A NOTE ON QUANTUM IMMANANTS AND THE CYCLE BASIS OF THE QUANTUM PERMUTATION SPACE

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ABSTRACT. There are many combinatorial expressions for evaluating characters of the Hecke algebra of type A. However, with rare exceptions, they give simple results only for permutations that have minimal length in their conjugacy class. For other permutations, a recursive formula has to be applied. Consequently, quantum immanants are complicated objects when expressed in the standard basis of the quantum permutation space.
In this paper, we introduce another natural basis of the quantum permutation space, and we prove that coefficients of quantum immanants in this basis are class functions.

1. The symmetric group and immanants

Denote by $\mathfrak{S}_n$ the symmetric group of n, i.e. the group of permutations of the set $\{1,\ldots,n\}$. We write permutations in the one-line notation: $v = v_1v_2\cdots v_n$ means that $v(i) = v_i$. We multiply permutations from the right: $24315 \cdot 53241 = 53412$. We will often use the cycle notation $24315 = (124)(35)$. We will always write the smallest element of the cycle first, and order the cycles so that the first elements form an increasing sequence. We define the cycle type $\mu(v)$ as the sequence of lengths of these cycles. Note that it is a composition, not a partition; permutations $(124)(35)$ and $(14)(253)$ have a different cycle type. An *inversion* of a permutation v is a pair (i,j) satisfying $i < j$ and $v_i > v_j$. Denote by $\mathrm{inv}(v)$ the number of inversions of v. We denote the identity permutation by id.

The symmetric group $\mathfrak{S}_n$ is generated by *simple transpositions* $s_i = (i, i+1)$, $1 \le i \le n-1$, which satisfy the relations

$$\begin{aligned} s_i^2 &= 1 && \text{for } i = 1,\ldots,n-1,\\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} && \text{if } |i-j| = 1,\\ s_is_j &= s_js_i && \text{if } |i-j| \ge 2. \end{aligned}$$

An expression $v = s_{i_1}s_{i_2}\cdots s_{i_k}$, $1 \le i_j \le n-1$, is *reduced* if it is the shortest such expression for v, and we have $k = \mathrm{inv}(v)$. We call k the *length* of v. All reduced expressions contain the same generators, see [BB05, Corollary 1.4.8 (ii)].

A *(virtual) character* of a group $\mathfrak{G}$ is a linear function $\chi\colon \mathfrak{G} \to \mathbb{C}$ for which $\chi(ab) = \chi(ba)$ for all $a, b \in \mathfrak{G}$. For example, the trace of a representation $\rho\colon \mathfrak{G} \to GL_n$ is a character. The simplest character is the *trivial character* $\eta(v) = 1$. In the symmetric group, another important character is the *sign character* $\epsilon(v) = (-1)^{\mathrm{inv}(v)}$.

Choose commutative variables x_{ij}, $1 \le i,j \le n$. Denote by $\mathcal{A}_n$ the vector space of all polynomials in x_{ij} generated by monomials of the form $x_v = x_{1v_1}x_{2v_2}\cdots x_{nv_n}$ for a permutation $v \in \mathfrak{S}_n$, and call $\mathcal{A}_n$ the permutation space. We will also use notation

$x_{u,v} = x_{u_1v_1}x_{u_2v_2}\cdots x_{u_nv_n}$, where $u, v \in \mathfrak{S}_n$. For a character $\chi\colon \mathfrak{S}_n \to \mathbb{C}$, define the χ-*immanant* $\operatorname{Imm}_\lambda X \in \mathcal{A}_n$ by

$$\operatorname{Imm}_\chi X = \sum_{v \in \mathfrak{S}_n} \chi(v) x_v.$$

For example, $\operatorname{Imm}_\eta X$ is the permanent $\operatorname{per} X$ of the matrix $X = (x_{ij})_{n\times n}$, and $\operatorname{Imm}_\epsilon X$ is the determinant $\det X$.

2. The Hecke algebra and quantum immanants

A beautiful quantization of the symmetric group is $H_n(q)$, the Hecke algebra of type A. Here $q \in \mathbb{C} \setminus \{0\}$. It is defined as the $\mathbb{C}$-algebra generated by the set of *modified natural generators* $\{\widetilde{T}_{s_j}\colon 1 \le j \le n-1\}$ subject to the relations

$$\begin{aligned} \widetilde{T}_{s_i}^2 &= 1 + (q^{1/2} - q^{-1/2})\widetilde{T}_{s_i} && \text{for } i = 1, \ldots, n-1, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_{i+1}}\widetilde{T}_{s_i} &= \widetilde{T}_{s_{i+1}}\widetilde{T}_{s_i}\widetilde{T}_{s_{i+1}} && \text{for } i = 1, \ldots, n-2, \\ \widetilde{T}_{s_i}\widetilde{T}_{s_j} &= \widetilde{T}_{s_j}\widetilde{T}_{s_i} && \text{for } |i-j| \ge 2 \end{aligned}$$

Remark 1 In other contexts, *natural generators* $T_w = q^{1/2}\widetilde{T}_w$ are often used instead of $\widetilde{T}_w$.

If $s_{i_1}\cdots s_{i_k}$ is a reduced expression for v of length $k = \operatorname{inv}(v)$, we define

$$\widetilde{T}_v = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_k}}.$$

This is well defined (say, by Matsumoto's theorem, see [GP00, Theorem 1.2.2]). The elements $\widetilde{T}_v$, $v \in \mathfrak{S}_n$, form a basis of the algebra $H_n(q)$ by Bourbaki's theorem, see for example [GP00, Theorem 4.4.6].

If $\operatorname{inv}(s_iw) = \operatorname{inv}(w) - 1$, we have $w = s_i(s_iw)$ and therefore

$$\widetilde{T}_{s_i}\widetilde{T}_w = \widetilde{T}_{s_i}\widetilde{T}_{s_i(s_iw)} = \widetilde{T}_{s_i}^2\widetilde{T}_{s_iw} = (1 + (q^{1/2} - q^{-1/2})\widetilde{T}_{s_i})\widetilde{T}_{s_iw} = \widetilde{T}_{s_iw} + (q^{1/2} - q^{-1/2})\widetilde{T}_w.$$

Thus

$$\widetilde{T}_{s_i}\widetilde{T}_w = \begin{cases} \widetilde{T}_{s_iw} & : \ \operatorname{inv}(s_iw) = \operatorname{inv}(w) + 1 \\ \widetilde{T}_{s_iw} + (q^{1/2} - q^{-1/2})\widetilde{T}_w & : \ \operatorname{inv}(s_iw) = \operatorname{inv}(w) - 1 \end{cases},$$

and similarly

$$\widetilde{T}_w\widetilde{T}_{s_i} = \begin{cases} \widetilde{T}_{ws_i} & : \ \operatorname{inv}(ws_i) = \operatorname{inv}(w) + 1 \\ \widetilde{T}_{ws_i} + (q^{1/2} - q^{-1/2})\widetilde{T}_w & : \ \operatorname{inv}(ws_i) = \operatorname{inv}(w) - 1 \end{cases}.$$

A *character* of $H_n(q)$ is a linear functional $\chi\colon H_n(q) \to \mathbb{C}$ satisfying

$$\chi(\widetilde{T}_w\widetilde{T}_v) = \chi(\widetilde{T}_v\widetilde{T}_w). \tag{1}$$

Example 2 Let us prove that the linear map $\eta\colon H_n(q) \to \mathbb{C}$ defined by $\eta(\widetilde{T}_v) = q^{\operatorname{inv}(v)/2}$ is a character by showing that $\eta(\widetilde{T}_w\widetilde{T}_v) = q^{(\operatorname{inv}(w)+\operatorname{inv}(v))/2}$ for all w, v. This is obviously true if $v = \mathrm{id}$, assume that it holds for all w, v with $\operatorname{inv}(v) = k-1$, and assume $\operatorname{inv}(v) = k$. We have $v = sv'$ for some $s \in \{s_1, \ldots, s_{n-1}\}$, $\operatorname{inv}(v') = k-1$. If $\operatorname{inv}(ws) = \operatorname{inv}(w) + 1$, then

$$\eta(\widetilde{T}_w\widetilde{T}_v) = \eta(\widetilde{T}_w\widetilde{T}_s\widetilde{T}_{v'}) = \eta(\widetilde{T}_{ws}\widetilde{T}_{v'}) = q^{(\operatorname{inv}(ws)+\operatorname{inv}(v'))/2} = q^{(\operatorname{inv}(w)+\operatorname{inv}(v))/2},$$

and if $\mathrm{inv}(ws) = \mathrm{inv}(w) - 1$, then

$$\eta(\widetilde{T}_w \widetilde{T}_v) = \eta(\widetilde{T}_w \widetilde{T}_s \widetilde{T}_{v'}) = \eta((\widetilde{T}_{ws} + (q^{1/2} - q^{-1/2})\widetilde{T}_w)\widetilde{T}_{v'}) =$$
$$= \eta(\widetilde{T}_{ws}\widetilde{T}_{v'}) + (q^{1/2} - q^{-1/2})\eta(\widetilde{T}_w \widetilde{T}_{v'}) = q^{(\mathrm{inv}(ws)+\mathrm{inv}(v'))/2} + (q^{1/2} - q^{-1/2})q^{(\mathrm{inv}(w)+\mathrm{inv}(v'))/2} =$$
$$= q^{(\mathrm{inv}(w)+\mathrm{inv}(v))/2-1} + (q^{1/2} - q^{-1/2})q^{(\mathrm{inv}(w)+\mathrm{inv}(v)-1)/2} = q^{(\mathrm{inv}(w)+\mathrm{inv}(v))/2}.$$

This character is called *trivial*. We can similarly prove that $\epsilon\colon H_n(q) \to \mathbb{C}$, defined by $\epsilon(\widetilde{T}_v) = (-q^{-1/2})^{\mathrm{inv}(v)}$, is a character, we call it the *sign character*.

We have the following relation for characters of $H_n(q)$.

Proposition 3 *Take $v \in \mathfrak{S}_n$, $s = s_i$ for some $i \in \{1, \ldots, n-1\}$, and a character χ of $H_n(q)$. Then:*

(a) *if $\mathrm{inv}(svs) = \mathrm{inv}(v)$, then $\chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v)$;*
(b) *if $\mathrm{inv}(svs) = \mathrm{inv}(v) + 2$, then $\chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{sv}) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{vs})$;*
(c) *if $\mathrm{inv}(svs) = \mathrm{inv}(v) - 2$, then $\chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v) - (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{sv}) = \chi(\widetilde{T}_v) - (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{vs})$;*

Proof. Assume that $\mathrm{inv}(sv) = \mathrm{inv}(v) - 1$ and $\mathrm{inv}(svs) = \mathrm{inv}(v)$. Then

$$\chi(\widetilde{T}_s \widetilde{T}_v \widetilde{T}_s) = \chi((\widetilde{T}_{sv} + (q^{1/2} - q^{-1/2})\widetilde{T}_v)\widetilde{T}_s) = \chi(\widetilde{T}_{svs}) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_v \widetilde{T}_s)$$

and, by (1),

$$\chi(\widetilde{T}_s \widetilde{T}_v \widetilde{T}_s) = \chi(\widetilde{T}_v \widetilde{T}_s \widetilde{T}_s) = \chi(\widetilde{T}_v(1 + (q^{1/2} - q^{-1/2})\widetilde{T}_s)) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_v \widetilde{T}_s),$$

so $\chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v)$. If $\mathrm{inv}(sv) = \mathrm{inv}(v) + 1$ and $\mathrm{inv}(svs) = \mathrm{inv}(v)$, then

$$\chi(\widetilde{T}_s \widetilde{T}_v \widetilde{T}_s) = \chi(\widetilde{T}_{sv} \widetilde{T}_s) = \chi(\widetilde{T}_{svs}) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{sv})$$

and, by (1),

$$\chi(\widetilde{T}_s \widetilde{T}_v \widetilde{T}_s) = \chi(\widetilde{T}_s \widetilde{T}_s \widetilde{T}_v) = \chi((1 + (q^{1/2} - q^{-1/2})\widetilde{T}_s)\widetilde{T}_v) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{sv}).$$

This proves (a). Let us prove (b). If $\mathrm{inv}(svs) = \mathrm{inv}(v) + 2$, then $\mathrm{inv}(sv) = \mathrm{inv}(vs) = \mathrm{inv}(v) + 1$, and so

$$\chi(\widetilde{T}_s \widetilde{T}_v \widetilde{T}_s) = \chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v \widetilde{T}_s \widetilde{T}_s) =$$
$$= \chi(\widetilde{T}_v(1 + (q^{1/2} - q^{-1/2})\widetilde{T}_s)) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{vs}),$$

and since $\chi(\widetilde{T}_s \widetilde{T}_v) = \chi(\widetilde{T}_v \widetilde{T}_s)$, we have $\chi(\widetilde{T}_{svs}) = \chi(\widetilde{T}_v) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_{sv})$. Swapping the roles of v and svs, we get (c) from (b). □

The *quantum polynomial ring* is generated by n^2 variables x_{ij}, $1 \le i, j \le n$, subject to the relations

$$\begin{aligned} x_{il}x_{ik} &= q^{1/2}x_{ik}x_{il}, \\ x_{jk}x_{ik} &= q^{1/2}x_{ik}x_{jk}, \\ x_{jk}x_{il} &= x_{il}x_{jk}, \\ x_{jl}x_{ik} &= x_{ik}x_{jl} + (q^{1/2} - q^{-1/2})x_{il}x_{jk} \end{aligned} \tag{2}$$

for all indices $i < j$, $k < l$. Denote by $\mathcal{A}_n(q)$ the subspace generated by monomials $x_{u,v} = x_{u_1v_1}x_{u_2v_2}\cdots x_{u_nv_n}$, where $u, v \in \mathfrak{S}_n$, and call it the *quantum permutation space*.

We will also use notation $x_v = x_{1v_1}x_{2v_2}\cdots x_{nv_n}$, where $v \in \mathfrak{S}_n$. The set $\{x_v \colon v \in \mathfrak{S}_n\}$ is a basis of $\mathcal{A}_n(q)$, we call it the *natural basis*. For a character $\chi\colon H_n(q) \to \mathbb{C}$, define the *modified χ-immanant* $\operatorname{Imm}_\lambda X \in \mathcal{A}_n(q)$ by

$$\operatorname{Imm}_\chi X = \sum_{v\in\mathfrak{S}_n} \chi(\widetilde{T}_v)x_v.$$

We call

$$\operatorname{Imm}_\eta X = \sum_{v\in\mathfrak{S}_n} q^{\operatorname{inv}(v)/2}x_v$$

the *modified quantum permanent*, and

$$\operatorname{Imm}_\epsilon X = \sum_{v\in\mathfrak{S}_n} (-q^{-1/2})^{\operatorname{inv}(v)}x_v$$

is the *modified quantum determinant*.

Remark 4 We use the word modified to distinguish this object from the sum

$$\sum_{v\in\mathfrak{S}_n} \chi(T_v)x_v.$$

See [KS] for other results on quantum immanants and for further references.

3. Cycle basis of the quantum permutation space and the main results

Given a permutation v, write it in cycle notation

$$v = (i_1^1,\ldots,i_{\mu_1}^1)(i_1^2,\ldots,i_{\mu_2}^2)\cdots(i_1^r,\ldots,i_{\mu_r}^r)$$

so that i_1^j is the smallest element of the j-th cycle and so that $i_1^1 < i_1^2 < \ldots < i_1^r$. Define the v *cycle monomial*, denoted x^v, to be the product

$$(x_{i_1^1 i_2^1}x_{i_2^1 i_3^1}\cdots x_{i_{\mu_1}^1 i_1^1})(x_{i_1^2 i_2^2}x_{i_2^2 i_3^2}\cdots x_{i_{\mu_2}^2 i_1^2})\cdots(x_{i_1^r i_2^r}x_{i_2^r i_3^r}\cdots x_{i_{\mu_r}^r i_1^r}).$$

For example, consider the permutation 45213, which is (14)(253) in cycle notation. Then $x_{45213} = x_{14}x_{25}x_{32}x_{41}x_{53}$ and $x^{45213} = x_{14}x_{41}x_{25}x_{53}x_{32}$.

Proposition 5 *The monomials $\{x^v \colon v \in \mathfrak{S}_n\}$ form a basis of $\mathcal{A}_n(q)$. Furthermore, the transition matrix relating this basis to the natural basis $\{x_v \colon v \in \mathfrak{S}_n\}$ is unitriangular.*

We will give the proof in the next section.

There are several results that give combinatorial descriptions of families of characters of the Hecke algebra $H_n(q)$ (see [Ram91], [RR97] and [Kon, §3]. However, neither of these results gives a description of $\chi(T_v)$ or $\chi(\widetilde{T}_v)$ for *all* $v \in \mathfrak{S}_n$ except in the simplest of cases (namely, the trivial and sign characters). For v which is not of minimal length in its conjugacy class, we have to use Proposition 3 to find $\chi(\widetilde{T}_v)$. Consequently, there are no simple formulas for immanants, with the exception of the modified quantum permanent and determinant.

The main result of this paper gives the expansion of modified quantum immanants in the cycle basis.

Example 6 Take the modified quantum permanent for $n=3$. Since

$$x_{21}x_{32}+q^{1/2}x_{22}x_{31}=x_{32}x_{21}+q^{-1/2}x_{22}x_{31}=x_{32}x_{21}+q^{-1/2}x_{31}x_{22},$$

we have

$$x_{11}x_{22}x_{33}+q^{1/2}x_{11}x_{23}x_{32}+q^{1/2}x_{12}x_{21}x_{33}+qx_{12}x_{23}x_{31}+qx_{13}x_{21}x_{32}+q^{3/2}x_{13}x_{22}x_{31}=$$
$$=x_{11}x_{22}x_{33}+q^{1/2}x_{11}x_{23}x_{32}+q^{1/2}x_{12}x_{21}x_{33}+qx_{12}x_{23}x_{31}+qx_{13}x_{32}x_{21}+q^{1/2}x_{13}x_{31}x_{22}.$$

Note that in this case, the coefficients of the cycle basis elements in the modified quantum permanent depend solely on the cycle type of the permutation. This is, in fact, true for all modified quantum immanants, as the main theorem shows.

Choose a composition $\mu=(\mu_1,\dots,\mu_p)$ of n. Denote the permutation

$$(1,2,\dots,\mu_1)(\mu_1+1,\mu_1+2,\dots,\mu_1+\mu_2)\cdots$$

by γ_μ. Recall that if v is a permutation, we denote by $\mu(v)$ its cycle type (see the first paragraph of Section 1).

The following is obvious.

Proposition 7 *A permutation v is of the form γ_μ for some μ if and only if $v(i)\leq i+1$ for all i.* □

The main theorem tells us that the set $\{x^v\colon v\in\mathfrak{S}_n\}$, which is, by Proposition 5, a basis, is in a certain sense superior to the usual basis $\{x_v\colon v\in\mathfrak{S}_n\}$.

Main theorem *For a character χ of $H_n(q)$, we have*

$$\mathrm{Imm}_\lambda X=\sum_{v\in\mathfrak{S}_n}\chi(\widetilde{T}_{\gamma_{\mu(v)}})x^v.$$

Remark 8 The quantum permutation space is isomorphic as a vector space to the Hecke algebra of type A via the isomorphism $x_{\alpha,\beta}\mapsto\widetilde{T}_\beta\widetilde{T}_{\alpha^{-1}}$. Therefore the main theorem gives us a new basis of the Hecke algebra of type A.

4. Proofs

We will make use of the following procedure, which takes a permutation v as an input and produces three sequences α_k^v, β_k^v and γ_k^v of permutations and two sequences i_k^v and j_k^v of integers.

(1) Set $\alpha_0^v=\mathrm{id}$, $\beta_0^v=v$, $k=0$.
(2) Repeat the following. Take $\pi_k^v=(\alpha_k^v)^{-1}\beta_k^v$ and let i_k^v be the least index for which $\pi_k^v(i_k^v)>i_k^v+1$. If no such index exists, set $\alpha_{k+1}^v=\alpha_k^v$, $\beta_{k+1}^v=\beta_k^v$, $\pi_{k+1}^v=\pi_k^v$, and terminate the sequences i^v and j^v. Otherwise
   (a) Set $j_k^v=\pi_k^v(i_k^v)-1$.
   (b) Set $\alpha_{k+1}^v=\alpha_k^v s_{j_k^v}$, $\beta_{k+1}^v=\beta_k^v s_{j_k^v}$.
   (c) Increment k.

We will denote by p^v the smallest index for which $\pi_{p^v}^v(i)\leq i+1$ for all i; note that we have $\alpha_{p^v}^v=\alpha_{p^v+1}^v=\alpha_{p^v+2}^v=\dots$, $\beta_k^v=\beta_{p^v+1}^v=\beta_{p^v+2}^v=\dots$, $\pi_{p^v}^v=\pi_{p^v+1}^v=\pi_{p^v+2}^v=\dots$ In theory, we could have $p^v=\infty$, but we will prove in Lemma 9 that this is not the case. Also note that

$$\pi_{k+1}^v=(\alpha_{k+1}^v)^{-1}\beta_{k+1}^v=s_{j_k^v}(\alpha_k^v)^{-1}\beta_k^v s_{j_k^v}=s_{j_k^v}\pi_k^v s_{j_k^v}$$

if $k < p^v$.

As an example, consider the permutation $v = 45132$. Applying the procedure, we have

k	0	1	2	3	4
α_k^v	12345	12435	14235	14325	14325
β_k^v	45132	45312	43512	43152	43152
π_k^v	45132	35412	24513	23154	23154
π_k^v, cycle notation	(143)(25)	(134)(25)	(124)(35)	(123)(45)	(123)(45)
i_k^v	1	1	2		
j_k^v	3	2	3		

Let us prove a series of lemmas about these sequences.

Lemma 9 *For every permutation v, the sequence $i_0^v, i_1^v, i_2^v, \ldots$ is weakly increasing. Furthermore, if $i_{k-1}^v = i_k^v$ for some $k < p^v$, then $j_{k-1}^v = j_k^v + 1$. Consequently, $p^v \leq \binom{n-1}{2}$.*

Proof. Throughout the proof, we will omit the superscript v.
We want to prove that $i_{k-1} \leq i_k$. Since i_k is the smallest index for which $\pi_k(i_k) > i_k + 1$, it is enough to prove that $\pi_k(i) \leq i+1$ when $i < i_{k-1}$. For such i, we have $i < i_{k-1} < j_{k-1}$, so $s_{j_{k-1}}(i) = i$. Furthermore, $\pi_{k-1}(i) \leq i+1$ by definition of i_{k-1}. But then $\pi_{k-1}(i) \leq i+1 \leq i_{k-1} < j_{k-1}$ and $s_{j_{k-1}}\pi_{k-1}(i) = \pi_{k-1}(i)$. In other words, we have proved that

$$\pi_k(i) = s_{j_{k-1}}\pi_{k-1}s_{j_{k-1}}(i) = s_{j_{k-1}}\pi_{k-1}(i) = \pi_{k-1}(i) \leq i+1.$$

Assume that $i_{k-1} = i_k$. Then

$$j_k+1 = \pi_k(i_k) = \pi_k(i_{k-1}) = s_{j_{k-1}}\pi_{k-1}s_{j_{k-1}}(i_{k-1}) = s_{j_{k-1}}\pi_{k-1}(i_{k-1}) = s_{j_{k-1}}(j_{k-1}+1) = j_{k-1},$$

where we used the fact that $j_{k-1} > i_{k-1}$ and therefore $s_{j_{k-1}}(i_{k-1}) = i_{k-1}$.
That means that $(i_k^v, j_k^v) \neq (i_l^v, j_l^v)$ if $k \neq l$. Since $1 \leq i_k < j_k < n$, we have at most $\binom{n-1}{2}$ such pairs. □

Lemma 10 *Take $v \in \mathfrak{S}_n$ and $k < p^v$. If $i > i_k^v$ and $l \leq k$, then $s_{j_l^v}s_{j_{l+1}^v}\cdots s_{j_{k-1}^v}(i) > i_l^v$.*

Proof. Throughout the proof, we will omit the superscript v.
By induction, it is enough to prove this statement for $l = k-1$. By the previous lemma, we have $i_k \geq i_{k-1}$. If $i_k > i_{k-1}$, then $i \geq i_{k-1}+2$. Multiplicating a permutation π by a simple transposition changes the value $\pi(i)$ by at most 1 for any i; in particular, $s_{j_{k-1}}(i) \geq i_{k-1}+1 > i_{k-1}$. If $i_k = i_{k-1}$ and $i \geq i_k+2$, the reasoning is the same. So it remains to prove that if $i_k = i_{k-1}$, then $s_{j_{k-1}}(i_k+1) > i_{k-1}$. But $j_{k-1} = j_k+1$ by the previous lemma and $j_k+1 > i_k+1$, so

$$s_{j_{k-1}}(i_k+1) = s_{j_k+1}(i_k+1) = i_k+1 = i_{k-1}+1 > i_{k-1},$$

which finishes the proof. □

Lemma 11 *For all v and $k < p^v$, we have $\alpha_k^v(j_k^v) < \alpha_k^v(j_k^v+1)$. Furthermore, if $i < j$ and $\alpha_k^v(i) > \alpha_k^v(j)$, then $\alpha_k^v(i) = \beta_l^v(i_l^v)$ for some $l < k$.*

Proof. Throughout the proof, we will omit the superscript v.
Let us first prove the second statement by induction on k. For $k = 0$, $\alpha_k = \mathrm{id}$ and there is nothing to prove. Now assume that the statement holds for $k-1$. We have $\alpha_k = \alpha_{k-1}s_{j_{k-1}}$. Take $i < j$ with $\alpha_k(i) > \alpha_k(j)$. We have the following possible cases:

- $i, j \neq j_{k-1}, j_{k-1}+1$. Then $\alpha_k(i) = \alpha_{k-1}(i)$ and $\alpha_k(j) = \alpha_{k-1}(j)$ and, by the induction hypothesis, $\alpha_k(i) = \beta_l(i_l)$ for some $l < k-1 < k$.
- $i < j_{k-1}, j = j_{k-1}$ or $i < j_{k-1}, j = j_{k-1}+1$. Then $\alpha_k(i) = \alpha_{k-1}(i)$, $\alpha_k(j) = \alpha_{k-1}(j+1)$ (respectively, $\alpha_k(j) = \alpha_{k-1}(j-1)$) and $\alpha_k(i) = \beta_l(i_l)$ for some $l < k-1 < k$ by the induction hypothesis.
- $i = j_{k-1}, j > j_{k-1}+1$ or $i = j_{k-1}+1, j > j_{k-1}+1$. Then $\alpha_k(i) = \alpha_{k-1}(i+1)$ (respectively, $\alpha_k(i) = \alpha_{k-1}(i-1)$) and $\alpha_k(j) = \alpha_{k-1}(j)$. By the induction hypothesis, $\alpha_{k-1}(i+1) = \alpha_k(i) = \beta_l(i_l)$ for some $l < k-1 < k$ (respectively, $\alpha_{k-1}(i-1) = \alpha_k(i) = \beta_l(i_l)$ for some $l < k-1 < k$).
- $i = j_{k-1}, j = j_{k-1}+1$. In this case, $\alpha_k(i) = \alpha_{k-1}(i+1) = \alpha_{k-1}(j_{k-1}+1) = \alpha_{k-1}(\pi_{k-1}(i_{k-1})) = \beta_{k-1}(i_{k-1})$.

Now assume that $\alpha_k(j_k) > \alpha_k(j_k+1)$. We now know that this implies that $\alpha_k(j_k) = \beta_l(i_l)$ for some $l < k$. Furthermore, $\beta_k(i_l) = \beta_l s_{j_l} \cdots s_{j_{k-1}}(i_l) = \beta_l(i_l)$ because $j_l > i_l$, $j_{l+1} > i_{l+1} \geq i_l$, etc. If $i_l < i_k$, then $j_k = \alpha_k^{-1}\beta_l(i_l) = \alpha_k^{-1}\beta_k(i_l) = \pi_k(i_l) =\leq i_l + 1 < i_k + 1 \leq j_k$, a contradiction. On the other hand, $i_l = i_k$ implies $j_k = \alpha_k^{-1}\beta_l(i_l) = \alpha_k^{-1}\beta_k(i_l) = \pi_k(i_k) = j_k + 1$, again a contradiction. Therefore we must have $\alpha_k(j_k) < \alpha_k(j_k+1)$. □

Lemma 12 *For all v and $k < p^v$, we have $(\pi_k^v)^{-1}(j_k^v) > (\pi_k^v)^{-1}(j_k^v+1)$. Furthermore, $\beta_k^v(j_k^v) < \beta_k^v(j_k^v+1)$ if and only if $s_{j_k^v}\pi_k^v(j_k^v) < s_{j_k^v}\pi_k^v(j_k^v+1)$, and if and only if $\pi_k^v(j_k^v) < \pi_k^v(j_k^v+1)$.*

Proof. Throughout the proof, we will omit the superscript v.
We know that $(\pi_k)^{-1}(j_k+1) = i_k$. Therefore $(\pi_k)^{-1}(j_k) < i_k$ would, by definition of i_k, imply $j_k = \pi_k((\pi_k)^{-1}(j_k)) \leq (\pi_k)^{-1}(j_k) + 1 \leq i_k < j_k$, a contradiction. That proves the first statement.
Denote $s_{j_k}\pi_k(j_k) = s_{j_k}\alpha_k^{-1}\beta_k(j_k) = \alpha_{k+1}^{-1}\beta_k(j_k)$ by i and $s_{j_k}\pi_k(j_k+1) = \alpha_{k+1}^{-1}\beta_k(j_k+1)$ by j. If $i < j$ and $\alpha_{k+1}(i) = \beta_k(j_k) > \alpha_{k+1}(j) = \beta_k(j_k+1)$, then Lemma 11 implies that $\alpha_{k+1}(i) = \beta_k(j_k) = \beta_l(i_l)$ for some $l < k+1$. Like in the proof of the previous lemma, $\beta_k(i_l) = \beta_l(i_l)$. Then $\beta_k(j_k) = \beta_k(i_l)$ implies $j_k = i_l \leq i_k < j_k$, a contradiction. Similarly, $i > j$ and $\alpha_{k+1}(i) = \beta_k(j_k) < \alpha_{k+1}(j) = \beta_k(j_k+1)$ implies $\alpha_{k+1}(j) = \beta_k(j_k+1) = \beta_l(i_l) = \beta_k(i_l)$ for some $l < k+1$, and $j_k + 1 = i_l \leq i_k < j_k$ gives the desired contradiction. Therefore $\beta_k^v(j_k^v) < \beta_k^v(j_k^v+1)$ if and only if $s_{j_k^v}\pi_k^v(j_k^v) < s_{j_k^v}\pi_k^v(j_k^v+1)$.
The proof of the last statement is almost the same (with α_{k+1} replaced by α_k, and $l < k+1$ replaced by $l < k$). □

Lemma 13 *For every permutation v, we have $\pi_k^v = \gamma_{\mu(v)}$ for $k \geq p^v$.*

Proof. Throughout the proof, we will omit the superscript v.
We claim that for every k, π_k and π_{k+1} have the same cycle type. If $k \geq p$, $\pi_k = \pi_{k+1}$ and the statement is obvious. Otherwise, we have

$$\pi_k = (1,\ldots,\mu_1)(\mu_1+1,\ldots,\mu_1+\mu_2)\cdots(\ldots,i_k-1,i_k,j_k+1,\ldots \Big[)\cdots(\Big] \ldots,j_k,\ldots)\cdots$$

for $j_k > i_k$, where the part in brackets either appears (if j_k and j_k+1 are in different cycles) or not. Then

$$\pi_{k+1} = s_{j_k}\pi_k s_{j_k} = (1,\ldots,\mu_1)\cdots(\ldots,i_k-1,i_k,j_k,\ldots \Big[)\cdots(\Big] \ldots,j_k+1,\ldots)\cdots$$

has the same type as π_k. Indeed, j_k+1 is not the first element of its cycle in π_k; j_k is the first element of its cycle in π_k if and only j_k+1 is the first element of its cycle in π_{k+1}, and if this is the case, the relative position of the cycle starting with j_k in π_k is the same as the relative position of the cycle starting with j_k+1 in π_{k+1}.
This implies that $v=\pi_0$ and π_p have the same cycle type. Furthermore, $\pi_p(i)\leq i+1$ for all i, so $\pi_p=\gamma_{\mu(\pi_p)}=\gamma_{\mu(v)}$. Since $\pi_k=\pi_p$ for $k\geq p$, $\pi_k^v=\gamma_{\mu(v)}$ for $k\geq p$. □

Lemma 14 *For every permutation v, we have $x_{\alpha_k^v,\beta_k^v}=x^v$ for $k\geq p^v$.*

Proof. Obviously, it is enough to prove the statement for $k=p^v$. Throughout the proof, we will omit the superscript v.
For every $k<p$, $\beta_{k+1}\alpha_{k+1}^{-1}=\beta_k s_{j_k}s_{j_k}\alpha_k^{-1}=\beta_k\alpha_k^{-1}$. That implies that $\beta\alpha^{-1}=v$, where we write $\alpha=\alpha_p$ and $\beta=\beta_p$. That means that the monomial $x_{\alpha,\beta}$ is a rearrangement of the monomial x.
On the other hand, $\pi(i)=\alpha^{-1}\beta(i)\leq i+1$ (where $\pi=\pi_p$) means that in $x_{\alpha,\beta}$, the variable $x_{iv(i)}$ appears either immediately before or after $x_{v(i)v^2(i)}$. That means that $x_{\alpha,\beta}$ is indeed a product of "cycles" $x_{iv(i)}x_{v(i)v^2(i)}\cdots x_{v^{c-1}(i)i}$, and it remains to show that for every such monomial, $i<v(i),v^2(i),\ldots$, and that if $x_{iv(i)}x_{v(i)v^2(i)}\cdots x_{v^{c-1}(i)i}$ appears to the left of $x_{i'v(i')}x_{v(i')v^2(i')}\cdots x_{v^{c'-1}(i')i'}$, then $i<i'$.
Note that for every $k<p$, $j_k>i_k\geq 1$. Therefore $\alpha(1)=s_{j_0}s_{j_1}\cdots s_{j_{p-1}}(1)=1$. In other words, the first variable of $x_{\alpha,\beta}$ is indeed $x_{1v(1)}$. That means that the first "cycle" of $x_{\alpha,\beta}$ is $x_{1v(1)}x_{v(1)v^2(1)}\cdots x_{v^{c-1}(1)1}$, which satisfies the above conditions. Furthermore, if $i_k\leq c$, $i<j$ and $\alpha_k(i)>\alpha_k(j)$, then by Lemma 11 we have $\alpha_k(i)=\beta_l(i_l)$ for some $l<k$. Then

$$\beta_l(i_l)=v\alpha_l(i_l)=v\alpha_l\pi_l(i_l-1)=v^2\alpha_l(i_l-1)=\ldots=v^{i_l}(1).$$

That means that in the one-line notation of α_k for $i_k\leq c$, the elements that are not in $\{1,v(1),v^2(1),\ldots,v^l(1)\}$ are written in increasing order. Induction on the number of cycles of v finishes the proof. □

Lemma 15 *Take $v\in\mathfrak{S}_n$ and $k<p^v$. Then there exists (a unique) $w\in\mathfrak{S}_n$ such that:*

- $k<p^w$
- $i_l^w=i_l^v$ *for* $l=0,1,\ldots,k$
- $j_l^w=j_l^v$ *for* $l=0,1,\ldots,k$
- $\alpha_k^w=\alpha_k^v$
- $\beta_k^w=\beta_k^v s_{j_k^v}$

Proof. In the previous lemma, we proved that $\beta_k^v(\alpha_k^v)^{-1}=v$ for every v and k. Therefore the only possible candidate for such w is $w=\beta_k^v s_{j_k^v}(\alpha_k^v)^{-1}$. Let us prove that this permutation indeed satisfies all the conditions of the lemma.
We want to prove that $k<p^w$, $i_l^w=i_l^v$ and $j_l^w=j_l^v$ for all $0\leq l\leq k$. Note that it is enough to prove that for all $l=0,\ldots,k$, we have $\pi_l^w(i)=\pi_l^v(i)$ for $i\leq i_l^v$. Assume by induction that this holds for $0,\ldots,l-1$. Then

$$\pi_l^w=s_{j_{l-1}^w}\cdots s_{j_0^w}\beta_k^v s_{j_k^v}(\alpha_k^v)^{-1}s_{j_0^w}\cdots s_{j_{l-1}^w}=s_{j_{l-1}^v}\cdots s_{j_0^v}\beta_k^v s_{j_k^v}(\alpha_k^v)^{-1}s_{j_0^v}\cdots s_{j_{l-1}^v}$$

and

$$\pi_l^v=s_{j_{l-1}^v}\cdots s_{j_0^v}\beta_k^v(\alpha_k^v)^{-1}s_{j_0^v}\cdots s_{j_{l-1}^v},$$

so we have to prove that

$$s_{j_k^v}(\alpha_k^v)^{-1}s_{j_0^v}\cdots s_{j_{l-1}^v}(i)=(\alpha_k^v)^{-1}s_{j_0^v}\cdots s_{j_{l-1}^v}(i)$$

for $i\le i_l^v$. This is equivalent to

$$(\alpha_k^v)^{-1}s_{j_0^v}\cdots s_{j_{l-1}^v}(i)\neq j_k^v,j_k^v+1$$

for $i\le i_l^v$, and this is equivalent to

$$i_l^v<s_{j_{l-1}^v}\cdots s_{j_0^v}\alpha_k^v(j_k^v),s_{j_{l-1}^v}\cdots s_{j_0^v}\alpha_k^v(j_k^v+1).$$

Since $\alpha_k^v=s_{j_0^v}\cdots s_{j_{k-1}^v}$, this is equivalent to

$$s_{j_l^v}s_{j_{l+1}^v}\cdots s_{j_{k-1}^v}(j_k^v),s_{j_l^v}s_{j_{l+1}^v}\cdots s_{j_{k-1}^v}(j_k^v+1)>i_l^v.$$

Since $j_k^v>i_k^v$, this follows from Lemma 10.
Also, $\alpha_k^w=s_{j_0^w}\cdots s_{j_{k-1}^w}=s_{j_0^v}\cdots s_{j_{k-1}^v}=\alpha_k^v$ and $\beta_k^w=w\alpha_k^w=\beta_k^v s_{j_k^v}(\alpha_k^v)^{-1}\alpha_k^w=\beta_k^v s_{j_k^v}$. That means that w has all the necessary properties. □

Lemma 16 *For every character χ of $H_n(q)$ and all indices $k\ge 0$, we have*

$$\sum_{v\in\mathfrak{S}_n}\chi(\widetilde{T}_{\pi_k^v})x_{\alpha_k^v,\beta_k^v}=\sum_{v\in\mathfrak{S}_n}\chi(\widetilde{T}_{\pi_{k+1}^v})x_{\alpha_{k+1}^v,\beta_{k+1}^v}. \tag{3}$$

Proof. Fix k in and consider the left-hand side

$$\sum_{v\in\mathfrak{S}_n}\chi(\widetilde{T}_{\pi_k^v})x_{\alpha_k^v,\beta_k^v}. \tag{4}$$

Take $v\in\mathfrak{S}_n$. If $k\ge p^v$, then $\alpha_{k+1}^v=\alpha_k^v$, $\beta_{k+1}^v=\beta_k^v$ and $\pi_{k+1}^v=\pi_k^v$. Therefore

$$\chi(\widetilde{T}_{\pi_{k+1}^v})x_{\alpha_{k+1}^v,\beta_{k+1}^v}=\chi(\widetilde{T}_{\pi_k^v})x_{\alpha_k^v,\beta_k^v}.$$

On the other hand, if $k<p^v$, take w from the last lemma. Write $\alpha=\alpha_k^v$, $\beta=\beta_k^v$, $\pi=\pi_v^k$, $j=j_k^v$. We know that $\alpha_k^w=\alpha$, $\beta_k^w=\beta s_j$, $\pi_k^w=(\alpha_k^w)^{-1}\beta_k^w=\pi s_j$, $\alpha_{k+1}^v=\alpha s_j$, $\beta_{k+1}^v=\beta s_j$, $\pi_{k+1}^v=(\alpha_{k+1}^v)^{-1}\beta_{k+1}^v=s_j\pi s_j$, $\alpha_{k+1}^w=\alpha_k^w s_{j_k^w}=\alpha s_j$, $\beta_{k+1}^w=\beta_k^w s_{j_k^w}=(\beta s_j)s_j=\beta$ and $\pi_{k+1}^w=(\alpha_{k+1}^w)^{-1}\beta_{k+1}^w=s_j\pi$. We also know that both $x_{\alpha_k^v,\beta_k^v}=x_{\alpha,\beta}$ and $x_{\alpha_k^w,\beta_k^w}=x_{\alpha,\beta s_j}$ appear in (4), the former with coefficient $\chi(\widetilde{T}_\pi)$ and the latter with coefficient $\chi(\widetilde{T}_{\pi s_j})$.
Note that $\mathrm{inv}(\sigma s_j)=\mathrm{inv}(\sigma)+1$ if and only if $\sigma(j)<\sigma(j+1)$. Since, by Lemma 12, we have $\pi^{-1}(j)>\pi^{-1}(j+1)$, that implies that $\mathrm{inv}(s_j\pi)=\mathrm{inv}(\pi^{-1}s_j)=\mathrm{inv}(\pi)-1$.
If $\beta(j)<\beta(j+1)$, we have $s_j\pi(j)<s_j\pi(j+1)$ by Lemma 12 and therefore $\mathrm{inv}(s_j\pi s_j)=\mathrm{inv}(s_j\pi)+1=\mathrm{inv}(\pi)$. Also by Lemma 12, we have $\pi(j)<\pi(j+1)$ and $\mathrm{inv}(\pi s_j)=\mathrm{inv}(\pi)+1$. If, on the other hand, $\beta(j)>\beta(j+1)$, we have $s_j\pi(j)>s_j\pi(j+1)$ and $\pi(j)>\pi(j+1)$ by the same lemma and $\mathrm{inv}(s_j\pi s_j)=\mathrm{inv}(s_j\pi)-1=\mathrm{inv}(\pi)-2$, $\mathrm{inv}(\pi s_j)=\mathrm{inv}(\pi)-1$.
Note that the first statement of Lemma 11 tells us that $\alpha(j)<\alpha(j+1)$. Let us study the two possible cases individually:

- $\alpha(j)<\alpha(j+1)$, $\beta(j)<\beta(j+1)$, $\mathrm{inv}(s_j\pi)=\mathrm{inv}(\pi)-1$, $\mathrm{inv}(\pi s_j)=\mathrm{inv}(\pi)+1$, $\mathrm{inv}(s_j\pi s_j)=\mathrm{inv}(\pi)$. By Proposition 3, $\mathrm{inv}(\pi s_j)=\mathrm{inv}(s_j(s_j\pi)s_j)=\mathrm{inv}(s_j\pi)+2$ implies

  $$\chi(\widetilde{T}_{\pi s_j})=\chi(\widetilde{T}_{s_j\pi})+(q^{1/2}-q^{-1/2})\chi(\widetilde{T}_\pi).$$

  Therefore

  $$\chi(\widetilde{T}_\pi)x_{\alpha_j\beta_j}x_{\alpha_{j+1}\beta_{j+1}}+\chi(\widetilde{T}_{\pi s_j})x_{\alpha_j\beta_{j+1}}x_{\alpha_{j+1}\beta_j}=$$

$$= \chi(\widetilde{T}_\pi) x_{\alpha_j\beta_j} x_{\alpha_{j+1}\beta_{j+1}} + \left(\chi(\widetilde{T}_{s_j\pi}) + (q^{1/2} - q^{-1/2})\chi(\widetilde{T}_\pi)\right) x_{\alpha_j\beta_{j+1}} x_{\alpha_{j+1}\beta_j} =$$

$$= \chi(\widetilde{T}_\pi)\left(x_{\alpha_j\beta_j} x_{\alpha_{j+1}\beta_{j+1}} + (q^{1/2} - q^{-1/2}) x_{\alpha_j\beta_{j+1}} x_{\alpha_{j+1}\beta_j}\right) + \chi(\widetilde{T}_{s_j\pi}) x_{\alpha_j\beta_{j+1}} x_{\alpha_{j+1}\beta_j}.$$

By (2), and because, by Proposition 3, $\chi(\widetilde{T}_\pi) = \chi(\widetilde{T}_{s_j\pi s_j})$, this is equal to

$$\chi(\widetilde{T}_{s_j\pi s_j}) x_{\alpha_{j+1}\beta_{j+1}} x_{\alpha_j\beta_j} + \chi(\widetilde{T}_{s_j\pi}) x_{\alpha_{j+1}\beta_j} x_{\alpha_j\beta_{j+1}}.$$

If we multiply the equality

$$\chi(\widetilde{T}_\pi) x_{\alpha_j\beta_j} x_{\alpha_{j+1}\beta_{j+1}} + \chi(\widetilde{T}_{\pi s_j}) x_{\alpha_j\beta_{j+1}} x_{\alpha_{j+1}\beta_j} =$$

$$= \chi(\widetilde{T}_{s_j\pi s_j}) x_{\alpha_{j+1}\beta_{j+1}} x_{\alpha_j\beta_j} + \chi(\widetilde{T}_{s_j\pi}) x_{\alpha_{j+1}\beta_j} x_{\alpha_j\beta_{j+1}}$$

on the left by $x_{\alpha_1\beta_1}\cdots x_{\alpha_{j-1}\beta_{j-1}}$ and on the right by $x_{\alpha_{j+2}\beta_{j+2}}\cdots x_{\alpha_n\beta_n}$, we get

$$\chi(\widetilde{T}_\pi) x_{\alpha,\beta} + \chi(\widetilde{T}_{\pi s_j}) x_{\alpha,\beta s_j} = \chi(\widetilde{T}_{s_j\pi s_j}) x_{\alpha s_j,\beta s_j} + \chi(\widetilde{T}_{s_j\pi}) x_{\alpha s_j,\beta}.$$

But this can also be written as

$$\chi(\widetilde{T}_{\pi_k^v}) x_{\alpha_k^v,\beta_k^v} + \chi(\widetilde{T}_{\pi_k^w}) x_{\alpha_k^w,\beta_k^w} = \chi(\widetilde{T}_{\pi_{k+1}^v}) x_{\alpha_{k+1}^v,\beta_{k+1}^v} + \chi(\widetilde{T}_{\pi_{k+1}^w}) x_{\alpha_{k+1}^w,\beta_{k+1}^w}.$$

- $\alpha(j) < \alpha(j+1)$, $\beta(j) > \beta(j+1)$, $\operatorname{inv}(s_j\pi) = \operatorname{inv}(\pi) - 1$, $\operatorname{inv}(\pi s_j) = \operatorname{inv}(\pi) - 1$, $\operatorname{inv}(s_j\pi s_j) = \operatorname{inv}(\pi) - 2$. If we reverse the roles of v and w, we get the previous case. Therefore we also have

$$\chi(\widetilde{T}_{\pi_k^v}) x_{\alpha_k^v,\beta_k^v} + \chi(\widetilde{T}_{\pi_k^w}) x_{\alpha_k^w,\beta_k^w} = \chi(\widetilde{T}_{\pi_{k+1}^v}) x_{\alpha_{k+1}^v,\beta_{k+1}^v} + \chi(\widetilde{T}_{\pi_{k+1}^w}) x_{\alpha_{k+1}^w,\beta_{k+1}^w}.$$

This finishes the proof. □

Proof of Proposition 5. Recall that the *Bruhat order* on $\mathfrak{S}_n$ is the partial order generated by the relations $v < v \cdot (i,j)$ for $\operatorname{inv}(v) < \operatorname{inv}(v \cdot (i,j))$ (see [BB05, Chapter 2]). Let us prove by induction on k that for every $v \in \mathfrak{S}_n$,

$$x_{\alpha_k^v,\beta_k^v} = x_v + \sum_{z>v} c_z^{v,k} x_z$$

for some $c_z^{v,k} \in \mathbb{C}$. We have $x_{\alpha_0^v,\beta_0^v} = x_v$, so this is true for $k = 0$, assume that the statement holds for k. Write $\alpha = \alpha_k^v$, $\beta = \beta_k^v$, $j = j_k^v$. By definition, $\alpha_{k+1}^v = \alpha s_j$, $\beta_{k+1}^v = \beta s_j$. We know (see Lemma 11) that $\alpha(j) < \alpha(j+1)$. There are two possible cases:

- $\beta(j) < \beta(j+1)$. By (2), we have

$$x_{\alpha_{j+1}\beta_{j+1}} x_{\alpha_j\beta_j} = x_{\alpha_j\beta_j} x_{\alpha_{j+1}\beta_{j+1}} + (q^{1/2} - q^{-1/2}) x_{\alpha_j\beta_{j+1}} x_{\alpha_{j+1}\beta_j},$$

if we multiply this equation on the left by $x_{\alpha_1\beta_1}\cdots x_{\alpha_{j-1}\beta_{j-1}}$ and on the right by $x_{\alpha_{j+2}\beta_{j+2}}\cdots x_{\alpha_n\beta_n}$, we get

$$x_{\alpha_{k+1}^v,\beta_{k+1}^v} = x_{\alpha,\beta} + (q^{1/2} - q^{-1/2}) x_{\alpha,\beta s_j}.$$

By Lemma 15, we have $\alpha = \alpha_k^w$ and $\beta s_j = \beta_k^w$ for $w = \beta s_j \alpha^{-1}$. Since $w = v \cdot (\alpha(j), \alpha(j+1))$, $\alpha(j) < \alpha(j+1)$ and $v(\alpha(j)) = \beta(j) < \beta(j+1) = v(\alpha(j+1))$, we have $w > v$ and by induction

$$x_{\alpha_{k+1}^v,\beta_{k+1}^v} = x_v + \sum_{z>v} c_z^{v,k} x_z + (q^{1/2} - q^{-1/2}) x_w + (q^{1/2} - q^{-1/2}) \sum_{z>w} c_z^{w,k} x_z = x_v + \sum_{z>v} c_z^{v,k+1} x_z.$$

- $\beta(j) > \beta(j+1)$. By (2), we have

$$x_{\alpha_{j+1}\beta_{j+1}} x_{\alpha_j\beta_j} = x_{\alpha_j\beta_j} x_{\alpha_{j+1}\beta_{j+1}}$$

and hence

$$x_{\alpha^v_{k+1},\beta^v_{k+1}} = x_{\alpha,\beta} = x_v + \sum_{z>v} c_z^{v,k} x_z = x_v + \sum_{z>v} c_z^{v,k+1} x_z.$$

by the induction hypothesis.

Lemma 14 states that $x^v = x_{\alpha^v_{p^v},\beta^v_{p^v}}$, so

$$x^v = x_v + \sum_{z>v} c_z^{v,p^v} x_z.$$

Take any linear extension of the Bruhat order. The matrix corresponding to the linear transformation $x_v \mapsto x^v$ is square, lower triangular, and has 1's on the diagonal. This proves that $\{x^v : v \in \mathfrak{S}_n\}$ is a basis, and that the change of basis matrix is unitriangular. □

Proof of the main theorem. We have

$$\mathrm{Imm}_\chi X = \sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_v) x_v = \sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\pi^v_0}) x_{\alpha^v_0,\beta^v_0}$$

by construction of α^v_0, β^v_0. By the last lemma, we have

$$\sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\pi^v_0}) x_{\alpha^v_0,\beta^v_0} = \sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\pi^v_1}) x_{\alpha^v_1,\beta^v_1} = \ldots = \sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\pi^v_k}) x_{\alpha^v_k,\beta^v_k}$$

for every $k \geq 0$. If $k \geq \max_v p^v$, then

$$\sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\pi^v_k}) x_{\alpha^v_k,\beta^v_k} = \sum_{v \in \mathfrak{S}_n} \chi(\widetilde{T}_{\gamma_{\mu(v)}}) x^v$$

by Lemmas 13 and 14. □

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