Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Non-commutative extensions of classical determinantal identities

Matjaž Konvalinka and Igor Pak

Abstract. We present several non-commutative extensions of MacMahon Master Theorem and Sylvester's Identity. The proofs are combinatorial and new even in the classical (commutative) cases.

Résumé. Nous présentons plusieurs extensions noncommutatives du Master théorème de MacMahon et de l'identité de Sylvester. Même dans les cas classiques (commutatifs), les démonstrations sont nouvelles et de nature combinatoire.

Introduction

Combinatorial Linear Algebra is a beautiful and underdeveloped part of Enumerative Combinatorics. The underlying idea is very simple: one takes a matrix identity and views it as an algebraic result over a (possibly non-commutative) ring. Once the identity is translated into the language of words, Lothaire style, an explicit bijection or an involution is employed to prove the result. The resulting combinatorial proofs are often insightful and lead to extensions and generalizations of the original identities, often in unexpected directions.

Now, it is not surprising that quantum linear algebra identities can also be established by combinatorial means. On the contrary, it is perhaps surprising that so little work has been done in this direction. Given the large body of q-results as well as (totally) non-commutative results, one would expect the quantum generalizations to play an important role in modern developments.

In this paper we establish a general framework of quantum and more general non-commutative generalizations of classical determinantal identities. We restrict ourselves to two identities: the *MacMahon Master Theorem* and the *Sylvester's determinant identity.* Both have been thoroughly studied and have a number of connections and applications to combinatorics and representation theory. In fact, both have been recently generalized to quantum matrices [**GLZ, KL**]. We find a far-reaching (q_{ij})-extensions of both results as well as a number of intermediate generalizations.

Our technique is based on explicit combinatorial arguments rather than algebra. We adopt the fundamental philosophy of *quasi-determinants* due to Gelfand and Retakh [**GR**] (see also [**G+**]) and restate the identities in the language of lattice paths (i.e. positive sums of certain words), by using the inverse matrix elements rather than determinant themselves. We then are able to prove bijectively the resulting equivalent versions of classical identities. These bijections are new in both cases and are of independent interest. We then show that the form of these identities and the structure of the bijections are such that they are easily amenable to advanced generalizations, with little change in the proof. In fact, the bijections themselves are *exactly the same*, but there is a fair amount of bookkeeping required to establish the refined results.

This extended abstract is constructed as follows. We start with the general algebraic framework and describe various classes of quantum matrices, quantum determinants, as well as some combinatorics of words. The main part of the paper is then split into two sections where we discuss MacMahon's Master Theorem and Sylvester's identity. Both parts proceed along parallel lines, but can be read independently. In each case we state the most general result and describe a bijection proving the classical result. We then briefly outline

2000 *Mathematics Subject Classification.* Primary 05A30; Secondary 15A15.
Key words and phrases. q-analogues, determinants, MacMahon Master Theorem, Sylvester's identity.

the necessary bookkeeping in each case, skipping most details for the lack of space. We conclude with final remarks and open problems.

This extended abstract is based on two papers [**K1, KP**] which contain complete proofs of all results, further extensions and applications. They will appear as separate publications.

Historical remarks. Let us give a brief overview of the wealth of previous results in the subject. This work being in the intersection of several fields, is strongly connected to several streams of recent developments on both algebraic and combinatorial side.

First, there is a great deal of the combinatorics of words approach to the linear algebra identities, their various extensions and applications. We refer to [**Z**] for an accessible introduction and basic references, and to [**L**] for an overview of the field.

Second, a tremendous body of literature exists on quantum groups and quantum linear algebra. Without going into history and technical details let us mention Manin's works [**M2, M1**] where the (q_{ij})-analogues were obtained. While this is not the last step in a long chain of generalizations, this version motivated our ultimate generalizations.

Third, the Gelfand-Retakh's non-commutative approach established a direction in which the identities can be generalized [**GR, G+**]. In particular, they showed that a large number of 'generalized determinants' are special cases of the general quasi-determinants they defined. A subsequent work [**ER**] further generalized this set of examples.

Now, specifically on MacMahon Master Theorem (MMT), the classical works of Cartier and Foata reproved the theorem by using the combinatorics of words [**CF, F1, F2**]. By doing so they explicitly extended it to what we call Cartier-Foata (partially commutative) matrices. Most recently, there has been a large number of extensions and generalizations. The turning point was the Garoufalidis-Lê-Zeilberger paper [**GLZ**] which proved a quantum analogue of the MMT. In fact, having restated the MMT in linear algebra form it opened a room for generalizations:

$$\text{(MMT)} \qquad \sum_{k=0}^{\infty} \operatorname{tr}(S^k A) \,=\, \frac{1}{\det(I-A)}.$$

In a series or papers [**FH1, FH2, FH3**], Foata and Han reproved the quantum MMT, found interesting further extensions and an important '$1 = q$' principle which allows easy algebraic proof of certain q-equation (implicitly based on the Gröbner bases of the underlying quadratic algebras). In a different direction, Hai and Lorenz established the quantum MMT by using the Koszul duality [**HL**], thus suggesting that MMT can be further extended to Koszul quadratic algebras with a large group of (quantum) symmetries. We refer to [**KP**], the basis of this abstract, for further references and details.

The Sylvester's determinant identity (SDI) has also been intensely studied, mostly in the algebraic rather than combinatorial context. The crucial step was made by Krob and Leclerc [**KL**] who found a quantum version of SDI. Since then, Molev found several far-reaching extensions of the SDI to Yangians, including other root systems [**Mo1, Mo2**] (see also [**HM**]). We refer to [**K1**] for further details and references.

1. Algebraic framework

1.1. Matrices and words. We work in the $\mathbb{C}$-algebra $\mathcal{A}$ of formal power series in non-commuting variables a_{ij}, $1 \le i,j \le m$. Elements of $\mathcal{A}$ are infinite linear combinations of words in variables a_{ij} (with coefficients in $\mathbb{C}$). In most cases we take elements of $\mathcal{A}$ modulo some ideal $\mathcal{I}$ generated by a finite number of quadratic relations.

We consider *lattice steps* of the form $(x,i) \to (x+1,j)$ for some $x,i,j \in \mathbb{Z}$, $1 \le i,j \le m$. We think of x being drawn along x-axis, increasing from left to right, and refer to i and j as the *starting height* and the *ending height*, respectively. We identitfy the step $(x,i) \to (x+1,j)$ with a_{ij}. Similarly, we identify a finite sequence of steps with a word in the alphabet $\{a_{ij} \colon 1 \le i,j \le m\}$, i.e. with an element of the algebra $\mathcal{A}$. If each step in a sequence starts at the ending point of the previous step, we call such a sequence a *lattice path.* A lattice path with starting height i and ending height j will be called *a path from i to j.*

We abbreviate the product $a_{\lambda_1\mu_1} \cdots a_{\lambda_\ell\mu_\ell}$ to $a_{\lambda,\mu}$ for $\lambda = \lambda_1 \cdots \lambda_\ell$ and $\mu = \mu_1 \cdots \mu_\ell$, where λ and μ are regarded as words in the alphabet $\{1,\ldots,m\}$. For such a word $\nu = \nu_1 \cdots \nu_\ell$, define the *set of inversions of* ν

$$\mathcal{I}(\nu) \,=\, \{(i,j) \colon i<j, \nu_i > \nu_j\},$$

and let $\operatorname{inv}\nu = |\mathcal{I}(\nu)|$ be the *number of inversions of* ν.

Let $B = (b_{ij})_{n\times n}$ be a square matrix with entries in $\mathcal{A}$, i.e. b_{ij}'s are linear combinations of words in $\mathcal{A}$. To define the determinant of B, expand the terms of

$$\sum_{\sigma\in S_n} (-1)^{\operatorname{inv}(\sigma)} b_{\sigma_1 1}\cdots b_{\sigma_n n}, \tag{1.1}$$

and weight a word $a_{\lambda,\mu}$ with a certain weight $w(\lambda,\mu)$. The resulting expression is called the *determinant of* B *with respect to* $\mathcal{A}$. In the usual commutative case, all weights are equal to 1.

In all cases we set $w(\varnothing,\varnothing) = 1$. We have

$$\frac{1}{\det(I-A)} = \frac{1}{1-\Sigma} = 1 + \Sigma + \Sigma^2 + \ldots,$$

where Σ is a certain finite weighted sum of words in a_{ij}. Note that both left and right inverse of $\det(I-A)$ are equal to the infinite sum on the right. Throughout the paper we use the fraction notation as above in non-commutative situations.

The (i,j)-th entry of A^k is the sum of all paths of length k from i to j. Since

$$(I-A)^{-1} = I + A + A^2 + \ldots,$$

the (i,j)-th entry of $(I-A)^{-1}$ is the sum of all paths (of any length) from i to j.

1.2. Classes of non-commutative matrices. For a matrix $A = (a_{ij})_{m\times m}$ we say that A is:

(1) *commutative* if

$$a_{jl}a_{ik} = a_{ik}a_{jl} \quad \text{for all } i,j,k,l;$$

(2) *Cartier-Foata* if

$$a_{jl}a_{ik} = a_{ik}a_{jl} \quad \text{for all } i,j,k,l,\ i\neq j;$$

(3) *right-quantum* if

$$\begin{aligned} a_{jk}a_{ik} &= a_{ik}a_{jk} \quad \text{for all } i,j,k \\ a_{ik}a_{jl} - a_{jk}a_{il} &= a_{jl}a_{ik} - a_{il}a_{jk} \quad \text{for all } i,j,k,l \end{aligned}$$

(4) *q-Cartier-Foata* if

$$\begin{aligned} a_{jl}a_{ik} &= a_{ik}a_{jl} \quad \text{for all } i<j, k<l, \\ a_{jl}a_{ik} &= q^2\, a_{ik}a_{jl} \quad \text{for all } i<j, k>l, \\ a_{jk}a_{ik} &= q\, a_{ik}a_{jk} \quad \text{for all } i<j; \end{aligned}$$

(5) *q-right-quantum* if

$$\begin{aligned} a_{jk}a_{ik} &= q\, a_{ik}a_{jk} \quad \text{for all } i<j, \\ a_{ik}a_{jl} - q^{-1}\, a_{jk}a_{il} &= a_{jl}a_{ik} - q\, a_{il}a_{jk} \quad \text{for all } i<j, k<l; \end{aligned}$$

(6) **q**-*Cartier-Foata* if

$$\begin{aligned} a_{jl}a_{ik} &= q_{kl}^{-1}q_{ij}\, a_{ik}a_{jl} \quad \text{for all } i<j, k<l \\ a_{jl}a_{ik} &= q_{ij}q_{lk}\, a_{ik}a_{jl} \quad \text{for all } i<j, k>l \\ a_{jk}a_{ik} &= q_{ij}\, a_{ik}a_{jk} \quad \text{for all } i<j; \end{aligned}$$

(7) **q**-*right-quantum* if

$$\begin{aligned} a_{jk}a_{ik} &= q_{ij}\, a_{ik}a_{jk} \quad \text{for all } i<j \\ a_{ik}a_{jl} - q_{ij}^{-1}\, a_{jk}a_{il} &= q_{kl}q_{ij}^{-1}\, a_{jl}a_{ik} - q_{kl}\, a_{il}a_{jk} \quad \text{for all } i<j, k<l. \end{aligned}$$

Here q, q_{ij} for $i<j$, are some fixed non-zero complex numbers.

We have the following implications:

$$\begin{array}{ccccccc} (7) & \Rightarrow & (5) & \Rightarrow & (3) & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ (6) & \Rightarrow & (4) & \Rightarrow & (2) & \Rightarrow & (1) \end{array}$$

For example, by (7) ⇒ (6) we mean that if a statement is true for all **q**-right-quantum matrices, it is also true for all **q**-Cartier-Foata matrices. Equivalently, every **q**-Cartier-Foata matrix is also **q**-right-quantum.

We denote the ideals of $\mathcal{A}$ generated by relations in (1)–(7) by

$$\mathcal{I}_{\mathrm{comm}},\ \mathcal{I}_{\mathrm{cf}},\ \mathcal{I}_{\mathrm{rq}},\ \mathcal{I}_{q-\mathrm{cf}},\ \mathcal{I}_{q-\mathrm{rq}},\ \mathcal{I}_{\mathbf{q}-\mathrm{cf}},\ \mathcal{I}_{\mathbf{q}-\mathrm{rq}}$$

respectively.

1.3. Non-commutative determinants. In cases (1)–(3), define the weight of $a_{\lambda,\mu}$ to be

$$w(\lambda,\mu)=1.$$

In cases (4)–(5), take

$$w(\lambda,\mu)=q^{\operatorname{inv}\mu-\operatorname{inv}\lambda},$$

and in cases (6)–(7), take

$$w(\lambda,\mu)=\prod_{(i,j)\in\mathcal{I}(\mu)}q_{\mu_j\mu_i}\prod_{(i,j)\in\mathcal{I}(\lambda)}q^{-1}_{\lambda_j\lambda_i}.$$

In other words, we keep track of the number of inversions in cases (4)–(5), and of actual inversions in (6)–(7).

The determinant with respect to $\mathcal{A}$ of the matrix $I-A$ plays an important role. In cases (1)–(3), we have:

$$\det(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det A_J,$$

where

$$\det A_J=\det(a_{ij})_{i,j\in J}=\sum_{\sigma\in S_J}(-1)^{\operatorname{inv}\sigma}a_{\sigma(j_1)j_1}\cdots a_{\sigma(j_k)j_k}$$

for $J=\{j_1<j_2<\ldots<j_k\}$. Similarly, in cases (4)–(5), we have:

$$\det_q(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det_q A_J,$$

where

$$\det_q A_J=\det_q(a_{ij})_{i,j\in J}=\sum_{\sigma\in S_J}(-q)^{-\operatorname{inv}\sigma}a_{\sigma(j_1)j_1}\cdots a_{\sigma(j_k)j_k}$$

for $J=\{j_1<j_2<\ldots<j_k\}$. Finally, in cases (6)–(7), we have:

$$\det_{\mathbf{q}}(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det_{\mathbf{q}} A_J,$$

where

$$\det_q A_J=\det_{\mathbf{q}}(a_{ij})_{i,j\in J}=\sum_{\sigma\in S_J}\left(\prod_{(j_s,j_t)\in\mathcal{I}(\sigma)}(-q_{\sigma(j_t)\sigma(j_s)})^{-1}\right)a_{\sigma(j_1)j_1}\cdots a_{\sigma(j_k)j_k}$$

for $J=\{j_1<j_2<\ldots<j_k\}$.

1.4. Matrix inverse formulas. Recall that if D is an invertible matrix with commuting entries, we have:

$$D^{-1}_{ij}=(-1)^{i+j}\frac{\det D^{ji}}{\det D}, \tag{1.2}$$

where D^{ji} denotes the matrix D without j-th row and i-th column. This matrix inverse formula can also be extended to cases (2)–(7) as follows.

In cases (2)–(3), when $A=(a_{ij})_{m\times m}$ is a Cartier-Foata matrix or a right-quantum matrix, we have:

$$\left(\frac{1}{I-A}\right)_{ij}=(-1)^{i+j}\frac{1}{\det(I-A)}\cdot\det{(I-A)}^{ji}\quad\text{for all}\quad 1\le i,j\le m.$$

The proof is a straightforward linear algebra manipulation, see e.g. [**KP**, §12].

In cases (4)–(5), when $A = (a_{ij})_{m\times m}$ is a q-Cartier-Foata or a q-right-quantum matrix, we have

$$\left(\frac{1}{I - A_{[ij]}}\right)_{ij} = (-1)^{i+j}\frac{1}{\det_q(I-A)} \cdot \det_q\,(I-A)^{ji} \quad \text{for all} \quad 1 \le i, j \le m,$$

where

$$A_{[ij]} = \begin{pmatrix} q^{-1}a_{11} & \cdots & q^{-1}a_{1j} & a_{1,j+1} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q^{-1}a_{i-1,1} & \cdots & q^{-1}a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,m} \\ a_{i1} & \cdots & a_{ij} & qa_{i,j+1} & \cdots & qa_{i,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & qa_{m,j+1} & \cdots & qa_{mm} \end{pmatrix}.$$

This follows from the "$1 = q$" principle [**FH1**, §3] and is given in [**KP**, §12] and [**K1**, §11].

Finally, in cases (6)–(7), when $A = (a_{ij})_{m\times m}$ is a $\mathbf{q}$-Cartier-Foata matrix or a $\mathbf{q}$-right-quantum matrix, we have:

$$\left(\frac{1}{I - A_{[ij]}}\right)_{ij} = (-1)^{i+j}\frac{1}{\det_{\mathbf{q}}(I-A)} \cdot \det_{\mathbf{q}}\,(I-A)^{ji} \quad \text{for all} \quad 1 \le i, j \le m,$$

where

$$A_{[ij]} = \begin{pmatrix} q_{1i}^{-1}a_{11} & \cdots & q_{1i}^{-1}a_{1j} & q_{1i}^{-1}q_{j,j+1}a_{1,j+1} & \cdots & q_{1i}^{-1}q_{jm}a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{i-1,i}^{-1}a_{i-1,1} & \cdots & q_{i-1,i}^{-1}a_{i-1,j} & q_{i-1,i}^{-1}q_{j,j+1}a_{i-1,j+1} & \cdots & q_{i-1,i}^{-1}q_{jm}a_{i-1,m} \\ a_{i1} & \cdots & a_{ij} & q_{j,j+1}a_{i,j+1} & \cdots & q_{jm}a_{i,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & q_{j,j+1}a_{m,j+1} & \cdots & q_{jm}a_{mm} \end{pmatrix}.$$

This follows from the "$1 = q_{ij}$" principle, see [**KP**, §12] and [**K1**, §11].

Alternatively, matrix inverse formulas can also be proved combinatorially, see [**K2**].

2. MacMahon Master Thorem

2.1. Main result. Assume that the variables $x_1, \ldots, x_m$ commute with all a_{ij} and that they satisfy the commutation relation

$$x_j x_i = q_{ij}\; x_i x_j \quad \text{for} \quad i < j$$

for some non-zero complex numbers q_{ij}. Choose $k_1, \ldots, k_m \ge 0$, and expand the product

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}.$$

For every term, use the commutation relations to move all x_i's to the right of a_{ij}'s and to rearrange them so that their indices are non-decreasing. Along the way, we exchange pairs of variables x_i and x_j, producing a product of q_{ij}'s. Denote by $G(k_1, \ldots, k_m)$ the coefficient at $x_1^{k_1}\cdots x_m^{k_m}$. Each such coefficient is a finite sum of words $a_{i_1j_1}\cdots a_{i_\ell j_\ell}$ weighted by monomials in q_{ij}'s, $1 \le i < j \le m$; here we have $i_1 \le \ldots \le i_\ell$, the number of variables $a_{i,*}$ is equal to k_i, and the number of variables $a_{*,j}$ is equal to k_j. Our main result is the following theorem.

Theorem 2.1 ($\mathbf{q}$-right-quantum MacMahon master theorem). *Let $A = (a_{ij})_{m\times m}$ be a $\mathbf{q}$-right-quantum matrix. Denote the coefficient of $x_1^{k_1}\cdots x_m^{k_m}$ in*

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i},$$

by $G(k_1, \ldots, k_m)$. Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m) = \frac{1}{\det_{\mathbf{q}}(I-A)}, \tag{2.1}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$.

The classical MacMahon master theorem states the same for A a complex matrix and the $\mathbf{q}$-determinant replaced by the usual determinant.

2.2. Cartier-Foata case. Assume first that A is a Cartier-Foata matrix.

Define a *balanced sequence* (*b-sequence*) to be a finite sequence of steps

$$\alpha = \{(0, i_1) \to (1, j_1), (1, i_2) \to (2, j_2), \ldots, (\ell - 1, i_\ell) \to (\ell, j_\ell)\},$$

such that the number of steps starting at height i is equal to the number of steps ending at height i, for all i. We denote this number by k_i, and call $(k_1, \ldots, k_m)$ the *type* of the b-sequence. Clearly, the total number of steps in the path is $\ell = k_1 + \ldots + k_m$.

Define an *ordered sequence* (*o-sequence*) of type $(k_1, \ldots, k_m)$ to be a b-sequence of k_1 steps starting at height 1, then k_2 steps starting at height 2, etc., so that k_i steps end at height i. Denote by $\mathbf{O}(k_1, \ldots, k_m)$ the set of all o-sequences of type $(k_1, \ldots, k_m)$.

Now consider a lattice path from $(0, 1)$ to $(x_1, 1)$ that never goes below $y = 1$ or above $y = m$, then a lattice path from $(x_1, 2)$ to $(x_2, 2)$ that never goes below $y = 2$ or above $y = m$, etc.; in the end, take a straight path from (x_{m-1}, m) to (x_m, m). We call this a *path sequence* (*p-sequence*). Observe that every p-sequence is also a b-sequence. Denote by $\mathbf{P}(k_1, \ldots, k_m)$ the set of all p-sequences of type $(k_1, \ldots, k_m)$.

EXAMPLE 2.2. Figure 1 presents an o-sequence and a p-sequence.

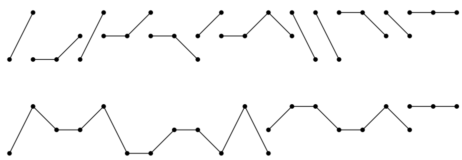

FIGURE 1. An o-sequence and a p-sequence of type $(4, 7, 8)$.

Observe that choosing a term of

$$\prod_{i=1}^{m} (a_{i1} x_1 + \ldots + a_{im} x_m)^{k_i}$$

means choosing a term $a_{1*} x_*$ k_1 times, then choosing a term $a_{2*} x_*$ k_2 times, etc., and then multiplying all these terms. In other words, each term in $G(k_1, \ldots, k_m)$ corresponds to an o-sequence in $\mathbf{O}(k_1, \ldots, k_m)$.

Let us define a bijection $\varphi : \mathbf{O}(k_1, \ldots, k_m) \to \mathbf{P}(k_1, \ldots, k_m)$ with the property that the word $\varphi(\alpha)$ is a rearrangement of the word α, for every o-sequence α.

Take an o-sequence α, and let $[0, x]$ be the maximal interval on which it is part of a p-sequence, i.e. the maximal interval $[0, x]$ on which the o-sequence has the property that if a step ends at level i, and the following step starts at level $j > i$, the o-sequence stays on or above height j afterwards. Let i be the height at x. Choose the step $(x', i) \to (x' + 1, i')$ in the o-sequence that is the first to the right of x that starts at level i (such a step exists because an o-sequence is a balanced sequence). Keep switching this step with the one to the left until it becomes the step $(x, i) \to (x + 1, i')$. The new object is part of a p-sequence at least on the interval $[0, x + 1]$. Continue this procedure until we get a p-sequence $\varphi(\alpha)$.

For example, for the o-sequence given in Figure 1 we have $x = 1$ and $i = 3$. The step we choose then is $(12, 3) \to (13, 1)$, i.e. $x' = 12$. The following result is straightforward.

LEMMA 2.3. *The map* $\varphi : \mathbf{O}(k_1, \ldots, k_m) \to \mathbf{P}(k_1, \ldots, k_m)$ *constructed above is a bijection.*

EXAMPLE 2.4. Figure 2 shows the switches for an o-sequence of type $(3, 1, 1)$, and the p-sequence in Figure 1 is the result of applying this procedure to the o-sequence in the same figure (we need 33 switches).

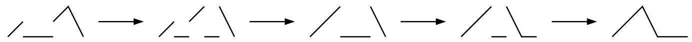

FIGURE 2. Transforming an o-sequence into a p-sequence.

Note that the sum of the paths from 1 to 1 is enumerated by $\left((I-A)^{-1}\right)_{11}$. By the matrix inverse formula in case (2), the sum of all p-sequences is equal to

$$\left(\frac{1}{I-A}\right)_{11}\left(\frac{1}{I-A^{11}}\right)_{22}\left(\frac{1}{I-A^{12,12}}\right)_{33}\cdots\frac{1}{1-a_{mm}}=$$

$$=\left(\det{}^{-1}(I-A)\cdot\det(I-A^{11})\right)\cdot\left(\det{}^{-1}(1-A^{11})\cdot\det{}^{-1}(I-A^{12,12})\right)\cdots=\frac{1}{\det(I-A)}.$$

All the steps we switched had different starting heights, so $\varphi(\alpha)=\alpha$ modulo the ideal $\mathcal{I}_{\mathrm{cf}}$. This completes the proof of Theorem 2.1 for A a Cartier-Foata matrix.

2.3. Right-quantum case. A slightly more involved proof proves the same theorem in the right-quantum case; here we have to perform the switches simultaneously. Figure 3 shows that the sum over all elements of $\mathbf{O}(3,1,1)$ is equal to the sum over all elements of $\mathbf{P}(3,1,1)$ modulo the ideal $\mathcal{I}_{\mathrm{rq}}$. Here p-sequences are drawn in bold, an arrow from a sequence α to a sequence α' means that we get α' from α by performing a switch and that $\alpha'=\alpha \mod \mathcal{I}_{\mathrm{rq}}$, and arrows from q-sequences α,β to q-sequences α',β' whose intersection is marked by a dot mean that we get α' (resp. β) from α (resp. β) by performing a switch, and $\alpha'+\beta'=\alpha+\beta \mod \mathcal{I}_{\mathrm{rq}}$.

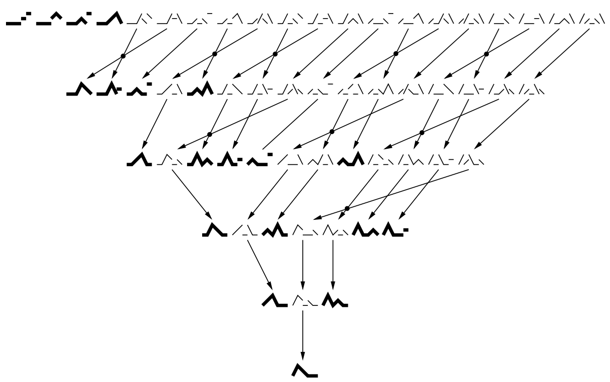

FIGURE 3. Transforming o-sequences into p-sequences via a series of simultaneous switches.

2.4. q-right-quantum case. Let us sketch the proof in the weighted case (5), i.e. when $x_jx_i=qx_ix_j$ for $i<j$, and the matrix A is q-right-quantum. When we expand

$$\prod_{i=1}^{m}(a_{i1}x_1+\ldots+a_{im}x_m)^{k_i},$$

a term $a_{\lambda,\mu}$ in $G(k_1,\ldots,k_m)$ has weight $q^{\operatorname{inv}\mu}=w(\lambda,\mu)$ (since λ is non-decreasing and $\operatorname{inv}\lambda=0$). Performing the switches changes the weight, but at each point the term $a_{\lambda,\mu}$ has weight $w(\lambda,\mu)$. For example, if $i<j$ and $k<l$, $\lambda=\lambda_1ij\lambda_2$, $\mu=\mu_1kl\mu_2$, $\lambda'=\lambda_1ji\lambda_2$, $\mu=\mu_1lk\mu_2$, then $\operatorname{inv}\lambda'=\operatorname{inv}\lambda+1$, $\operatorname{inv}\mu'=\operatorname{inv}\mu+1$ and so the relation $a_{jl}a_{ik}=a_{ik}a_{jl}$ gives

$$q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}a_{\lambda,\mu} \quad=\quad q^{\operatorname{inv}\mu'-\operatorname{inv}\lambda'}a_{\lambda',\mu'} \qquad \mod\ \mathcal{I}_{q-\mathrm{cf}}.$$

Therefore, the sum of all o-sequences with corresponding weights is equal to the sum of all p-sequences with corresponding weights modulo $\mathcal{I}_{q-\mathrm{rq}}$. By the matrix inversion formula, the latter is equal to $\det_q{}^{-1}(I-A)$ modulo $\mathcal{I}_{q-\mathrm{rq}}$.

2.5. q-right-quantum case. We assume that $x_j x_i = q_{ij} x_i x_j$ for $i < j$, and that the matrix A is $\mathbf{q}$-right-quantum. When we expand

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i},$$

a term $a_{\lambda,\mu}$ in $G(k_1, \ldots, k_m)$ has weight

$$\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j \mu_i} = w(\lambda, \mu).$$

While performing the switches, the weight of a term $a_{\lambda,\mu}$ is $w(\lambda, \mu)$ at every point, so the sum of all o-sequences with corresponding weights is equal to the sum of all p-sequences with corresponding weights modulo $\mathcal{I}_{\mathbf{q}-\mathrm{rq}}$. The latter is equal to $\det_{\mathbf{q}}{}^{-1}(I - A)$ by the matrix inversion formula.

3. Sylvester's determinant identity

3.1. Main result. Consider a matrix $A = (a_{ij})_{m\times m}$. Choose $n < m$, and denote by $\widehat{A}$ the submatrix $(a_{ij})_{1\le i,j\le n}$. Also define

$$a_{i*} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}, \quad a_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

The main result is the following theorem.

THEOREM 3.1 ($\mathbf{q}$-right-quantum Sylvester's determinant identity). *Let $A = (a_{ij})_{m\times m}$ be a $\mathbf{q}$-right-quantum matrix, and choose $n < m$. Let $\widehat{A}, a_{i*}, a_{*j}$ be defined as above, and let*

$$c_{ij}^{\mathbf{q}} = -\det\nolimits_{\mathbf{q}}^{-1}(I - \widehat{A}) \cdot \det\nolimits_{\mathbf{q}} \begin{pmatrix} I - \widehat{A} & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}, \quad C^{\mathbf{q}} = (c_{ij})_{n+1\le i,j\le m}.$$

Suppose $q_{ij} = q_{i'j'}$ for all $i, i' \le n$ and $j, j' > n$. Then

$$\det\nolimits_{\mathbf{q}}^{-1}(I - \widehat{A}) \cdot \det\nolimits_{\mathbf{q}}(I - A) = \det\nolimits_{\mathbf{q}}(I - C^{\mathbf{q}}).$$

Here $\det_{\mathbf{q}}(I - C^{\mathbf{q}})$ is defined with respect to the algebra $\mathcal{C}^{\mathbf{q}}$ generated by $c_{ij}^{\mathbf{q}}$, $n + 1 \le i, j \le m$, and the other determinants are defined with respect to $\mathcal{A}$.

3.2. Non-commutative Sylvester's identity. First, let us present a combinatorial proof of the so-called non-commutative Sylvester's identity [**GR**].

THEOREM 3.2 (Gelfand-Retakh). *Consider the matrix $C = (c_{ij})_{i,j=n+1}^{m}$, where*

$$c_{ij} = a_{ij} + a_{i*}(I - \widehat{A})^{-1} a_{*j}.$$

Then

$$(I - A)_{ij}^{-1} = (I - C)_{ij}^{-1}.$$

PROOF. Take a lattice path $a_{ii_1}a_{i_1i_2}\cdots a_{i_{\ell-1}j}$ with $i, j > n$. Clearly it can be uniquely divided into paths $P_1, P_2, \ldots P_p$ with the following properties:

- the ending height of P_r is the starting height of P_{r+1}
- the starting and the ending heights of all P_r are strictly greater than n
- all intermediate heights are less than or equal to n

Next, note that for $i, j > n$, the sum over all non-trivial paths with starting height i, ending height j, and intermediate heights $\le n$ is equal to

$$a_{ij} + \sum_{k,l\le n} a_{ik}(I + \widehat{A} + \widehat{A}^2 + \ldots)_{kl} a_{lj} = a_{ij} + a_{i*}(I - \widehat{A})^{-1} a_{*j} = c_{ij}.$$

The decomposition therefore proves Theorem 3.2. □

Example 3.3. Figure 4 depicts a path from 4 to 4 with a dotted line between heights n and $n+1$, and the corresponding decomposition, for $n=3$.

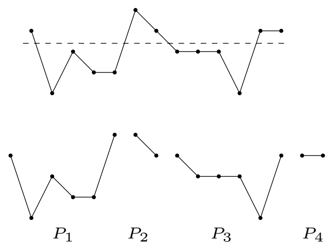

Figure 4. The decomposition $(a_{41}a_{13}a_{32}a_{22}a_{25})(a_{54})(a_{43}a_{33}a_{33}a_{31}a_{14})(a_{44})$.

The theorem implies that

$$(I-A)^{-1}_{n+1,n+1}(I-A^{n+1,n+1})^{-1}_{n+2,n+2}\cdots\left(I-\begin{pmatrix}\widehat{A} & a_{*m}\\ a_{m*} & a_{mm}\end{pmatrix}\right)^{-1}_{mm}= \tag{3.1}$$

$$=(I-C)^{-1}_{n+1,n+1}(I-C^{n+1,n+1})^{-1}_{n+2,n+2}\cdots(1-c_{mm})^{-1}.$$

In all the cases (1)–(7), both the left-hand side and the right-hand side of this equation can be written in terms of non-commutative determinants.

3.3. Cartier-Foata case. Assume that A is Cartier-Foata. By the matrix inverse formula, the left-hand side of (3.1) is equal to

$$\det{}^{-1}(I-\widehat{A})\cdot\det(I-A).$$

The following lemma implies that the right-hand side of (3.1) is equal to $\det{}^{-1}(I-C)$.

Lemma 3.4. *If A is a Cartier-Foata matrix, then C is a right-quantum matrix.*

Proof. The proof involves a switching procedure similar to the one in the proof of MacMahon Master Theorem. The product $c_{ik}c_{jk}$ is the sum of terms of the form

$$a_{ii_1}a_{i_1i_2}\cdots a_{i_pk}a_{jj_1}a_{j_1j_2}\cdots a_{j_rk}$$

for $p,r\geq 0$, $i_1,\ldots,i_p,j_1,\ldots,j_r\leq n$. We can transform this term into a term of the form

$$a_{jj'_1}a_{j'_1j'_2}\cdots a_{j'_tk}a_{ii'_1}a_{i'_1i'_2}\cdots a_{i'_sk}$$

without changing it modulo $\mathcal{I}_{\mathrm{cf}}$. This means that $c_{ik}c_{jk}=c_{jk}c_{ik}$ modulo $\mathcal{I}_{cf}$. The proof of the other right-quantum relation is similar. □

For example, take $m=5$, $n=2$, $i=3$, $j=5$, $k=4$ and the term $a_{31}a_{12}a_{24}a_{52}a_{22}a_{24}$; Figure 5 shows the steps that transform it into $a_{52}a_{24}a_{31}a_{12}a_{22}a_{24}$.

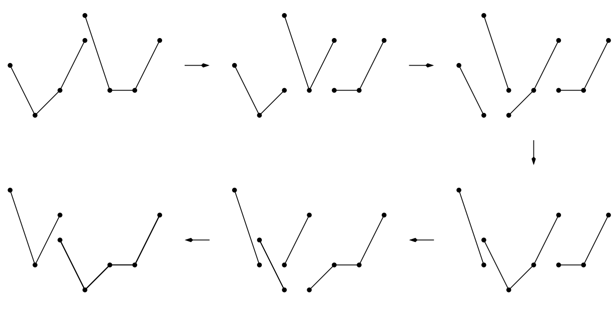

Figure 5. Transforming $a_{31}a_{12}a_{24}a_{52}a_{22}a_{24}$ into $a_{52}a_{24}a_{31}a_{12}a_{22}a_{24}$.

By using the matrix inverse formula, we can also prove that if A is Cartier-Foata, then

$$c_{ij} = -\det{}^{-1}(I - \widehat{A}) \cdot \det \begin{pmatrix} I - \widehat{A} & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix}.$$

This finishes the proof of the Cartier-Foata Sylvester's identity.

3.4. Other cases. The proofs in cases (3)–(7) follow the same pattern. For (3), we prove similarly that C is right-quantum when A is right-quantum. The rest of the proof can be repeated verbatim. In cases (4)–(7), the matrix C has elements

$$c_{ij} = a_{ij} + a_{i*}(I - q^{-1}\widehat{A})^{-1}(q^{-1}a_{*j}),$$

where $q_{ij} = q$ for $i \le n$ and $j > n$ in the **q**-Cartier-Foata and **q**-right-quantum cases. Essentially the same proof as above shows that C is q-right-quantum (resp. **q**-right-quantum) if A is q-Cartier-Foata or q-right-quantum (resp. **q**-Cartier-Foata or **q**-right-quantum). Now the corresponding matrix inverse formula implies Theorem 3.1.

4. Final remarks

4.1. MacMahon's original proof of the MacMahon Master Theorem and numerous application to binomial identities can be found in [**MM**]. The standard analytic proof of MMT usually involves the Lagrange inversion. Let us mention here a number of papers on the q-Lagrange inversion (see references in [**KP**]) as well as non-commutative lagrange inversion (see [**Ge, PPR**]). Interestingly, none of the proofs extends to this case.

4.2. In [**KS**], Krattenthaler and Schlosser found a different kind of q-extension of MMT. In [**KP**] we show how this formula follows from the Cartier-Foata version.

4.3. Most recently, Martin Lorenz, reported to the authors that he found a Koszul duality proof of our (q_{ij})-extension of MMT. In a different direction, Etingof and Pak found an unusual algebraic extension of the MMT by using the generalized Koszul dualty by Berger [**EP**]. It would be interesting to find a combinatorial proof of this result.

4.4. Sylvester's determinant identity is usually given in the form used by Bareiss [**B**]:

$$\det A \cdot \left(\det \widehat{A}\right)^{m-n-1} = \det B$$

for

$$b_{ij} = \det \begin{pmatrix} \widehat{A} & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{i,j=n+1}^{m}.$$

Our Theorem 3.1 in the commutative case is easily equivalent to this version. The proof in [**B**] is a straighforward linear algebra argument. We refer to [**MG, AAM**] for other proofs and mild (commutative) generalizations.

4.5. It would be nice to obtain generalizations of the MMT and Sylvester's determinant identity to other root systems, as suggested in [**Mo2**]. One would have to substitute the determinants with the *Sklyanin minors*, but a combinatorial interpretations is yet to be found.

Acknowledgements. We are grateful to George Andrews, Agriano Garsia, Pavel Etingof, Dominique Foata, Christian Krattenthaler, Martin Lorenz, Yuri Manin, Alexander Molev, Vladimir Retakh, Richard Stanley and Doron Zeilberger for interesting discussions and help with the references. The second author was partially supported by the NSF.

References

[AAM] A. G. Akritas, E. K. Akritas and G. I. Malaschonok, Various proofs of Sylvester's (determinant) identity, *Math. Comput. Simulation* **42** (1996), 585–593

[B] E. H. Bareiss, Sylvester's identity and multistep integer-preserving Gaussian elimination, *Math. Comput.* **22** (1968), 565–578

[CF] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lecture Notes in Mathematics **85**, Springer, Berlin, 1969

[EP] P. Etingof and I. Pak, An algebraic extension of the MacMahon Master Theorem, arXive:math.CO/0608005

[ER] P. Etingof and V. Retakh, Quantum determinants and quasideterminants, *Asian J. Math.* **3** (1999), 345–351

[F1] D. Foata, Étude algébrique de certains problèmes d'analyse combinatoire et du calcul des probabilités, *Publ. Inst. Statist. Univ. Paris* **14** (1965), 81–241; available at `http://www.mat.univie.ac.at/~slc/books/cartfoa.html`

[F2] D. Foata, A Noncommutative Version of the Matrix Inversion Formula, *Adv. Math.* **31** (1979), 330–349

[FH1] D. Foata and G.-N. Han, A basis for the right quantum algebra and the "$1 = q$" principle, arXive:math.CO/0603463

[FH2] D. Foata and G.-N. Han, A new proof of the Garoufalidis-Lê-Zeilberger Quantum MacMahon Master Theorem, arXive:math.CO/0603464

[FH3] D. Foata and G.-N. Han, Specializations and Extensions of the quantum MacMahon Master Theorem, arXive:math.CO/0603466

[GLZ] S. Garoufalidis, T. Tq Lê and D. Zeilberger, The Quantum MacMahon Master Theorem, to appear in *Proc. Natl. Acad. of Sci.*

[GR] I. M. Gelfand and V. S. Retakh, A theory of non-commutative determinants and characteristic functions of graphs, *Funct. Anal. Appl.* **26** (1992), no. 4, 1–20

[G+] I. Gelfand, S. Gelfand, V. Retakh and R. L. Wilson, Quasideterminants, *Adv. Math.* **193** (2005), 56–141

[Ge] I. Gessel, A noncommutative generalization and q-analog of the Lagrange inversion formula, *Trans. Amer. Math. Soc.* **257** (1980), 455–482.

[GJ] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, John Wiley, New York, 1983

[HL] P. H. Hai and M. Lorenz, Koszul algebras and the quantum MacMahon master theorem, arXive:math.QA/0603169

[HM] M. J. Hopkins and A. I. Molev, A q-analogue of the centralizer construction and skew representations of the quantum affine algebra, arXive:QA/0606121

[K1] M. Konvalinka, *Non-commutative Sylvester's determinantal identity*, preprint (2007)

[K2] M. Konvalinka, *A generalization of Foata's fundamental transformation and its applications to the right-quantum algebra*, preprint (2007)

[KP] M. Konvalinka and I. Pak, *Non-commutative extensions of MacMahon's Master Theorem*, arXive:math.CO/0607737

[KS] C. Krattenthaler and M. Schlosser, A new multidimensional matrix inverse with applications to multiple q-series, *Disc. Math.* **204** (1999), 249–279

[KL] D. Krob and B. Leclerc, Minor Identities for Quasi-Determinants and Quantum Determinants, *Comm. Math. Phys.* **169** (1995), 1–23

[L] M. Lothaire, *Algebraic combinatorics on words*, Cambridge U. Press, Cambridge, UK, 2002.

[MM] P. A. MacMahon, *Combinatory Analysis*, 2 vols., Cambridge U. Press, 1915 and 1916; reprinted in one volume by Chelsea, New York, 1960

[M1] Yu. I. Manin, Multiparameter quantum deformations of the linear supergroup, *Comm. Math. Phys.* **123** (1989), 163–175

[M2] Yu. I. Manin, *Quantum groups and noncommutative geomtry*, CRM, Université de Montréal, QC, 1988

[Mo1] A. I. Molev, Yangians and transvector algebras, *Discrete Math.* **246** (2002), 231–253

[Mo2] A. I. Molev, Skew representations of twisted Yangians, *Selecta Math. (N.S.)* **12** (2006), 1–38

[MG] G. Mühlbach and M. Gasca, A Generalization of Sylvester's Identity on Determinants and Some Applications, *Linear Algebra Appl.* **66** (1985), 221–234

[PPR] I. Pak, A. Postnikov and V. Retakh, Non-commutative Lagrange inversion, preprint, 1995; available electronically at `http://www-math.mit.edu/~pak/research.html`

[V] G. X. Viennot, Heaps of pieces. I: Basic definitions and combinatorial lemmas, in *Lecture Notes in Math.* **1234**, Springer, Berlin (1986), 321–350

[Z] D. Zeilberger, A combinatorial approach to matrix algebra, *Discrete Math.* **56** (1985), 61–72

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: `konvalinka@math.mit.edu`
URL: `http://www-math.mit.edu/~konvalinka/`

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: `pak@math.mit.edu`
URL: `http://www-math.mit.edu/~pak/`