ON GOULDEN-JACKSON'S DETERMINANTAL FORMULA FOR THE IMMANANT

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Abstract. In 1992, Goulden and Jackson found a beautiful determinantal expression for the immanant of a matrix. This paper proves the same result combinatorially. We also present a β-extension of the theorem and a simple determinantal expression for the irreducible characters of the symmetric group.

1. Introduction

Take a matrix $A = (a_{ij})_{i,j=1}^m$ for complex variables a_{ij}, its characteristic polynomial $\chi_A(t) = \det(A - tI)$ and its eigenvalues $\omega_1, \ldots, \omega_m$. Vieta's formulas tell us that the elementary symmetric functions

$$e_i(\omega_1, \ldots, \omega_m) = \sum_{j_1 < \ldots < j_i} \omega_{j_1} \cdots \omega_{j_i}$$

are easily expressible in terms of a_{ij}:

$$e_0 t^m - e_1 t^{m-1} + \ldots + (-1)^m e_m = (t - \omega_1) \cdots (t - \omega_m) =$$

$$= \det(tI - A) = \sum_{i=0}^m (-1)^i t^{m-i} \sum_{J \in \binom{[m]}{i}} \det A_J,$$

with $A_J = (a_{ij})_{i,j \in J}$, i.e.

$$e_i(\omega_1, \ldots, \omega_m) = \sum_{J \in \binom{[m]}{i}} \det A_J.$$

The complete homogeneous symmetric functions

$$h_i(\omega_1, \ldots, \omega_m) = \sum_{j_1 \leq \ldots \leq j_i} \omega_{j_1} \cdots \omega_{j_i}$$

are also easy. We know that

$$\sum_i h_i t^i = \frac{1}{\sum_i (-1)^i e_i t^i} = \frac{1}{t^m \sum_i (-1)^i e_i t^{i-m}} = \frac{1}{t^m \det(t^{-1} I - A)} = \frac{1}{\det(I - tA)}$$

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and by MacMahon Master Theorem [Mac15, page 98]

$$h_i(\omega_1,\dots,\omega_m)=\sum_{\lambda} a_{\overline{\lambda}_1\lambda_1}a_{\overline{\lambda}_2\lambda_2}\cdots a_{\overline{\lambda}_i\lambda_i}, \tag{1}$$

where:

- $\lambda=\lambda_1\cdots\lambda_i$ runs over all sequences of i letters from $[m]$, and
- $\overline{\lambda}=\overline{\lambda}_1\cdots\overline{\lambda}_i$ is the non-decreasing rearrangement of λ.

Goulden and Jackson [GJ92a] proved the following.

Theorem 1 *Abbreviate $h_i(\omega_1,\dots,\omega_m)$ to h_i, choose a partition $\lambda=(\lambda_1,\dots,\lambda_p)$ of m, and write $\mathbf{a}_\pi=a_{1\pi(1)}a_{2\pi(2)}\cdots a_{m\pi(m)}$ for a permutation $\pi\in S_m$. Then*

$$[\mathbf{a}_\pi]\det(h_{\lambda_i-i+j})_{p\times p}=\chi^\lambda(\pi), \tag{2}$$

where $[\mathbf{a}_\pi]E$ is the coefficient of the basis element $\mathbf{a}_\pi$ in E, and χ^λ is the irreducible character of the symmetric group S_m corresponding to λ.

The characters of the symmetric group are class functions in the sense that $\chi^\lambda(\pi)$ depends only on the cycle type of π. That means that it makes sense to define $\chi^\lambda(\mu)$ for partitions λ,μ of the same size.

Note that $\det(h_{\lambda_i-i+j})$ can be expressed as

$$s_\lambda(\omega_1,\dots,\omega_m)$$

by the Jacobi-Trudi identity (see for example [Sta99, Theorem 7.16.1]). Here s_λ is the Schur symmetric function corresponding to the partition λ.

Goulden and Jackson's result is stated in the (clearly equivalent) language of immanants. Theirs is one of the many papers in the early 1990's that brought about fascinating conjectures and results on immanants; see for example [GJ92b], [Gre92], [SS93], [Ste91], [Hai93] for details and further references.

We will give two more proofs of this result. The first gives a recursion that specializes to Murnaghan-Nakayama's rule, and the second is a simple combinatorial proof of a statement equivalent to (2).

2. A recursive proof of Theorem 1

Let us first give an expression for the left-hand side of (2) which is recursive in the sense that it expresses the desired coefficient in terms of coefficients of determinants of smaller matrices.

Note first that all the terms $\mathbf{a}=a_{1*}\cdots a_{1*}a_{2*}\cdots$ of

$$s_\lambda(\omega_1,\dots,\omega_m)=\det(e_{\lambda'_i-i+j})$$

are *balanced* in the sense that each i appears as many times among a_{i*} as among a_{*i} (this also follows from (1) and the formula $s_\lambda=\det(h_{\lambda_i-i+j})$); here we are using the dual Jacobi-Trudi identity [Sta99, Corollary 7.16.2].

Suppose we want to find the coefficient of $\mathbf{a} = a_{1*} \cdots a_{1*} a_{2*} \cdots$. Assume that $C = \{1, \ldots, k\}$, $D = \{k+1, \ldots, m\}$ and that $\mathbf{a}$ does not contain a_{ij} for $i \in C, j \in D$ or $i \in D, j \in C$. The coefficient of $\mathbf{a}$ does not change if we set all a_{ij} that do not appear in $\mathbf{a}$ equal to 0; we may therefore assume that the matrix A has a block diagonal form $A_1 \oplus A_2$, and if $\xi_1, \ldots, \xi_k$ are the eigenvalues of A_1 and $\zeta_{k+1}, \ldots, \zeta_m$ are the eigenvalues of A_2, the eigenvalues of A are $(\omega_1, \ldots, \omega_m) = (\xi_1, \ldots, \xi_k, \zeta_{k+1}, \ldots, \zeta_m)$. By definition,

$$s_\lambda(\omega_1, \ldots, \omega_m),$$

is the sum of $\omega^T = \omega_1^{\alpha_1(T)} \omega_2^{\alpha_2(T)} \cdots$ over all semistandard Young tableau (SSYT) T of shape λ; here $\alpha_i(T)$ is the number of i's in T. See [Sta99, §7.10] for definitions and details. In every such T, the numbers $1, \ldots, k$ form a SSYT of some shape $\nu \subseteq \lambda$, and the numbers $k+1, \ldots, m$ form a SSYT of shape λ/ν. Therefore

$$[\mathbf{a}] s_\lambda(\omega_1, \ldots, \omega_m) = \sum_{\nu \subseteq \lambda} [\mathbf{a}_1] s_\nu(\xi_1, \ldots, \xi_k) [\mathbf{a}_2] s_{\lambda/\nu}(\zeta_{k+1}, \ldots, \zeta_m), \tag{3}$$

where $\mathbf{a}_1$ (respectively $\mathbf{a}_2$) is the product of the terms of $\mathbf{a}$ with indices in C (respectively D). Since $s_{\lambda/\nu}$ is homogeneous of degree $|\lambda| - |\nu|$, we can restrict the sum to partitions $\nu \vdash k$.

The factors in this sum can either be calculated explicitly (for example using the dual Jacobi-Trudi identity) or recursively.

Example 2 Let us calculate the coefficient of $a_{11} a_{12} a_{21} a_{22}^2 a_{34} a_{43}$. By (3), we have to find the coefficient of $a_{34} a_{43}$ in $s_{\lambda/\nu}$ for $\nu = 32, 311, 221$.
We have

$$s_{322/32}(\zeta_3, \zeta_4) = s_2(\zeta_3, \zeta_4) = h_2(\zeta_3, \zeta_4) = a_{33}^2 + a_{33} a_{44} + a_{34} a_{43} + a_{44}^2,$$
$$s_{322/311}(\zeta_3, \zeta_4) = s_{11}(\zeta_3, \zeta_4) = e_2(\zeta_3, \zeta_4) = a_{33} a_{44} - a_{34} a_{43},$$
$$s_{322/221}(\zeta_3, \zeta_4) = s_{21/1}(\zeta_3, \zeta_4) = e_1^2(\zeta_3, \zeta_4) = a_{33}^2 + 2 a_{33} a_{44} + a_{44}^2$$

and therefore

$$[a_{11} a_{12} a_{21} a_{22}^2 a_{34} a_{43}] s_{322}(\omega_1, \ldots, \omega_4) = [a_{11} a_{12} a_{21} a_{22}^2] \left(s_{32}(\zeta_1, \zeta_2) - s_{311}(\zeta_1, \zeta_2) \right).$$

Furthermore,

$$s_{32}(\zeta_1, \zeta_2) = \begin{vmatrix} e_2 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & e_0 & e_1 \end{vmatrix} = e_2^2 e_1 = (a_{11} a_{22} - a_{12} a_{21})^2 (a_{11} + a_{22})$$

and

$$s_{311}(\zeta_1, \zeta_2) = \begin{vmatrix} 0 & 0 & 0 \\ 1 & e_1 & 0 \\ 0 & 1 & e_1 \end{vmatrix} = 0.$$

Therefore

$$[a_{11}a_{12}a_{21}a_{22}^2a_{34}a_{43}]s_{322}(\omega_1,\dots,\omega_4) = -2.$$

Let us prove that this recursion specializes to the Murnaghan-Nakayama's rule. In order to do that, assume that $\pi = \pi_1 \cdot (k+1,k+2,\dots,m)$ for $\pi_1 \in S_k$. In this case, $\mathbf{a}_2$ is of the form $a_{k+1,k+2}a_{k+2,k+3}\cdots a_{m,k+1} = b_1\cdots b_l$. The corresponding matrix A_2 is

$$\begin{pmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_{l-1} \\ b_l & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4}$$

and its characteristic polynomial is $(-1)^l(t^l - b_1b_2\cdots b_l)$. If the zeros are denoted $\eta_1,\dots,\eta_l$, then $e_0(\eta_1,\dots,\eta_l) = 1$, $e_l(\eta_1,\dots,\eta_l) = (-1)^{l-1}b_1\cdots b_l$, $e_i(\eta_1,\dots,\eta_l) = 0$ for $i \neq 0, l$.

For a partition $\lambda = (\lambda_1,\dots,\lambda_p)$, define the conjugate partition λ' by $\lambda'_i = |\{j\colon \lambda_j \geq i\}|$. A border strip is a connected skew shape with no 2×2 square, and the height ht of a border strip is defined to be one less than the number of rows. The following result is well known; we include the proof for the sake of completeness. It will imply that the only non-zero terms in the sum (3) are indexed by border strips.

Lemma 3 *If $\lambda = (\lambda_1,\dots,\lambda_p)$ is a partition and $\nu \subseteq \lambda$ is a subpartition with $|\lambda|-|\nu| = l$, then:*

1. *λ/ν is a border strip if and only if its conjugate λ'/ν' is a border strip;*
2. *λ/ν is a border strip if and only if $\lambda_1 + p = l+1$;*
3. *λ/ν is a border strip if and only if $\lambda_i = \nu_{i-1}+1$ for $2 \leq i \leq p$.*

Proof: Let (a,b) be the entry in row a and column b. (1) is obvious. (2) The squares of a border strip λ/μ form a NE-path from $(1,p)$ to $(\lambda_1,1)$. Each such path has $p-1+\lambda_1-1+1$ squares. Conversely, a partition containing $(1,p)$, $(\lambda_1,1)$ and λ_1+p-3 other squares must be a NE-path and hence its squares form a border strip. (3) A 2×2 square in rows $i-1,i$ implies that $\lambda_i \geq \nu_{i-1}+2$. It is clear that if the squares of a shape λ/μ form a NE-path, we have $\lambda_i = \nu_{i-1}+1$. □

Proposition 4 *If the zeros of $t^l - b_1b_2\cdots b_l$ are $\eta_1,\dots,\eta_l$, we have $s_{\lambda/\nu}(\eta_1,\dots,\eta_l) = 0$ unless λ/ν is a border strip, and $s_{\lambda/\nu}(\eta_1,\dots,\eta_l) = (-1)^{\mathrm{ht}(\lambda/\nu)}b_1b_2\cdots b_l$ for a border strip λ/ν,*

Proof: We can assume that $\lambda_i \neq \nu_i$ and that $\nu_p = 0$. The indices of the entries of the matrix

$$\det(e_{\lambda'_i-\nu'_j-i+j})_{\lambda_1\times\lambda_1}$$

are strictly increasing in rows; the largest possible index is $\lambda_1 + p - 1 = l$; and the indices of the diagonal elements are $\lambda'_i - \nu'_i$ with $1 \leq \lambda'_i - \nu'_i \leq l$ (and $\lambda'_i - \nu'_i = l$ only in the trivial case $\lambda_1 = l$, $\nu = \emptyset$). Note that this determinant is equal to $s_{\lambda/\nu}$. By the lemma and the fact that $e_k = 0$ for $k \neq 0, l$, the matrix is 0 on and above the diagonal unless λ/ν is a border strip. If it is a border strip, the matrix entries in positions (i,j), $i \leq j < \lambda_1$, are 0 (in particular, all the entries in the first row are 0 except the last, which is e_l), and by the lemma, the subdiagonal entries are 1 and the element $(1, \lambda_1)$ is $(-1)^{l-1} b_1 \cdots b_l$. Therefore $s_{\lambda/\nu}(\eta_1, \ldots, \eta_l) = (-1)^{l-1+\lambda_1-1} b_1 b_2 \cdots b_l = (-1)^{p-1} b_1 b_2 \cdots b_l$, with $p - 1 = \operatorname{ht}(\lambda/\nu)$. $\square$

In other words, if $\pi = \pi_1 \cdot (k+1, k+2, \ldots, m)$ for $\pi_1 \in S_k$, then by (3),

$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \sum_\nu (-1)^{\operatorname{ht}(\lambda/\nu)} [\mathbf{a}_1] s_\nu(\xi_1, \ldots, \xi_k),$$

where $\mathbf{a}_1 = \mathbf{a}_{\pi_1}$ and the sum is over all partitions $\nu \subseteq \mu$ for which μ/ν is a border strip. This is precisely the Murnaghan-Nakayama's rule, see [Sag01, Theorem 4.10.2]. Together with the fact that Murnaghan-Nakayama's rule completely determines the irreducible characters χ^λ, this shows that

$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \chi^\lambda(\pi).$$

Note that this also gives us the coefficient of $\mathbf{a}_\pi = a_{1\pi(1)} \cdots a_{m\pi(m)}$ in $p_\lambda(\omega_1, \ldots, \omega_m)$: we know that

$$p_\lambda = \sum_\mu \chi^\mu(\lambda) s_\mu$$

and hence

$$[\mathbf{a}_\pi] p_\lambda(\omega_1, \ldots, \omega_m) = \sum_\mu \chi^\mu(\lambda) \chi^\mu(\pi)$$

is (by the orthogonality of the columns of the table of characters) equal to

$$z_\lambda = 1^{j_1} j_1! 2^{j_2} j_2! \cdots ,$$

where $\lambda = \langle 1^{j_1} 2^{j_2} \cdots \rangle$ is the cycle type of π, and 0 otherwise; see e.g. [Sta99, Proposition 7.17.6].

3. A proof of Theorem 1 via scalar product

Let us find the coefficient of $\mathbf{a}_\pi$ in $e_\lambda(\omega_1, \ldots, \omega_m) = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_p}$. If we pick $a_{i\pi(i)}$ from e_{λ_1}, we must also pick $a_{\pi(i)j}$ from e_{λ_1} because every term in e_{λ_1} is balanced. But since $a_{\pi(i)\pi^2(i)}$ is the only term of $\mathbf{a}_\pi$ with $\pi(i)$ as the first index, we must have $j = \pi^2(i)$. In other words, each of the cycles of π must be chosen from one of the e_{λ_i}'s. We know that for $J = \{j_1 < j_2 < \ldots < j_i\}$ and τ a permutation of $j_1, \ldots, j_i$, $a_{j_1\tau(j_1)} \cdots a_{j_i\tau(j_i)}$ appears in e_i with coefficient equal to the sign of τ.

This reasoning implies that the coefficient of $\mathbf{a}_\pi = a_{1\pi(1)} \cdots a_{m\pi(m)}$ with π of cycle type $\mu = (\mu_1, \ldots, \mu_q)$ in e_λ is equal to $\varepsilon_\mu R_{\mu\lambda}$, where

- ε_μ is equal to $(-1)^{j_2+j_4+\cdots}$ for $\mu = \langle 1^{j_1} 2^{j_2} \cdots \rangle$, and
- $R_{\mu\lambda}$ is the number of ordered partitions $(B_1, \ldots, B_p)$ of the set $\{1, \ldots, q\}$ such that
$$\lambda_j = \sum_{i \in B_j} \mu_i$$
for $1 \leq j \leq p$.

But we know that if $\langle \cdot, \cdot \rangle$ is the standard scalar product in the space of symmetric functions defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ and ω is the standard (scalar product preserving) involution given by $\omega(h_\lambda) = e_\lambda$, then $p_\mu = \sum_\nu R_{\mu\nu} m_\nu$ and $\omega(p_\mu) = \varepsilon_\mu p_\mu$ (see [Sta99, §7.4 – §7.9]), so
$$\langle e_\lambda, p_\mu \rangle = \langle \omega(e_\lambda), \omega(p_\mu) \rangle = \varepsilon_\mu \langle h_\lambda, p_\mu \rangle = \varepsilon_\mu R_{\mu\lambda}.$$
Since e_λ form a vector-space basis of the space of symmetric functions and since both the scalar product and the operator $[\mathbf{a}_\pi]$ are linear, we have proved the following.

Proposition 5 *For any symmetric function f and for π a permutation of cycle type μ, we have*
$$[\mathbf{a}_\pi] f(\omega_1, \ldots, \omega_m) = \langle f, p_\mu \rangle$$
In particular,
$$[\mathbf{a}_\pi] s_\lambda(\omega_1, \ldots, \omega_m) = \langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu). \qquad \square$$
The proposition of course also implies that
$$[\mathbf{a}_\pi] p_\lambda(\omega_1, \ldots, \omega_m) = \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu},$$
which is what we proved at the end of the last section.

4. A β-extension

The β-extension of the MacMahon master theorem given in [FZ] yields the following extension of Theorem 1.

Theorem 6 *Denote by h_i^β the coefficient of t^i in*
$$\left(\frac{1}{\det(I - tA)} \right)^\beta.$$
Choose a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of m, and write $\mathbf{a}_\pi = a_{1\pi(1)} a_{2\pi(2)} \cdots a_{m\pi(m)}$ for a permutation $\pi \in S_m$. Then
$$[\mathbf{a}_\pi] \det(h^\beta_{\lambda_i - i + j})_{p \times p} = \beta^{\operatorname{cyc} \pi} \chi^\lambda(\pi), \tag{5}$$
where χ^λ is the irreducible character of the symmetric group S_m corresponding to λ, and $\operatorname{cyc} \pi$ is the number of cycles of π.
In particular, for $\beta = -1$, the formula (5) gives the relation
$$\chi^{\lambda'}(\pi) = (-1)^{\operatorname{cyc} \pi + |\lambda|} \chi^\lambda(\pi). \tag{6}$$

Sketch of proof: Each term in the expansion of the determinant on the left-hand side of (5) is a product of certain h_i^β. The same reasoning as in the previous section shows that for each cycle $(j, \pi(j), \pi^2(j), \ldots)$, the variables $a_{j,\pi(j)}, a_{\pi(j),\pi^2(j)}, \ldots$ must be chosen from the same h_i^β. It follows immediately from [FZ, §3] that if

$$\left(a_{j_1,\pi(j_1)}a_{\pi(j_1),\pi^2(j_1)}\cdots\right)\left(a_{j_2,\pi(j_2)}a_{\pi(j_2),\pi^2(j_2)}\cdots\right)\cdots\left(a_{j_s,\pi(j_s)}a_{\pi(j_s),\pi^2(j_s)}\cdots\right)$$

is the disjoint cycle decomposition of a permutation π, then the coefficient of $\mathbf{a}_\pi$ in h_i^β is β^s. It follows that the coefficient of $\mathbf{a}_\pi$ in $\det(h^\beta_{\lambda_i-i+j})_{p\times p}$ is equal to $\beta^{\operatorname{cyc}\pi}$ times the coefficient of $\mathbf{a}_\pi$ in $\det(h_{\lambda_i-i+j})_{p\times p}$, which is precisely (5). □

5. An application to irreducible characters of S_m

It is worthwhile to note the following determinantal description of irreducible characters of the symmetric group.

Corollary 7 *Let λ, μ be partitions of m and define $f_0, \ldots, f_m$ via the formula*

$$(t^{\mu_1}-u_1)(t^{\mu_2}-u_2)\cdots(t^{\mu_q}-u_q) = f_0t^m - f_1t^{m-1} + \ldots + (-1)^m f_m.$$

Then

$$\chi^\lambda(\mu) = [u_1\cdots u_q]\det(f_{\lambda'_i-i+j}) = (-1)^{|\lambda|+q}[u_1\cdots u_q]\det(f_{\lambda_i-i+j}).$$

Proof: Take the permutation $\pi = (1,2,\ldots,\mu_1)(\mu_1+1,\mu_1+2,\ldots,\mu_1+\mu_2)\cdots$ and form the block diagonal $A = A_1\oplus\ldots\oplus A_q$ with blocks of the form (4) corresponding to cycles of π. For example, for $\mu = (3,2,2)$, the permutation is $(123)(45)(67)$ and the matrix is

$$\begin{pmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{67} \\ 0 & 0 & 0 & 0 & 0 & a_{76} & 0 \end{pmatrix}.$$

The characteristic polynomial of A is

$$(t^{\mu_1}-u_1)(t^{\mu_2}-u_2)\cdots(t^{\mu_q}-u_q)$$

for $u_1 = a_{12}a_{23}\cdots a_{\mu_1 1}$, $u_2 = a_{\mu_1+1,\mu_1+2}a_{\mu_1+2,\mu_1+3}\cdots a_{\mu_1+\mu_2,\mu_1+1}$, etc. In other words, if $\omega_1,\ldots,\omega_m$ are the eigenvalues of A, then $e_i(\omega_1,\ldots,\omega_m) = f_i$. But then $\det(f_{\lambda'_i-i+j}) = s_\lambda(\omega_1,\ldots,\omega_m)$ and equation (2) implies

$$[u_1\cdots u_q]\det(f_{\lambda'_i-i+j}) = [\mathbf{a}_\pi]s_\lambda(\omega_1,\ldots,\omega_m) = \chi^\lambda(\mu).$$

The second equality is a corollary of (6). □

Example 8 Let $\lambda = (2,2,2,1)$ and $\mu = (3,2,2)$. Then
$(t^3 - u_1)(t^2 - u_2)(t^2 - u_3) = t^7 - (u_2 + u_3)t^5 - u_1 t^4 + u_2 u_3 t^3 + (u_1 u_2 + u_1 u_3)t^2 - u_1 u_2 u_3$
and so $f_0 = 1$, $f_1 = 0$, $f_2 = -u_2 - u_3$, $f_3 = u_1$, $f_4 = u_2 u_3$, $f_5 = -u_1 u_2 - u_1 u_3$, $f_6 = 0$, $f_7 = u_1 u_2 u_3$. Hence

$$\chi^{2221}(322) = [u_1 u_2 u_3] \begin{vmatrix} f_4 & f_5 \\ f_2 & f_3 \end{vmatrix} = [u_1 u_2 u_3](f_4 f_3 - f_2 f_5) = 1 - 2 = -1. \qquad \square$$

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