = (x, (l_2 + 1, l_1 - 1))$ without causing confusion.

The following generalizes the construction of Problem 1; indeed, for $n = 2$ the construction gives a sijection from $[a_1, b_1]$ to $[a_1, x] \sqcup (-[b_1 + 1, x])$.

Problem 2. Given $\mathbf{a} = (a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$, $\mathbf{b} = (b_1, \dots, b_{n-1}) \in \mathbb{Z}^{n-1}$, $x \in \mathbb{Z}$, construct a normal sijection

$$\beta = \beta_{\mathbf{a}, \mathbf{b}, x}: \underline{[a_1, b_1]} \times \cdots \times \underline{[a_{n-1}, b_{n-1}]} \Rightarrow \bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{[l_1, l_2]} \times \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]},$$

where $\underline{S_i} = (\{a_i\}, \emptyset) \sqcup (\emptyset, \{b_i + 1\})$

Note that $(\{a_i\}, \emptyset) \sqcup (\emptyset, \{b_i + 1\})$ can be identified with $(\{a_i\}, \{b_i + 1\})$ if $a_i \neq b_i + 1$.

Construction. The proof is by induction, with the case $n = 1$ being trivial and the case $n = 2$ was constructed in Problem 1. Now, for $n \geq 3$,

$$\begin{aligned} \underline{[a_1, b_1]} \times \cdots \times \underline{[a_{n-1}, b_{n-1}]} &\approx \underline{[a_1, b_1]} \times \bigsqcup_{(l_2, \dots, l_{n-1}) \in \underline{S_2} \times \cdots \times \underline{S_{n-1}}} \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]} \\ &\approx \left(\underline{[a_1, b_1]} \times \bigsqcup_{(l_3, \dots, l_{n-1}) \in \underline{S_3} \times \cdots \times \underline{S_{n-1}}} \underline{[a_2, l_3]} \times \cdots \times \underline{[l_{n-1}, x]} \right) \\ &\quad \sqcup \left(\underline{[a_1, b_1]} \times \bigsqcup_{(l_3, \dots, l_{n-1}) \in \underline{S_3} \times \cdots \times \underline{S_{n-1}}} \underline{(-[b_2 + 1, l_3])} \times \cdots \times \underline{[l_{n-1}, x]} \right), \end{aligned}$$

where we used induction for the first equivalence, and distributivity and the fact that $S_2 = (\{a_2\}, \emptyset) \sqcup (\emptyset, \{b_2 + 1\})$ for the second equivalence. By Problem 1 and Proposition 2 (2), there exists a sijection from the last expression to

$$\begin{aligned} &\left(\left(\underline{[a_1, a_2]} \sqcup \underline{(-[b_1 + 1, a_2])} \right) \times \bigsqcup_{(l_3, \dots, l_{n-1}) \in \underline{S_3} \times \cdots \times \underline{S_{n-1}}} \underline{[a_2, l_3]} \times \cdots \times \underline{[l_{n-1}, x]} \right) \\ &\quad \sqcup \left(\left(\underline{[a_1, b_2 + 1]} \sqcup \underline{(-[b_1 + 1, b_2 + 1])} \right) \times \bigsqcup_{(l_3, \dots, l_{n-1}) \in \underline{S_3} \times \cdots \times \underline{S_{n-1}}} \underline{(-[b_2 + 1, l_3])} \times \cdots \times \underline{[l_{n-1}, x]} \right) \\ &\approx \bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{[l_1, l_2]} \times \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]}, \end{aligned}$$

where for the last equivalence we have again used distributivity. Normality follows from the normality of all the sijections involved in the construction. \square

Problem 3. Given $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, construct a normal sijection

$$\begin{aligned} \gamma &= \gamma_{\mathbf{k}, x}: \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\ &\Rightarrow \bigsqcup_{i=1}^n \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, x + n - i]} \times \underline{[x + n - i, k_{i+1}]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\ &\quad \sqcup \bigsqcup_{i=1}^{n-2} \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1} + 1, x + n - i - 1]} \times \underline{[k_{i+1}, x + n - i - 2]} \times \underline{[k_{i+2}, k_{i+3}]} \times \cdots. \end{aligned}$$

Construction. The proof is by induction with respect to n . The case $n = 1$ is trivial, and $n = 2$ is Problem 1. Now take $n > 2$. By the induction hypothesis (for (k_1, \dots, k_{n-1}) and

$x + 1$), we have

$$\begin{aligned} \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]} &\approx \left(\bigsqcup_{i=1}^{n-1} \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, x + n - i]} \times \underline{[x + n - i, k_{i+1}]} \times \cdots \times \underline{[k_{n-2}, k_{n-1}]} \right. \\ &\left. \sqcup \bigsqcup_{i=1}^{n-3} \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i+1} + 1, x + n - i - 1]} \times \underline{[k_{i+1}, x + n - i - 2]} \times \cdots \times \underline{[k_{n-2}, k_{n-1}]} \right) \times \underline{[k_{n-1}, k_n]}. \end{aligned}$$

We use distributivity. We keep all terms except the one corresponding to $i = n - 1$ in the first part. Because

$$\begin{aligned} \underline{[k_{n-2}, x + 1]} \times \underline{[k_{n-1}, k_n]} &\approx \underline{[k_{n-2}, x + 1]} \times (\underline{[k_{n-1}, x]} \sqcup \underline{[x + 1, k_n]}) \\ &\approx (\underline{[k_{n-2}, k_{n-1}]} \sqcup \underline{[k_{n-1} + 1, x + 1]}) \times \underline{[k_{n-1}, x]} \sqcup \underline{[k_{n-2}, x + 1]} \times \underline{[x + 1, k_n]} \\ &\approx \underline{[k_{n-2}, k_{n-1}]} \times \underline{[k_{n-1}, x]} \sqcup \underline{[k_{n-1} + 1, x + 1]} \times \underline{[k_{n-1}, x]} \sqcup \underline{[k_{n-2}, x + 1]} \times \underline{[x + 1, k_n]}, \end{aligned}$$

we obtain the required Cartesian products for the first term on the right-hand side at $i = n$, the second term at $i = n - 2$, and the first term at $i = n - 1$. Again, normality follows from the fact that α is normal. \square

4. GELFAND-TSETLIN PATTERNS

Using our definition of a disjoint union of signed sets, it is easy to define generalized Gelfand-Tsetlin patterns, or GT patterns for short (compare with [Fis05]).

Definition 5. For $k \in \mathbb{Z}$, define $\underline{\text{GT}}(k) = (\{\cdot\}, \emptyset)$, and for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, define recursively

$$\underline{\text{GT}}(\mathbf{k}) = \underline{\text{GT}}(k_1, \dots, k_n) = \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(l_1, \dots, l_{n-1}).$$

In particular, $\underline{\text{GT}}(a, b) \approx \underline{[a, b]}$.

Of course, one can think of an element of $\underline{\text{GT}}(\mathbf{k})$ in the usual way, as a triangular array $A = (A_{i,j})_{1 \leq j \leq i \leq n}$ of $\binom{n+1}{2}$ numbers, arranged as

$$\begin{array}{cccccccc} & & & & & & & A_{1,1} \\ & & & & & & & A_{2,1} & A_{2,2} \\ & & & & & & & A_{3,1} & A_{3,2} & A_{3,3} \\ & & & & & & & \ddots & \vdots & \ddots \\ & & & & & & & \vdots & \ddots & \vdots & \ddots \\ & & & & & & & A_{n,1} & A_{n,2} & \dots & A_{n,n}, \end{array}$$

so that $A_{i+1,j} \leq A_{i,j} \leq A_{i+1,j+1}$ or $A_{i+1,j} > A_{i,j} > A_{i+1,j+1}$ for $1 \leq j \leq i < n$, and $A_{n,i} = k_i$. The sign of such an array is $(-1)^m$, where m is the number of (i, j) with $a_{i,j} > a_{i,j+1}$.

Some crucial bijections for GT patterns are given by the following constructions.

Problem 4. Given $\mathbf{a} = (a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$, $\mathbf{b} = (b_1, \dots, b_{n-1}) \in \mathbb{Z}^{n-1}$, $x \in \mathbb{Z}$, construct a bijection

$$\rho = \rho_{\mathbf{a}, \mathbf{b}, x}: \bigsqcup_{\mathbf{l} \in \underline{[a_1, b_1]} \times \cdots \times \underline{[a_{n-1}, b_{n-1}]}} \underline{\text{GT}}(\mathbf{l}) \Rightarrow \bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{\text{GT}}(l_1, \dots, l_{n-1}, x),$$

where $\underline{S}_i = (\{a_i\}, \emptyset) \sqcup (\emptyset, \{b_i + 1\})$.

Construction. In Problem 2, we constructed a normal sijection

$$\underline{[a_1, b_1]} \times \cdots \times \underline{[a_{n-1}, b_{n-1}]} \Rightarrow \bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{[l_1, l_2]} \times \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]}.$$

By Proposition 2 (3) (see the comment at the end of Section 2), this gives a sijection

$$\bigsqcup_{\mathbf{l} \in \underline{[a_1, b_1]} \times \cdots \times \underline{[a_{n-1}, b_{n-1}]}} \underline{\text{GT}}(\mathbf{l}) \Rightarrow \bigsqcup_{\mathbf{m} \in \bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{[l_1, l_2]} \times \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]}} \underline{\text{GT}}(\mathbf{m}).$$

By basic sijection constructions, we get that this is equivalent to

$$\bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \bigsqcup_{\mathbf{m} \in \underline{[l_1, l_2]} \times \underline{[l_2, l_3]} \times \cdots \times \underline{[l_{n-2}, l_{n-1}]} \times \underline{[l_{n-1}, x]}} \underline{\text{GT}}(\mathbf{m}),$$

and by definition of $\underline{\text{GT}}$, this is equal to $\bigsqcup_{(l_1, \dots, l_{n-1}) \in \underline{S_1} \times \cdots \times \underline{S_{n-1}}} \underline{\text{GT}}(l_1, \dots, l_{n-1}, x)$. \square

The result is important because while it adds a dimension to GT patterns, it (typically) greatly reduces the size of the indexing signed set. In fact, there is an analogy to the fundamental theorem of calculus: instead of extending the disjoint union over the entire signed box, it suffices to consider the boundary; x corresponds in a sense to the constant of integration.

Problem 5. Given $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and i , $1 \leq i \leq n-1$, construct a sijection

$$\pi = \pi_{\mathbf{k}, i}: \underline{\text{GT}}(k_1, \dots, k_n) \Rightarrow -\underline{\text{GT}}(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \dots, k_n).$$

Given $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$ such that for some i , $1 \leq i \leq n-1$, we have $a_{i+1} = a_i - 1$ and $b_{i+1} = b_i - 1$, construct a sijection

$$\sigma = \sigma_{\mathbf{a}, \mathbf{b}, i}: \bigsqcup_{\mathbf{l} \in \underline{[a_1, b_1]} \times \cdots \times \underline{[a_n, b_n]}} \underline{\text{GT}}(\mathbf{l}) \Rightarrow \underline{\emptyset}.$$

Construction. The proof is by induction, with the induction step for π using σ and vice versa. For $n = 1$, there is nothing to prove. For $n = 2$ and $i = 1$, the existence of π follows from the statement $\underline{[k_1, k_2]} = -\underline{[k_2 + 1, k_1 - 1]}$, and σ was constructed in Example 4. Assume that $n > 2$ and $1 < i < n-1$. We have

$$\underline{\text{GT}}(k_1, \dots, k_n) = \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_{i+1}, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}).$$

By using $\text{id} \times \cdots \times \text{id} \times \alpha_{k_{i-1}, k_{i+1} + 1, k_i} \times \text{id} \times \alpha_{k_{i+1}, k_i - 2, k_{i+2}} \times \text{id} \times \cdots \times \text{id}$ and distributivity, we get a normal sijection

$$\begin{aligned} & \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_{i+1}, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]} \Rightarrow \\ & \quad \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_{i+1} + 1]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_i - 1, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\ & \sqcup \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_{i+1} + 1]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_{i+1}, k_i - 2]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\ & \quad \sqcup \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i+1} + 2, k_i]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_i - 1, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\ & \quad \sqcup \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i+1} + 2, k_i]} \times \underline{[k_{i+1} + 1, k_i - 1]} \times \underline{[k_{i+1}, k_i - 2]} \times \cdots \times \underline{[k_{n-1}, k_n]} \end{aligned}$$

By Proposition 2 (3), this gives a sijection

$$\begin{aligned}
& \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1}+1, k_i-1]} \times \underline{[k_{i+1}, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}) \Rightarrow \\
& \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_{i+1}+1]} \times \underline{[k_{i+1}+1, k_i-1]} \times \underline{[k_i-1, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}) \\
& \sqcup \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_{i+1}+1]} \times \underline{[k_{i+1}+1, k_i-1]} \times \underline{[k_{i+1}, k_i-2]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}) \\
& \sqcup \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i+1}+2, k_i]} \times \underline{[k_{i+1}+1, k_i-1]} \times \underline{[k_i-1, k_{i+2}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}) \\
& \sqcup \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i+1}+2, k_i]} \times \underline{[k_{i+1}+1, k_i-1]} \times \underline{[k_{i+1}, k_i-2]} \times \cdots \times \underline{[k_{n-1}, k_n]}} -\underline{\text{GT}}(l_1, \dots, l_{n-1}).
\end{aligned}$$

By definition, the first signed set on the right-hand side is $-\underline{\text{GT}}(k_1, \dots, k_{i-1}, k_{i+1} + 1, k_i - 1, k_{i+2}, \dots, k_n)$. The other three disjoint unions all satisfy the condition needed for the existence of σ (for i , for $i - 1$ and for both, $i - 1$ and i , respectively), and hence we can sject them to \emptyset .

If $i = 1$ or $i = n - 1$, the proof is similar but easier (as we only have to use α once, and we get only two factors after using distributivity). Details are left to the reader.

Now take $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{l}' = (l_1, \dots, l_{i-1}, l_{i+1} + 1, l_i - 1, l_{i+2}, \dots, l_n)$. The sijection σ can then be defined as

$$\sigma_{\mathbf{a}, \mathbf{b}, i}(A, \mathbf{l}) = \begin{cases} (\pi_{1,i}(A), \mathbf{l}) & \text{if } \pi_{1,i}(A) \in \underline{\text{GT}}(\mathbf{l}) \\ (\pi_{1,i}(A), \mathbf{l}') & \text{if } \pi_{1,i}(A) \in \underline{\text{GT}}(\mathbf{l}') \end{cases}.$$

It is easy to check that this is a sijection. Compare with Example 4. \square

Problem 6. Given $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, construct a sijection

$$\tau = \tau_{\mathbf{k}, x}: \underline{\text{GT}}(k_1, \dots, k_n) \Rightarrow \bigsqcup_{i=1}^n \underline{\text{GT}}(k_1, \dots, k_{i-1}, x + n - i, k_{i+1}, \dots, k_n).$$

Construction. In Problem 3, we constructed a normal sijection

$$\begin{aligned}
& \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]} \Rightarrow \bigsqcup_{i=1}^n \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, x + n - i]} \times \underline{[x + n - i, k_{i+1}]} \times \cdots \times \underline{[k_{n-1}, k_n]} \\
& \sqcup \bigsqcup_{i=1}^{n-2} \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1} + 1, x + n - i - 1]} \times \underline{[k_{i+1}, x + n - i - 2]} \times \underline{[k_{i+2}, k_{i+3}]} \times \cdots,
\end{aligned}$$

which gives a sijection

$$\begin{aligned}
& \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(\mathbf{l}) \Rightarrow \bigsqcup_{i=1}^n \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, x+n-i]} \times \underline{[x+n-i, k_{i+1}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(\mathbf{l}) \\
& \sqcup \bigsqcup_{i=1}^{n-2} \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, k_i]} \times \underline{[k_{i+1}+1, x+n-i-1]} \times \underline{[k_{i+1}, x+n-i-2]} \times \underline{[k_{i+2}, k_{i+3}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(\mathbf{l}).
\end{aligned}$$

All disjoint unions in the second term satisfy the conditions for the existence of σ from Problem 5, so we can sject them to \emptyset . This gives a sijection

$$\bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(\mathbf{l}) \Rightarrow \bigsqcup_{i=1}^n \bigsqcup_{\mathbf{l} \in \underline{[k_1, k_2]} \times \cdots \times \underline{[k_{i-1}, x+n-i]} \times \underline{[x+n-i, k_{i+1}]} \times \cdots \times \underline{[k_{n-1}, k_n]}} \underline{\text{GT}}(\mathbf{l}),$$

which is, by the definition of $\underline{\text{GT}}$, a sijection $\underline{\text{GT}}(k_1, \dots, k_n) \Rightarrow \bigsqcup_{i=1}^n \underline{\text{GT}}(k_1, \dots, k_{i-1}, x + n - i, k_{i+1}, \dots, k_n)$. \square

5. COMBINATORICS OF THE MONOTONE TRIANGLE RECURSION

Monotone triangles. Suppose that $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{l} = (l_1, \dots, l_{n-1})$ are two sequences of integers. We say that \mathbf{l} *interlaces* \mathbf{k} , $\mathbf{l} < \mathbf{k}$, if the following holds:

- (1) for every i , $1 \leq i \leq n - 1$, l_i is in the closed interval between k_i and k_{i+1} ;
- (2) if $k_{i-1} \leq k_i \leq k_{i+1}$ for some i , $2 \leq i \leq n - 1$, then l_{i-1} and l_i cannot both be k_i ;
- (3) if $k_i > l_i = k_{i+1}$, then $i \leq n - 2$ and $l_{i+1} = l_i = k_{i+1}$;
- (4) if $k_i = l_i > k_{i+1}$, then $i \geq 2$ and $l_{i-1} = l_i = k_i$.

For example, if $k_1 < k_2 < \dots < k_n$, then $l_i \in [k_i, k_{i+1}]$ and $l_1 < l_2 < \dots < l_{n-1}$.

A *monotone triangle of size n* is a map $T: \{(i, j): 1 \leq j \leq i \leq n\} \rightarrow \mathbb{Z}$ so that line $i - 1$ (i.e. the sequence $T_{i-1,1}, \dots, T_{i-1,i-1}$) interlaces line i (i.e. the sequence $T_{i,1}, \dots, T_{i,i}$).

Example 6. *The following is a monotone triangles of size 5:*

$$\begin{array}{cccccc}
 & & & & & 4 \\
 & & & & & 3 & 5 \\
 & & & & & 3 & 4 & 5 \\
 & & & & & 3 & 3 & 4 & 5 \\
 & & & & & 5 & 3 & 1 & 4 & 6
 \end{array}$$

This notion of (generalized) monotone triangle was introduced in [Rie13]. Other notions appeared in [Fis12].

The *sign* of a monotone triangle T is $(-1)^r$, where r is the sum of:

- the number of strict descents in the rows of T , i.e. the number of pairs (i, j) so that $1 \leq j < i \leq n$ and $T_{i,j} > T_{i,j+1}$, and
- the number of (i, j) so that $1 \leq j \leq i - 2$, $i \leq n$ and $T_{i,j} > T_{i-1,j} = T_{i,j+1} = T_{i-1,j+1} > T_{i,j+2}$.

The sign of our example is -1 .

We denote the signed set of all monotone triangles with bottom row \mathbf{k} by $\underline{\text{MT}}(\mathbf{k})$.

It turns out that $\underline{\text{MT}}(\mathbf{k})$ satisfies a recursive “identity”. Let us define the signed set of *arrow rows of order n* as

$$\underline{\text{AR}}_n = (\{\nearrow, \nwarrow\}, \{\not\leftrightarrow\})^n.$$

Alternatively, we can think of them as rows of length n with elements $\nwarrow, \nearrow, \not\leftrightarrow$, where the positive elements are precisely those with an even number of $\not\leftrightarrow$'s.

The role of an arrow row μ of order n is that it induces a deformation of $\underline{[k_1, k_2]} \times \underline{[k_2, k_3]} \times \dots \times \underline{[k_{n-1}, k_n]}$ as follows. Consider

$$\begin{array}{ccccccc}
 \underline{[k_1, k_2]} & \underline{[k_2, k_3]} & \dots & \dots & \underline{[k_{n-2}, k_{n-1}]} & \underline{[k_{n-1}, k_n]} \\
 \mu_1 & \mu_2 & \mu_3 & \dots & \mu_{n-1} & \mu_n,
 \end{array}$$

and if $\mu_i \in \{\nwarrow, \not\leftrightarrow\}$ (that is we have an arrow pointing towards $\underline{[k_{i-1}, k_i]}$) then k_i is decreased by 1 in $\underline{[k_{i-1}, k_i]}$, while there is no change for this k_i if $\mu_i = \nearrow$. If $\mu_i \in \{\nearrow, \not\leftrightarrow\}$ (that is we have an arrow pointing towards $\underline{[k_i, k_{i+1}]}$) then k_i is increased by 1 in $\underline{[k_i, k_{i+1}]}$, while there is no change for this k_i if $\mu_i = \nwarrow$.

For a more formal description, we let $\delta_{\varkappa}(\varkappa) = \delta_{\varkappa}(\varnothing) = \delta_{\nearrow}(\nearrow) = \delta_{\nearrow}(\varnothing) = 1$ and $\delta_{\varkappa}(\nearrow) = \delta_{\nearrow}(\varkappa) = 0$, and we define

$$e(\mathbf{k}, \mu) = [k_1 + \delta_{\nearrow}(\mu_1), k_2 - \delta_{\varkappa}(\mu_2)] \times \dots \times [k_{n-1} + \delta_{\nearrow}(\mu_{n-1}), k_n - \delta_{\varkappa}(\mu_n)].$$

for $\mathbf{k} = (k_1, \dots, k_n)$ and $\mu \in \underline{\mathbb{A}\mathbb{R}}_n$.

Problem 7. Given $\mathbf{k} = (k_1, \dots, k_n)$, construct a sijection

$$\Xi = \Xi_{\mathbf{k}}: \underline{\mathbb{M}\mathbb{T}}(\mathbf{k}) \Rightarrow \bigsqcup_{\mu \in \underline{\mathbb{A}\mathbb{R}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k}, \mu)} \underline{\mathbb{M}\mathbb{T}}(\mathbf{l}).$$

Construction. All elements on the left are mapped to the right with Ξ , while there are quite a few cancellations on the right-hand side. More specifically, take a monotone triangle T with bottom row \mathbf{k} . Then $\Xi(T) = ((T', \mathbf{l}), \mu)$, where T' is the monotone triangle we obtain from T by deleting the last row, \mathbf{l} is the bottom row of T' , and $\mu = (\mu_1, \dots, \mu_n)$ is the arrow row defined as follows:

- $\mu_1 = \varkappa$;
- $\mu_n = \nearrow$;
- for $1 < i < n$, μ_i is determined as follows:
 - (1) if $k_{i-1} \leq l_{i-1} = k_i$, take $\mu_i = \nearrow$;
 - (2) if $k_{i-1} > l_{i-1} = k_i = l_i > k_{i+1}$, take $\mu_i = \varnothing$;
 - (3) otherwise, take $\mu_i = \varkappa$.

It is easy to check that \mathbf{l} is indeed in $e(\mathbf{k}, \mu)$. Note that in (1) and (2) of the third bullet point, μ_i is forced if we require $\mathbf{l} \in e(\mathbf{k}, \mu)$. In (3), $\mu_i = \varnothing$ would also be possible if and only if $\mu_i = \nearrow$ would also be possible.

On the other hand, for $((T', \mathbf{l}), \mu)$, define $\Xi((T', \mathbf{l}), \mu)$ as follows. For the construction it is useful to keep in mind that $\mathbf{l} \in e(\mathbf{k}, \mu)$ implies that conditions (1) and (2) for $\mathbf{l} < \mathbf{k}$ are satisfied.

- if $\mu_1 \neq \varkappa$, take $\Xi((T', \mathbf{l}), \mu) = ((T', \mathbf{l}), \mu')$, where we obtain μ' from μ by replacing \varnothing in position 1 by \nearrow and vice versa;
- if $\mu_1 = \varkappa$ and $\mu_n \neq \nearrow$, take $\Xi((T', \mathbf{l}), \mu) = ((T', \mathbf{l}), \mu')$, where we obtain μ' from μ by replacing \varnothing in position n by \varkappa and vice versa;
- if $\mu_1 = \varkappa$ and $\mu_n = \nearrow$, and $\mathbf{l} \not< \mathbf{k}$, find the smallest i between 2 and $n-1$ such that:
 - condition (3) of $\mathbf{l} < \mathbf{k}$ is not satisfied at i , i.e. $k_{i-1} > l_{i-1} = k_i \neq l_i$ (which implies $\mu_i \in \{\varkappa, \varnothing\}$), or
 - condition (4) of $\mathbf{l} < \mathbf{k}$ is not satisfied at i , i.e. $l_{i-1} \neq k_i = l_i > k_{i+1}$ (which implies $\mu_i \in \{\nearrow, \varnothing\}$).

Then take $\Xi((T', \mathbf{l}), \mu) = ((T', \mathbf{l}), \mu')$, where we obtain μ' from μ by replacing \varnothing in position i by \varkappa and vice versa in the first case, and replacing \varnothing in position i by \nearrow and vice versa in the second case;

- if $\mu_1 = \varkappa$ and $\mu_n = \nearrow$, and $\mathbf{l} < \mathbf{k}$, find the smallest i for an instance of (3) of the third bullet point in the first paragraph of the proof with $\mu_i \neq \varkappa$ (if such an i exists). Then take $\Xi((T', \mathbf{l}), \mu) = ((T', \mathbf{l}), \mu')$, where we obtain μ' from μ by replacing \varnothing in position i by \varkappa and vice versa.

If no such i exists, we take $\Xi((T', \mathbf{l}), \mu) = T$, where we obtain T from T' by adding \mathbf{k} as the last row. It is easy to see that this is a well-defined sijection. \square

Remark 7. The previous construction could have been avoided by using alternative extensions of monotone triangles provided in [Fis12]. However, the advantage of the definition used in

this paper is that it is more reduced than the others in the sense that it can be obtained from these by cancelling elements using certain sign-reversing involutions.

Arrow patterns and shifted GT patterns. Define the signed set of arrow patterns of order n as

$$\underline{\text{AP}}_n = (\{\swarrow, \searrow\}, \{\bowtie\})^{(n)}.$$

Alternatively, we can think of an arrow pattern of order n as a triangular array $T = (t_{p,q})_{1 \leq p < q \leq n}$ arranged as

$$T = \begin{array}{cccccccc} & & & & t_{1,n} & & & \\ & & & & t_{1,n-1} & t_{2,n} & & \\ & & & t_{1,n-2} & \dots & t_{2,n-1} & t_{3,n} & \\ & & \dots & \vdots & \dots & \vdots & \dots & \\ t_{1,2} & \dots & t_{2,3} & \dots & \dots & \dots & \dots & t_{n-1,n} \end{array},$$

with $t_{p,q} \in \{\swarrow, \searrow, \bowtie\}$, and the sign of an arrow pattern is 1 if the number of \bowtie 's is even and -1 otherwise.

The role of an arrow pattern of order n is that it induces a deformation of (k_1, \dots, k_n) , which can be thought of as follows. Add k_1, \dots, k_n as bottom row of T (i.e., $t_{i,i} = k_i$), and for each \swarrow or \bowtie which is in the same \swarrow -diagonal as k_i add 1 to k_i , while for each \searrow or \bowtie which is in the same \searrow -diagonal as k_i subtract 1 from k_i . More formally, letting $\delta_{\swarrow}(\swarrow) = \delta_{\swarrow}(\bowtie) = \delta_{\searrow}(\searrow) = \delta_{\searrow}(\bowtie) = 1$ and $\delta_{\swarrow}(\searrow) = \delta_{\searrow}(\swarrow) = 0$, we set

$$c_i(T) = \sum_{j=i+1}^n \delta_{\swarrow}(t_{i,j}) - \sum_{j=1}^{i-1} \delta_{\searrow}(t_{j,i}) \text{ and } d(\mathbf{k}, T) = (k_1 + c_1(T), k_2 + c_2(T), \dots, k_n + c_n(T))$$

for $\mathbf{k} = (k_1, \dots, k_n)$ and $T \in \underline{\text{AP}}_n$.

For $\mathbf{k} = (k_1, \dots, k_n)$ define *shifted Gelfand-Tsetlin patterns*, or SGT patterns for short, as the following disjoint union of GT patterns over arrow patterns of order n :

$$\underline{\text{SGT}}(\mathbf{k}) = \bigsqcup_{T \in \underline{\text{AP}}_n} \underline{\text{GT}}(d(\mathbf{k}, T))$$

Considering that $|(\{\swarrow, \searrow\}, \{\bowtie\})| = 1$ and therefore $|\underline{\text{AP}}_n| = 1$, the following is not surprising.

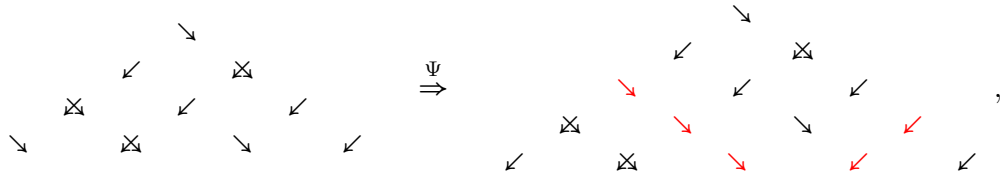
Problem 8. Given n and i , $1 \leq i \leq n$, construct a bijection

$$\Psi = \Psi_{n,i}: \underline{\text{AP}}_{n-1} \Rightarrow \underline{\text{AP}}_n.$$

Construction. For $T \in \underline{\text{AP}}_{n-1}$, take $\Psi(T) = (t'_{p,q})_{1 \leq p < q \leq n}$ to be the arrow pattern defined by

$$t'_{p,q} = \begin{cases} t_{p,q} & \text{if } p < q < i \\ t_{p,q-1} & \text{if } p < i < q \\ t_{p-1,q-1} & \text{if } i < p < q \\ \searrow & \text{if } p < q = i \\ \swarrow & \text{if } i = p < q \end{cases}.$$

An example for $n = 6$ and $i = 4$ is



where the new arrows are indicated in red. If $T \in \underline{\text{AP}}_n$, $t_{p,i} = \searrow$ for $p = 1, \dots, i-1$, $t_{i,q} = \swarrow$ for $q = i+1, \dots, n$, take $\Psi(T) = (t'_{p,q})_{1 \leq p < q \leq n-1}$, where

$$t'_{p,q} = \begin{cases} t_{p,q} & \text{if } p < q < i \\ t_{p,q+1} & \text{if } p < i \leq q \\ t_{p+1,q+1} & \text{if } i \leq p < q \end{cases}.$$

Otherwise, there either exists p so that $t_{p,i} \neq \searrow$, or there exists q so that $t_{i,q} \neq \swarrow$. In the first case, define $\Psi(T) = (t'_{p,q})_{1 \leq p < q \leq n}$, where $t'_{p,i} = \swarrow$ if $t_{p,i} = \bowtie$ and $t'_{p,i} = \bowtie$ if $t_{p,i} = \swarrow$, and all other array elements are equal. In the second case, define $\Psi(T) = (t'_{p,q})_{1 \leq p < q \leq n}$, where $t'_{i,q} = \searrow$ if $t_{i,q} = \bowtie$ and $t'_{i,q} = \bowtie$ if $t_{i,q} = \searrow$, and all other array elements are equal. It is easy to see that this is a sijection. \square

The difficult part of this paper is to prove that $\underline{\text{SGT}}$ satisfies the same ‘‘recursion’’ as $\underline{\text{MT}}$. While the proof of the recursion was easy for monotone triangles, it is very involved for shifted GT patterns, and needs almost all the sijections we have constructed in this and previous sections.

Problem 9. *Given $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, construct a sijection*

$$\Phi = \Phi_{\mathbf{k},x}: \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k},\mu)} \underline{\text{SGT}}(\mathbf{l}) \Rightarrow \underline{\text{SGT}}(\mathbf{k}).$$

Construction. To make the construction of Φ a little easier, we will define it as the composition of several sijections. The first one will reduce the indexing sets (from a signed box to its ‘‘corners’’) using Problem 4. The second one increases the order of the arrow patterns using the sijection from Problem 8. The third one further reduces the indexing set (from a signed set with 2^{n-1} elements to $[1, n]$). The last one gets rid of the arrow row and then uses Problem 6.

For $\mu \in \underline{\text{AR}}_n$, define $\underline{S}_i = (\{k_i + \delta_{\swarrow}(\mu_i)\}, \emptyset) \sqcup (\emptyset, \{k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + 1\})$. Then Φ is the composition of sijections

$$\begin{aligned} & \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k},\mu)} \underline{\text{SGT}}(\mathbf{l}) \\ & \xrightarrow{\Phi_1} \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \bigsqcup_{\mathbf{m} \in \underline{S}_1 \times \dots \times \underline{S}_{n-1}} \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x) \\ & \xrightarrow{\Phi_2} \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_n} \bigsqcup_{\mathbf{m} \in \underline{S}_1 \times \dots \times \underline{S}_{n-1}} \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x) \\ & \xrightarrow{\Phi_3} \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_n} \bigsqcup_{i=1}^n \underline{\text{GT}}(\dots, k_{i-1} + \delta_{\swarrow}(\mu_{i-1}) + c_{i-1}(T), x + n - i, k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + c_i(T), \dots) \\ & \xrightarrow{\Phi_4} \underline{\text{SGT}}(\mathbf{k}), \end{aligned}$$

where Φ_1 , Φ_2 , Φ_3 , and Φ_4 are constructed as follows.

Construction of Φ_1 . By definition of $\underline{\text{SGT}}$, we have

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k},\mu)} \underline{\text{SGT}}(\mathbf{l}) = \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k},\mu)} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \underline{\text{GT}}(d(\mathbf{l}, T)).$$

By switching the inner disjoint unions, we get a sijection to

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \bigsqcup_{\mathbf{l} \in e(\mathbf{k},\mu)} \underline{\text{GT}}(d(\mathbf{l}, T)).$$

There is an obvious sijection from this signed set to

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \bigsqcup_{\mathbf{l} \in d(\epsilon(\mathbf{k}, \mu), T)} \underline{\text{GT}}(\mathbf{l}),$$

by abuse of notation setting

$$d(\underline{[x_1, y_1]} \times \dots \times \underline{[x_{n-1}, y_{n-1}]}, T) = \underline{[x_1 + c_1(T), y_1 + c_1(T)]} \times \dots \times \underline{[x_{n-1} + c_{n-1}(T), y_{n-1} + c_{n-1}(T)]}.$$

Now for each μ and T , use the map ρ from Problem 4 for $a_i = k_i + \delta_{\nearrow}(\mu_i) + c_i(T)$, $b_i = k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + c_i(T)$, and x . We get a sijection to

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \bigsqcup_{\mathbf{m} \in \underline{S}'_1 \times \dots \times \underline{S}'_{n-1}} \underline{\text{GT}}(m_1, \dots, m_{n-1}, x),$$

where $\underline{S}'_i = (\{k_i + \delta_{\nearrow}(\mu_i) + c_i(T)\}, \emptyset) \sqcup (\emptyset, \{k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + c_i(T) + 1\})$. Finally, there is an obvious sijection from this signed set to

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_{n-1}} \bigsqcup_{\mathbf{m} \in \underline{S}_1 \times \dots \times \underline{S}_{n-1}} \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x).$$

Construction of Φ_2 . In Problem 8, we constructed sijections $\Psi_{n,i}: \underline{\text{AP}}_{n-1} \Rightarrow \underline{\text{AP}}_n$. We construct Φ_2 by using Proposition 2 (3) for $\psi = \Psi_{n,n}$, $\underline{T} = \underline{\text{AP}}_{n-1}$, $\underline{\tilde{T}} = \underline{\text{AP}}_n$,

$$\underline{S}_T = \bigsqcup_{\mathbf{m} \in \underline{S}_1 \times \dots \times \underline{S}_{n-1}} \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x) \quad \text{for } T \in \underline{\text{AP}}_{n-1} \sqcup \underline{\text{AP}}_n$$

and $\varphi_T = \underline{\text{id}}$. This is well defined because $c_i(T) = c_i(\Psi_{n,n}(T))$ for $T \in \underline{\text{AP}}_{n-1} \sqcup \underline{\text{AP}}_n$ and $i = 1, \dots, n-1$.

Construction of Φ_3 . Let η be the involution that maps $\swarrow \leftrightarrow \nearrow$, $\searrow \leftrightarrow \nwarrow$, $\bowtie \leftrightarrow \boxtimes$. The elements of the signed set $\underline{S} = \underline{S}_1 \times \dots \times \underline{S}_{n-1}$ are $(n-1)$ -tuples of elements that are either $(k_i + \delta_{\nearrow}(\mu_i), 0)$ or $(k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + 1, 1)$. Define \underline{S}' as the subset of \underline{S} containing tuples of the form $(\dots, (m_i, 1), (m_{i+1}, 0), \dots)$, i.e. the ones where we choose $k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + 1$ in position i and $k_{i+1} + \delta_{\nearrow}(\mu_{i+1})$ in position $i+1$ for some i . Then we can define a sijection

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_n} \bigsqcup_{\mathbf{m} \in \underline{S}'} \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x) \Rightarrow \underline{\emptyset}$$

as follows: given $\mu \in \underline{\text{AR}}_n$, $T \in \underline{\text{AP}}_n$, $\mathbf{m} = (\dots, k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + 1, k_{i+1} + \delta_{\nearrow}(\mu_{i+1}), \dots)$ (and i is the smallest index where this happens), $A \in \underline{\text{GT}}(m_1 + c_1(T), \dots, m_{n-1} + c_{n-1}(T), x)$, map $((A, \mathbf{m}), T, \mu)$ to $((A', \mathbf{m}'), T', \mu')$, where:

- $A' = \pi_{i,n}(A)$;
- T' is T if A' has the same bottom row as A ; otherwise, T' is obtained from T by interchanging $t_{i,j}$ and $t_{i+1,j}$ for $j > i+1$ as well as $t_{j,i}$ and $t_{j,i+1}$ for $j < i$, and setting $t'_{i,i+1} = \eta(\mu_{i+1})$;
- μ' is μ if A' has the same bottom row as A ; otherwise, μ' is obtained from μ by replacing μ_{i+1} with $\eta(t_{i,i+1})$;
- $\mathbf{m}' = (\dots, k_{i+1} - \delta_{\searrow}(\mu'_{i+1}) + 1, k_{i+1} + \delta_{\nearrow}(\mu'_{i+1}), \dots)$.

What remains is

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{T \in \underline{\text{AP}}_n} \bigsqcup_{i=1}^n (-1)^{n-i} \underline{\text{GT}}(\dots, k_{i-1} + \delta_{\nearrow}(\mu_{i-1}) + c_{i-1}(T), k_{i+1} - \delta_{\searrow}(\mu_{i+1}) + c_i(T) + 1, \dots, x),$$

and we can now apply $\pi_{i,n} \circ \pi_{i,n} \circ \dots \circ \pi_{n-1,n}$ to obtain what is claimed.

Construction of Φ_4 . By switching the order in which we do disjoint unions, we arrive at

$$\bigsqcup_{T \in \underline{\text{AP}}_n} \bigsqcup_{i=1}^n \bigsqcup_{\mu \in \underline{\text{AR}}_n} \underline{\text{GT}}(\dots, k_{i-1} + \delta_{\nearrow}(\mu_{i-1}) + c_{i-1}(T), x + n - i, k_{i+1} - \delta_{\nwarrow}(\mu_{i+1}) + c_i(T), \dots).$$

Let us define a sijection $\Lambda_{n,i}: \underline{\text{AR}}_n \Rightarrow (\{\cdot\}, \{\cdot\})$. For $\mu' = (\nwarrow, \dots, \nwarrow, \nearrow, \dots, \nearrow)$ (with i \nwarrow 's), take $\Lambda_{n,i}(\mu') = \cdot$ and $\Lambda_{n,i}(\cdot) = \mu'$. For every other μ , take the smallest p so that $\mu_p \neq \mu'_p$. If $p \leq i$, replace \nearrow with \nwarrow and vice versa in position p to get $\Lambda_{n,i}(\mu)$ from μ , and if $p > i$, replace \nwarrow with \nearrow and vice versa in position p to get $\Lambda_{n,i}(\mu)$ from μ . If $\mu \neq \mu'$, $(\delta_{\nearrow}(\mu_1), \dots, \delta_{\nearrow}(\mu_{i-1}), \delta_{\nwarrow}(\mu_{i+1}), \dots, \delta_{\nwarrow}(\mu_n))$ are unaffected by this sijection, so it induces a sijection

$$\bigsqcup_{\mu \in \underline{\text{AR}}_n} \underline{\text{GT}}(\dots, k_{i-1} + \delta_{\nearrow}(\mu_{i-1}) + c_{i-1}(T), x + n - i, k_{i+1} - \delta_{\nwarrow}(\mu_{i+1}) + c_i(T), \dots) \\ \Rightarrow \underline{\text{GT}}(\dots, k_{i-1} + c_{i-1}(T), x + n - i, k_{i+1} + c_i(T), \dots).$$

We switch disjoint unions again, and we get

$$\bigsqcup_{i=1}^n \bigsqcup_{T \in \underline{\text{AP}}_n} \underline{\text{GT}}(k_1 + c_1(T), \dots, k_{i-1} + c_{i-1}(T), x + n - i, k_{i+1} + c_i(T), \dots, k_n + c_{n-1}(T)).$$

For chosen i , use Proposition 2 (3) for $\psi = \Psi_{n,i} \circ \Psi_{n,n}^{-1}$ and $\varphi_t = \text{id}$. We get a sijection to

$$\bigsqcup_{i=1}^n \bigsqcup_{T \in \underline{\text{AP}}_n} \underline{\text{GT}}(k_1 + c_1(T), \dots, k_{i-1} + c_{i-1}(T), x + n - i, k_{i+1} + c_{i+1}(T), \dots, k_n + c_n(T)).$$

If we switch disjoint unions one last time, we can use the sijection τ^{-1} (see Problem 6), and we get

$$\bigsqcup_{T \in \underline{\text{AP}}_n} \underline{\text{GT}}(d(\mathbf{k}, T)) = \underline{\text{SGT}}(\mathbf{k}).$$

This completes the construction of Φ_4 and therefore of Φ . \square

Problem 10. Given $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $x \in \mathbb{Z}$, construct a sijection

$$\Gamma = \Gamma_{\mathbf{k},x}: \underline{\text{MT}}(\mathbf{k}) \Rightarrow \underline{\text{SGT}}(\mathbf{k}).$$

Construction. The proof is by induction on n . For $n = 1$, both sides consist of one (positive) element, and the sijection is obvious. Once we have constructed Γ for all lists of length less than n , we can construct $\Gamma_{\mathbf{k},x}$ as the composition of sijections

$$\underline{\text{MT}}(\mathbf{k}) \xrightarrow{\Xi_{\mathbf{k}}} \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k}, \mu)} \underline{\text{MT}}(\mathbf{l}) \xrightarrow{\sqcup \sqcup \Gamma} \bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k}, \mu)} \underline{\text{SGT}}(\mathbf{l}) \xrightarrow{\Phi_{\mathbf{k},x}} \underline{\text{SGT}}(\mathbf{k}),$$

where $\sqcup \sqcup \Gamma$ means $\bigsqcup_{\mu \in \underline{\text{AR}}_n} \bigsqcup_{\mathbf{l} \in e(\mathbf{k}, \mu)} \Gamma_{\mathbf{l},x}$. \square

Running the code shows that the main sijection Γ indeed depends on the choice x . As an example, take $\mathbf{k} = (1, 2, 3)$. In this case, $\underline{\text{MT}}(\mathbf{k})$ has 7 positive elements, and $\underline{\text{SGT}}(\mathbf{k})$ has 10 positive and 3 negative elements. For $x = 0$, the sijection is given by

$$\begin{array}{ccc} \begin{array}{l} 1 \\ 12 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 1 \\ 11 \\ 111 \end{array}, \begin{array}{c} \searrow \\ \searrow \searrow \end{array} \right) \end{array} & \begin{array}{l} 2 \\ 12 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 2 \\ 12 \\ 122 \end{array}, \begin{array}{c} \searrow \\ \searrow \swarrow \end{array} \right) \end{array} & \begin{array}{l} 1 \\ 13 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 1 \\ 12 \\ 122 \end{array}, \begin{array}{c} \searrow \\ \searrow \swarrow \end{array} \right) \end{array} \\ \begin{array}{l} 2 \\ 13 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 2 \\ 23 \\ 223 \end{array}, \begin{array}{c} \swarrow \\ \searrow \swarrow \end{array} \right) \end{array} & \begin{array}{l} 3 \\ 13 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 3 \\ 23 \\ 223 \end{array}, \begin{array}{c} \swarrow \\ \searrow \swarrow \end{array} \right) \end{array} & \begin{array}{l} 2 \\ 23 \\ 123 \end{array} \leftrightarrow \begin{array}{l} \left(\begin{array}{c} 2 \\ 22 \\ 312 \end{array}, \begin{array}{c} \swarrow \\ \nwarrow \searrow \end{array} \right) \end{array} \end{array}$$

$$\begin{array}{c} 3 \\ 23 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 3 \\ 33 \\ 333 \end{array}, \begin{array}{l} \swarrow \\ \swarrow \swarrow \end{array} \right) \quad \left(\begin{array}{c} 2 \\ 22 \\ 223 \end{array}, \begin{array}{l} \swarrow \\ \searrow \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \boxtimes \\ \searrow \swarrow \end{array} \right)$$

$$\left(\begin{array}{c} 2 \\ 22 \\ 231 \end{array}, \begin{array}{l} \searrow \boxtimes \\ \swarrow \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \swarrow \swarrow \\ \searrow \boxtimes \end{array} \right) \quad \left(\begin{array}{c} 2 \\ 22 \\ 122 \end{array}, \begin{array}{l} \searrow \swarrow \\ \searrow \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \searrow \swarrow \\ \boxtimes \swarrow \end{array} \right)$$

while for $x = 1$, it is given by

$$\begin{array}{c} 1 \\ 12 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 1 \\ 11 \\ 111 \end{array}, \begin{array}{l} \searrow \\ \searrow \searrow \end{array} \right) \quad \begin{array}{c} 2 \\ 12 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 223 \end{array}, \begin{array}{l} \swarrow \\ \searrow \swarrow \end{array} \right) \quad \begin{array}{c} 1 \\ 13 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 1 \\ 12 \\ 122 \end{array}, \begin{array}{l} \searrow \\ \searrow \swarrow \end{array} \right)$$

$$\begin{array}{c} 2 \\ 13 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 23 \\ 223 \end{array}, \begin{array}{l} \swarrow \\ \searrow \swarrow \end{array} \right) \quad \begin{array}{c} 3 \\ 13 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 3 \\ 23 \\ 223 \end{array}, \begin{array}{l} \swarrow \\ \searrow \swarrow \end{array} \right) \quad \begin{array}{c} 2 \\ 23 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 122 \end{array}, \begin{array}{l} \searrow \\ \searrow \swarrow \end{array} \right)$$

$$\begin{array}{c} 3 \\ 23 \\ 123 \end{array} \leftrightarrow \left(\begin{array}{c} 3 \\ 33 \\ 333 \end{array}, \begin{array}{l} \swarrow \\ \swarrow \swarrow \end{array} \right) \quad \left(\begin{array}{c} 2 \\ 12 \\ 122 \end{array}, \begin{array}{l} \searrow \\ \searrow \swarrow \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \boxtimes \\ \searrow \swarrow \end{array} \right)$$

$$\left(\begin{array}{c} 2 \\ 22 \\ 312 \end{array}, \begin{array}{l} \swarrow \swarrow \\ \searrow \boxtimes \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \swarrow \swarrow \\ \searrow \boxtimes \end{array} \right) \quad \left(\begin{array}{c} 2 \\ 22 \\ 231 \end{array}, \begin{array}{l} \searrow \swarrow \\ \swarrow \boxtimes \end{array} \right) \leftrightarrow \left(\begin{array}{c} 2 \\ 22 \\ 222 \end{array}, \begin{array}{l} \searrow \swarrow \\ \boxtimes \swarrow \end{array} \right)$$

6. CONCLUDING REMARKS

Future work. In this article, we have presented the first bijective proof of the operator formula. The operator formula is the main tool for non-combinatorial proofs of several results where alternating sign matrix objects are related to plane partition objects, or simply for showing that $n \times n$ ASMs are enumerated by (1).

- The operator formula was used in [Fis07] to show that $n \times n$ ASMs are counted by (1) and, more generally, to count ASMs with respect to the position of the unique 1 in the top row.
- While working on this project, we actually realized that the final calculation in [Fis07] also implies that ASMs are equinumerous with DPPs without having to use Andrews' result [And79] on the number of DPPs; more generally, we can even obtain the equivalence of the refined count of $n \times n$ ASMs with respect to the position of the unique 1 in the top row and the refined count of DPPs with parts no greater than n with respect to the number of parts equal to n . This was conjectured in [MRR83] and first proved in [BDFZJ12].
- In [Fis19b], the operator formula was used to show that ASTs with n rows are equinumerous with TSSCPPs in a $2n \times 2n \times 2n$ -box. Again we do not rely on Andrews' result [And94] on the number of TSSCPPs and we were actually able to deal with a refined count again (which has also the same distribution as the position of the unique 1 in the top row of an ASM).
- In [Fis19a], we have considered alternating sign trapezoids (which generalize ASTs) and, using the operator formula, we have shown that they are equinumerous with objects generalizing DPPs. These objects were already known to Andrews and he actually enumerated them in [And79]. Later Krattenthaler [Kra06] realized that

these more general objects are (almost trivially) equivalent to cyclically symmetric lozenge tilings of a hexagon with a triangular hole in the center. Again we do not rely on Andrews' enumeration of these generalized DPPs, and in this case we were able to include three statistics.

We plan to work on converting the proofs just mentioned into bijective proofs. For those mentioned in the first and second bullet point, this has already been worked out. The attentive reader will have noticed that working out all of them will link all four known classes of objects that are enumerated by (1).

Computer code. As mentioned before, we consider computer code for the constructed sijections an essential part of this project. The code (in python) is available at <https://www.fmf.uni-lj.si/~konvalinka/asmcode.html>. All the constructed sijections are quite efficient. If run with pypy, checking that $\Gamma_{(1,2,3,4,5),0}$ is a sijection between $\underline{MT}(1, 2, 3, 4, 5)$ (with 429 positive elements and no negative elements) and $\underline{SGT}(1, 2, 3, 4, 5)$ (with 18913 positive elements and 18484 negative elements) takes less than a minute. Of course, the sets involved can be huge, so checking that $\Gamma_{(1,2,3,4,5,6),0}$ is a sijection between $\underline{MT}(1, 2, 3, 4, 5, 6)$ (with 7436 positive elements and no negative elements) and $\underline{SGT}(1, 2, 3, 4, 5, 6)$ (with 11167588 positive elements and 11160152 negative elements) took almost 20 hours.

REFERENCES

- [ABF16] A. Ayer, R. Behrend, and I. Fischer. Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order, 2016. preprint.
- [And79] G. Andrews. Plane partitions. III. The weak Macdonald conjecture. *Invent. Math.*, 53(3):193–225, 1979.
- [And86] George E. Andrews. *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, volume 66 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.
- [And94] G. Andrews. Plane partitions. V. The TSSCPP conjecture. *J. Combin. Theory Ser. A*, 66(1):28–39, 1994.
- [Bau] Andrej Bauer. What is an explicit bijection? lecture available at http://videlectures.net/FPSAC2019_bauer_explicit_bijection/.
- [BC16] Philippe Biane and Hayat Cheballah. Gog and GOGam pentagons. *J. Combin. Theory Ser. A*, 138:133–154, 2016.
- [BDFZJ12] R. Behrend, P. Di Francesco, and P. Zinn-Justin. On the weighted enumeration of alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 119(2):331–363, 2012.
- [BDFZJ13] R. Behrend, P. Di Francesco, and P. Zinn-Justin. A doubly-refined enumeration of alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 120(2):409–432, 2013.
- [BFK17] R. Behrend, I. Fischer, and M. Konvalinka. Diagonally and antidiagonally symmetric alternating sign matrices of odd order. *Adv. Math.*, 315:324–365, 2017.
- [BP99] D. Bressoud and J. Propp. How the alternating sign matrix conjecture was solved. *Notices Amer. Math. Soc.*, 46(6):637–646, 1999.
- [Bre99] D. Bressoud. *Proofs and confirmations. The story of the alternating sign matrix conjecture*. MAA Spectrum. Mathematical Association of America and Cambridge University Press, Washington, DC and Cambridge, 1999.
- [CS11] Luigi Cantini and Andrea Sportiello. Proof of the Razumov-Stroganov conjecture. *J. Combin. Theory Ser. A*, 118(5):1549–1574, 2011.
- [EKLP92a] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. Alternating-sign matrices and domino tilings. I. *J. Algebraic Combin.*, 1(2):111–132, 1992.
- [EKLP92b] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp. Alternating-sign matrices and domino tilings. II. *J. Algebraic Combin.*, 1(3):219–234, 1992.

- [Fis05] I. Fischer. A method for proving polynomial enumeration formulas. *J. Combin. Theory Ser. A*, 111:37–58, 2005.
- [Fis06] I. Fischer. The number of monotone triangles with prescribed bottom row. *Adv. in Appl. Math.*, 37(2):249–267, 2006.
- [Fis07] I. Fischer. A new proof of the refined alternating sign matrix theorem. *J. Combin. Theory Ser. A*, 114(2):253–264, 2007.
- [Fis10] I. Fischer. The operator formula for monotone triangles—simplified proof and three generalizations. *J. Combin. Theory Ser. A*, 117(8):1143–1157, 2010.
- [Fis12] I. Fischer. Sequences of labeled trees related to gelfand-tsetlin patterns. *Adv. in Appl. Math.*, 49:165–195, 2012.
- [Fis16] I. Fischer. Short proof of the asm theorem avoiding the six-vertex model. *J. Combin. Theory Ser. A*, 144:139–156, 2016.
- [Fis19a] I. Fischer. A constant term approach to enumerating alternating sign trapezoids. *Adv. Math.*, 356, 2019.
- [Fis19b] I. Fischer. Enumeration of alternating sign triangles using a constant term approach. *Trans. Amer. Math. Soc.*, 372:1485–1508, 2019.
- [FZJ08] T. Fonseca and P. Zinn-Justin. On the doubly refined enumeration of alternating sign matrices and totally symmetric self-complementary plane partitions. *Electron. J. Combin.*, 15:Research Paper 81, 35 pp. (electronic), 2008.
- [GC50] I. M. Gel’fand and M. L. Cetlin. Finite-dimensional representations of the group of unimodular matrices. *Doklady Akad. Nauk SSSR (N.S.)*, 71:825–828, 1950.
- [GM81] A. M. Garsia and S. C. Milne. A Rogers-Ramanujan bijection. *J. Combin. Theory Ser. A*, 31(3):289–339, 1981.
- [Kra96] C. Krattenthaler. A gog-magog conjecture, 1996.
- [Kra06] C. Krattenthaler. Descending plane partitions and rhombus tilings of a hexagon with a triangular hole. *European J. Combin.*, 27(7):1138–1146, 2006.
- [Kra16] C. Krattenthaler. Plane partitions in the work of Richard Stanley and his school. In *The mathematical legacy of Richard P. Stanley*, pages 231–261. Amer. Math. Soc., Providence, RI, 2016.
- [Kup96] G. Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, (3):139–150, 1996.
- [MRR83] W. Mills, D. Robbins, and H. Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.
- [MRR86] W. Mills, D. Robbins, and H. Rumsey, Jr. Self-complementary totally symmetric plane partitions. *J. Combin. Theory Ser. A*, 42(2):277–292, 1986.
- [Oka06] S. Okada. Enumeration of symmetry classes of alternating sign matrices and characters of classical groups. *J. Algebraic Combin.*, 23(1):43–69, 2006.
- [Rie13] Lukas Riegler. Generalized monotone triangles: an extended combinatorial reciprocity theorem. In *25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013)*, Discrete Math. Theor. Comput. Sci. Proc., AS, pages 647–658. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2013.
- [RR86] David P. Robbins and Howard Rumsey, Jr. Determinants and alternating sign matrices. *Adv. in Math.*, 62(2):169–184, 1986.
- [Sta99] R. Stanley. *Enumerative combinatorics. Volume 2*. Cambridge Studies in Advanced Mathematics 62. Cambridge University Press, Cambridge, 1999.
- [Str04] Y. Stroganov. A new way to deal with izergin-korepin determinant at root of unity, 2004. preprint.
- [Voe] Vladimir Voevodsky. C-systems defined by universe categories: presheaves. arXiv = 1706.03620.
- [Wie00] Benjamin Wieland. A large dihedral symmetry of the set of alternating sign matrices. *Electron. J. Combin.*, 7:Research Paper 37, 13, 2000.
- [Zei96a] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Combin.*, 3(2):Research Paper 13, 84 pp. (electronic), 1996. The Foata Festschrift.
- [Zei96b] D. Zeilberger. Proof of the refined alternating sign matrix conjecture. *New York J. Math.*, 2:59–68 (electronic), 1996.

UNIVERSITY OF VIENNA, AUSTRIA

URL: <https://www.mat.univie.ac.at/~ifischer/>

FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, AND INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, SLOVENIA

URL: <http://www.fmf.uni-lj.si/~konvalinka/>