PARABOLIC DOUBLE COSETS IN COXETER GROUPS

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ABSTRACT. Parabolic subgroups W_I of Coxeter systems (W, S), as well as their ordinary and double quotients W/W_I and $W_I \backslash W/W_J$, appear in many contexts in combinatorics and Lie theory, including the geometry and topology of generalized flag varieties and the symmetry groups of regular polytopes. The set of ordinary cosets wW_I, for $I \subseteq S$, forms the Coxeter complex of W, and is well-studied. In this article we look at a less studied object: the set of all double cosets $W_I w W_J$ for $I, J \subseteq S$. Double coset are not uniquely presented by triples (I, w, J). We describe what we call the lex-minimal presentation, and prove that there exists a unique such object for each double coset. Lex-minimal presentations are then used to enumerate double cosets via a finite automaton depending on the Coxeter graph for (W, S). As an example, we present a formula for the number of parabolic double cosets with a fixed minimal element when W is the symmetric group S_n (in this case, parabolic subgroups are also known as Young subgroups). Our formula is almost always linear time computable in n, and we show how it can be generalized to any Coxeter group with little additional work. We spell out formulas for all finite and affine Weyl groups in the case that w is the identity element.

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1. Introduction

Let G be a group with subgroups H and K. The group G is partitioned by the *double quotient*

$$H\backslash G/K = \{HgK \mid g \in G\},$$

i.e., the collection of *double cosets* HgK. If G is finite, then the number of double cosets in $H\backslash G/K$ is the inner product of the characters of the two trivial representations on H and K respectively, induced up to G [19, Exercise 7.77a]. Double cosets are usually more complicated than one-sided cosets. For instance, unlike one-sided cosets, two double cosets need not have the same size.

In this article, we investigate the parabolic double cosets of a finitely generated Coxeter group. That is, given a Coxeter system (W, S) of finite rank $|S|$, we consider cosets

$$W_I w W_J,$$

where I and J are subsets of the generating set S, and

$$W_I = \langle s : s \in I \rangle$$

denotes the standard *parabolic subgroup* of W generated by the subset I. These cosets are elements of the double quotient $W_I\backslash W/W_J$, though a given coset can be a member of more than one such quotient.

The parabolic double cosets are natural objects of study in many contexts. For example, they play a prominent role in the paper of Solomon that first defines the descent algebra of a Coxeter group [17]. For finite Coxeter groups, Kobayashi showed that these double cosets are intervals in Bruhat order, and these intervals have a rank-symmetric generating function with respect to length [11]. (While rank-symmetric, there exist parabolic double cosets in S_5 that are not self-dual.) Geometrically, these intervals correspond to the cell decomposition of certain rationally smooth Richardson varieties.

If we fix I and J, then the structure of the double quotient

$$W_I\backslash W/W_J$$

is also well-studied. For example, Stanley [18] shows the Bruhat order on such a double quotient is strongly Sperner (for finite W), and Stembridge [21] has characterized when the natural root coordinates corresponding to elements in the quotient give an order embedding of the Bruhat order (for any finitely generated W). As cited above, the number of elements in the double quotient is a product of characters,

$$|W_I\backslash W/W_J| = \left\langle \operatorname{ind}_{W_I}^W 1_{W_I}, \operatorname{ind}_{W_J}^W 1_{W_J} \right\rangle, \tag{1}$$

where 1_{W_J} denotes the trivial character on W_J. For fixed I and varying J, Garsia and Stanton connect parabolic double cosets to basic sets for the Stanley-Reisner rings of Coxeter complexes [8].

In this paper, we are interested in a basic problem about parabolic double cosets that appears to have been unexamined until now: how many distinct double cosets W_IwW_J does W have as I and J range across subsets of S? This question is partly motivated by the analogous problem for ordinary cosets, where the set $\{wW_I : I \subseteq S, w \in W\}$ is equal to the set of cells of the Coxeter complex. When W is the symmetric group S_n, the number of such cells is the nth ordered Bell number [13, A000670]. One fact that makes the one-sided case substantially simpler than the two-sided version is that each ordinary parabolic coset has the form wW_I for a unique subset $I \subseteq S$. If we take w to be the minimal element in the coset, then the choice of w is also unique. While double cosets do have unique minimal elements, different pairs of sets I and J often give the same double coset. For example, if e denotes the identity element of $W = (W, S)$, we have $W = W_SeW_J = W_IeW_S$ for *any* subsets I and J of S. Thus we cannot count distinct double parabolic cosets simply by summing Equation (1) over all I and J.

As mentioned earlier, every double coset has a unique minimal element, and we use this fact to recast our motivating question: for any $w \in W$, how many distinct double cosets does W have with minimal element $w \in W$?

Definition 1.1. Let (W, S) be a Coxeter system of finite rank $|S|$, and fix an element $w \in W$. Set

$$c_w := \#\{\text{double cosets with minimal element } w\}.$$

Because W has finite rank, c_w is always finite. Our first result is a formula for c_w when $W = S_n$, based on what we call the *marine model*, introduced in Section 3.4.

Theorem 1.2. *There is a finite family of sequences of positive integers b_m^K, $m \geq 0$, such that the number of parabolic double cosets with minimal element $w \in S_n$ is*

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \prod_{\mathcal{R} \in \operatorname{Rafts}(w)} b_{|\mathcal{R}|}^{K(\mathcal{R},T)}.$$

The sets $\operatorname{Floats}(w)$, $\operatorname{Tethers}(w)$, and $\operatorname{Rafts}(w)$ are subsets of the left and right ascents of w, and will be defined precisely in Section 3, as will the sequences b_m^K. Given a particular $w \in S_n$, Theorem 1.2 enables fast computations for two reasons. First, the sequences b_m^K satisfy a linear recurrence, and thus can be easily computed in time linear in m. Second, tethers are rare, and hence the sum typically has only one term, which corresponds to the empty set. In fact, the expected number of tethers in S_n is approximately $1/n$.

Proposition 1.3. *For all $n \geq 1$, the expected value of $|\operatorname{Tethers}(w)|$, over all $w \in S_n$ chosen uniformly, is given by*

$$\frac{1}{n!} \sum_{w \in S_n} |\operatorname{Tethers}(w)| = \frac{(n-3)(n-4)}{n(n-1)(n-2)}.$$

Theorem 1.2 allows us to calculate

$$p_n = \sum_{w \in S_n} c_w,$$

the total number of distinct double cosets in S_n. Although this requires summing $n!$ terms, the approach seems to be a significant improvement over what was previously known. The initial terms of the sequence $\{p_n\}$ are

$$1, 3, 19, 167, 1791, 22715, 334031, 5597524, 105351108, 2200768698, \\ 50533675542, 1265155704413, 34300156146805. \tag{2}$$

The sequence $\{p_n\}$ did not appear in the OEIS before our work, and it can now be found at [13, A260700]. We also make the following conjecture.

Conjecture 1.4. *There exists a constant K so that*

$$\frac{p_n}{n!} \sim \frac{K}{\log^{2n} 2}.$$

From the enumeration for $n \leq 13$, we observe that the constant K seems to be close to 0.4.

As predicted, summing Equation (1) over I and J overcounts the double cosets. In particular, this would produce

$$1, 5, 33, 281, 2961, 37277, \ldots,$$

which counts "two-way contingency tables" ([13, A120733], [19, Exercise 7.77], [6], [7, Section 5]). This sequence also enumerates cells in a two-sided analogue of the Coxeter complex recently studied by the third author [14].

The key to proving the formula in Theorem 1.2 is a condition on pairs of sets (I, J) guaranteeing that each double coset $W_I w W_J$ arises exactly once. In other words, we identify a canonical presentation $W_I w W_J$ for each double coset with minimal element w. The canonical presentation we found most useful for enumeration is what we call the *lex-minimal presentation*. This presentation can be easily defined for any Coxeter group.

Definition 1.5 (Lex-minimal presentation)**.** Let C be a parabolic double coset of some Coxeter system (W, S). A *presentation* of C is a choice of $I, J \subseteq S$ and $w \in W$ such that $C = W_I w W_J$. A presentation is *lex-minimal* if w is the minimal element of C, and $(|I|, |J|)$ is lexicographically minimal among all presentations of C. When w is fixed, we will abuse terminology slightly to call (I, J) a *lex-minimal pair* for w if $W_I w W_J$ is a lex-minimal presentation.

In other words, if $C = W_I w W_J$ is lex-minimal, and $C = W_{I'} w' W_{J'}$ is another presentation, then $w \leq w'$, and if $w = w'$ then either $|I| < |I'|$, or $|I| = |I'|$ and $|J| \leq |J'|$.

In Section 3.3 and Section 4, we show that every parabolic double coset has a unique lex-minimal presentation (the former section treats only the symmetric group, while the latter studies general Coxeter groups). Thus we focus our attention on counting the lex-minimal presentations. The main point of this paper is to show that there exists a finite state automaton that encodes the lex-minimal conditions along rafts as allowable words in the automaton. Moreover, *this same automaton can be used for all Coxeter systems.*

By the transfer matrix method (see [20, Section 4.7]), the number of allowable words of a given length in a language given by such an automaton has a rational generating function. Hence the sequences we use to count lex-minimal presentations satisfy finite linear recurrence relations. (In fact, we will see that there are only four different recurrences, the longest of which has six terms.) As in Theorem 1.2, this allows us to compute c_w more efficiently, and we get an enumerative formula that generalizes Theorem 1.2 to any Coxeter group.

Theorem 1.6. *There exists a finite family of sequences of positive integers b_m^K for $m \geq 0$, each determined by a linear homogeneous constant-coefficient recurrence relation, such that for any Coxeter group W and any $w \in W$, the number of parabolic double cosets with minimal element w is*

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \sum_{U \subseteq \operatorname{Wharfs}(w)} \prod_{\mathcal{R} \in \operatorname{Rafts}(w)} b_{|\mathcal{R}|}^{K(\mathcal{R},T,U)}.$$

Here $\operatorname{Wharfs}(w)$ is another subset of ascents of w, defined in Section 4. As evidence that the method is effective, in Section 4 we give formulae for c_w when $w = e$ is the identity element in each irreducible Weyl group of finite and affine type.

As a byproduct of the proofs of Theorems 1.2 and 1.6, we introduce the w-ocean graph for each $w \in W$. This graph encodes all presentations (I, w, J) for parabolic double cosets with minimal element w. In Theorem 4.15, we show that $W_I w W_J = W_{I'} w W_{J'}$ if and only if (I, J) can be obtained from (I', J') by plank moves which are defined in terms of moving connected components on the w-ocean. The lex-mininal presentations are also characterized in terms of plank moves.

The paper is organized as follows. In Section 2 we give an overview of parabolic double cosets for Coxeter groups. The enumeration and the marine model for S_n is described in Section 3. In Section 4, we develop the marine model including the w-ocean in the general Coxeter group setting.

We finish with Section 5, which describes a way to think about enumeration of cosets with minimal element $w \in W$ as equivalent to enumeration of cosets with minimal element $e \in \overline{W} \supset W$, with restricted reflections on the left and right. That is, we can translate the general problem of counting parabolic double cosets to the problem of counting double cosets containing the identity, with the trade-off being that we have to work in a larger Coxeter group, and we may have to restrict our allowable set of reflections. This suggests some further directions.

2. Background

In this section, we give definitions and relevant background information on the objects of interest in this paper. We begin with a discussion of Coxeter groups, and focus our attention first on parabolic cosets, and then on parabolic double cosets of these groups.

Coxeter groups are a broad family containing the symmetric groups, as well as Weyl groups of Kac-Moody groups and all finite real reflection groups.

Definition 2.1. A *Coxeter group* is a group W with a presentation

$$\langle s \in S : s^2 = e \text{ for all } s \in S, \text{ and } (st)^{m(s,t)} = e \text{ for all } s, t \in S \rangle,$$

where e is the identity element, and $m(s,t) = m(t,s)$ are values in the set $\{2, 3, \ldots\} \cup \{+\infty\}$. (If $m(s,t) = +\infty$, then the corresponding relation is omitted.) The elements of S are the *simple reflections* of W. The *reflections* of W are the conjugates of elements of S. The set of reflections is usually denoted by T, so $T = \{wsw^{-1} : s \in S, w \in W\}$.

In this paper, we study *finitely generated* Coxeter groups. That is, we assume that the set S is finite. In this case, $|S|$ is called the *rank* of the group.

A Coxeter group W can have more than one presentation. However, once a set S of simple reflections is chosen, the presentation is unique. Consequently, a pair (W, S) is referred to as a *Coxeter system*. The data of the presentation corresponding to a Coxeter system can be encoded in an edge-labeled graph.

Definition 2.2. Given a Coxeter system (W, S), the *Coxeter graph* has vertex set S, and edge set $\{\{s,t\} : m(s,t) > 2\}$. If $m(s,t) > 3$ then the edge $\{s,t\}$ is labeled by $m(s,t)$.

Note that pairs of nonadjacent vertices in a Coxeter graph correspond to pairs of commuting simple reflections.

Definition 2.3. A *parabolic subgroup* $W_I \subseteq W$ is a subgroup generated by a subset $I \subseteq S$.

Parabolic subgroups are Coxeter groups in their own right.

Example 2.4. For the symmetric group S_n, the simple reflections are usually chosen to be the adjacent transpositions $(i \leftrightarrow i+1)$ for $1 \leq i < n$. The Coxeter graph for S_n is a path with $n-1$ vertices labeled $1, 2, \ldots, n-1$, consecutively. The reflections in S_n are the transpositions $(i \leftrightarrow j)$ for $1 \leq i < j \leq n$. The parabolic subgroups of S_n, which are also known as *Young subgroups*, are always products of smaller symmetric groups.

Because the simple reflections in a Coxeter system generate the Coxeter group, every element in the group can be written as a product of these generators.

Definition 2.5. For a Coxeter system (W, S) and an element $w \in W$, the *length* $\ell(w)$ of w is the minimum number of simple reflections needed to produce w; that is, $\ell(w)$ is minimal so that $w = s_1 \cdots s_{\ell(w)}$ for $s_i \in S$. Such a word $s_1 \cdots s_{\ell(w)}$ is a *reduced expression* for w.

The *Bruhat order* on W is defined by taking the transitive closure of the relations $w < wt$, where $t \in T$ is a reflection of W and $\ell(w) < \ell(wt)$. Bruhat order is ranked by the length function, and can be understood in terms of subwords of reduced expressions: if $v = s_1 \cdots s_k$ is a reduced expression, then $u \leq v$ if and only if $u = s_{i_1} \cdots s_{i_l}$ for some $1 \leq i_1 < \cdots < i_l \leq k$.

The computations in this paper are stated in terms of the "marine model", which we introduce in the next section. A marine model is in turn made up of the following objects:

Definition 2.6. A simple reflection $s \in S$ is a *right ascent* of w if $\ell(ws) > \ell(w)$. Similarly, if $\ell(sw) > \ell(w)$, then s is a *left ascent*. By default, *ascents* will refer to right ascents. We denote the set of (right) ascents by

$$\mathrm{Asc}(w) := \mathrm{Asc}_R(w) = \{s \in S : \ell(ws) > \ell(w)\}.$$

Similarly, $\mathrm{Asc}_L(w) = \{s \in S : \ell(sw) > \ell(w)\}$. Every reduced expression for w^{-1} is the reverse of a reduced expression for w, so $\mathrm{Asc}_L(w) = \mathrm{Asc}_R(w^{-1})$. An element of S that is not an ascent is a *descent*. Again, by default *descents* will refer to right descents. We denote the set of (right) descents by

$$\mathrm{Des}(w) := \mathrm{Des}_R(w) = S \setminus \mathrm{Asc}_R(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

Similarly, the set of left descents is denoted $\mathrm{Des}_L(w)$, and $\mathrm{Des}_L(w) = \mathrm{Des}_R(w^{-1})$.

We refer the reader to [1, 10] for further background on Coxeter groups.

2.1. **Parabolic cosets of Coxeter groups.** In this work, we are concerned with parabolic double cosets. To give that study some context, and to emphasize the complexity of those objects, we first briefly state some facts about one-sided parabolic cosets.

Let W_J be a parabolic subgroup of W. The left cosets in the quotient W/W_J each have a unique minimal-length element, and thus W/W_J can be identified with the set W^J of all minimal-length left W_J-coset representatives. An element $w \in W$ belongs to W^J if and only if J contains no right descents of w; that is, if and only if $J \subseteq \mathrm{Asc}(w)$. When S is fixed, we write $J^c = S - J$ for the complement of J. Thus we can also say that $w \in W^J$ if and only if $\mathrm{Des}(w) \subseteq J^c$.

Every element $w \in W$ can be written uniquely as $w = uv$, where $u \in W^J$ and $v \in W_J$. This is the *parabolic decomposition* of w [10, Section 5.12]. The product $w = uv$ is a *reduced factorization*, meaning that $\ell(w) = \ell(u) + \ell(v)$. As a poset under Bruhat order, every coset wW_J is isomorphic to W_J. Consequently, if W_J is finite, then every coset wW_J is also finite. In addition, this set has a unique maximal element, implying that wW_J is a Bruhat interval.

Analogous statements can be made for right cosets. We use the notation ${}^IW$ for the set of minimal-length right coset representatives for $W_I\backslash W$.

2.2. **Parabolic double cosets of Coxeter groups.** A *parabolic double coset* is a subset $C \subseteq W$ of the form $C = W_IwW_J$ for some $w \in W$ and $I, J \subseteq S$. Parabolic double cosets inherit some of the nice properties of one-sided parabolic cosets mentioned previously, including the following:

Proposition 2.7. *Let (W, S) be a Coxeter system, and fix $I, J \subseteq S$.*

(a) Every parabolic double coset in $W_I\backslash W/W_J$ has a unique minimal element with respect to Bruhat order. As Bruhat order is graded by length, this element is also the unique element of minimal length.

(b) An element $w \in W$ is the minimal-length element of a double coset in $W_I\backslash W/W_J$ if and only if w belongs to both ${}^IW$ and W^J. Thus $W_I\backslash W/W_J$ can be identified with

$${}^IW^J := {}^IW \cap W^J = \{w \in W : I \subseteq \mathrm{Asc}_L(w) \text{ and } J \subseteq \mathrm{Asc}_R(w)\}.$$

(c) The parabolic double cosets in $W_I\backslash W/W_J$ are finite if and only if W_I and W_J are both finite. In this case, each $C \in W_I\backslash W/W_J$ has a unique maximal-length element which is also the unique maximal element with respect to Bruhat order. In particular, if C is finite then it is a Bruhat interval.

For the proof of Proposition 2.7, see [2] or [11]. The following statement is a consequence of Proposition 2.7. It has appeared, for instance, in [8, 5].

Corollary 2.8 (Double Parabolic Decomposition)**.** *Fix $I, J \subseteq S$ and $w \in {}^IW^J$. Set*

$$H := I \cap (wJw^{-1}).$$

Then $uw \in W^J$ for $u \in W_I$ if and only if $u \in W_I^H$, the minimal-length coset representatives in W_I/W_H. Consequently, every element of W_IwW_J can be written uniquely as uwv, where $u \in W_I^H$, $v \in W_J$, and $\ell(uwv) = \ell(u) + \ell(w) + \ell(v)$.

If W_I and W_J are finite parabolic subgroups, then Corollary 2.8 gives a bijective proof of the double coset formula $|W_IwW_J| = |W_I||W_J|/|W_H|$ [9, Theorem 2.5.1, and Exercise 40 on page 49].

Lemma 2.9. *Let C be a parabolic double coset in $W_I\backslash W/W_J$ with minimal element $w \in {}^IW^J$. For $y \in C$, fix a reduced factorization $y = uwv$ where $u \in W_I$, $v \in W_J$. For all $x \in W$, the following are equivalent:*

(i) $x \in C$ and $x \leq y$ in Bruhat order,
(ii) $w \leq x \leq y$, and
(iii) $x = u'wv'$ for some $u' \leq u$ and $v' \leq v$.

Proof. It is straightforward to check that (i) implies (ii) and (iii) implies (i). We need to show that (ii) implies (iii).

Take a reduced expression $y = s_1 \cdots s_k$, where $u = s_1 \cdots s_a$, $w = s_{a+1} \cdots s_b$, and $v = s_{b+1} \cdots s_k$. This means that $s_i \in I$ for all $i \in [1, a]$, and $s_i \in J$ for all $j \in [b+1, k]$, while $s_{a+1} \notin I$ and $s_b \notin J$.

If $x \leq y$ then x has a reduced expression $s_{i_1} \cdots s_{i_l}$ for $1 \leq i_1 < i_2 < \ldots < i_l \leq k$. If $w \leq x$ then w has a reduced expression $w = s_{i_{j_1}} \cdots s_{i_{j_m}}$, where $1 \leq j_1 < \ldots < j_m \leq l$. Since $w \in {}^I W^J$, we must have $s_{i_{j_1}} \notin I$ and $s_{i_{j_m}} \notin J$. Thus $i_{j_1} \geq a+1$ and $i_{j_m} \leq b$. In other words,

$$i_{j_1 - 1} < a + 1 \leq i_{j_1} < \cdots < i_{j_m} \leq b < i_{j_m + 1}.$$

Since $m = \ell(w) = b - a$, and there are only $b - a$ letters from $a+1$ to b, we have

$$\{i_{j_1} < \cdots < i_{j_m}\} = \{a + 1 < \cdots < b\},$$

and $u' := s_{i_1} \cdots s_{i_{j_1 - 1}} \in W_I$, $v' := s_{i_{j_m + 1}} \cdots s_{i_l} \in W_J$, as desired. □

In the remainder of the paper, we will assume that we know the unique minimal-length element of a parabolic double coset. The following corollary shows that this is computationally easy to find from any presentation of the coset. The algorithm can also be used to test if an arbitrary Bruhat interval is a parabolic double coset.

Corollary 2.10. *Given any parabolic double coset $C = W_I w W_J$ with w not necessarily minimal, one can find the unique minimal element in C by applying a simple greedy algorithm to w. The algorithm proceeds by recursively multiplying w by either s_i on the left for any $s_i \in I \cap \mathrm{Des}_L(w)$, or s_j on the right for any $s_j \in J \cap \mathrm{Des}_R(w)$. The algorithm terminates in at most $\ell(w)$ steps with an element $\min(C) = s_{i_p} \cdots s_{i_1} w s_{j_1} \cdots s_{j_q}$ that has no left descents in I, nor right descents in J.*

If C is finite, the maximal element $\max(C)$ of C can be found in the analogous way by using ascent sets instead of descent sets for w.

That $\min(C)$ (respectively, $\max(C)$) is the unique minimal (respectively, maximal) element of C follows from Corollary 2.9.

Corollary 2.11. *Let $[u, v]$ be a finite interval in Bruhat order in any Coxeter group W. Then $[u, v]$ is a parabolic double coset if and only if $u = \min(C)$ where*

$$\begin{aligned} C &= W_I v W_J, \\ I &= \mathrm{Asc}_L(u) \cap \mathrm{Des}_L(v), \\ J &= \mathrm{Asc}_R(u) \cap \mathrm{Des}_R(v), \ \text{and} \end{aligned}$$

$\min(C)$ is found via the greedy algorithm described in Corollary 2.10, starting at v.

Another property of finite intervals $[u, v]$ that are parabolic double cosets is that they are rank-symmetric (see [11]). It is natural to wonder if this rank-symmetry follows because the interval is self-dual, but this is not generally true. For example, the interval from the identity to the permutation $54312 = s_3 s_2 s_1 s_4 s_3 s_2 s_4 s_3 s_4$ in the symmetric group S_5 can be written as

$$[e, 54312] = W_{\{s_2, s_3, s_4\}} e W_{\{s_1, s_2, s_3\}},$$

where s_i denotes the ith adjacent transposition. Computer verification shows this interval is not self-dual.

3. Parabolic double cosets in the symmetric group

In this section we describe one of the main tools and results of this paper: the marine model for S_n, and the accompanying formula for the number c_w of parabolic double cosets with a fixed minimal permutation $w \in S_n$. At this stage, we focus on motivating the marine model, and consequently some aspects are discussed only informally. All the facts used in the enumeration will be discussed more formally in the next section on general Coxeter groups.

3.1. **Ascents and descents in the symmetric group.** By Proposition 2.7, we know that w is the minimal element of a parabolic double coset $W_I w W_J$ if and only if $I \subseteq \mathrm{Asc}_L(w)$ and $J \subseteq \mathrm{Asc}_R(w)$. In the symmetric group, ascents have a well-known combinatorial description which we now describe.

First, recall from Example 2.4 that the symmetric group S_n is Coxeter group with generating set $S = \{s_1, \ldots, s_{n-1}\}$, where s_i denotes the ith adjacent transposition. Elements of S_n are encoded by permutations of the set $\{1, 2, \ldots, n\}$. We will usually write a permutation $w \in S_n$ in one-line notation $w = w(1) \cdots w(n)$. Thus right action permutes positions, while left action permutes values.

Example 3.1. If $w = 7123546 \in S_7$, then $ws_6 = 7123564$ and $s_6 w = 6123547$.

The length function for S_n is the inversion statistic for permutations

$$\ell(w) = \mathrm{inv}(w) := \#\{(i,j) : i < j \text{ and } w(i) > w(j)\}.$$

Since w and ws_j differ only in that the letters $w(j)$ and $w(j+1)$ have swapped positions, we have $\ell(ws_j) > \ell(w)$ if and only if $w(j+1) > w(j)$. Thus, in a standard abuse of notation, we can write

$$\mathrm{Asc}_R(w) = \{1 \leq j \leq n-1 : w(j) < w(j+1)\},$$

and similarly

$$\mathrm{Des}_R(w) = \{1 \leq j \leq n-1 : w(j) > w(j+1)\}.$$

In the symmetric group, then, we can think of ascents and descents in terms of *positions* in permutations, in addition to the standard interpretation in terms of simple generators. The study of these combinatorial notions of ascents and descents goes (at least) as far back as the work of MacMahon in the early twentieth century (see, for example, [3, 12]).

Left multiplication by s_i swaps the positions of the letters i and $i+1$, so s_i is a left ascent of w if and only if the value i appears to the left of $i+1$ in the word $w(1) \cdots w(n)$, and we can think of $\mathrm{Asc}_L(w) = \mathrm{Asc}_R(w^{-1})$ and $\mathrm{Des}_L(w) = \mathrm{Des}_R(w^{-1})$ as values of the permutation.

To understand parabolic double cosets, we need to look at a particular kind of ascent which is, in a sense, "small."

Definition 3.2. A *small right ascent* of w is an index j such that $w(j+1) = w(j) + 1$. If j is a small right ascent, then $ws_j w^{-1} = s_i$, where $i = w(j)$, and we say that i is a *small left ascent* of w. Any ascent that is not a small ascent is a *large ascent*.

Example 3.3. For $w = 7123546$, we have $\mathrm{Asc}_R(w) = \{2, 3, 4, 6\}$, $\mathrm{Des}_R(w) = \{1, 5\}$, $\mathrm{Asc}_L(w) = \{1, 2, 3, 5\}$, and $\mathrm{Des}_L(w) = \{4, 6\}$. The small right ascents of w are $\{2, 3\}$, and the large right ascents are $\{4, 6\}$. The small left ascents of w are $\{1, 2\}$, and the large left ascents are $\{3, 5\}$.

3.2. **Balls in boxes.** Parabolic double cosets in the symmetric group can be represented by *balls-in-boxes pictures*, in which a number of balls are placed in a two-dimensional grid of boxes separated by some solid vertical and horizontal "walls." This idea, attributed to Nantel Bergeron, appears in work of Diaconis and Gangolli [6, Proof of Theorem 3.1] (see also [14, 15]).

We construct the balls-in-boxes picture for a permutation w by placing balls as one would do in a permutation matrix. To be precise, if $w(a) = b$, we put a ball in column a of row b, where columns are labeled left-to-right and rows are labeled bottom-to-top in Cartesian coordinates. The symmetric group acts on the left by permuting rows of such pictures, and on the right by permuting columns.

We can consider a parabolic double coset $C = W_I w W_J$ as a collection of pictures. While C is not invariant under the full action of the symmetric group, it is invariant under the left action of W_I and the right action of W_J. Thus walls are added to the picture to indicate which simple transpositions are allowed to act on C. For $i \in I^c$, the complement of I, we put a horizontal wall between rows i and $i+1$. For $j \in J^c$, we put a vertical wall between columns j and $j+1$. We also draw walls around the boundary of the entire permutation because no simple transpositions act in those positions. If no sets I and J are specified, then we will assume that $I = \mathrm{Asc}_L(w)$ and $J = \mathrm{Asc}_R(w)$; that is, we will draw walls in the left and right descent positions. Thus C can be represented by the balls-in-boxes pictures with walls, for (I, w, J).

Example 3.4. Continuing Example 3.3, the permutation $w = 7123546$ is the minimal-length representative for any parabolic double coset $W_I w W_J$ with $I \subseteq \{1, 2, 3, 5\} = \mathrm{Asc}_L(w)$ and $J \subseteq \{2, 3, 4, 6\} = \mathrm{Asc}_R(w)$. Figure 1 depicts the balls-in-boxes model for $w = 7123546$, with $I = \{1, 2, 3, 5\}$ and $J = \{2, 3, 4, 6\}$.

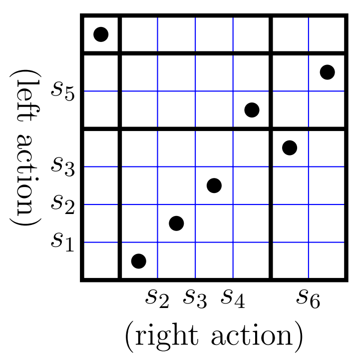


FIGURE 1. The balls-in-boxes model for a parabolic double coset. The light blue lines indicate which simple transpositions preserve the parabolic double coset.

Given a parabolic double coset $C = W_I w W_J$, we would like to find its minimal and maximal elements as in Corollary 2.10. The balls-in-boxes picture can help with this task: The walls in the picture partition the $n \times n$ grid into rectangular enclosures. By using the adjacent transpositions of rows and columns that are not separated by walls, sort the balls between each parallel pair of adjacent walls so that they create no inversions. In this sorting process, no ball leaves its enclosure; that is, the number of balls contained in any given enclosure does not change. The resulting picture will represent (I, u, J) where u is the minimal length representative for the parabolic double coset $W_I w W_J$. Similarly, if we sort the balls so to maximize the number of inversions between each parallel pair of adjacent walls, we get the maximal-length representative v for $W_I w W_J$. The parabolic double coset is the Bruhat interval $[u, v]$ with u and v as just defined.

Example 3.5. Let $w = 3512467$, with $I = \{1, 3, 4\}$ and $J = \{2, 3, 5, 6\}$. Figure 2 shows the balls-in-boxes pictures for the triples (I, w, J), (I, u, J), and (I, v, J), where $u = 3124567$ is the minimal element of the parabolic double coset and $v = 5421763$ is the maximal element.

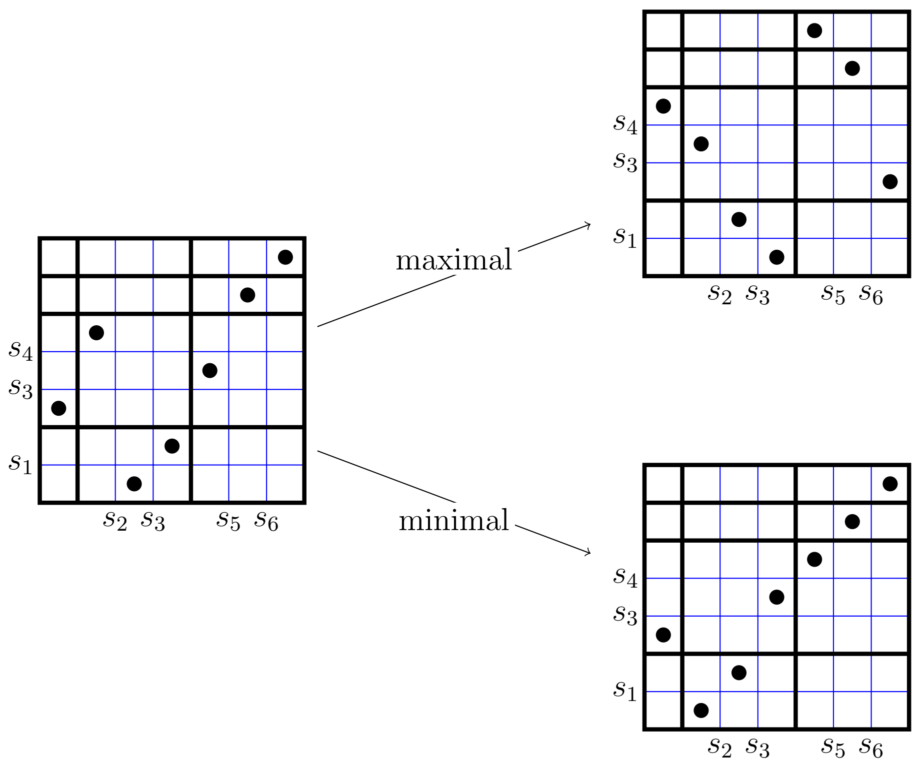


FIGURE 2. A parabolic double coset, with its minimal and maximal representatives. Again, lines of reflection that represent allowable simple transpositions are drawn in blue.

3.3. **Identifying canonical presentations in the symmetric group.** Before we count parabolic double cosets having w as a minimal element, we need to know how to identify a canonical presentation. Parabolic double cosets are intervals in the Bruhat order, so they are uniquely identified by their maximal and minimal elements. In Theorem 3.9, we will characterize the lex-minimal presentation of a parabolic double coset in the symmetric group. This is foreshadows a more general result for all Coxeter groups, appearing in Theorem 4.9, and it is this lex-minimal presentation that we will use in the enumeration in Theorem 1.2. First, though, we will take a few moments to describe, quite simply, a canonical presentation for a parabolic double coset that is, in a sense, "maximal." The generalized version of this will be discussed in Proposition 4.3.

Definition 3.6. For a parabolic double coset C in S_n, let its minimal and maximal elements be $w_{C\text{-min}}$ and $w_{C\text{-max}}$, respectively, and set

$$M_L(C) := \mathrm{Asc}_L(w_{C\text{-min}}) \cap \mathrm{Des}_L(w_{C\text{-max}}) \text{ and}$$
$$M_R(C) := \mathrm{Asc}_R(w_{C\text{-min}}) \cap \mathrm{Des}_R(w_{C\text{-max}}).$$

Proposition 3.7. *Let C be a parabolic double coset in S_n. There is a presentation*

$$C = W_{M_L(C)} w W_{M_R(C)},$$

and this is the largest possible presentation for C, in the sense that if $C = W_I w' W_J$, then $I \subseteq W_{M_L(C)}$ and $J \subseteq W_{M_R(C)}$.

Proof. Since S_n is finite, the coset C is a finite Bruhat interval, and Corollary 2.11 gives the desired presentation.

Now suppose that $C = W_I w W_J$ is another presentation of this parabolic double coset. Since $w_{C\text{-min}}$ is the minimal element of C, we must have $I \subseteq \mathrm{Asc}_L(w_{C\text{-min}})$ and $J \subseteq \mathrm{Asc}_R(w_{C\text{-min}})$. Similarly, since $w_{C\text{-max}}$ is the maximal element of C, we must have $I \subseteq \mathrm{Des}_L(w_{C\text{-max}})$ and $J \subseteq \mathrm{Des}_R(w_{C\text{-max}})$. (In other words, no element of I or J can take us up in Bruhat order.) Hence, $I \subseteq \mathrm{Asc}_L(w_{C\text{-min}}) \cap \mathrm{Des}_L(w_{C\text{-max}})$ and $J \subseteq \mathrm{Asc}_R(w_{C\text{-min}}) \cap \mathrm{Des}_R(w_{C\text{-max}})$, as desired. $\square$

The maximal presentation

$$C = W_{M_L(C)} w W_{M_R(C)}$$

of a parabolic double coset C is quite natural. In the balls-in-boxes context, it describes the minimum number of walls necessary to produce the desired coset.

Example 3.8. Continuing Example 3.5, the parabolic double coset C has $w_{C\text{-min}} = 3124567$ and $w_{C\text{-max}} = 5421763$, and $M_L(C) = \{1, 3, 4, 6\}$ and $M_R(C) = \{2, 3, 5, 6\}$. The balls-in-boxes pictures for the triples

$$(M_L(C), w, M_R(C)),\ (M_L(C), w_{C\text{-min}}, M_R(C)),\ \text{and } (M_L(C), w_{C\text{-max}}, M_R(C))$$

appear in Figure 3.

The enumeration in Theorem 1.2 will be done in terms of another canonical presentation of a parabolic double coset; namely, the lex-minimal presentation (see Definition 1.5). We spend a few moments now exploring some features of that presentation. We postpone a completely rigorous analysis of lex-minimal presentations until Section 4.

Suppose we begin with the balls-in-boxes picture of a parabolic double coset $C = W_I w W_J$. Assume that $I \subseteq \mathrm{Asc}_L(w)$ and $J \subseteq \mathrm{Asc}_R(w)$, so that w is minimal. We can then ask whether any other walls could be inserted without changing C.

Suppose that w has consecutive small left ascents $\{a, a+1, \ldots, b\} \subseteq I$, while $\{a-1, b+1\} \cap I = \emptyset$. Thus there are horizontal walls in positions $a-1$ and $b+1$ of the balls-in-boxes picture, and a southwest-to-northeast diagonal of $b-a+1$ balls appears between them, spanning exactly $b-a+1$ columns. Suppose, further, that no vertical walls appear between these balls. In other words, the small right ascents $\{w^{-1}(a), w^{-1}(a+1) = w^{-1}(a)+1, \ldots, w^{-1}(b)\}$

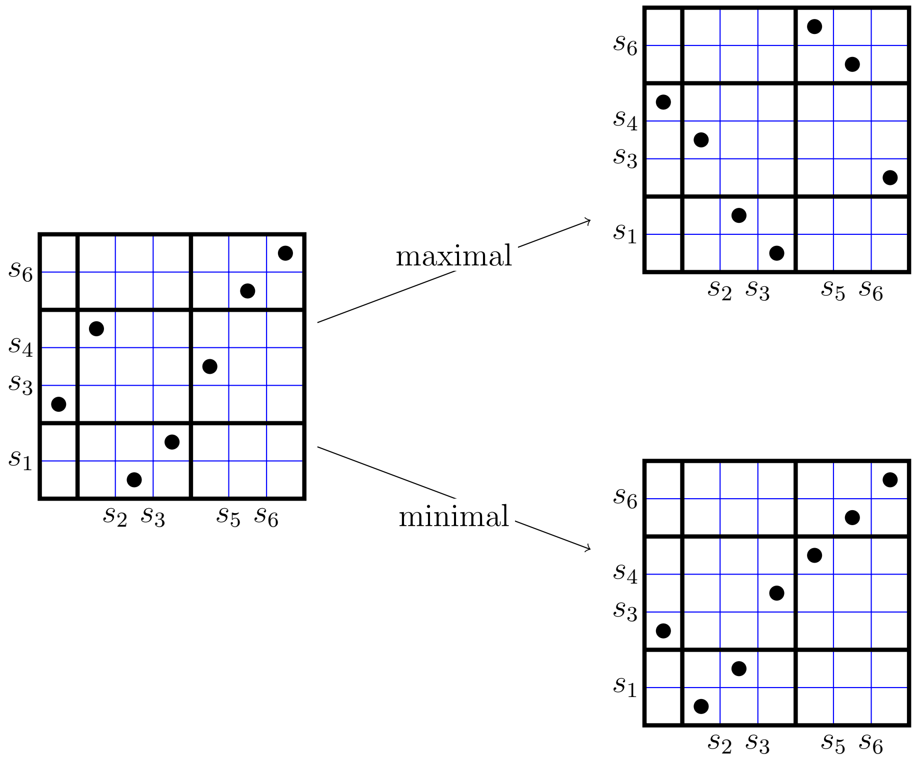
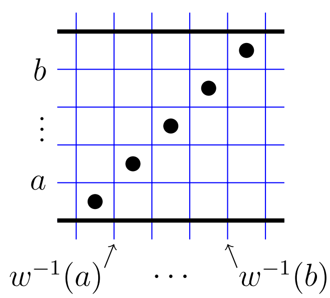
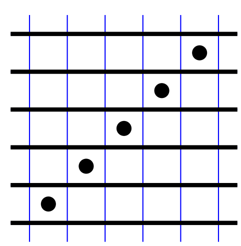


FIGURE 3. The maximal presentation of a parabolic double coset, with its minimal and maximal representatives. (Cf. Figure 2.)

are in the set J. This is illustrated in the figure below.



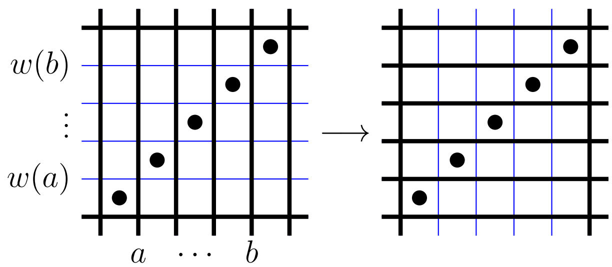
Now consider the same balls-in-boxes picture, but with $I' = I \setminus \{a, \ldots, b\}$. Locally, this appears as follows.



The new horizontal walls clearly do not impede the movement of balls outside this portion of the picture under either the left or right actions. Thus to determine if the two balls-in-boxes pictures represent the same parabolic double coset, it suffices to check that the balls in the picture can be sorted in the same way. In both pictures, right actions are enough to sort the balls into decreasing order. Hence both parabolic double cosets $W_I w W_J$ and $W_{I'} w W_J$ have the same maximal element, and so $W_I w W_J = W_{I'} w W_J$. Because $|I'| < |I|$, we conclude from this that the presentation $W_I w W_J$ is not lex-minimal.

Swapping the roles of I and J and of "left" and "right" yields an analogous conclusion about non lex-minimality when small right ascents are squeezed between consecutive vertical walls.

Another scenario we can easily analyze involves both removing and inserting walls. Suppose that $a, a+1, \ldots, b$ is a sequence of consecutive small right ascents of w, none of which are in J. Suppose, further, that neither $a-1$ nor $b+1$ are in J. In other words, there are vertical walls in each of the gaps from $a-1$ to $b+1$. Now suppose that the corresponding left ascents, $w(a), \ldots, w(b)$ are in I, but $w(a-1)$ and $w(b+1)$ are not in I. Consider removing all the vertical walls in $a, \ldots, b$ and inserting horizontal walls in $w(a), \ldots, w(b)$. This is illustrated below.



This change has no impact on the balls outside of this local area, and in both cases, we can sort the balls in decreasing order. Hence the two parabolic double cosets are the same. Let $I' = I \setminus \{w(a), \ldots, w(b)\}$ and let $J' = J \cup \{a, \ldots, b\}$. The picture on the left is $W_I w W_J$, while the picture on the right is $W_{I'} w W_{J'}$. Because $|I'| < |I|$, the presentation $W_I w W_J$ is not lex-minimal.

In fact, the preceding analysis characterizes lex-minimal presentations for S_n.

Theorem 3.9. *Let $w \in S_n$ and let I and J be subsets of the left and right ascent sets of w, respectively. Then $W_I w W_J$ is a lex-minimal presentation of a parabolic double coset of S_n if and only if*

- *if $\{a-1, a, \ldots, b, b+1\} \cap I = \{a, \ldots, b\}$ and these are all small left ascents, then $\{w^{-1}(a), \ldots, w^{-1}(b)\} \not\subseteq J$ and $\{w^{-1}(a)-1, w^{-1}(a), \ldots, w^{-1}(b), w^{-1}(b)+1\} \cap J \neq \emptyset$, and*
- *if $\{a-1, a, \ldots, b, b+1\} \cap J = \{a, \ldots, b\}$ and these are all small right ascents, then $\{w(a), \ldots, w(b)\} \not\subseteq I$.*

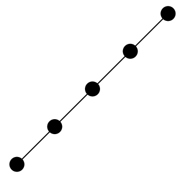
One direction of the proof of Theorem 3.9 was discussed above. The other direction is a specialization of the upcoming result, Theorem 4.9, for general Coxeter groups.

3.4. **The marine model in the symmetric group.** We are now ready to introduce the marine model. For S_n, we illustrate this model using balls-in-boxes pictures. Later, in Definition 3.19, we will use a different method of illustration that applies to any Coxeter group.

Definition 3.10. The following objects comprise the *marine model* for a permutation $w \in S_n$. For each, we will give both a combinatorial and a pictorial description.

- A *raft* of w is an interval $[a,b]=\{a,\ldots,b\}$ such that $a,a+1,\ldots,b$ are all small right ascents of w, while $a-1$ and $b+1$ are not. In other words, a raft is a maximal increasing run of consecutive values: $w(a), w(a+1)=w(a)+1,\ldots,w(b)=w(a)+(b-a)$. The *size* of such a raft is $|[a,b]|=b-a+1$.

  In a balls-in-boxes picture, a raft looks like a copy of an identity permutation, and we will connect the balls of a raft as follows.

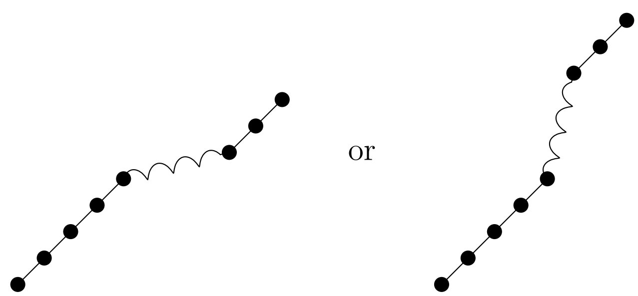


  We write $\mathrm{Rafts}(w)$ for the set of intervals of small (right) ascents that make up the rafts of w.

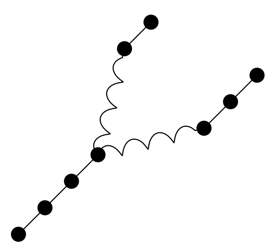
The ends of a raft may connect to other balls in various ways.

- A *tether* of w is an ascent that is adjacent to two rafts. Because rafts are maximal, tethers are necessarily large ascents. If desired, we can specify a "left" tether or a "right" tether, corresponding to whether the large ascent in question is a left or a right ascent. We let $\mathrm{Tethers}_L(w)$ and $\mathrm{Tethers}_R(w)$ denote the sets of positions of the left and right tethers of w, respectively.

  In pictures, we draw tethers with squiggly lines. The left-hand picture below depicts a left tether, and the right-hand figure depicts a right tether.

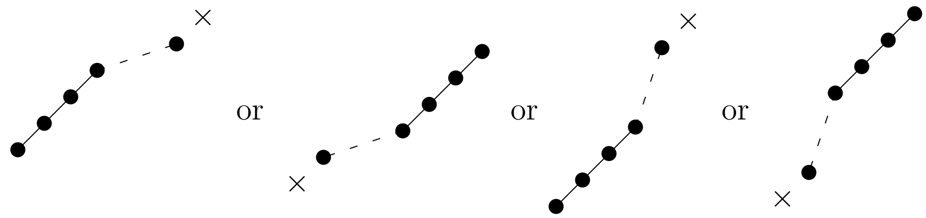


  Note that while two rafts can be connected by at most one tether, it is possible to have a raft with both types of tethers emanating from the same ball, as shown below.



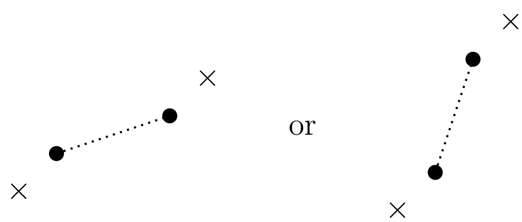
- A *rope* of w is a large ascent that is adjacent to exactly one raft. Again, we can specify that a given rope is a "left" rope or a "right" rope. We let $\mathrm{Ropes}_L(w)$ and $\mathrm{Ropes}_R(w)$ denote the sets of positions of the left and right ropes, respectively.

  In pictures, we draw ropes with dashed lines. The two leftmost figures below depict left ropes, whereas the two rightmost depict right ropes. Because a rope is adjacent to exactly one raft, the "×" symbols in the figures below indicate locations where balls may not appear.



- A *float* of w is a large ascent that is not adjacent to any rafts. Once again, we can specify that a given float is a "left" float or a "right" float. The sets of positions of left and right floats are denoted by $\mathrm{Floats}_L(w)$ and $\mathrm{Floats}_R(w)$, respectively.

  In pictures, a float connects two isolated balls, and we draw floats with a dotted line. As was the case for ropes, we mark locations that cannot have a ball by "$\times$" symbols. The following figures depict left and right floats, respectively.



Before examining the utility of this model, we describe two permutations whose marine models are, in a sense, extreme.

Example 3.11.

- If $w \in S_n$ is the identity permutation, in which all positions are small ascents, then w has one raft, $[1, n-1]$, and no floats, ropes, or tethers.
- If $w \in S_n$ is the longest permutation, in which no positions are ascents, then w has no rafts, floats, ropes, or tethers.

To streamline notation, we will write left large ascents with tick marks $1', 2', \ldots$ and set

$$\begin{aligned} \mathrm{Tethers}(w) &:= \{i' : i \in \mathrm{Tethers}_L(w)\} \cup \mathrm{Tethers}_R(w), \\ \mathrm{Ropes}(w) &:= \{i' : i \in \mathrm{Ropes}_L(w)\} \cup \mathrm{Ropes}_R(w), \text{ and} \\ \mathrm{Floats}(w) &:= \{i' : i \in \mathrm{Floats}_L(w)\} \cup \mathrm{Floats}_R(w). \end{aligned}$$

Example 3.12. Continuing Example 3.4, for the permutation $w = 7123546$, the marine model is overlaid on the balls-in-boxes picture of $w = 7123546$ in Figure 4. We have $\mathrm{Rafts}(w) = \{[2, 3]\}$, $\mathrm{Tethers}(w) = \emptyset$, $\mathrm{Ropes}(w) = \{3', 4\}$, and $\mathrm{Floats}(w) = \{5', 6\}$.

3.5. **Starting to count.** We now illustrate how the marine model enables enumeration of parabolic double cosets with a given minimal element.

We start from the idea that rafts are, in a sense, well-behaved, since they look like copies of the identity permutation. Suppose we know the number of parabolic double cosets for an identity permutation (we will study this number in Section 3.7). We would next want to identify how any relationships between rafts and isolated balls will affect the total number of parabolic double cosets whose minimal element is the permutation w.

Consider the balls-in-boxes picture for w, with $I = \mathrm{Asc}_L(w)$ and $J = \mathrm{Asc}_R(w)$ as large as possible. The parabolic double coset $W_I w W_J$ will contain all other cosets for which w is

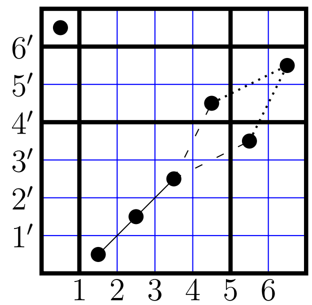


FIGURE 4. The permutation 7123546 with its lone raft, two ropes, and two floats marked. There are no tethers. Walls are drawn in left and right descent positions.

the minimal element. We want to insert walls in this picture, yielding all the lex-minimal representations of these cosets.

We start to understand how to do this with a simple observation.

Lemma 3.13. *If two balls are connected by a tether, rope, or float, then the balls occupy different boxes; that is, there is a wall between them. If the connector is of "left" type, then the balls are separated by a vertical wall; if the connector is of "right" type, then the balls are separated by a horizontal wall.*

Floats exhibit particularly interesting behavior, which we highlight here.

Lemma 3.14. *If two nodes are connected by a float, then inserting a wall in this position will result in a different parabolic double coset, independent of all other choices for the walls.*

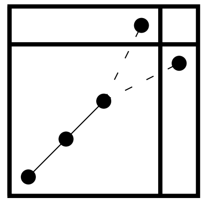
Proof. Suppose, without loss of generality, that two balls are connected by a right float. Then Lemma 3.13 says there is a horizontal wall between them. Suppose the balls correspond to $w(i)$ and $w(i+1)$, with $w(i) < w(i+1)$. If there is no vertical wall between them, then the maximal element for this parabolic double coset, call it v, has $v(i) > v(i+1)$, obtained by acting on the right by s_i at some point to swap the columns these balls occupy. If, on the other hand, there is a vertical wall in position i, then the maximal element must have $v(i) < v(i+1)$, since there are both vertical and horizontal bars between the two balls. $\square$

Lemma 3.14 means that floats are independent actors in our counting. In other words, if w has m floats, then the formula for c_w has the form

$$c_w = 2^m \cdot (\text{something}).$$

If w has no tethers, then each raft contributes independently to the formula, and that "something" will be a product of terms related to each raft. If w does have tethers, then these contributions will vary depending on whether we choose to place a wall in a tether position.

Example 3.15. Continuing Example 3.12, the permutation $w = 7123546$ has two floats, so c_w is a multiple of four. If we ignore these floats, then we have only three potential horizontal walls (in positions $1', 2', 3'$) and three potential vertical walls (in positions $2, 3, 4$) whose insertion can possibly give rise to new cosets. This is equivalent to counting the parabolic double cosets for $x = 12354$, pictured below.



Through brute force, we find that $c_x = 36$. Thus w is the minimal element of $2^2 \cdot 36 = 144$ parabolic double cosets in S_7.

We next consider interplay between rafts.

Definition 3.16. When multiple rafts are connected by tethers of the same orientation, the resulting structure is a (horizontal or vertical) *flotilla*.

Example 3.17. Let

$$w = 1\ 3\ 4\ 5\ 7\ 8\ 2\ 6\ 14\ 15\ 16\ 9\ 10\ 11\ 12\ 13 \in S_{16}.$$

We see the balls-in-boxes picture of the marine model for w in Figure 5. We have $\mathrm{Rafts}(w) = \{[2,3],[5,5],[9,10],[12,15]\}$, $\mathrm{Tethers}(w) = \{8',4\}$, $\mathrm{Ropes}(w) = \{5',1,8\}$, and $\mathrm{Floats}(w) = \{1',7\}$.

Label the rafts A, B, C, and D from left to right in Figure 5. There is a horizontal flotilla consisting of rafts B and D, and a vertical flotilla consisting of rafts A and B.

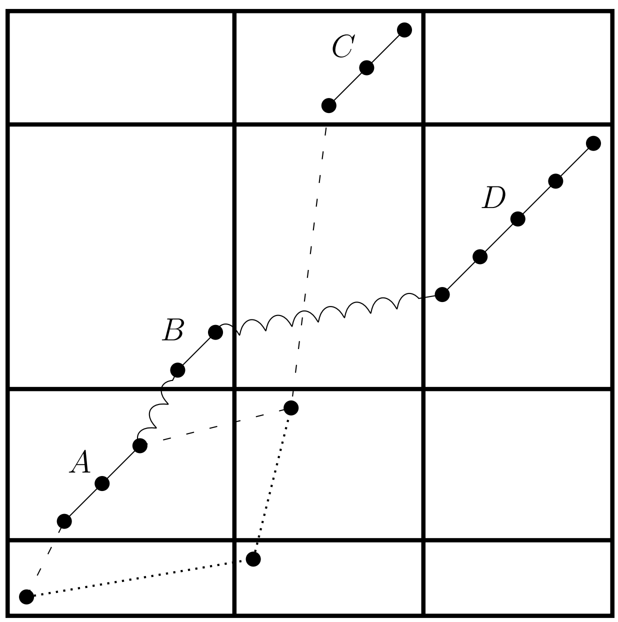


FIGURE 5. The permutation 1 3 4 5 7 8 2 6 14 15 16 9 10 11 12 13 with its (four) rafts, (one right and one left) tethers, (two right and one left) ropes, and (one right and one left) floats marked.

We now examine two adjacent rafts in a flotilla, and the tether connecting them. For example, consider the rafts A and B in Figure 5. In the maximal representative of the corresponding parabolic double coset, the balls of raft B will be sorted above and to the left of the balls of raft A. On the other hand, if a vertical wall is inserted between these two rafts, as depicted in Figure 6, then the balls of raft B will appear above and to the right of the balls in raft A in the maximal representative of the corresponding parabolic double coset. Thus the two parabolic double cosets are distinct.

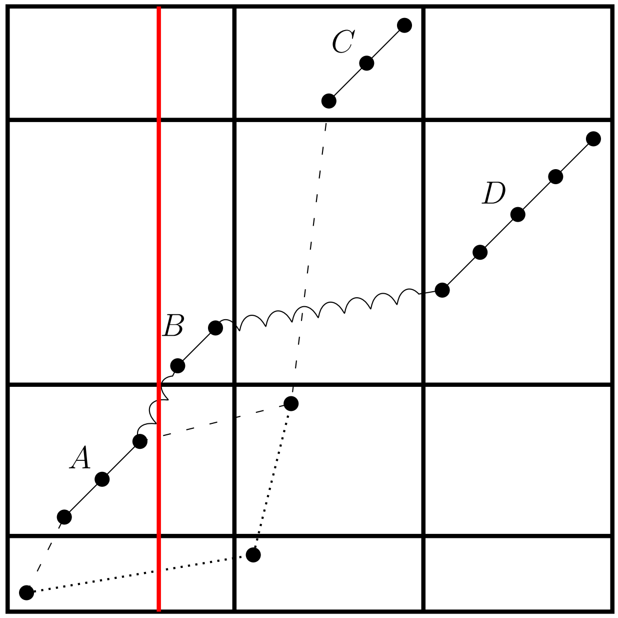


FIGURE 6. Inserting a vertical wall (marked in red) through the vertical tether between rafts A and B in Figure 5.

In fact, the same argument can be used to show this phenomenon holds generally.

Lemma 3.18. *If two rafts are connected by a left (respectively, right) tether, then inserting a horizontal (respectively, vertical) wall in that position will result in a different parabolic double coset, independent of all other choices for the walls.*

Despite the similarities between Lemmas 3.14 and 3.18, the contribution from the rafts can be different depending on the subset of tethers that are cut by walls. This is because each tether is incident to two rafts, whereas a float is incident to none. This gives the enumeration of c_w the form:

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} (\text{something depending on } T),$$

where the "something" will look like a product of contributions from the rafts.

The only piece of the marine model that we have yet to discuss is a rope. Ropes occur at the end of a raft. These are better behaved, in the sense that choosing whether or not to cut a rope only affects one raft's boundary.

3.6. **The Coxeter-theoretic picture.** We step briefly away from enumeration to expand our combinatorial model.

Foreshadowing our treatment of parabolic double cosets in other Coxeter groups, it will be useful to be able to represent the marine model in terms of the simple reflections that generate S_n. This more general visual representation will be based on the Coxeter graph

G of the group, which, in the case of the symmetric group S_n, is a path of vertices labeled $\{1, \ldots, n-1\}$. We now revisit the objects in Definition 3.10.

Definition 3.19. To a permutation $w \in S_n$, we associate a diagram called the *w-ocean*, formed as follows.

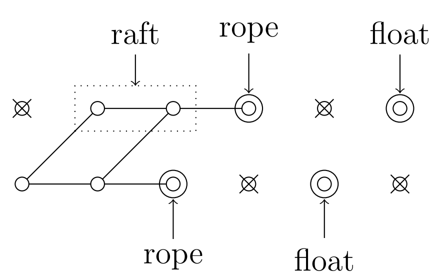
- Draw two rows of $n-1$ vertices, represented as open dots. We will think of these as being labeled $1, 2, \ldots, n-1$ from left to right, representing two copies of the set of adjacent transpositions that generates S_n.
- Cross out each dot in the top (respectively, bottom) row that corresponds to a right (respectively, left) descent of w. Thus the remaining dots correspond to left or right ascents of w.
- Circle each dot in the top (respectively, bottom) row that corresponds to a large right (respectively, left) ascent in w.
- For each k which appears in a raft of w, draw a line from the kth dot in the top row to the $w(k)$th dot in the bottom row. Such lines are *planks*.
- Draw horizontal lines connecting consecutive small ascents.
- If the ith dot in the top row is a right rope or tether, then draw an edge(s) horizontally from it to its adjacent small ascent(s). Do the same in the bottom row for the left ropes and tethers.

In the w-ocean, we can represent a pair (I, J), where $I \subset \mathrm{Asc}_L(w)$ and $J \subset \mathrm{Asc}_R(w)$, by filling in the corresponding open dots. More precisely, elements of I are filled in the bottom row, and elements of J are filled in the top row. Such fillings will be the basis for our enumeration of lex-minimal pairs for w.

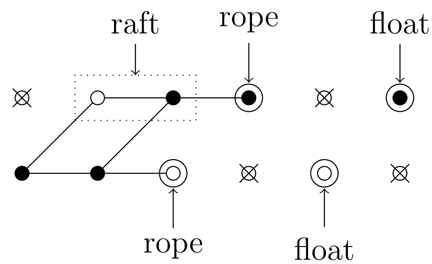
As we mentioned at the end of Section 3.5, the enumeration of lex-minimal fillings reduces to individual rafts. For each raft, the number of lex-minimal fillings is controlled by the fillings of adjacent ropes and tethers. The only nodes of a raft which can be adjacent to a rope or tether are those nodes in the boundary of the raft. Hence we make the following (somewhat informal) definition:

Definition 3.20. A *boundary apparatus* for a particular raft consists of the arrangement of ropes and tethers adjacent to the raft, along with a choice of fillings for these ropes and tethers.

Example 3.21. Continuing Example 3.15, the w-ocean for $w = 7123546$ is shown below. There is a raft of size 2, with one rope on each row on the right-hand side. There are also two floats.



To represent the triple (I, w, J) with $I = \{1, 2\}$ and $J = \{3, 4, 6\}$, we fill the appropriate dots in the w-ocean. In this case, the boundary apparatus of the raft consists of the filled in rope attached to the upper right corner, and the unfilled rope attached to the lower right corner.



To fully appreciate the marine model as represented in the w-ocean, we consider a larger example.

Example 3.22. Continuing Example 3.17, Figure 7 shows the w-ocean for the permutation $w = 1\ 3\ 4\ 5\ 7\ 8\ 2\ 6\ 14\ 15\ 16\ 9\ 10\ 11\ 12\ 13 \in S_{16}$. Compare with Figure 5.

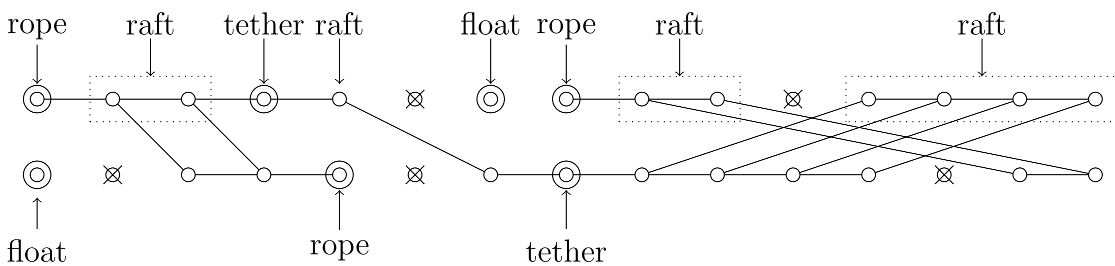


FIGURE 7. The w-ocean of rafts, tethers, floats, and ropes for the permutation $w = 1\ 3\ 4\ 5\ 7\ 8\ 2\ 6\ 14\ 15\ 16\ 9\ 10\ 11\ 12\ 13 \in S_{16}$.

To enumerate the lex-minimal presentations of parabolic double cosets with a fixed $w \in S_n$ as the minimal element, we want to count the ways of choosing subsets I and J from the lower and upper row of the w-ocean, respectively, such that the conditions in Theorem 3.9 are satisfied.

Definition 3.23. Let $\mathcal{R}$ be a raft in a w-ocean. A *lex-minimal filling* of $\mathcal{R}$ is a lex-minimal filling of the w-ocean such that the only filled vertices belong either to $\mathcal{R}$, or to the ropes and tethers adjacent to $\mathcal{R}$.

It should be clear that the number of lex-minimal fillings of a raft does not depend on the full permutation w, only on the length of the raft and the adjacent ropes and tethers.

For the next lemma, we need one more temporary definition. Let (I, J) be a filling of the w-ocean. The *restriction* of (I, J) to a raft $\mathcal{R}$ is a new filling in which all nodes outside of $\mathcal{R}$ and its boundary apparatus are left unfilled (and the nodes in $\mathcal{R}$ and its boundary apparatus are left unchanged).

Lemma 3.24. *A filling of the w-ocean is lex-minimal if and only if the restriction to every raft $\mathcal{R}$ is a lex-minimal filling of $\mathcal{R}$.*

We can now prove the first version of our main enumeration formula, leaving the number of lex-minimal fillings as a black box.

Proposition 3.25. *For any $w \in S_n$,*

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \sum_{S \subseteq \operatorname{Ropes}(w)} \prod_{\mathcal{R} \in \operatorname{Rafts}(w)} a(\mathcal{R}, S, T) \tag{3}$$

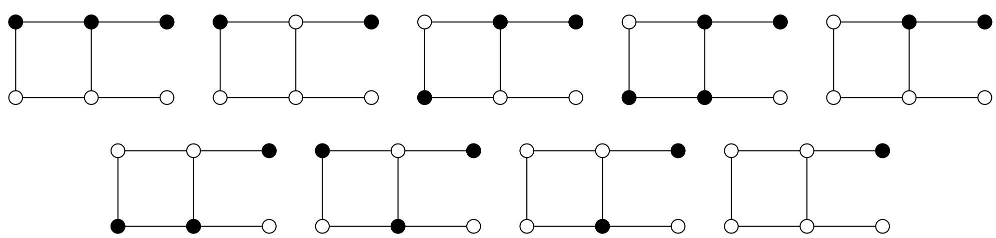
where $a(\mathcal{R}, S, T)$ is the number of lex-minimal fillings of raft $\mathcal{R}$ with the boundary apparatus determined by S and T.

Proof. As described above, we want to choose appropriate subsets I and J from the lower and upper row of the w-ocean, as required by Theorem 3.9. We can think of such pairs (I, J) as arising from an arbitrary choice of any of the floats, ropes, and tethers for w (that is, any large ascents), followed by an appropriate filling of the vertices in the rafts. Once the floats, ropes, and tethers are specified, the vertices on each raft are filled in such a way that they satisfy the lex-minimal conditions of Theorem 3.9. By Lemma 3.24, the choices made on each raft are independent of the choice made on other rafts.

□

We have reduced the problem of calculating c_w to finding a way to compute $a(\mathcal{R}, S, T)$ for a single raft $\mathcal{R}$ with fixed apparatus on its boundary.

Example 3.26. Recall the w-ocean for $w = 7123546$ shown in Example 3.21, which has a rope on each row. If we choose the upper rope, but not the lower rope, then there are nine ways to fill in vertices on the raft with lex-minimal presentation. These nine filling are in bijection with the following pictures.

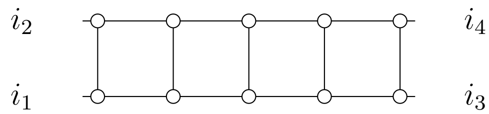


This proves that $a(\mathcal{R}, \{4\}, \emptyset) = 9$. The reader is encouraged to verify that $a(\mathcal{R}, \{3'\}, \emptyset) = 9$, $a(\mathcal{R}, \{4, 3'\}, \emptyset) = 12$, and $a(\mathcal{R}, \emptyset, \emptyset) = 6$ using the conditions in Theorem 3.9. Thus,

$$\begin{aligned} c_w &= 2^2\big(a(\mathcal{R}, \{4\}, \emptyset) + a(\mathcal{R}, \{3'\}, \emptyset) + a(\mathcal{R}, \{4, 3'\}, \emptyset) + a(\mathcal{R}, \emptyset, \emptyset)\big) \\ &= 4 \cdot (9 + 9 + 12 + 6) \\ &= 144. \end{aligned}$$

3.7. **Enumeration for rafts.** Fix a raft $\mathcal{R}$ and an accompanying boundary apparatus, composed of the selected ropes S and tethers T. To compute $a(\mathcal{R}, S, T)$, we need to consider all pairs of subsets (I, J) of the lower and upper vertices of the raft that satisfy the lex-minimal conditions in Theorem 3.9. We will show that these lex-minimal subsets can be recognized by a finite state automaton, pictured in Figure 10. As a result, the transfer-matrix method can be used to find a recurrence for the number of choices for I and J, depending on the size of the raft $\mathcal{R}$ and the boundary apparatus. Recall that the size of a raft is equal to the number of vertices in the raft.

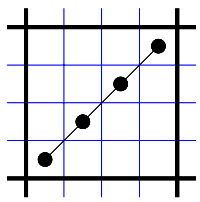
To make this idea precise, suppose that the tuple (i_1, i_2, i_3, i_4) represents the apparatus attached to the lower-left, upper-left, lower-right, and upper-right outer corners, respectively, of a raft as follows.



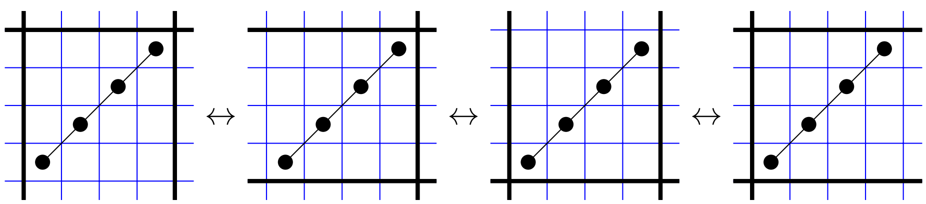
The indicators are 1 (if selected and filled) or 0 (if not selected and not filled), depending on whether or not a rope or tether attached at that point appears in S or T. In terms of the balls-in-boxes picture, $i_1 = 0$ indicates the presence of a horizontal wall immediately below a raft, $i_2 = 0$ indicates the presence of a vertical wall immediately to the left of the raft, $i_3 = 0$ indicates the presence of a horizontal wall immediately above, and $i_4 = 0$ indicates the presence of a vertical wall immediately to the right.

Up to symmetry, there are seven cases:

- $i_1 + i_2 + i_3 + i_4 = 0$, i.e., four walls,

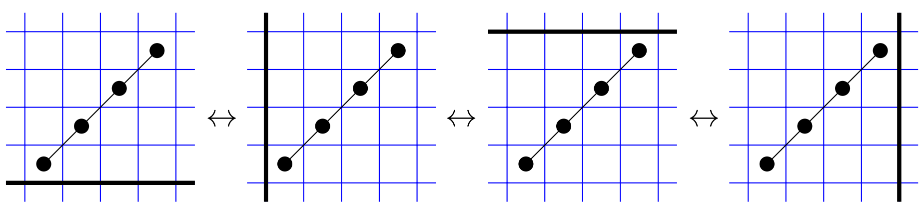


- $i_1 + i_2 + i_3 + i_4 = 1$, i.e., three walls,



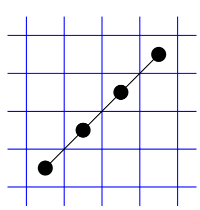
$i_1 = 0$ $\qquad$ $i_2 = 0$ $\qquad$ $i_3 = 0$ $\qquad$ $i_4 = 0$

- $i_1 + i_2 + i_3 + i_4 = 3$, i.e., one wall,

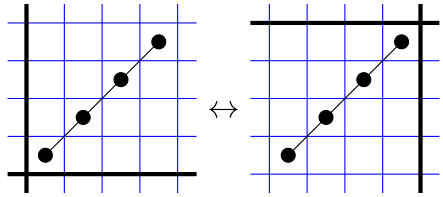


$i_1 = 1$ $\qquad$ $i_2 = 1$ $\qquad$ $i_3 = 1$ $\qquad$ $i_4 = 1$

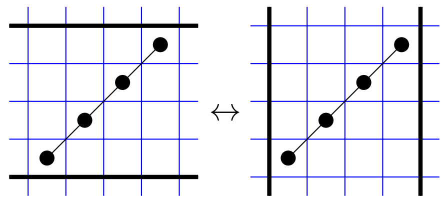
- $i_1 + i_2 + i_3 + i_4 = 4$, i.e., zero walls,



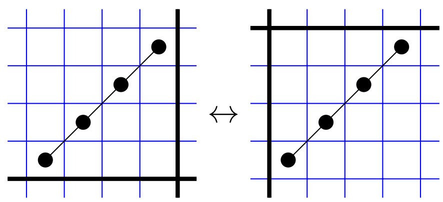
- and $i_1 + i_2 + i_3 + i_4 = 2$, i.e., two walls, in three distinct ways.



$i_1 = i_2 = 1$ $\qquad$ $i_3 = i_4 = 1$



$i_1 = i_3 = 1$ $\qquad$ $i_2 = i_4 = 1$



$i_1 = i_4 = 1$ $\qquad$ $i_2 = i_3 = 1$

These seven symmetry types can be encoded by the function

$$k(i_1, i_2, i_3, i_4) = \begin{cases} i_1 + i_2 + i_3 + i_4 & \text{if } i_1 + i_2 + i_3 + i_4 \neq 2, \\ 2 & \text{if } i_1 + i_2 + i_3 + i_4 = 2 \text{ and } i_1 = i_2, \\ 2' & \text{if } i_1 + i_2 + i_3 + i_4 = 2 \text{ and } i_1 = i_3, \text{ and} \\ 2'' & \text{if } i_1 + i_2 + i_3 + i_4 = 2 \text{ and } i_1 = i_4. \end{cases}$$

In the following statement, the seven possibilities for $k := k(i_1, i_2, i_3, i_4)$ each determine an integer sequence denoted $(a^k_m)_{m \in \mathbb{N}}$, which we call the *a-sequences*. These sequences are at the core of our enumerative results — not just for symmetric groups, but for general Coxeter groups as well.

Theorem 3.27. *Consider a raft $\mathcal{R}$ of size $m > 0$ and its boundary apparatus, composed of the selected ropes S and tethers T. Let (i_1, i_2, i_3, i_4) represent this choice of apparatus attached to the lower-left, upper-left, lower-right, and upper-right outer corners of the raft, respectively, with $k := k(i_1, i_2, i_3, i_4)$. Then*

$$a(\mathcal{R}, S, T) = a^k_m,$$

where the family of sequences $(a^k_m)_{m \in \mathbb{N}}$, for $k \in \{0, 1, 2, 2', 2'', 3, 4\}$, are defined by the recurrence

$$a_m = 6a_{m-1} - 13a_{m-2} + 16a_{m-3} - 11a_{m-4} + 4a_{m-5} \quad \textit{for} \quad m \geq 5,$$

with initial conditions given in Table 1.

Before proving Theorem 3.27, we pause for commentary and an example.

$m =$	0	1	2	3	4
$k = 0$	1	2	6	20	66
1	1	3	9	28	89
2	1	4	12	36	112
$2'$	1	3	11	37	119
$2''$	1	4	12	37	118
3	1	4	14	46	148
4	1	4	16	56	184

Table 1. Initial conditions for the a-sequences a^k_m.

Remark 3.28. The characteristic polynomial corresponding to the recurrence is

$$1 - 6t + 13t^2 - 16t^3 + 11t^4 - 4t^5 = (1 - t + t^2)(1 - 5t + 7t^2 - 4t^3).$$

In fact, the sequences $(a^k_m)_{m\in\mathbb{N}}$ for $k = 0, 1, 2, 3, 4$ (but not for $k = 2', 2''$) actually satisfy a recurrence of order 3:

$$a_m = 5a_{m-1} - 7a_{m-2} + 4a_{m-3} \quad \text{for} \quad m \geq 3.$$

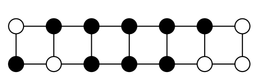
For the sake of brevity, however, we opt for stating the result in terms of a single (higher order) recurrence.

To prove Theorem 3.27, we will use a finite automaton to recognize pairs (I, J) that give *lex-minimal fillings* for rafts with a given boundary apparatus. Our automaton will read the raft from left to right, one vertical pair of nodes at a time, so the alphabet will be the set of tiles

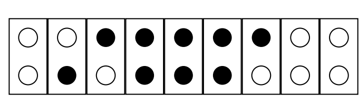
$$\mathcal{A} = \left\{ \boxed{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}, \boxed{\begin{smallmatrix}\bullet\\ \circ\end{smallmatrix}}, \boxed{\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}}, \boxed{\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}} \right\}. \tag{4}$$

The first tile represents the boundary apparatus on the left of the raft, and the last tile represents the apparatus on the right.

Example 3.29. Consider a raft of length $m = 7$ with no apparatus on either side, and with $I = \{1, 3, 4, 5\}$ and $J = \{2, 3, 4, 5, 6\}$. This is drawn as:



or



where prefix and suffix tiles have been appended to indicate the (lack of an) apparatus. Notice that this presentation is not lex-minimal, because the interval $[3, 5] \subset I$ is a proper subset of J.

A priori, from Theorem 3.9, we can realize any lex-minimal pair (I, J) for a raft of size m by a walk on the graph in Figure 8. However, not every walk will correspond to a lex-minimal pair.

We are now ready to prove the recurrence relation for $a(\mathcal{R}, S, T)$.

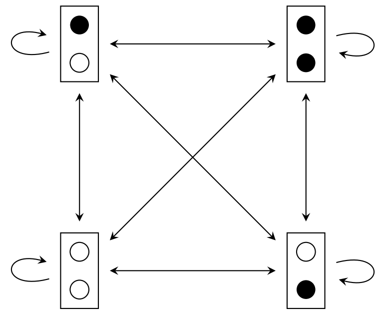


FIGURE 8. The finite state automaton for all words on the alphabet $\mathcal{A}$ given in Equation (4).

Proof of Theorem 3.27. Recall from Proposition 3.25 that $a(\mathcal{R}, S, T)$ is the number of ways to fill the vertices in the upper and lower rows of raft $\mathcal{R}$ with the boundary apparatus determined by S and T in a lex-minimal presentation. Since $a(\mathcal{R}, S, T)$ only depends on the boundary apparatus in $S \cup T$ and the size of $\mathcal{R}$, we will simplify the notation by setting $a_m(s,t) := a(\mathcal{R}, S, T)$ where m is the size of $\mathcal{R}$ and $s, t \in \mathcal{A}$ are chosen accordingly. Set $a_0(s,t) = 1$ for all $s, t \in \mathcal{A}$. This defines 4×4 auxiliary sequences $a_m(s,t)$ for $m \geq 0$.

Theorem 3.9 gives local conditions whose avoidance characterizes lex-minimal pairs. There are three types of local configurations to avoid, shown in Figure 9. In each type, the ellipsis represents arbitrarily long repetition of the adjacent tile. Types (i) and (ii) correspond to part (a) of Theorem 3.9, where $\{a, \ldots, b\}$ is an interval of small left ascents in I, with neither $a-1$ nor $b+1$ in I. Type (iii) corresponds to part (b), in which $\{a, \ldots, b\}$ is an interval of small right ascents in J, with neither $a-1$ nor $b+1$ in J. In the pictures for Types (i) and (iii) of Figure 9, the pattern is forbidden whether or not the nodes marked "$\star$" are filled.

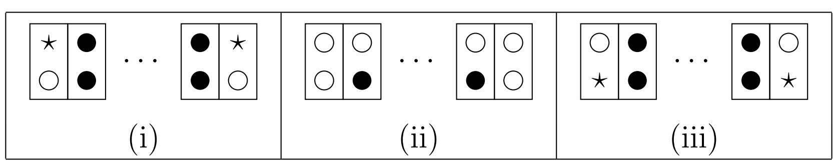


FIGURE 9. Patterns that are forbidden for lex-minimal presentations.

In order to construct an automaton for the allowed configurations, we need a more refined graph (as opposed to the graph in Figure 8) labeled by the tiles of $\mathcal{A}$. The first and last tiles can be any one of the four tiles in $\mathcal{A}$ corresponding to the boundary apparatus. The tiles in between these two must not introduce any forbidden configurations. This involves some tedious case analysis, but the result is given in Figure 10. While there are still only four tile types, there are eight states, reflecting the need for allowed walks to avoid introducing any of the patterns shown in Figure 9.

Having built the automaton, enumerating walks in this graph is now a straightforward application of the transfer matrix method. See [20, Section 4.7]. The matrix of adjacencies

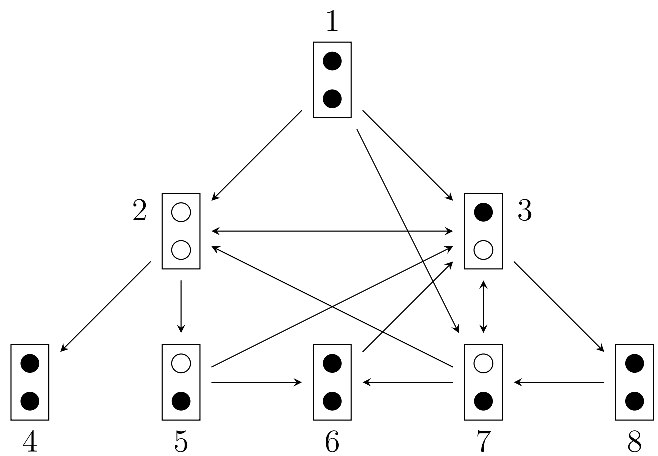


Figure 10. The finite automaton for lex-minimal presentations for a raft. Loops are allowed at each node, and have only been omitted for the sake of readability.

for the automaton is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Denote the generating function for walks that begin at node i and end at node j in the automaton by $R_{i,j}$. That is,

$$R_{i,j}(t) = \sum_{m \geq 0} r(i,j;m)t^m,$$

where $r(i,j;m)$ denotes the number of walks of length m that begin at node i and end at node j. It is a well-known result that

$$R_{i,j}(t) = (-1)^{i+j} \frac{\det(\mathcal{I} - tA : j, i)}{\det(\mathcal{I} - tA)}, \tag{5}$$

where $\mathcal{I}$ denotes the identity matrix and $\det(B : r, s)$ denotes the determinant of the matrix B after deleting row r and column s.

Consider walks of length $m+1$ on the graph given in Figure 10, including loops which are not drawn, starting in states 1, 2, 3, or 7, and ending at any tile. We claim that such walks are in bijective correspondence with lex-minimal fillings of rafts of length m with fixed boundary apparatus. The starting state, 1, 2, 3, or 7, is uniquely determined by the starting tile $s \in \mathcal{A}$. The final state can be any one of the eight states, but it must correspond with the raft's boundary apparatus, meaning $t \in \mathcal{A}$. The bijection sends such a walk $v_0 v_1 \ldots v_{m+1}$ with $v_0 = s$ and $v_{m+1} = t$ to (I, J), with $k \in I$ if and only if the node in the bottom row of v_k is filled, and $k \in J$ if and only if the node in the top row of v_k is filled, for all $1 \leq k \leq m$. The claim now follows from the fact that the forbidden configurations will never occur in such a walk, and conversely a sequence of tiles avoiding the forbidden configurations can be realized by such a walk.

Thus, Table 2 expresses the generating functions for the 4×4 sequences $a_m(s,t)$ for $s,t \in \mathcal{A}$, in terms of the finite automaton, where we abbreviate

$$R_{i,j_1\ldots j_k} := R_{i,j_1}(t) + \cdots + R_{i,j_k}(t)$$

for legibility. Each sequence is shifted by a factor of t because we are relating walks of length $m+1$ to fillings of rafts of size m. Along the diagonal, we first subtract the constant term because we are only considering walks of length at least 1. Note that the forbidden configurations in Figure 9 are symmetric under reversal, so $a_m(s,t) = a_m(t,s)$. Therefore, we only have to describe ten sequences.

	$\boxed{\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\bullet\\ \circ\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}$
$\boxed{\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}}$	$\frac{R_{2,2}-1}{t}$	$\frac{R_{2,3}}{t}$	$\frac{R_{2,57}}{t}$	$\frac{R_{2,468}}{t}$
$\boxed{\begin{smallmatrix}\bullet\\ \circ\end{smallmatrix}}$		$\frac{R_{3,3}-1}{t}$	$\frac{R_{3,57}}{t}$	$\frac{R_{3,468}}{t}$
$\boxed{\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}}$			$\frac{R_{7,57}-1}{t}$	$\frac{R_{7,468}}{t}$
$\boxed{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}$				$\frac{R_{1,1468}-1}{t}$

TABLE 2. Generating functions for the sequences $a_m(s,t)$.

By applying Equation (5), we can compute the explicit form for each rational generating function $R_{i,j}$ and use that to compute each $a_m(s,t)$ in Table 2. We leave the computation to the reader, and instead we give the first terms and recurrences which is an equivalent formulation. All $a_m(s,t)$ sequences satisfy one of the following recurrences:

R1: $a_m = 5a_{m-1} - 7a_{m-2} + 4a_{m-3}$ for $m \geq 3$
R2: $a_m = 6a_{m-1} - 13a_{m-2} + 16a_{m-3} - 11a_{m-4} + 4a_{m-5}$ for $m \geq 5$

The recurrences and initial conditions are shown in Table 3.

	$\boxed{\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\bullet\\ \circ\end{smallmatrix}}$	$\boxed{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}$
$\boxed{\begin{smallmatrix}\circ\\ \circ\end{smallmatrix}}$	**R1** 1,2,6	**R1** 1,3,9	**R1** 1,3,9	**R1** 1,4,12
$\boxed{\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}}$		**R2** 1,3,11,37,119	**R2** 1,4,12,37,118	**R1** 1,4,14
$\boxed{\begin{smallmatrix}\bullet\\ \circ\end{smallmatrix}}$			**R2** 1,3,11,37,119	**R1** 1,4,14
$\boxed{\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}}$				**R1** 1,4,16

TABLE 3. Recurrence relations and initial conditions for the sequences $a_m(s,t)$.

One can observe from this table that

$$
\begin{aligned}
a_m\left(\boxed{\substack{\circ\\ \circ}}, \boxed{\substack{\circ\\ \bullet}}\right) &= a_m\left(\boxed{\substack{\circ\\ \circ}}, \boxed{\substack{\bullet\\ \circ}}\right),\\
a_m\left(\boxed{\substack{\circ\\ \bullet}}, \boxed{\substack{\bullet\\ \bullet}}\right) &= a_m\left(\boxed{\substack{\bullet\\ \circ}}, \boxed{\substack{\bullet\\ \bullet}}\right), \text{ and}\\
a_m\left(\boxed{\substack{\circ\\ \bullet}}, \boxed{\substack{\circ\\ \bullet}}\right) &= a_m\left(\boxed{\substack{\bullet\\ \circ}}, \boxed{\substack{\bullet\\ \circ}}\right),
\end{aligned}
$$

so, in fact, the original sixteen sequences $a_m(s,t)$ fall into seven distinct families.

Comparing Table 3 to the statement in Theorem 3.27 and Remark 3.28 finishes the proof. $\square$

3.8. **Finishing the enumeration.** We now have a complete answer for how many lex-minimal presentations a raft can have, given a fixed choice of selected nodes on its boundary. Whether these choices are available depends on the immediate neighborhood of the raft in the w-ocean. We can lump together some of these boundary cases, which differ only by the selection of ropes.

Suppose that $\mathcal{R}$ is a raft of size m in the w-ocean for some permutation w, and (i_1, i_2, i_3, i_4) are the indicators on the boundary of the raft. For a fixed set of selected tethers $T \subseteq \mathrm{Tethers}(w)$ on the boundary of $\mathcal{R}$, consider

$$
b_m(\mathcal{R}, T) = \sum_S a(\mathcal{R}, S, T) = \sum a_m^{k(i_1,i_2,i_3,i_4)},
$$

where the first sum is over all possible selections from the ropes S adjacent to $\mathcal{R}$ in the w-ocean, and the second sum is over all encodings for the corresponding triples $(\mathcal{R}, S, T)$. Note there are at most sixteen terms in the sum. We encode the terms in $b_m(\mathcal{R}, T)$ by defining indicator sets $K(\mathcal{R}, T) = (I_1, I_2, I_3, I_4)$ where

$$
I_r = \begin{cases} \{0,1\} & i_r \in \mathrm{Ropes}(w), \\ \{1\} & i_r \in T, \text{ and} \\ \{0\} & i_r \notin T \cup \mathrm{Ropes}(w). \end{cases}
$$

Thus,

$$
b_m(\mathcal{R}, T) = \sum_{\substack{(i_1,i_2,i_3,i_4)\\ \in I_1\times I_2\times I_3\times I_4}} a_m^{k(i_1,i_2,i_3,i_4)}.
$$

Since $b_m(\mathcal{R}, T)$ only depends on m, and $K(\mathcal{R}, T) = (I_1, I_2, I_3, I_4)$, we define $81 = 3^4$ sequences known as the b-*sequences*:

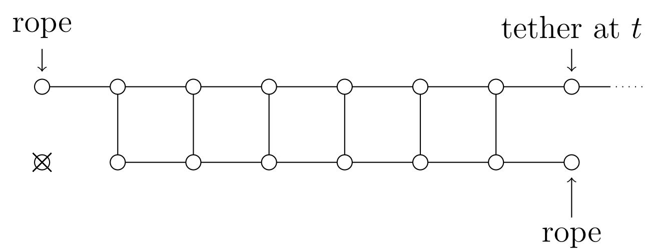
$$
b_m^{(I_1,I_2,I_3,I_4)} = \sum_{\substack{(i_1,i_2,i_3,i_4)\\ \in I_1\times I_2\times I_3\times I_4}} a_m^{k(i_1,i_2,i_3,i_4)}, \tag{6}
$$

where each $I_r \in \{\{0\}, \{1\}, \{0,1\}\}$. Similar to the a-sequences, we can use open or filled dots to denote the possible apparatus on either end of a raft. Let the symbols $\circ$, $\bullet$, and $\odot$ represent the three options $\{0\}$, $\{1\}$, and $\{0,1\}$, respectively. Then the apparatus at each end of a raft with a selection of specified tethers can be represented by one of the nine tiles in the alphabet

$$
\mathcal{B} = \left\{\boxed{\substack{\circ\\ \circ}},\ \boxed{\substack{\circ\\ \bullet}},\ \boxed{\substack{\bullet\\ \circ}},\ \boxed{\substack{\bullet\\ \bullet}},\ \boxed{\substack{\odot\\ \circ}},\ \boxed{\substack{\circ\\ \odot}},\ \boxed{\substack{\odot\\ \bullet}},\ \boxed{\substack{\bullet\\ \odot}},\ \boxed{\substack{\odot\\ \odot}}\right\}.
$$

Thus, each $b_m^{(I_1,I_2,I_3,I_4)}$ can be denoted by $b_m(s,t)$ for $s,t \in \mathcal{B}$. Up to symmetry among the a-sequences, there are only 27 different b-sequences. Initial terms in each case appear in the Appendix to this article.

Example 3.30. Suppose $\mathcal{R}$ is the following raft of length $m = 6$.



If the tether at i_4 in this raft is not selected, then, by considering whether or not the ropes are selected, we conclude that

$$b_6(\mathcal{R},\emptyset) = b_6^{(0,01,01,0)} = b_6\left(\boxed{\substack{\odot\\ \circ}},\boxed{\substack{\circ\\ \odot}}\right) = a_6\left(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \circ}}\right) + a_6\left(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \bullet}}\right) + a_6\left(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\circ\\ \circ}}\right) + a_6\left(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\circ\\ \bullet}}\right) = 3732$$

lex-minimal presentations for this raft, where we have abbreviated $\{0\}$ by "0" and $\{0,1\}$ by "01" for the sake of legibility. On the other hand, selecting the tether at i_4 yields

$$b_6(\mathcal{R},\{t\}) = b_6^{(0,01,01,1)} = b_6\left(\boxed{\substack{\odot\\ \circ}},\boxed{\substack{\bullet\\ \odot}}\right) = a_6\left(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\bullet\\ \circ}}\right) + a_6\left(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\bullet\\ \bullet}}\right) + a_6\left(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\bullet\\ \circ}}\right) + a_6\left(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\bullet\\ \bullet}}\right) = 4788$$

lex-minimal presentations for $\mathcal{R}$, with analogous abbreviations.

Corollary 3.31. *The sequences $b_m^{(I_1,I_2,I_3,I_4)}$ satisfy the linear recurrence*

$$b_m = 6b_{m-1} - 13b_{m-2} + 16b_{m-3} - 11b_{m-4} + 4b_{m-5} \quad \textit{for} \quad m \geq 5,$$

with initial conditions depending on (I_1,I_2,I_3,I_4), which can be deduced from the initial conditions for the a-sequences given in Table 1.

Proof. The recurrence relation follows from the fact that the b-sequences are each defined as a fixed finite sum of the a-sequences, and the a-sequences all satisfy the same recurrence, as shown in Theorem 3.27. $\square$

Now that we have developed all of the notation, we can prove our main enumeration theorem for S_n, originally given in Theorem 1.2. We restate the theorem here for the reader's convenience before giving the proof.

Theorem 1.2. The number of parabolic double cosets with minimal element $w \in S_n$ is

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{T \subseteq \operatorname{Tethers}(w)} \prod_{\mathcal{R} \in \operatorname{Rafts}(w)} b_{|\mathcal{R}|}^{K(\mathcal{R},T)}.$$

Proof. By Proposition 3.25, we reduce the computation of c_w to finding $a(\mathcal{R},S,T)$, the number of lex-minimal fillings of the raft $\mathcal{R}$ with boundary apparatus apparatus determined by S and T. By definition of the b-sequences, we collect the terms in each sum corresponding to $\operatorname{Ropes}(w)$ to get the stated formula. $\square$

We finish our discussion of the symmetric group case by calculating c_w for a large, somewhat generic example.

Example 3.32. Continuing Example 3.17, let

$$w = 1\ 3\ 4\ 5\ 7\ 8\ 2\ 6\ 14\ 15\ 16\ 9\ 10\ 11\ 12\ 13 \in S_{16},$$

whose balls-in-boxes picture is shown in Figure 5, and whose w-ocean is shown in Figure 7. Recall that $\mathrm{Floats}(w) = \{1', 7\}$, $\mathrm{Tethers} = \{8', 4\}$, and there are four rafts:

$$A = [2,3],\ B = [5,5],\ C = [9,10],\ \text{and } D = [12,15].$$

Raft A has ropes in positions i_2 and i_3, raft B has no ropes, raft C has a rope in position i_2, and raft D has no ropes. Thus

$$\begin{aligned} c_w &= 2^{|\,\mathrm{Floats}(w)|} \sum_{T\subseteq \mathrm{Tethers}(w)} \prod_{\mathcal{R}\in \mathrm{Rafts}(w)} b_{\mathcal{R}}(T), \\ &= 4 \cdot \sum_{T\subseteq\{8',4\}} b_A(T) b_B(T) b_C(T) b_D(T). \end{aligned}$$

We now consider each subset of T. If $T = \emptyset$, then

$$\begin{aligned} b_A(\emptyset) &= b_2^{(0,01,01,0)} = a_2^0 + 2a_2^1 + a_2^{2''} = 6 + 2(9) + 12 = 36, \\ b_B(\emptyset) &= b_1^{(0,0,0,0)} = a_1^0 = 2, \\ b_C(\emptyset) &= b_2^{(0,01,0,0)} = a_2^0 + a_2^1 = 6 + 9 = 15, \text{ and} \\ b_D(\emptyset) &= b_4^{(0,0,0,0)} = a_4^0 = 66, \end{aligned}$$

and so

$$\prod_{\mathcal{R}\in\mathrm{Rafts}(w)} b_{\mathcal{R}}(\emptyset) = 71280.$$

For $T = \{8'\}$, we find

$$\begin{aligned} b_A(\{8'\}) &= b_2^{(0,01,01,0)} = a_2^0 + 2a_2^1 + a_2^{2''} = 6 + 2(9) + 12 = 36, \\ b_B(\{8'\}) &= b_1^{(0,0,1,0)} = a_1^1 = 3, \\ b_C(\{8'\}) &= b_2^{(0,01,0,0)} = a_2^0 + a_2^1 = 6 + 9 = 15, \text{ and} \\ b_D(\{8'\}) &= b_4^{(1,0,0,0)} = a_4^1 = 89, \end{aligned}$$

and so

$$\prod_{\mathcal{R}\in\mathrm{Rafts}(w)} b_{\mathcal{R}}(\{8'\}) = 144180.$$

When $T = \{4\}$, we have

$$\begin{aligned} b_A(\{4\}) &= b_2^{(0,01,01,1)} = a_2^1 + a_2^2 + a_2^{2'} + a_2^3 = 9 + 12 + 11 + 14 = 46, \\ b_B(\{4\}) &= b_1^{(0,1,0,0)} = a_1^1 = 3, \\ b_C(\{4\}) &= b_2^{(0,01,0,0)} = a_2^0 + a_2^1 = 6 + 9 = 15, \text{ and} \\ b_D(\{4\}) &= b_4^{(0,0,0,0)} = a_4^0 = 66, \end{aligned}$$

and so

$$\prod_{\mathcal{R}\in\mathrm{Rafts}(w)} b_{\mathcal{R}}(\{4\}) = 136620.$$

Finally, $T = \{8', 4\}$ yields

$$\begin{aligned}
b_A(\{8',4\}) &= b_2^{(0,01,01,1)} = a_2^1 + a_2^2 + a_2^{2'} + a_2^3 = 9 + 12 + 11 + 14 = 46,\\
b_B(\{8',4\}) &= b_1^{(0,1,1,0)} = a_1^{2''} = 4,\\
b_C(\{8',4\}) &= b_2^{(0,01,0,0)} = a_2^0 + a_2^1 = 6 + 9 = 15, \text{ and}\\
b_D(\{8',4\}) &= b_4^{(1,0,0,0)} = a_4^1 = 89,
\end{aligned}$$

and so

$$\prod_{\mathcal{R} \in \mathrm{Rafts}(w)} b_{\mathcal{R}}(\{8',4\}) = 245640.$$

Therefore, the number of parabolic double cosets for which this w is the minimal element is

$$\begin{aligned}
c_w &= 4(71280 + 144180 + 136620 + 245640)\\
&= 2{,}390{,}880.
\end{aligned}$$

3.9. **Expected number of tethers.** Given a particular w, Theorem 1.2 can, in general, be computed quickly using the linear recurrence relation for the b-sequences given in Corollary 3.31. However, one might be concerned about computing the sum over all subsets of $\mathrm{Tethers}(w)$. Recall the claim, made in the introduction, that tethers are rare (and hence the sum has few terms). In fact, as stated in Proposition 1.3, an exact formula for the expected number of tethers for $w \in S_n$ for all $n > 2$ is

$$\frac{1}{n!} \sum_{w \in S_n} |\,\mathrm{Tethers}(w)| = \frac{(n-3)(n-4)}{n(n-1)(n-2)}.$$

Proof of Proposition 1.3. For w to have a right tether at position k means that

$$(w(k-1), w(k), w(k+1), w(k+2)) = (i, i+1, j, j+1),$$

where $i + 1 < j - 1$. Let $RT_k(w) \in \{0, 1\}$ be the number of right tethers of w in position k. The expected value of RT_k as a random variable can be computed by counting the number of pairs i, j for which $1 \le i < j - 2 \le n - 3$, multiplied by the number of permutations containing $(i, i+1, j, j+1)$ as consecutive values with $i + 1$ in position k. Namely,

$$\mathbb{E}(RT_k) = \frac{\binom{n-3}{2}(n-4)!}{n!} = \frac{(n-4)}{2n(n-1)(n-2)}.$$

for all $2 \le k \le n - 2$. Expectation of random variables is linear, so the expected number of right tethers is

$$\mathbb{E}(RT_2) + \cdots + \mathbb{E}(RT_{n-2}) = \frac{(n-3)(n-4)}{2n(n-1)(n-2)}.$$

An analogous argument proves that the expected number of left tethers has the same value, so summing the two contributions proves the formula. □

While the set $\mathrm{Tethers}(w)$ is typically small (on the order of $1/n$), there are some permutations for which $|\,\mathrm{Tethers}(w)|$ can be quite large, as seen in the following example.

Example 3.33. The permutation

$$w = 1\ 2\ 17\ 18\ 3\ 4\ 19\ 20\ \ldots\ 15\ 16\ 31\ 32$$

has fourteen left tethers and eight right tethers (see Figure 11), leading to a sum with 2^{22} terms. In this case, the value $c_w = 632{,}371{,}867{,}544{,}102$ can be determined on a computer within a few minutes.

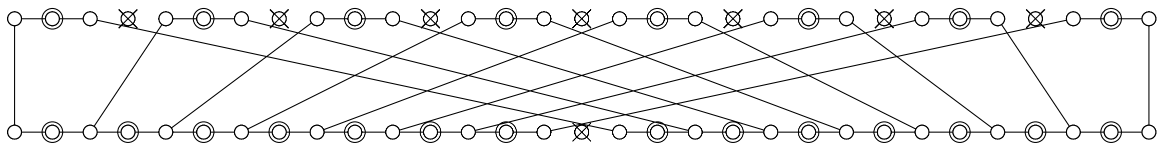


FIGURE 11. The w-ocean for the permutation of Example 3.33. We see that w has 22 large ascents denoted by two concentric circles, each of which is a tether

4. Parabolic double cosets for Coxeter groups

We now turn to the general setting of Coxeter groups, where our first task is to extend the characterization of lex-minimal presentations in Theorem 3.9 to this context.

Fix a Coxeter system (W, S) and recall the notation and terminology of Section 2. We first observe that if a coset has two different presentations, we get a third presentation by taking the union of the generators acting on the left and the right.

Lemma 4.1. *Suppose a parabolic double coset C has two different presentations $W_I w W_J = C = W_{I'} w W_{J'}$. Then $W_{I\cup I'} w W_{J\cup J'}$ is also a presentation for C.*

Proof. The lemma is an instance of a fact about general groups: if H_i, K_i, $i = 1, 2$ are subgroups of a group G such that $C := H_1 g K_1 = H_2 g K_2$ for some $g \in G$, then $C = HgK$, where $H = \langle H_1, H_2\rangle$ is the subgroup generated by H_1 and H_2, and $K = \langle K_1, K_2\rangle$ is the subgroup generated by K_1 and K_2.

To prove this fact, observe that $H_i C K_i = H_i g K_i = C$. But this means that $C = H_1 H_2 g K_2 K_1$. Repeating this idea again, we get that $C = H_2 C K_2 = H_2 H_1 H_2 g K_2 K_1 K_2$. By induction, we conclude that $C = (H_1 H_2)^k g (K_2 K_1)^k$ for all $k \geq 1$. Taking the union across k, we get that $C = HgK$.

To finish the lemma, note that if $H_1 = W_I$ and $H_2 = W_{I'}$, then $\langle H_1, H_2\rangle = W_{I\cup I'}$. ∎

Definition 4.2. Given a parabolic double coset C, set

$$M_L(C) := \bigcup_{\substack{(I,w,J)\,:\\ C = W_I w W_J}} I \quad \text{and} \quad M_R(C) := \bigcup_{\substack{(I,w,J)\,:\\ C = W_I w W_J}} J.$$

Proposition 4.3. *Let C be a parabolic double coset.*

(a) C has a presentation $C = W_{M_L(C)} w W_{M_R(C)}$, and this is the largest possible presentation for C, in the sense that if $C = W_I w' W_J$ then $I \subseteq M_L(C)$ and $J \subseteq M_R(C)$.

(b) The sets $M_L(C)$ and $M_R(C)$ can be determined by

$$M_L(C) = \{s \in S : sx \in C \text{ for all } x \in C\} \text{ and}$$
$$M_R(C) = \{s \in S : xs \in C \text{ for all } x \in C\}.$$

Proof. Part (a) follows immediately from the definition and Lemma 4.1.

For part (b), let $I' = \{s \in S : sx \in C \text{ for all } x \in C\}$ and $J' = \{s \in S : xs \in C \text{ for all } x \in C\}$. Then $M_L(C) \subseteq I'$ and $M_R(C) \subseteq J'$. Hence $C \subseteq W_{I'}wW_{J'}$. At the same time, C is closed under left multiplication by members of I' and right multiplication by members of J'. Hence $W_{I'}wW_{J'} \subseteq C$. Thus $W_{I'}wW_{J'}$ is a presentation for C, which means $I' \subseteq M_L(C)$ and $J' \subseteq M_R(C)$. Putting this all together means $I' = M_L(C)$ and $J' = M_R(C)$, as desired. □

In light of Proposition 4.3, we introduce the following terminology.

Definition 4.4. The presentation $W_{M_L(C)}wW_{M_R(C)}$ appearing in Proposition 4.3 is the *maximal presentation* for C.

We will also want to identify presentations that are as small as possible.

Definition 4.5. A presentation $C = W_IwW_J$ is *minimal* if

(a) $w \in {}^IW^J$,
(b) no connected component of I is contained in $(wJw^{-1}) \cap S$, and
(c) no connected component of J is contained in $(w^{-1}Iw) \cap S$.

Note that this is not the same as lex-minimality, which was introduced in Definition 1.5.

If a connected component I_0 of I is contained in $(wJw^{-1}) \cap S$ for $w \in {}^IW^J$, then $W_IwW_J = W_{I\setminus I_0}wW_J$. A similar argument applies to subsets of J, so every presentation can be reduced to a minimal presentation. In Proposition 4.8, we will show that our nomenclature is appropriate; that is, minimal presentations have minimum size.

Lemma 4.6. *Let $C = W_IwW_J$ be a minimal presentation of C. Then*

$$M_L(C) = I \cup \{s \in (wJw^{-1}) \cap S : s \text{ is not adjacent to } I\} \text{ and}$$
$$M_R(C) = J \cup \{s \in (w^{-1}Iw) \cap S : s \text{ is not adjacent to } J\}.$$

Proof. That $M_L(C)$ includes its proposed reformulation is clear. For the other direction, suppose that $s \in M_L(C)$ and $s \notin I$. By Proposition 4.3(b), $w < sw \in C$, and by Corollary 2.8 it follows that $sw = wt$ for some $t \in J$.

We can extend this argument to show that s is not adjacent to any element of I. Indeed, suppose that s is adjacent to some element of r of I, and let I_0 be the connected component of r in I. We argue that $I_0 \subseteq wJw^{-1}$, in contradiction of our hypothesis. Any element r' of I_0 is connected to s by a simple path $s = s_0, s_1, \ldots, s_k = r'$, $k \geq 1$, in the Coxeter graph of W, where $s_1, \ldots, s_k$ is entirely contained in I_0. Using Proposition 4.3(b) and Corollary 2.8, we can write $s_0 \cdots s_kw = uwv$, where $u \in W_I$ and $v \in {}^{(w^{-1}Iw)\cap S}W_J$. Now, s_0 is the only left descent of $s_0 \cdots s_k$, and because $I \cup \{s_0\} \subset \mathrm{Asc}_L(w)$, we conclude that s_0 is the only left descent of $s_0 \cdots s_kw$ in $I \cup \{s_0\}$. But any left descent of u will be a left descent of $s_0 \cdots s_kw$ in I, so we conclude that u has no left descents, or in other words, $u = e$. As a result, $w^{-1}s_0 \cdots s_kw = v \in W_J$. The same argument shows that $w^{-1}s_0 \cdots s_{k-1}w \in W_J$. Hence we have $w^{-1}s_kw \in W_J$. Because $s_kw = w(w^{-1}s_kw)$, and $w \in {}^IW^J$, we conclude that $w^{-1}s_kw = w^{-1}r'w$ belongs to J. Thus $I_0 \subseteq wJw^{-1}$, yielding the desired contradiction.

The equality for $M_R(C)$ is analogous. □

Corollary 4.7. *Suppose that $C = W_IwW_J$ is a minimal presentation of C. If T is any connected subset of $M_L(C)$, then either $T \subseteq I$, or T is disjoint and non-adjacent to I and $T \subseteq (wJw^{-1}) \cap S$.*

Proof. Suppose $s \in T$ is not contained in I. By Lemma 4.6, this s must be non-adjacent to I and contained in wJw^{-1}. Because T is connected, iterating this argument for the neighbors of s yields the desired conclusion. $\square$

Given subsets $X, Y, Z \subseteq S$, write

$$X = Y \overset{\not\sim}{\sqcup} Z$$

to mean that X is the disjoint union of Y and Z, and Y and Z are non-adjacent. (In other words, the subgraph of the Coxeter graph induced by the vertex set X is isomorphic to the disjoint union of the vertex-induced subgraphs of Y and Z.) The following proposition is, roughly speaking, obtained by repeated application of Corollary 4.7.

Proposition 4.8. *Fix $w \in {}^IW^J$. A presentation $C = W_IwW_J$ is minimal if and only if $|I| + |J| \leq |I'| + |J'|$ for all other presentations $C = W_{I'}wW_{J'}$ of C. Furthermore, if $C = W_IwW_J$ and $C = W_{I'}wW_{J'}$ are both minimal presentations, then there are sequences of connected components $I_1, \ldots, I_m$ and $J_1, \ldots, J_n$ of the subgraphs induced by I and J, respectively, such that*

$$I' = \left(I \overset{\not\sim}{\sqcup} wJ_1w^{-1} \overset{\not\sim}{\sqcup} \cdots \overset{\not\sim}{\sqcup} wJ_nw^{-1} \right) \setminus \bigcup_{i=1}^{m} I_i \text{ and}$$

$$J' = \left(J \overset{\not\sim}{\sqcup} w^{-1}I_1w \overset{\not\sim}{\sqcup} \cdots \overset{\not\sim}{\sqcup} w^{-1}I_mw \right) \setminus \bigcup_{j=1}^{n} J_j.$$

Note that the order of operations in the identities of Proposition 4.8 is significant, in that wJ_jw^{-1} is required to be disjoint and non-adjacent to I_i for all i, j.

Proof of Proposition 4.8. We start by proving the second part of the proposition. Suppose that $C = W_IwW_J$ and $C = W_{I'}wW_{J'}$ are both minimal presentations, and let I_0 be a connected component of I. Then $I_0 \subseteq M_L(C)$, so by Corollary 4.7 either $I_0 \subseteq I'$, or I_0 is disjoint and non-adjacent to I'. In the former case, I_0 must be contained in a connected component I_0' of I'. Applying Corollary 4.7 again, we must have $I_0' \subseteq I$, and hence $I_0 = I_0'$. If I_0 is disjoint and non-adjacent to I', then Corollary 4.7 also tells us that $I_0 \subseteq (wJ_0'w^{-1}) \cap S$, where J_0' is a connected component of J'. Since $C = W_IwW_J$ is minimal, it is impossible for J_0' to be contained in J, because then we would have $I_0 \subseteq wJw^{-1}$. Applying Corollary 4.7 one last time, we see that J_0' is disjoint and non-adjacent to J, and $J_0' \subseteq w^{-1}I_1w$ for some connected component I_1 of I. But then $I_0 \subseteq I_1$, which means that $I_0 = I_1$ and $J_0' = w^{-1}I_0w$. Combining the two cases shows that either I_0 is a connected component of I', or $w^{-1}I_0w$ is a connected component of J', which is disjoint and non-adjacent to J. This proves the second part of the proposition.

For the first part of the proposition, let N be the minimum of $|\tilde{I}| + |\tilde{J}|$ across all presentations $C = W_{\tilde{I}}wW_{\tilde{J}}$ of the parabolic double coset C. Any presentation $C = W_{I'}wW_{J'}$ with $|I'| + |J'| = N$ is clearly minimal. If $C = W_IwW_J$ is another minimal presentation, then I, J and I', J' are related as in the second part of the proposition, and consequently $|I| + |J| = |I'| + |J'| = N$. $\square$

We now get the desired characterization of lex-minimal presentations as an immediate corollary of Proposition 4.8.

Theorem 4.9. *Fix $w \in {}^IW^J$. Then $C = W_IwW_J$ is lex-minimal if and only if*

(a) no connected component of J is contained in $(w^{-1}Iw) \cap S$, and
(b) if a connected component I_0 of I is contained in wSw^{-1}, then $w^{-1}I_0w$ is not contained in J, and there is an element of J adjacent to or contained in $w^{-1}I_0w$.

Furthermore, every parabolic double coset has a unique lex-minimal presentation.

When w is the identity, Theorem 4.9 implies that lex-minimal presentations are two-level staircase diagrams in the sense of recent work by Richmond and Slofstra [16]. Note that while [16] addresses the enumeration of staircase diagrams, two-level staircase diagrams were not considered.

4.1. **The marine model for Coxeter groups.** The enumeration formula in Theorem 1.2 extends to general Coxeter groups. To describe the formula in the general setting, we need to extend the marine model. In Theorem 4.15, we present a visual test for detecting when two parabolic double cosets with the same minimal element are equal.

Definition 4.10. For $w \in W$, an ascent $s \in \mathrm{Asc}_R(w)$ is a *small ascent* if $wsw^{-1} \in S$. Otherwise, this s is a *large ascent*.

Note that s is a small ascent of w if and only if wsw^{-1} is a small ascent of w^{-1}. Also, if s and t are both small ascents for w, then $m_{st} = m_{s't'}$, where $s' = wsw^{-1}$ and $t' = wtw^{-1}$.

Definition 3.19 introduced the w-ocean of a permutation, built out of two copies of the Coxeter graph of S_n. For general Coxeter groups, we build an analogous graph.

Definition 4.11. For $w \in W$, the *w-ocean* is the graph $G(W, S, w)$ whose vertices are

$$\big\{(s,1) : s \text{ a right ascent of } w\big\} \cup \big\{(s,0) : s \text{ a left ascent of } w\big\}.$$

There is an edge between vertices $(s,1)$ and $(t,1)$ (respectively, $(s,0)$ and $(t,0)$), if s and t are adjacent in the Coxeter graph of W, and at least one of them is a small ascent of w (respectively, w^{-1}). There is an edge between $(s,1)$ and $(t,0)$ if s is a small ascent of w and $t = wsw^{-1}$.

In terms of Definition 3.19, the vertices $(s,1)$ are the top row of the w-ocean, and the vertices $(s,0)$ are the bottom row.

Definition 4.12. Given a w-ocean $G = G(W, S, w)$, we identify certain vertices and induced subgraphs. Note how these generalize the classifications in Definition 3.10, and note also the new type "wharf."

- A *small ascent* of G is a vertex of the form $(s,1)$ or $(s,0)$ such that s is a small ascent of w or w^{-1} respectively.
- A *large ascent* of G is a vertex of the form $(s,1)$ or $(s,0)$ such that s is a large ascent of w or w^{-1} respectively.
- A *float* is an isolated vertex in G. They are all large ascents of G not adjacent to any small ascents.
- A *rope* is a large ascent of G which is adjacent to exactly one small ascent of G.
- A *tether* is a large ascent of G which is adjacent to at least two small ascents of G.
- A *plank* in G is an induced subgraph consisting of exactly two small ascents of the form $\{(s,1),(t,0)\}$ such that $t = wsw^{-1}$.
- A *wharf* is a plank in G such that at least one of its two vertices is adjacent to at least three other vertices of G, all on the same row, and at least two of the adjacent vertices are small ascents of G. In addition, any plank can be designated as a wharf

if at least one of its vertices is adjacent to two other small ascents of G (on the same row). If G contains cycles, then we will always choose a number of additional wharfs so that the graph with tethers and wharfs deleted is acyclic.

- A *raft* is a connected component of the subgraph of G induced by the planks which are not wharfs. The *size* of a raft is the number of planks it contains, or equivalently half the number of vertices. If a raft R has size m, then it is isomorphic to the w-ocean for $w = e \in S_{m+1}$.

Compared to the definition of the marine model in Section 3.4, floats, ropes and rafts are essentially the same as before. Tethers are also essentially the same, although they can be adjacent to more than two small ascents when the degree of the corresponding vertex of the Coxeter graph is at least three. It is also now possible to have small ascents that are connected to more than two small ascents, or to two small ascents and a large ascent, which is where wharfs come in. Examples of all of these objects are given in the next sections.

Remark 4.13. Observe that the construction of the w-ocean does not depend on edge labels $m(s,t)$ in the Coxeter graph of (W,S).

The w-ocean is a useful tool for studying the presentations (I,w,J) for all $(I,J) \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$. Each such pair (I,J) can be represented by the subgraph of the w-ocean containing the vertices $\{(i,0) : i \in I\} \cup \{(j,1) : j \in J\}$. We denote the selected vertices by filled dots and the unselected vertices by open dots in the w-ocean as in the type A case. By a slight abuse of notation, we will equate the subsets of $\{(i,0) : i \in I\}$ in the w-ocean with subsets of I by the natural bijection. In particular, the connected components of I as an induced subgraph of the Coxeter graph are in natural bijection with the components of $\{(i,0) : i \in I\}$ as an induced subgraph of the w-ocean. Similarly, there is a natural bijection between subsets of J and $\{(j,1) : j \in J\}$.

Definition 4.14. Assume $w \in W$ and $(I,J) \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$. Let I_0 be a connected component of I consisting entirely of small left ascents. Then, each vertex in I_0 is the endpoint of a plank in the w-ocean. Let $J_0 = w^{-1}I_0w$ be the corresponding connected set of endpoints on the top row of the w-ocean so J_0 consists entirely of small right ascents. We define the following three types of *plank moves* when applicable along with their analogs obtained from switching the roles of I and J.

- Contraction move: $(I,J) \to (I \setminus I_0, J)$ provided $J_0 \subset J$.
- Expansion move: $(I \setminus I_0, J) \to (I,J)$ provided $J_0 \subset J$.
- Slide move: $(I,J) \to (I \setminus I_0, J \cup J_0)$ provided $J \cap J_0 = \emptyset$ and J_0 is a connected component of $J \cup J_0$ on the top row of the w-ocean.

Theorem 4.15. *Let $(I,J),(I',J') \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$. Then, $W_IwW_J = W_{I'}wW_{J'}$ if and only if (I,J) can be obtained from (I',J') by plank moves.*

Proof. Let $(I,J) \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$, and let $I_0 \subset I$ be a connected component of I as an induced subgraph of the Coxeter graph. Then, the elements of $W_{I\setminus I_0}$ commute with W_{I_0} so $W_Iw = W_{I\setminus I_0}W_{I_0}w$. If in addition each vertex in I_0 is a small ascent or equivalently adjacent to a plank, then $W_Iw = W_{I\setminus I_0}wW_{J_0}$ where $J_0 = w^{-1}I_0w$ is the set of vertices on the other side of the planks attached to I_0. Thus, $W_IwW_J = W_{I\setminus I_0}wW_{J_0}W_J$. The product $W_{J_0}W_J$ equals the parabolic subgroup W_J if $J_0 \subset J$ (expansion/contraction move) or $W_{J_0\cup J}$ provided W_{J_0} and W_J are commuting subgroups (slide move). Recall, the ladder condition is equivalent to saying J_0 is a connected component of the induced subgraph of the w-ocean on

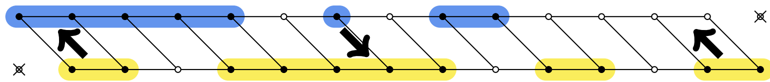


FIGURE 12. For Example 4.17, the connected components of (I, J) along with three possible plank moves are shown on the w-ocean.

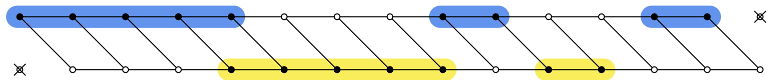


FIGURE 13. For Example 4.17, the connected components of (I', J') are shown on the w-ocean. This pair is lex-minimal.

vertices $J \cup J_0$ and $J \cap J_0 = \emptyset$. Thus, both slides and contractions preserve the corresponding double cosets so if (I', J') is connected to (I, J) by plank moves, then $W_I w W_J = W_{I'} w W_{J'}$.

Conversely, assume $W_I w W_J = W_{I'} w W_{J'}$. By applying all possible contraction moves on (I, J) and (I', J'), we arrive at two minimal presentations for the same double coset. Then by Proposition 4.8, these two minimal presentations are connected by slide moves. □

Corollary 4.16. *For $w \in W$ and $(I, J) \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$, the lex-minimal presentation of the parabolic double coset $W_I w W_J$ is obtained from (I, J) by applying all possible contraction moves and then applying all possible slide moves on the remaining components in the bottom row.*

Proof. The statement follows directly from Theorem 4.9 and Theorem 4.15. □

Example 4.17. Consider the permutation

$$w = 2\ 3\ 4\ \ldots\ 15\ 16\ 1\ \in S_{16}$$

and the parabolic double $W_I w W_J$ with

$$I = \{2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15\} \text{ and } J = \{1, 2, 3, 4, 5, 7, 9, 10\}.$$

Figure 12 shows the connected components of I and J as induced subgraphs of the w-ocean on the bottom and top rows respectively. The arrows indicate possible plank moves; from left to right we see two contractions and a slide. Applying these plank moves to (I, J) results in the pair (I', J') for

$$I' = \{5, 6, 7, 8, 9, 11, 12\} \text{ and } J' = \{1, 2, 3, 4, 5, 9, 10, 13, 14\}$$

whose connected components are shown in Figure 13. Thus, by Theorem 4.15, $W_I w W_J = W_{I'} w W_{J'}$. One can also verify that (I', J') is a lex-minimal pair for w by Corollary 4.16 or directly from Theorem 4.9.

4.2. **Lex-minimal presentations near wharfs in star Coxeter groups.** As a warm-up, let (W, S) be a Coxeter system whose Coxeter graph is a star, specifically s_1 is the central vertex and $s_2, \ldots, s_n$ for $n \geq 4$ are adjacent to s_1 and nothing else. Consider the identity element $e \in W$. Every $1 \leq i \leq n$ indexes a small ascent for e, and thus $\{(s_1, 0), (s_1, 1)\}$ is a wharf in the e-ocean. In this subsection, we study the enumeration of lex-minimal presentations (I, e, J) for such Coxeter groups in order to determine the valid neighborhoods for wharfs in lex-minimal presentations for any w and any Coxeter group.

Partition the set of lex-minimal presentations (I, e, J) according to which nodes are selected at the wharf. We have four choices: we either include or exclude the node on the upper level, and we either include or exclude the node on the lower level. Denote these four options by , , , where again a filled dot means that it is selected to be in I or J depending on if it is on the bottom or on the top, respectively.

By Theorem 4.9, the allowed pairs for lex-minimal pairs $(I, J) \subset [n]^2$ for e in the neighborhood of a wharf are characterized as follows.

(a) If $1 \in I$, then I and J must be incomparable in subset order.
(b) If $1 \notin I$, then J can be any subset, and either I is empty or I is not comparable to J.

Example 4.18. Consider the case of a star Coxeter graph with four vertices. If the edge labels are all 3, this Coxeter group is type D_4, and $c_e = 72$. Up to symmetry of the three leaves around the central vertex, there are twenty-four distinct types of allowable lex-minimal presentations (I, e, J) as shown in Figure 14.

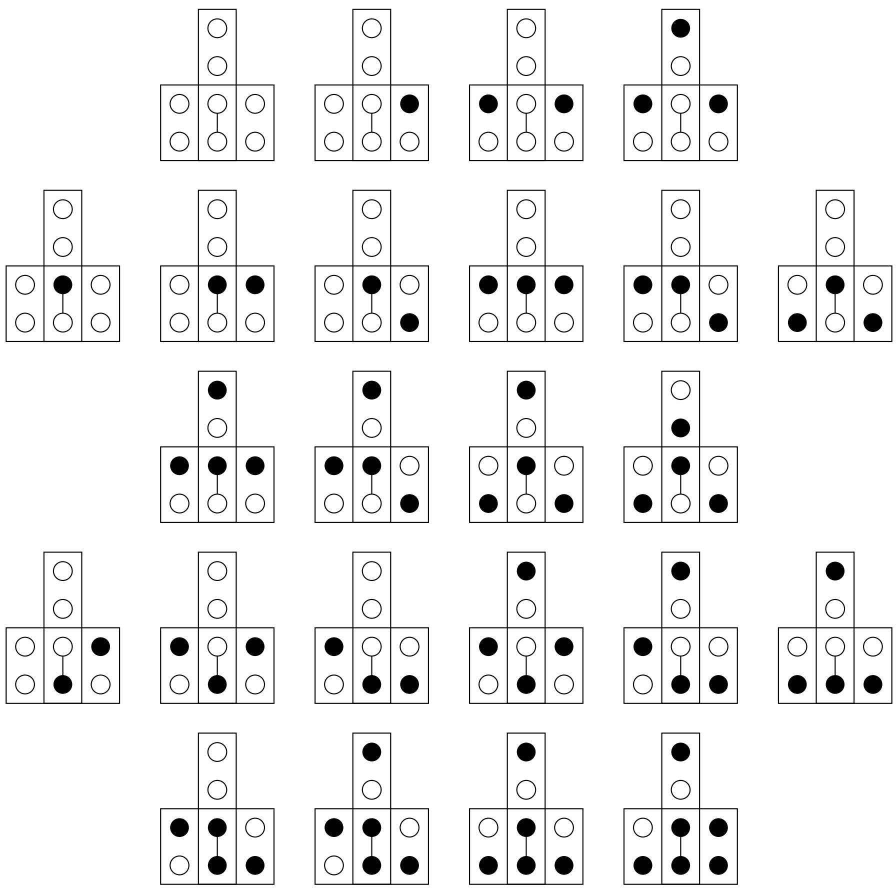


FIGURE 14. Lex-minimal pairs for D_4, up to symmetry around the central tile.

4.3. **Enumeration for Coxeter groups.** Given an element w in a general Coxeter group W, the w-ocean formed on its ascents consists of rafts, tethers, ropes, floats, and wharfs. To enumerate the lex-minimal pairs $(I, J) \in \mathrm{Asc}(w^{-1}) \times \mathrm{Asc}(w)$, we partition the set of lex-minimal pairs according to which tethers, ropes, and wharfs are included in I and J. Every possible selection of tethers, ropes, and wharfs leads to a nonempty set of lex-minimal pairs. After fixing such a selection, we further partition the set of lex-minimal pairs by adding local conditions around the wharfs which allow us to once again reduce the enumeration to the fillings of rafts avoiding the forbidden patterns in Figure 9 and the automaton in Figure 10.

The purpose of this section is to describe this process in detail, starting with the selection of tethers, ropes and wharfs. For each tether and rope, we have two choices: we either include it or not, as in the case of S_n. For each wharf, we have four choices: we either include or exclude the node in the upper row, and we either include or exclude the node in the lower row. As with S_n, we indicate our choices by filling in the vertices on the w-ocean. For the sake of discussion, we will denote our selection by C.

After choosing C, we need to make additional choices on the way that the rafts are filled near wharfs. These conditions are recorded on a new graph constructed from the w-ocean, which we call a harbor. Before we define a harbor, consider the rafts in the w-ocean. It is convenient to think of rafts as a path of planks, each of which has a node at the top and at the bottom. If a raft has size greater than one, then it has two distinct endplanks, and each of these endplanks can be adjacent to a wharf, or to some collection of ropes and tethers on the top and/or bottom. Note that, in this case, if one of these endplanks is adjacent to a wharf then it cannot be adjacent to a rope or a tether, since that would make it a wharf. For rafts consisting of a single plank, then that plank can either be adjacent to two wharfs, or to at most one wharf and some (possibly empty) collection of ropes and/or tethers. To treat rafts in a uniform fashion, we will think of rafts of size one as having two logical endplanks. In this way, we can divide adjacent wharfs, ropes, and tethers among the two logical endplanks so that no endplank is adjacent to more than one wharf, and no endplank is adjacent to both a wharf and a rope or tether.

Roughly speaking, a harbor graph is defined by thinking of the rafts as edges in a new graph. We now make this precise:

Definition 4.19. For a choice C of fillings of the vertices in the w-ocean corresponding to tethers, ropes, and wharfs, we define a simple graph H_C, called a *harbor*, as follows: First, the harbor has a vertex for each wharf of the w-ocean, and one vertex for each endplank of a raft which is not adjacent to a wharf. As mentioned above, we think of rafts of size one as having two logical endplanks; if such a raft is adjacent to only one wharf, then we add one endplank vertex, and if the raft is not adjacent to any wharf, then we add two endplank vertices. The harbor also has an edge for every raft in the w-ocean. This edge is incident to a vertex of the harbor if and only if the vertex corresponds to a wharf which is adjacent to the endplank of the raft in question, or the vertex corresponds to the endplank of the raft. In addition, the harbor has a vertex and edge for each pair (w, r), where w is a wharf and r is a selected rope or tether adjacent to w. The edge connects this additional vertex to the vertex corresponding to w. Finally, we connect two wharfs by an edge if they are adjacent in the w-ocean.

Next we describe the edge, vertex and half edge decorations on H_C. The edge and vertex decorations are completely determined by the w-ocean and the choice C. In contrast, there are different possible ways to decorate the half edges, and we will need to consider all of them for the main enumeration formula. This step will necessarily be more complicated. The reader may wish to look ahead to Theorem 4.26 and the example in Figure 15.

Observe that every vertex in the graph H_C is connected to at least one edge by construction. Each edge represents a raft of some size, possibly of size 0. Rafts of size 0 come from edges connecting two wharfs or a (w, r) pair. Every edge of the harbor H_C is decorated with the integer corresponding to the size of the corresponding raft.

The vertices of the harbor H_C are decorated with tiles from

$$\left\{\boxed{\substack{\circ\\ |\\ \circ}},\boxed{\substack{\circ\\ |\\ \bullet}},\boxed{\substack{\bullet\\ |\\ \circ}},\boxed{\substack{\bullet\\ |\\ \bullet}},\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \bullet}},\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\bullet\\ \bullet}}\right\}.$$

Wharfs are decorated with tiles $\boxed{\substack{\circ\\ |\\ \circ}}$, $\boxed{\substack{\circ\\ |\\ \bullet}}$, $\boxed{\substack{\bullet\\ |\\ \circ}}$, and $\boxed{\substack{\bullet\\ |\\ \bullet}}$ according to which nodes of the wharf are selected. Vertices corresponding to (w, r) are decorated with $\boxed{\substack{\circ\\ \bullet}}$ or $\boxed{\substack{\bullet\\ \circ}}$ depending on whether the selected rope or tether r is on the top or the bottom of the w-ocean. Endplanks of rafts are decorated with tiles $\boxed{\substack{\circ\\ \circ}}$, $\boxed{\substack{\circ\\ \bullet}}$, $\boxed{\substack{\bullet\\ \circ}}$, and $\boxed{\substack{\bullet\\ \bullet}}$, where the top (resp. bottom) node of the tile is filled if the endplank is adjacent to any selected rope or tether on the top (resp. bottom). In this way, a selected tether is split into many ropes, and then selected ropes adjacent to the same endplank are amalgamated.

For rafts of size one, we must again take special care for arbitrary Coxeter groups—if the raft is not adjacent to any wharf, then we can split adjacent ropes and tethers arbitrarily among the two endplanks before applying the above recipe without changing the lex-minimal conditions by Theorem 4.9. For instance, we can assign all ropes and tethers to one endplank, meaning that we label that endplank as above, and then label the other endplank by $\boxed{\substack{\circ\\ \circ}}$.

Each half-edge of the harbor H_C is decorated by one of the labels from the set $\mathcal{L} \cup \mathcal{L}'$ where

$$\mathcal{L} = \left\{\boxed{\substack{\circ\\ \circ}},\ \boxed{\substack{\bullet\\ \circ}},\ \boxed{\substack{\circ\\ \bullet}},\ \boxed{\substack{\bullet\\ \bullet}},\ \boxed{\substack{\circ\\ |\\ \odot}},\ \boxed{\substack{\circledcirc\\ |\\ \bullet}},\ \boxed{\substack{\odot\\ |\\ \bullet}},\ \boxed{\substack{\bullet\\ |\\ \odot}},\ \boxed{\substack{\bullet\\ |\\ \bullet}}\right\} \text{ and } \mathcal{L}' = \left\{\left(\substack{\circ\\ \bullet}\right),\ \left(\substack{\bullet\\ \bullet}\right)\right\}$$

Not all possible labellings of the half-edges are allowed for H_C. We refer to the labellings which are allowed as *legal labellings*. Legal labellings encode the local conditions on lex-minimal fillings in the neighborhoods of wharfs which are similar to those found in Section 4.2. The idea is that the half-edge labels are used to specify the boundary apparatus at each end of the corresponding raft in the w-ocean, and then the enumeration of lex-minimal fillings reduces to finding the number of lex-minimal fillings for each raft with the specified boundary apparatus. This in turn translates to a walk on the automata of Section 3.7. Furthermore, the half-edge labels indicate specific selections of nodes on the initial or final segment of rafts in certain lex-minimal pairs (I, J) consistent with the selection C. Thus, they also allow us to refer back to the "top" and "bottom" rows of the w-ocean, even though the harbor doesn't have "top" or "bottom" vertices.

We now explain what makes a labeling legal; this also gives us an opportunity to explain what each label means. Afterwards we give a more concise formal definition. Throughout this discussion we will assume (I, J) is a lex-minimal pair for w consistent with C and the given half edge labeling. Recall no contraction or upward slide moves apply to (I, J) by Corollary 4.16. The pair (I, J) determines an induced subgraph $G_C(I, J)$ of the w-ocean with vertices I on the bottom and J on the top, and we will frequently refer to the connected components of this subgraph. Also, it will be useful to recall the meaning of the tiles on planks in rafts from the type A case. For example, if we have a tile of the form $\boxed{\substack{\circ\\ \bullet}}$ on the first plank of a raft, that means that the lower endpoint of the plank is in I and its upper endpoint is not in J.

(1) Vertices of type $\boxed{\substack{\circ\\ |\\ \circ}}$ indicate a wharf with neither top nor bottom node chosen in C. Thus they behave just like the boundary apparatus $\boxed{\substack{\circ\\ \circ}}$ when filling the adjacent rafts. We label all of the half-edges emanating from a $\boxed{\substack{\circ\\ |\\ \circ}}$ vertex by $\boxed{\substack{\circ\\ \circ}}$. Recall that $\boxed{\substack{\circ\\ \circ}}$ is vertex 2 in the automaton from Section 3.7. Any walk in the automaton starting or ending at vertex 2 corresponds to an initial or final segment of a lex-minimal filling of an adjacent raft.

(2) Vertices of type $\boxed{\substack{\bullet \\ | \\ \circ}}$ indicate a wharf with only the top node selected. The selected node may join connected components of $G_C(I, J)$ on the top in adjacent rafts. Because the bottom node of the wharf is not selected, the connected component containing the wharf will not be contained in any connected component on the bottom (so the conditions of Theorem 4.9 are satisfied). Thus wharfs of this type behave just like the boundary apparatus $\boxed{\substack{\bullet \\ \circ}}$ when filling the adjacent rafts. Thus we label all of the half-edges emanating from a $\boxed{\substack{\bullet \\ | \\ \circ}}$ vertex by $\boxed{\substack{\bullet \\ \circ}}$. Recall that $\boxed{\substack{\bullet \\ \circ}}$ is vertex 3 in the automaton. Any walk in the automaton starting or ending at vertex 3 corresponds to an initial or final segment of a lex-minimal filling of an adjacent raft.

(3) Vertices of type $\boxed{\substack{\circ \\ | \\ \bullet}}$ indicate a wharf with only the bottom node selected. As in the previous case, this selected node can join connected components of $G_C(I, J)$ on the bottom in adjacent rafts. Since the top node of the wharf is not selected, the connected component of $G_C(I, J)$ containing this wharf is not contained in a connected component on the top. To prevent there being an available upward slide move, we must also show either that the planks with bottom node in the connected component contain or are adjacent to a selected node on top, or that the connected component contains a large ascent. In either case, the selected top node or large ascent might be connected to the wharf in question only after passing through another wharf or sequence of wharfs. To keep track of the different possibilities, we label the adjacent half-edges by $\boxed{\substack{\circ \\ | \\ \odot}}$, $\boxed{\substack{\circledcirc \\ | \\ \bullet}}$, or $\left(\substack{\circ \\ \bullet}\right)$. The meaning of these labels is as follows:

(a) A half-edge label $\boxed{\substack{\circ \\ | \\ \odot}}$ means that on that side of the corresponding raft, we have any nonnegative number of $\boxed{\substack{\circ \\ \bullet}}$ tiles emanating from the wharf, after which we have a tile $\boxed{\substack{\circ \\ \circ}}$ or $\boxed{\substack{\circ \\ | \\ \circ}}$. For example, we may choose three intermediate tiles $\boxed{\substack{\circ \\ | \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \circ}}$, or even zero intermediate tiles $\boxed{\substack{\circ \\ | \\ \bullet}}\boxed{\substack{\circ \\ \circ}}$, along the raft corresponding to a half-edge labeled $\boxed{\substack{\circ \\ | \\ \odot}}$. After the $\boxed{\substack{\circ \\ \circ}}$ tile, the allowed tiles follow the automaton again, starting with vertex 2. If the raft is oriented so that the half-edge is at the terminal end of a raft, then the label implies that the corresponding walks in the automaton terminate at vertex 5.

(b) A half-edge label $\boxed{\substack{\circledcirc \\ | \\ \bullet}}$ means that on that end of the corresponding raft, we have any nonnegative number of $\boxed{\substack{\circ \\ \bullet}}$ tiles moving away from the wharf, and then either a $\boxed{\substack{\bullet \\ \bullet}}$ tile, a $\boxed{\substack{\bullet \\ | \\ \bullet}}$ tile, or a $\boxed{\substack{\bullet \\ \circ}}$ tile. For example, we may choose three intermediate tiles $\boxed{\substack{\circ \\ | \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\bullet \\ \bullet}}$ or $\boxed{\substack{\circ \\ | \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\circ \\ \bullet}}\boxed{\substack{\bullet \\ \circ}}$, or even zero intermediate tiles $\boxed{\substack{\circ \\ | \\ \bullet}}\boxed{\substack{\bullet \\ \bullet}}$, along the raft corresponding to a half-edge labeled $\boxed{\substack{\circ \\ | \\ \odot}}$. Note that in the automaton, after visiting vertex 5 ($\boxed{\substack{\circ \\ \bullet}}$), a walk must either visit vertex 3 ($\boxed{\substack{\bullet \\ \circ}}$) or vertex 6 ($\boxed{\substack{\bullet \\ \bullet}}$), and then proceed to visit vertex 3 before going on to other vertices. Thus, this case is encoded in the automaton by starting with vertex 5 or ending at vertex 7.

(c) The half-edge label $\left(\substack{\circ \\ \bullet}\right)$ is special, and means that every plank of the raft corresponding to that edge is tiled by $\boxed{\substack{\circ \\ \bullet}}$, and the other vertex of the edge is labeled by $\boxed{\substack{\circ \\ \bullet}}$ or $\boxed{\substack{\circ \\ | \\ \bullet}}$. This can happen in two cases. Either:

(i) the edge connects two wharfs, both labeled by $\boxed{\substack{\circ \\ | \\ \bullet}}$, and both half-edges of the edge are labeled by $\left(\substack{\circ \\ \bullet}\right)$; or

(ii) the edge connects a wharf labeled by $\boxed{\substack{\circ \\ | \\ \bullet}}$ to a vertex labeled by $\boxed{\substack{\circ \\ \bullet}}$. The half-edge incident to $\boxed{\substack{\circ \\ | \\ \bullet}}$ is labeled by $\left(\substack{\circ \\ \bullet}\right)$, and the half-edge incident to $\boxed{\substack{\circ \\ \bullet}}$ is labeled by $\boxed{\substack{\circ \\ \bullet}}$.

We call a path in the harbor H_C a -*path* if at least one half-edge of every edge of the path is labeled by .

To be a legal labeling, every vertex labeled by must have an incident half-edge labeled by , be connected by a -path to a vertex with an incident half-edge labeled by , or be connected by a -path to a vertex labeled by . In the two former cases, the planks in the connected component contain or are adjacent to a selected top node, while in the latter case, the connected component will contain a large ascent.

(4) Vertices of type indicate a wharf with both nodes selected. In this case, this wharf will join connected components in adjacent rafts on both the top and the bottom. For the conditions of Theorem 4.9 to be satisfied, the connected component of vertices on the top must contain either a plank with only the top component selected, or a large ascent. The connected component of vertices on the bottom must satisfy the same condition. The wharf itself provides a plank whose bottom node is in the connected component on the bottom, and whose top node is selected, so this part of Theorem 4.9 is automatically satisfied. Once again, we have to consider several cases. The different cases are distinguished by the tiles , , and .

(a) A label means that on that side, we choose some number of doubly filled tiles, followed by or . Thus, this case is encoded in the automaton by starting with vertex 6 or ending at vertex 8.

(b) A label means that on that side, we choose some number of doubly filled tiles, followed by or . Thus, this case is encoded in the automaton by starting with vertex 8 or ending at vertex 6.

(c) A label means that on that side, we choose any nonnegative number of doubly filled tiles, followed by . Thus, if this label appears at the initial end of the edge, the corresponding lex-minimal conditions are encoded in the automaton by starting with some nonnegative number of tiles, followed by any walk starting at vertex 2. If this label appears at the terminal end of the edge, then the walks must all terminate at vertex 4.

(d) The half-edge label is special. It means that every plank along the corresponding edge is tiled by and the other vertex of the edge is labeled by or . Therefore, it occurs in exactly two cases. Either:

(i) it must connect a wharf of type to another wharf of the same type, and both halves of the edge must have the label; or

(ii) it must connect a wharf of type to a vertex of type , with the half-edge incident to labeled by , and the half-edge incident to labeled by .

We call a path in the harbor H_C a -*path* if every edge in the path contains at least one half-edge label .

To be a legal labeling, every vertex labeled by must have an incident half-edge labeled by , or be connected by a -path to a vertex labeled by with an incident half-edge labeled by , or be connected by an -path to a vertex labeled by . In the two former cases, the top component either contains a wharf with only the top node selected, or a large ascent (which one depends on where the or tile appears). In the latter case, the top component contains a large ascent. Similarly, every vertex labeled by must be incident to a half-edge labeled by , or be connected by a -path to a vertex labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$ with an incident half-edge labeled by $\boxed{\substack{\bullet\\|\\\odot}}$, or be connected by an $\left(\substack{\circ\\\bullet}\right)$-path to a vertex labeled by $\boxed{\substack{\bullet\\\bullet}}$.

(5) The vertices of types $\boxed{\substack{\circ\\\circ}}$, $\boxed{\substack{\circ\\\bullet}}$, $\boxed{\substack{\bullet\\\circ}}$, and $\boxed{\substack{\bullet\\\bullet}}$ all correspond to type A boundary apparatuses on rafts, and are adjacent to exactly one edge. These impose the same initial/final conditions for lex-minimal fillings as they did in Section 3.7, so their half-edges are labeled by the same symbol.

We summarize this in a formal definition:

Definition 4.20. Let C be a selection of nodes for the wharfs, tethers, and ropes of a w-ocean. A labeling L of the half-edges of the harbor H_C is a *legal labeling* if the following conditions are satisfied.

(1) The half-edge labels are compatible with the vertex labels:
   (a) a half-edge coming from a vertex of type $\boxed{\substack{\circ\\\circ}}$ or $\boxed{\substack{\circ\\|\\\circ}}$ is labeled by $\boxed{\substack{\circ\\\circ}}$,
   (b) a half-edge coming from a vertex of type $\boxed{\substack{\bullet\\\circ}}$ or $\boxed{\substack{\bullet\\|\\\circ}}$ is labeled by $\boxed{\substack{\bullet\\\circ}}$,
   (c) a half-edge coming from a vertex of type $\boxed{\substack{\circ\\\bullet}}$ is labeled by $\boxed{\substack{\circ\\\bullet}}$,
   (d) a half-edge coming from a vertex of type $\boxed{\substack{\bullet\\\bullet}}$ is labeled by $\boxed{\substack{\bullet\\\bullet}}$,
   (e) a half-edge coming from a vertex of type $\boxed{\substack{\circ\\|\\\bullet}}$ is labeled by $\boxed{\substack{\circ\\|\\\odot}}$, $\boxed{\substack{\circledcirc\\|\\\bullet}}$, or $\left(\substack{\circ\\\bullet}\right)$,
   (f) a half-edge coming from a vertex of type $\boxed{\substack{\bullet\\|\\\bullet}}$ is labeled by $\boxed{\substack{\odot\\|\\\bullet}}$, $\boxed{\substack{\bullet\\|\\\odot}}$, $\boxed{\substack{\bullet\\|\\\bullet}}$, or $\left(\substack{\bullet\\\bullet}\right)$.

(2) The special labels $\left(\substack{\circ\\\bullet}\right)$ and $\left(\substack{\bullet\\\bullet}\right)$ are used correctly:
   (a) if a half-edge is labeled by $\left(\substack{\circ\\\bullet}\right)$ then the other vertex is labeled by $\boxed{\substack{\circ\\|\\\bullet}}$ or $\boxed{\substack{\circ\\\bullet}}$, and the other half-edge is labeled by $\left(\substack{\circ\\\bullet}\right)$ or $\boxed{\substack{\circ\\\bullet}}$ respectively; and
   (b) if a half-edge is labeled by $\left(\substack{\bullet\\\bullet}\right)$ then the other vertex is labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$ or $\boxed{\substack{\bullet\\\bullet}}$, and the other half-edge is labeled by $\left(\substack{\bullet\\\bullet}\right)$ or $\boxed{\substack{\bullet\\\bullet}}$ respectively.

(3) Several conditions to ensure lex-minimality are satisfied:
   (a) for every vertex labeled by $\boxed{\substack{\circ\\|\\\bullet}}$, there must be a $\left(\substack{\circ\\\bullet}\right)$-path (possibly of length zero) either to a vertex labeled by $\boxed{\substack{\circ\\|\\\bullet}}$ with an outgoing half-edge labeled by $\boxed{\substack{\circledcirc\\|\\\bullet}}$, or to a vertex labeled by $\boxed{\substack{\circ\\\bullet}}$,
   (b) for every vertex labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$, there must be a $\left(\substack{\bullet\\\bullet}\right)$-path (possibly of length zero) either to a vertex labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$ with an outgoing half-edge labeled by $\boxed{\substack{\odot\\|\\\bullet}}$, or to a vertex labeled by $\boxed{\substack{\bullet\\\bullet}}$, and
   (c) for every vertex labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$, there must be a $\left(\substack{\bullet\\\bullet}\right)$-path (possibly of length zero) either to a vertex labeled by $\boxed{\substack{\bullet\\|\\\bullet}}$ with an outgoing half-edge labeled by $\boxed{\substack{\bullet\\|\\\odot}}$, or to a vertex labeled by $\boxed{\substack{\bullet\\\bullet}}$.

   As above, a $\left(\substack{\circ\\\bullet}\right)$-path (resp. $\left(\substack{\bullet\\\bullet}\right)$-path) is a path in the harbor in which every edge has at least one half-edge labeled by $\left(\substack{\circ\\\bullet}\right)$ (resp. $\left(\substack{\bullet\\\bullet}\right)$).

We also define what it means for a lex-minimal filling of the w-ocean to be *consistent* with a legal labeling. In the informal description of half-edge labels above, we used tiles to refer to the selections of planks, and we now formalize this with the notion of a *tile sequence* associated to an edge.

Definition 4.21. Let L be a legal labeling as outlined above, and let F be a (not necessarily lex-minimal) filling of the w-ocean consistent with C. To every oriented edge e (meaning an edge e with choice of orientation) in the harbor, we associate a sequence of tiles which begins with the label of the initial half-edge, and ends with the label of the final half-edge. The tiles in between are drawn from $\{\boxed{\substack{\circ\\\circ}},\boxed{\substack{\bullet\\\circ}},\boxed{\substack{\circ\\\bullet}},\boxed{\substack{\bullet\\\bullet}}\}$, and indicate the filling in F of the corresponding raft (so the overall sequence has $n+2$ tiles, where n is the label of the edge ε in the harbor).

We refer to this sequence as the *tile sequence associated to* ε. Using this terminology, we say that F is *consistent with* L if, for every oriented edge, the associated tile sequence satisfies the following conditions:

(1) If the sequence begins with a , then the remaining sequence starts with a non-negative number of 's, followed by a .
(2) If the sequence begins with a , then the remaining sequence starts with a non-negative number of 's, followed by a , , or .
(3) If the sequence begins with a , then every tile in the sequence, aside from the first and last, is a .
(4) If the sequence begins with a , then the remaining sequence starts with a non-negative number of 's, followed by a .
(5) If the sequence begins with a , then the remaining sequence starts with a non-negative number of 's, followed by a or .
(6) If the sequence begins with a , then the remaining sequence starts with a non-negative number of 's, followed by a .
(7) If the sequence begins with a , then every tile in the sequence, aside from the first and last, is a .

Note that F does not have to be lex-minimal in the above definition, and consistency with L alone is not enough to guarantee lex-minimality, since the conditions of Theorem 4.9 might not be satisfied in the middle of a raft.

Lemma 4.22. *Let L be a legal labeling, and suppose that F is a filling of the w-ocean consistent with L. Then F is lex-minimal if and only if for every oriented edge ε, the associated tile sequence $t_0 \cdots t_{n+1}$ satisfies two conditions:*

(a) If $t_i t_{i+1} t_{i+2} \cdots t_j$ is a subsequence with all top nodes selected, such that $1 \leq i, j \leq n$ and t_{i-1} and t_{j+1} do not have top nodes selected, then there is some $i \leq k \leq j$ such that the bottom node of t_k is not selected.

(b) If $t_i t_{i+1} t_{i+2} \cdots t_j$ is a subsequence with all bottom nodes selected, such that $1 \leq i, j \leq n$ and t_{i-1} and t_{j+1} do not have bottom nodes selected, then there is some $i \leq k \leq j$ such that the top node of t_k is not selected, and some $i-1 \leq k' \leq j+1$ such that the top node of $t_{k'}$ is selected.

Note that this criterion only looks at subsequences not containing the endplanks. In other words, if F is consistent with a legal labeling, then lex-minimality reduces to checking the conditions of Theorem 3.9 on the interior of tile sequences.

Proof. Follows from Theorem 4.9. The proof is given in the informal description of Definitions 4.20 and 4.21. $\square$

Finally, we define a generalization of the a-sequences from Section 3.6.

Definition 4.23. Let $u, v \in \mathcal{L} \cup \mathcal{L}'$, the set of half-edge labellings. We say that (u, v) is a *legal pair* if neither u or v belongs to $\mathcal{L}' = \{ , \}$, or one of u or v is (resp.) and the other is or (resp. or).

If u, v is a legal pair, and $m \geq 0$, then we define $a(u, v; m)$ to be the number of tile sequences $t_0 \cdots t_{m+1}$ such that:

(i) $t_0 = u$ and $t_{m+1} = v$,

(ii) both $t_0 \cdots t_{m+1}$ and $t_{m+1} t_m \cdots t_0$ satisfy conditions (1)-(7) of Definition 4.21, and
(iii) $t_0 \cdots t_{m+1}$ satisfies conditions (a) and (b) of Lemma 4.22.

Lemma 4.24. *The sequences $a(u, v; m)$ satisfy the following properties:*

- $a(u, v; m) = a(v, u; m)$.
- *If one of u or v is or then $a(u, v; m) = 1$ for all $m \geq 0$.*
- *If neither u or v is or , then the family of sequences $a(u, v; m)$ for $u, v \in \mathcal{L}$ are determined by the entries of Table 4, given in terms of the generating functions for walks on the automaton.*

Proof. The first two properties follow immediately by construction. For the third property, while we have added additional boundary conditions, the enumeration is otherwise the same as in the symmetric group case. In particular, each sequence is determined by a function of the $R_{i,j}$ generating functions given in Equation (5), as shown in Table 4. Note that along each row of the table, the initial index i in $R_{i,j}$ is the same. These values come from our description of the half-edge labels on harbors and/or the original assumption from type A that walks must begin in vertices 1, 2, 3, or 7, according to the label. The set of indices j that appear in each $R_{i,j}$ in any one column can be determined similarly, either by referring back to the type A case, or by using the descriptions of the half-edge labels.

The factor $1/(1-t)$ occurs when we can start to fill a raft moving away from a wharf with any number of tiles of a given type, and then we follow an allowed walk starting at a given vertex. We subtract 1 in cases where there is a constant term arising from $i = j$, because walks with 0 edges cannot occur in the enumeration of lex-minimal fillings of rafts since we always include two tiles representing the apparatus on either end of the raft. Finally, in the case of $a\big(\ ,\ ; m\big)$, the generating function $R_{6,6} - 1/(1-t)$ counts walks starting at vertex 6 and ending at vertex 6, except for the walk that never leaves vertex 6. That exception is an -path, and it is counted as a different labeling of the harbor.

The verification of the entries in Table 4 is now straightforward, although a bit tedious. ☐

Corollary 4.25. *Each of the generalized a-sequences from Table 4 satisfies one of the following recurrences:*

R1: $a_n = 5a_{n-1} - 7a_{n-2} + 4a_{n-3}$ *for* $n \geq 3$,
R2: $a_n = 6a_{n-1} - 12a_{n-2} + 11a_{n-3} - 4a_{n-4}$ *for* $n \geq 4$,
R3: $a_n = 6a_{n-1} - 13a_{n-2} + 16a_{n-3} - 11a_{n-4} + 4a_{n-5}$ *for* $n \geq 5$, *or*
R4: $a_n = 7a_{n-1} - 19a_{n-2} + 29a_{n-3} - 27a_{n-4} + 15a_{n-5} - 4a_{n-6}$ *for* $n \geq 6$.

The recurrences and initial conditions are shown in Table 5.

The initial conditions in Table 5 can all be verified from the generating functions or by considering the lex-minimal presentations in 3 families of cases.

(1) No wharfs: The w-ocean for $w = s_1 s_n = [2, 1, 3, 4, \ldots, n-1, n]$ in type A_n for $n > 4$ has 4 ropes at $(2, 0), (2, 1), (n-1, 0), (n-1, 1)$ attached to the 4 corners of one raft of size $n - 4$. Each rope can be selected independently.
(2) One wharf: In type D_n, say the unique leaf not connected to the branch node of the Coxeter graph is labeled n, and the branch node is labeled s_1. Then if $w = s_n$, the w-ocean has a wharf on vertices $\{(1, 0), (1, 1)\}$, ropes at $(n-1, 0)$ and $(n-1, 1)$ and each $\{(i, 0), (i, 1)\}$ for $1 < i < n-1$ is a plank in one of the 3 rafts.

	$\frac{R_{2,2}-1}{t}$	$\frac{R_{2,3}}{t}$	$\frac{R_{2,57}}{t}$	$\frac{R_{2,5}}{t}$	$\frac{R_{2,7}}{t}$	$\frac{R_{2,468}}{t}$	$\frac{R_{2,8}}{t}$	$\frac{R_{2,6}}{t}$	$\frac{R_{2,4}}{t}$
		$\frac{R_{3,3}-1}{t}$	$\frac{R_{3,57}}{t}$	$\frac{R_{3,5}}{t}$	$\frac{R_{3,7}}{t}$	$\frac{R_{3,468}}{t}$	$\frac{R_{3,8}}{t}$	$\frac{R_{3,6}}{t}$	$\frac{R_{3,4}}{t}$
			$\frac{R_{7,57}-1}{t}$	$\frac{R_{7,5}}{t}$	$\frac{R_{7,7}-1}{t}$	$\frac{R_{7,468}}{t}$	$\frac{R_{7,8}}{t}$	$\frac{R_{7,6}}{t}$	$\frac{R_{7,4}}{t}$
				$\frac{R_{2,5}}{1-t}$	$\frac{R_{2,7}}{1-t}$	$\frac{R_{2,468}}{1-t}$	$\frac{R_{2,8}}{1-t}$	$\frac{R_{2,6}}{1-t}$	$\frac{R_{2,4}}{1-t}$
					$\frac{R_{5,7}}{t}$	$\frac{R_{5,468}}{t}$	$\frac{R_{5,8}}{t}$	$\frac{R_{5,6}}{t}$	$\frac{R_{5,4}}{t}$
						$\frac{R_{1,1468}-1}{t}$	$\frac{R_{1,8}}{t}$	$\frac{R_{1,6}}{t}$	$\frac{R_{1,4}}{t}$
							$\frac{R_{6,8}}{t}$	$\frac{R_{6,6}-\frac{1}{1-t}}{t}$	$\frac{R_{6,4}}{t}$
								$\frac{R_{8,6}}{t}$	$\frac{R_{8,4}}{t}$
									$\frac{R_{2,4}}{1-t}$

TABLE 4. Generating functions of the sequences $a(u,v;n)$ in terms of the finite automaton.

(3) Two wharfs: In type $\widetilde{D}_n$, the identity element has two wharfs corresponding with the two branch nodes and every generator is a small ascent on the left and the right.

We can now state the main theorem in full generality.

Theorem 4.26. *Fix an arbitrary Coxeter group W and an element $w \in W$. The number of parabolic double cosets with minimal element w is*

$$c_w = 2^{|\operatorname{Floats}(w)|} \sum_{\substack{\text{choices } C \text{ of} \\ \text{tethers, ropes, and} \\ \text{wharfs for } w}} \; \sum_{\substack{\text{legal labelings } L \\ \text{of the harbor } H_C}} \; \prod_{\mathcal{R} \in \operatorname{Rafts}(w)} a(R,C,L),$$

where $a(R,C,L)$ is determined by the rational generating functions (or, equivalently, the linear recurrence relations) given in Table 4 (Table 5).

Proof. The proof follows from Definitions 4.20, 4.21 and 4.23, and Lemmas 4.22 and 4.24. □

We demonstrate this result for examples for Weyl groups and affine Weyl groups in the next subsections.

Remark 4.27. As in Section 3.8, we can sum together collections of a-sequences containing the option to select a given rope(s) or not. Thus, there exist b-sequences for all Coxeter

	R1 1,2,6	**R1** 1,3,9	**R1** 1,3,9	**R1** 1,2,4	**R1** 0,1,5	**R1** 1,4,12	**R1** 0,1,4	**R1** 0,1,4	**R1** 1,2,4
		R3 1,3,11,37,119	**R3** 1,4,12,37,118	**R1** 0,1,4	**R3** 1,3,8,24,77	**R1** 1,4,14	**R3** 1,2,5,16,53	**R3** 0,1,5,17,54	**R1** 0,1,4
			R3 1,3,11,37,119	**R1** 0,1,4	**R3** 1,2,7,24,78	**R1** 1,4,14	**R3** 0,1,5,17,54	**R3** 1,2,5,16,53	**R1** 0,1,4
				R2 0,1,3,7	**R2** 0,0,1,6	**R2** 0,1,5,17	**R1** 0,0,1	**R1** 0,0,1	**R2** 0,1,3,7
					R3 0,1,5,17,53	**R2** 1,3,9,29	**R3** 0,1,4,12,36	**R3** 1,2,4,11,35	**R2** 0,0,1,6
						R1 1,4,16	**R2** 0,1,5,19	**R2** 0,1,5,19	**R2** 0,1,5,17
							R3 0,1,3,8,24	**R4** 0,0,1,6,23,77	**R1** 0,0,1
								R3 0,1,3,8,24	**R1** 0,0,1
									R2 0,1,3,7

TABLE 5. Recurrence relations and initial conditions arising from the finite automaton.

groups as well. The proof of Theorem 1.6 now follows as a corollary to Theorem 4.26 provided we define Wharfs(w) to be the set of all possible choices of dots on all of the wharfs of w along with legal labels of all the half-edges emanating from the wharfs.

4.4. **Example: a wharf with three branches.** We now study the case of a wharf with three branches. These will have sizes i, j, and k, and each will end in a doubly unfilled tile. This structure describes the identity in D_n, E_n, $\widetilde{B}_n$, and $\widetilde{E}_n$. We denote the total number of parabolic double cosets in this case by

$$\mathrm{branch}(i,j,k).$$

There are four options for the wharf. Throughout the following computation, we will refer to Figure 15.

The blank wharf gives only one possible labeling (see Figure 15, drawing 1), and contributes

$$a(\ ,\ ;i)\,a(\ ,\ ;j)\,a(\ ,\ ;k). \tag{7}$$

The wharf also gives only one possible labeling (see Figure 15, drawing 2), and contributes

$$a(\ ,\ ;i)\,a(\ ,\ ;j)\,a(\ ,\ ;k). \tag{8}$$

The wharf gives seven possible labelings (see Figure 15, drawings 3–9): every half-edge can be either or , but at least one of them has to be labeled . The total contribution is therefore

$$
\begin{aligned}
(9)\qquad & a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k).
\end{aligned}
$$

Finally, the wharf gives twelve possible labelings (see Figure 15, drawings 10–21): every half-edge can be either , , or , at least one of them must be , and at least one of them must be . The total contribution is

$$
\begin{aligned}
(10)\qquad & a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) \\
& + a(\ ,\ ,;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k) + a(\ ,\ ;i)\, a(\ ,\ ;j)\, a(\ ,\ ;k).
\end{aligned}
$$

Therefore, $\mathrm{branch}(i,j,k)$ is the sum of expressions (7)–(10).

Example 4.28. For $n \geq 3$, consider the identity elements in D_n and $\widetilde{B}_{n-1}$. The number of parabolic double cosets whose minimal representative is one of these elements is $\mathrm{branch}(1,1,n-3)$. For $n = 3,\ldots,10$, this gives the sequence

$$20, 72, 234, 746, 2380, 7614, 24394, 78192,$$

and has generating function

$$\frac{t^3\,(20 - 28t + 14t^2)}{1 - 5t + 7t^2 - 4t^3}.$$

Example 4.29. Consider the identity element in E_n, for $n \geq 4$. For this, we compute $\mathrm{branch}(2,1,n-4)$. For $n = 4,\ldots,10$, this gives the sequence

$$66, 234, 750, 2376, 7566, 24198, 77532,$$

and has generating function

$$\frac{t^4\,(66 - 96t + 42t^2)}{1 - 5t + 7t^2 - 4t^3}.$$

Example 4.30. The number of parabolic double cosets whose minimal representative is the identity element in the affine Coxeter group $\widetilde{E}_n$, for $n = 6,7,8$, is

$$
\begin{aligned}
&\mathrm{branch}(2,2,2) = 2378,\\
&\mathrm{branch}(3,1,3) = 7514, \text{ and}\\
&\mathrm{branch}(2,1,5) = 24198.
\end{aligned}
$$

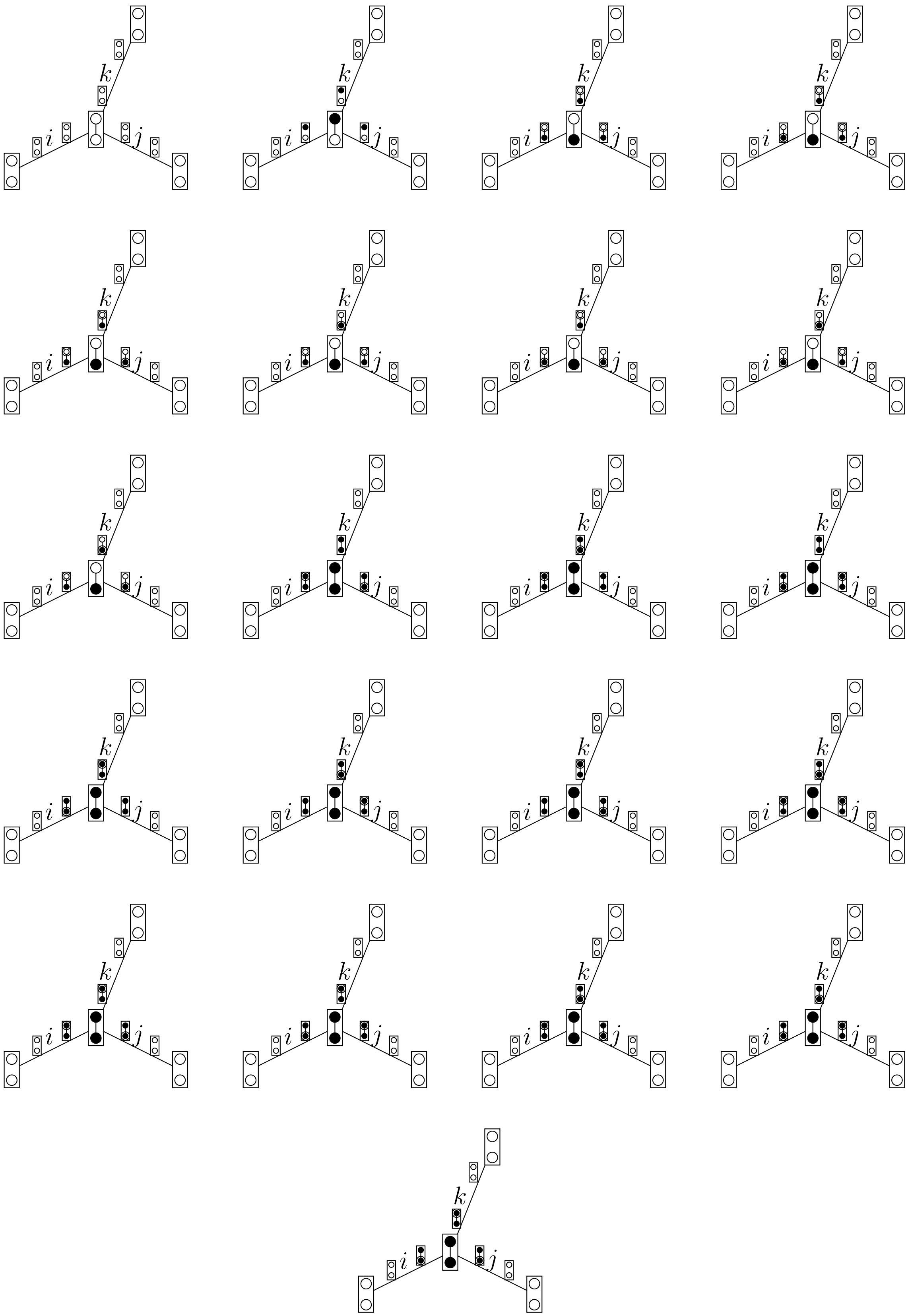


FIGURE 15. All 21 legally labeled, decorated, harbor graphs for the ocean with one wharf and three rafts of sizes i, j and k.

4.5. **Example: a circular raft.** We can enumerate parabolic double cosets in the affine group $\widetilde{A}_n$, for $n \geq 0$. The Coxeter graph in this case is a cycle with $n+1$ elements. To study the e-ocean, it is helpful to introduce an "artificial" wharf here, meaning that we pick arbitrary nodes (one on top of the other) and declare them to be a wharf. Again, we have four options for the wharf. The unfilled wharf $\boxed{\substack{\circ\\ \circ}}$ gives only one possible labeling (see Figure 16, drawing 1), the wharf $\boxed{\substack{\bullet\\ \circ}}$ also gives only one possible labeling (see Figure 16, drawing 2), the wharf $\boxed{\substack{\circ\\ \bullet}}$ has three possible labelings (see Figure 16, drawings 3–5), and the wharf $\boxed{\substack{\bullet\\ \bullet}}$ has two possible labelings (see Figure 16, drawings 6–7). Thus the number of parabolic double cosets is

$$a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \circ}};n\big)+a\big(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\bullet\\ \circ}};n\big)+a\big(\boxed{\substack{\circledcirc\\ \bullet}},\boxed{\substack{\circledcirc\\ \bullet}};n\big)+a\big(\boxed{\substack{\circledcirc\\ \bullet}},\boxed{\substack{\circ\\ \odot}};n\big)+a\big(\boxed{\substack{\circ\\ \odot}},\boxed{\substack{\circledcirc\\ \bullet}};n\big)+a\big(\boxed{\substack{\bullet\\ \odot}},\boxed{\substack{\odot\\ \bullet}};n\big)+a\big(\boxed{\substack{\odot\\ \bullet}},\boxed{\substack{\bullet\\ \odot}};n\big). \tag{11}$$

For $n = 0, \ldots, 10$, this gives $2, 6, 26, 98, 332, 1080, 3474, 11146, 35738, 114566, 367248$. The generating function is

$$\frac{2 - 8t + 22t^2 - 28t^3 + 20t^4 - 4t^5}{(1-t)\,(1-t+t^2)\,(1-5t+7t^2-4t^3)}.$$

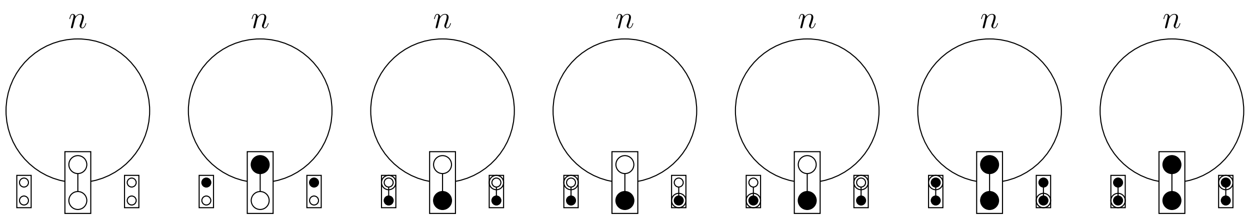


FIGURE 16. A circular raft, and all seven legal labelings.

4.6. **Example: two wharfs with three branches.** Our final example is the case of the identity for a Coxeter graph with two branch points connected by a path (of size k), and with two more branches (of sizes i_1, j_1, i_2, and j_2) coming out of each branch point. An example is the affine group $\widetilde{D}_n$, for $n \geq 4$ (with $k = n-4$ and $i_1 = j_1 = i_2 = j_2 = 1$).

There are now too many labelings to state in a concise manner. For each wharf, there are four choices, so we have 16 choices total. If both wharfs are either $\boxed{\substack{\circ\\ \circ}}$ or $\boxed{\substack{\bullet\\ \circ}}$, then there is only one labeling. For example, for left wharf $\boxed{\substack{\circ\\ \circ}}$ and right wharf $\boxed{\substack{\bullet\\ \circ}}$, the contribution is

$$a\big(\boxed{\substack{\circ\\ \circ}},,\boxed{\substack{\circ\\ \circ}};i_1\big)\,a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \circ}};j_1\big)\,a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\bullet\\ \circ}};k\big)\,a\big(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\circ\\ \circ}};i_2\big)\,a\big(\boxed{\substack{\bullet\\ \circ}},\boxed{\substack{\circ\\ \circ}};j_2\big)$$

(see Figure 17, drawing 1). But, for example, for left wharf $\boxed{\substack{\circ\\ \bullet}}$ and right wharf $\boxed{\substack{\circ\\ \bullet}}$, there are $49 + 15 = 64$ possible labelings (we can either label all three half-edges coming out of each wharf in one of 7 possible ways – the choices are $\boxed{\substack{\circ\\ \odot}}$ or $\boxed{\substack{\circledcirc\\ \bullet}}$ for each, and we cannot select $\boxed{\substack{\circ\\ \odot}}$ for all three – or we can label the edge between the wharfs by $\boxed{\substack{\circ\\ \bullet}}$, and then we can label the remaining four half-edges either by $\boxed{\substack{\circ\\ \odot}}$ or $\boxed{\substack{\circledcirc\\ \bullet}}$, but we cannot label them all $\boxed{\substack{\circ\\ \odot}}$). In Figure 17, drawings 2, we see a labeling that contribute $a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circledcirc\\ \bullet}};i_1\big)\,a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \odot}};j_1\big)\,a\big(\boxed{\substack{\circledcirc\\ \bullet}},\boxed{\substack{\circledcirc\\ \bullet}};k\big)\,a\big(\boxed{\substack{\circ\\ \odot}},\boxed{\substack{\circ\\ \circ}};i_2\big)\,a\big(\boxed{\substack{\circ\\ \odot}},\boxed{\substack{\circ\\ \circ}};j_2\big)$ while drawing 3 in Figure 17 contributes $a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circledcirc\\ \bullet}};i_1\big)\,a\big(\boxed{\substack{\circ\\ \circ}},\boxed{\substack{\circ\\ \odot}};j_1\big)\,a\big(\boxed{\substack{\circ\\ \odot}},\boxed{\substack{\circ\\ \circ}};i_2\big)\,a\big(\boxed{\substack{\circ\\ \odot}},\boxed{\substack{\circ\\ \circ}};j_2\big)$. In total, there are

$$(1+1+7+12) + (1+1+7+12) + (7+7+64+84) + (12+12+84+194) = 506$$

possible labelings, each contributing a product of four or five terms to the sum.

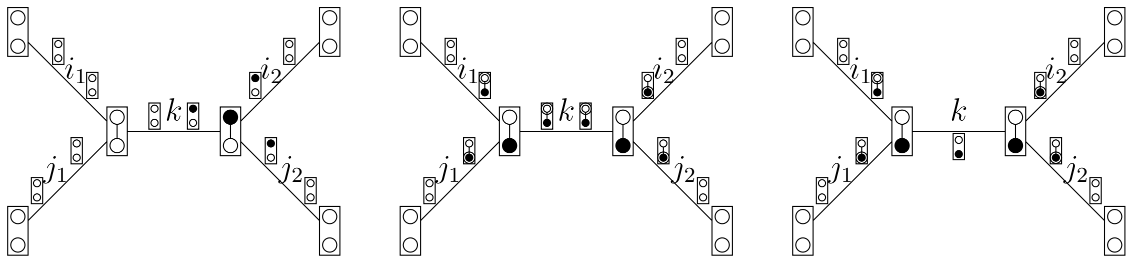


FIGURE 17. A harbor graph with two wharfs and some of its legal labelings.

Example 4.31. The number of parabolic double cosets whose minimal representative is the identity in $\widetilde{D}_n$, with $n = 5, \ldots, 14$,

$$814, 2558, 8176, 26230, 84150, 269844, 865090, 2773142, 8889456, 28495646, 91344606,$$

and the generating function is (the surprisingly simple)

$$\frac{t^4(814 - 1512t + 1084t^2)}{1 - 5t + 7t^2 - 4t^3}.$$

5. PARABOLIC DOUBLE COSETS WITH RESTRICTED SIMPLE REFLECTIONS

We finish with some remarks about our enumerative formulas. The formula for S_n and the formula for general Coxeter groups both center around the number of parabolic double cosets over rafts, subject to different boundary conditions. The boundary conditions do not change the underlying recurrence, only its initial conditions. The number of parabolic double cosets for a raft of size n, ignoring boundary conditions, is the same as the number of parabolic double cosets in S_{n+1} whose minimal element is the identity. Boundary conditions amount to enumerating parabolic double cosets with presentations $W_I e W_J$ where simple reflections s_1 and s_n may be forbidden from belonging to I or J.

This suggests looking at the problem of enumerating parabolic double cosets $W_I w W_J$ where certain simple reflections are not allowed to belong to I or J. It turns out that the characterization of lex-minimal elements in Theorem 4.9 also applies to this more general problem. By working in this framework, we get an intriguing structural explanation for our enumerative formulas: the set of parabolic double cosets with fixed minimal element w is in bijection with a restricted set of parabolic double cosets of the identity in a larger Coxeter group. This suggests that our enumerative formulas are somewhat natural, despite their apparent complexity.

Definition 5.1. Fix subsets $X_L, X_R \subseteq S$. A presentation $C = W_I w W_J$ of a parabolic double coset *avoids* X_L and X_R if $I \cap X_L = \emptyset$ and $J \cap X_R = \emptyset$. A parabolic double coset C *avoids* X_L and X_R if it has a presentation that avoids X_L and X_R.

In other words, a parabolic double coset C avoids X_L and X_R if C has a presentation which does not use any elements of X_L on the left, nor any elements of X_R on the right.

The natural question to ask is then: given an element $w \in W$, and sets $X_L, X_R \subseteq S$, how many parabolic double cosets with minimal element w avoid X_L and X_R? (To make the question interesting, we can assume that X_L and X_R are subsets of the left and right ascent set of w, respectively.)

Even if a parabolic double coset C avoids X_L and X_R, this does not mean that every presentation of C avoids X_L and X_R, nor even that the lex-minimal presentation avoids these sets. For instance, the parabolic double coset $C = W$ has two minimal presentations, $C = W_S e W_\emptyset = W_\emptyset e W_S$, with the latter being lex-minimal. If we set $X_L = \emptyset$ and $X_R = \{s\}$ for any $s \in S$, then $W_\emptyset e W_S$ does not avoid X_L and X_R. On the other hand, the former presentation, $W_S e W_\emptyset$, is now lex-minimal among all presentations that avoid X_L and X_R.

Fortunately, if C avoids X_L and X_R, then it clearly has some minimal presentation (in the sense of Definition 4.5) that avoids X_L and X_R, and we can characterize the unique lex-minimal presentation of this form.

Proposition 5.2. *Let X_L and X_R be subsets of the left and right ascent sets of $w \in W$, respectively. Let C be a parabolic double coset with minimal element w and a presentation $C = W_I w W_J$ that avoids X_L and X_R. This presentation is lex-minimal among all X_L- and X_R-avoiding presentations for C if and only if*

(a) no connected component of J is contained within $(w^{-1}Iw) \cap S$, and

(b) if a connected component I_0 of I is contained in wSw^{-1}, then either some element of $w^{-1}I_0w$ is contained in X_R, or some element of $w^{-1}I_0w$ is adjacent to but not contained in J.

Furthermore, every parabolic double coset avoiding X_L and X_R has a unique presentation that is lex-minimal among the coset's X_L- and X_R-avoiding presentations.

Proof. By Proposition 4.8, the minimal presentations of C differ only by switching the sides of certain connected components. If I_0 is a connected component of I with $I \cap (wX_Rw^{-1}) \neq \emptyset$, then I_0 must appear on the left in an X_L- and X_R-avoiding presentation. $\square$

It is much easier to work with the criteria in Theorem 4.9 and Proposition 5.2 if w is the identity. By allowing restricted simple reflections, we can expand the Coxeter graph to reduce to this case. To explain how this works, suppose we are given some element $w \in W$, and let G be the Coxeter graph of W. Now proceed as follows.

(1) Make a new Coxeter graph $G_L \sqcup G_R$, where G_L and G_R are each isomorphic to G, and $\sqcup$ refers to the disjoint union of graphs. The new Coxeter graph has vertex set $S_L \sqcup S_R$, where S_L (respectively, S_R) is the vertex set of G_L (respectively, G_R). Each set S_L and S_R is canonically identified with S via bijections $\phi^L : S \longrightarrow S_L$ and $\phi^R : S \longrightarrow S_R$.

(2) Delete the vertices in S_L (respectively, S_R) corresponding to left (respectively, right) descents of w. The functions ϕ^L and ϕ^R are now defined only on the left and right ascent sets $\mathrm{Asc}_L(w)$ and $\mathrm{Asc}_R(w)$ of w, respectively.

(3) If s is a left ascent of w such that $w^{-1}sw$ is a right ascent of w (which happens if and only if $w^{-1}sw \in S$), then identify the vertices $\phi^L(s)$ and $\phi^R(w^{-1}sw)$. Call the resulting graph $\overline{G}$. The induced functions $\overline{\phi^L}$ and $\overline{\phi^R}$ are still injective, but their images are no longer necessarily disjoint.

(4) Given sets $X_L \subseteq \mathrm{Asc}_L(w)$ and $X_R \subseteq \mathrm{Asc}_R(w)$, set

$$\overline{X}_L := \overline{\phi^L}(X_L) \cup \left(\overline{\phi^R}(\mathrm{Asc}_R(w)) \setminus \overline{\phi^L}(\mathrm{Asc}_L(w))\right) \text{ and}$$
$$\overline{X}_R := \overline{\phi^R}(X_R) \cup \left(\overline{\phi^L}(\mathrm{Asc}_L(w)) \setminus \overline{\phi^R}(\mathrm{Asc}_R(w))\right).$$

In other words, any right ascent of w that is not conjugate to a left ascent is not allowed to act on the left, and vice versa.

Proposition 5.3. *Given an element $w \in W$ and sets $X_L \subseteq \mathrm{Asc}_L(w)$ and $X_R \subseteq \mathrm{Asc}_R(w)$, define $\overline{G}$, $\overline{\phi^L}$, $\overline{\phi^R}$, $\overline{X}_L$, and $\overline{X}_R$ as above. Let $\overline{W}$ be the Coxeter group with Coxeter graph $\overline{G}$. If $I \subseteq \mathrm{Asc}_L(w)$ and $J \subseteq \mathrm{Asc}_R(w)$, then $C = W_I w W_J$ is lex-minimal among X_L- and X_R-avoiding presentations if and only if $(\overline{W}_{\overline{\phi^L}(I)})e(\overline{W}_{\overline{\phi^R}(J)})$ is lex-minimal among $\overline{X}_L$- and $\overline{X}_R$-avoiding presentations.*

Proof. The image of $\overline{\phi^L}(\mathrm{Asc}_L(w))$ avoids $\overline{\phi^R}(\mathrm{Asc}_R(w)) \setminus \overline{\phi^L}(\mathrm{Asc}_L(w))$, so I avoids X_L if and only if $\overline{\phi^L}(I)$ avoids $\overline{X}_L$. Similarly J avoids X_R if and only if $\overline{\phi^R}(I)$ avoids $\overline{X}_R$. The remainder of the proposition follows immediately from Proposition 5.2. □

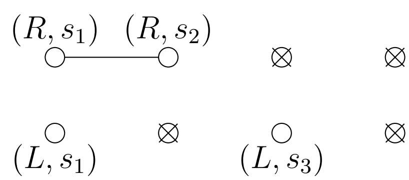
Corollary 5.4. *There is a bijection between X_L- and X_R-avoiding parabolic double cosets in W with minimal element w, and $\overline{X}_L$- and $\overline{X}_R$-avoiding parabolic double cosets in $\overline{W}$ with minimal element e.*

Proof. $\overline{\phi^L}$ induces a bijection between subsets of $\mathrm{Asc}_L(w)$ and subsets of the simple reflections of $\overline{W}$ that avoid

$$\left(\overline{\phi^R}(\mathrm{Asc}_R(w)) \setminus \overline{\phi^L}(\mathrm{Asc}_L(w))\right).$$

A similar statement can be made for $\overline{\phi^R}$. Thus the corollary follows from Propositions 5.2 and 5.3. □

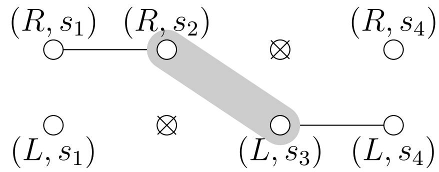
Example 5.5. Consider the permutation $w = 13542 \in S_5$, which has reduced expression $s_2s_3s_4s_3$. The left and right descent sets of w are $\{s_2, s_4\}$ and $\{s_3, s_4\}$ respectively, and there are no simple reflections s such that wsw^{-1} is also simple. Thus the Coxeter graph $\overline{G}$ consists of two copies of the Coxeter graph of S_5, with the descent sets deleted from each respective copy.



In this diagram, the vertices from $\overline{\phi^L}(\mathrm{Asc}_L(w))$ are labeled by (L, s_i) and the vertices from $\overline{\phi^R}(\mathrm{Asc}_R(w))$ are labeled by (R, s_i). After deleting descents, the nodes that remain are isomorphic to the Coxeter graph $\overline{G}$ of $\overline{W} \cong S_3 \times S_2 \times S_2$.

If we start with $X_L = X_R = \emptyset$, then $\overline{X}_L = \{(R, s_1), (R, s_2)\}$ and $\overline{X}_R = \{(L, s_1), (L, s_3)\}$. Thus the total number of parabolic double cosets with minimal element w is equal to the total number of parabolic double cosets $\overline{W}_I e \overline{W}_J$ where $I \subseteq \{(L, s_1), (L, s_3)\}$ and $J \subseteq \{(R, s_1), (R, s_2)\}$. Every such choice of I and J gives a distinct parabolic double coset, so there are 16 such cosets.

Example 5.6. Consider another permutation $w = 13425 \in S_5$, this one with reduced expression $w = s_2s_3$. The left and right descent sets of w are $\{s_2\}$ and $\{s_3\}$ respectively. Among the left ascents $w^{-1}s_3w = s_2$, so the Coxeter graph for $\overline{G}$ consists of two copies of the Coxeter graph of S_5, with s_2 deleted from the left copy, s_3 deleted from the right copy, and s_3 from the left copy identified with s_2 in the right copy:



In this case, then, $\overline{W} \cong S_4 \times S_2 \times S_2$. If we start with $X_L = X_R = \emptyset$, then we must avoid $\overline{X}_L = \{(R, s_1), (R, s_4)\}$ and $\overline{X}_R = \{(L, s_1), (L, s_4)\}$ in the new group.

This correspondence also preserves the Bruhat order on each individual coset.

Proposition 5.7. *Let C be a parabolic double coset in W with minimal element w, and let $\overline{C}$ be the corresponding parabolic double coset in $\overline{W}$. Then C and $\overline{C}$ are isomorphic as posets in Bruhat order.*

Proof. Continuing with the notation above, the parabolic subgroup W_I of W is isomorphic to the parabolic subgroup $\overline{W}_{\overline{\phi^L(I)}}$ of $\overline{W}$, and similarly with W_J. If $C = W_I w W_J$ for $w \in {}^I W^J$, then $\overline{C} = (W_{\overline{\phi^L(I)}}) e (W_{\overline{\phi^R(I)}})$. (Note that this does not depend on whether $C = W_I w W_J$ is a lex-minimal presentation, but we can choose a lex-minimal presentation if we wish to do so.) By the construction of $\overline{G}$ (specifically, the vertex identification), there is a well-defined bijection $C \longrightarrow \overline{C}$ sending uwv to $\overline{u}w\overline{v}$, where $\overline{u}$ is the element of $\overline{W}_{\overline{\phi^L(I)}}$ corresponding to u, and similarly for v and $\overline{v}$. By Proposition 2.7(b), this bijection is an order isomorphism with respect to Bruhat order. $\square$

Question 5.8. If C is finite, then $\overline{C}$ is the Bruhat interval between the identity and the maximal element of $\overline{C}$. Thus $\overline{C}$ can be considered as the 1-skeleton of a Schubert variety. Is there a geometric version of Proposition 5.7?

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7. Appendix

The b-sequences for the symmetric groups were defined in (6) on Page 29. Each such sequence is denoted by a superscript 4-tuple $(I_1, I_2, I_3, I_4) \in \{\{0\}, \{1\}, \{0,1\}\}$. The initial values for all 81 b-sequences are given below starting at $m = 0$. Note, there are only 27 distinct sequences due to the symmetries among the a-sequences.

For example, the 4-tuple $(\{0,1\}, \{1\}, \{0\}, \{1\})$, abbreviated $(01, 1, 0, 1)$, corresponds with the sequence $b_m^{(I_1,I_2,I_3,I_4)}$ with $I_1 = \{0,1\}$, $I_2 = \{1\}$, $I_3 = \{0\}$, $I_4 = \{1\}$. This sequence expands in terms of the a-sequences as $b_m^{(01,1,0,1)} = a_m(\boxed{\substack{\bullet\\ \bullet}}, \boxed{\substack{\bullet\\ \circ}}) + a_m(\boxed{\substack{\bullet\\ \circ}}, \boxed{\substack{\bullet\\ \circ}})$. This sequence has initial values $2, 7, 25, 83, 267, 854, 2734$ starting at $m = 0$ so for example $b_1^{(01,1,0,1)} = a_1(\boxed{\substack{\bullet\\ \bullet}}, \boxed{\substack{\bullet\\ \circ}}) + a_1(\boxed{\substack{\bullet\\ \circ}}, \boxed{\substack{\bullet\\ \circ}}) = 4 + 3 = 7$ using Table 3 and the symmetry property of the a-sequences.

Initial Values	4-tuples
$1, 2, 6, 20, 66, 214, 688:$	$(0,0,0,0)$
$1, 3, 9, 28, 89, 285, 914:$	$(1,0,0,0)(0,1,0,0)(0,0,1,0)(0,0,0,1)$
$1, 3, 11, 37, 119, 380, 1216:$	$(1,0,1,0)(0,1,0,1)$
$1, 4, 12, 36, 112, 356, 1140:$	$(1,1,0,0)(0,0,1,1)$
$1, 4, 12, 37, 118, 379, 1216:$	$(0,1,1,0)(1,0,0,1)$
$1, 4, 14, 46, 148, 474, 1518:$	$(1,1,1,0)(1,1,0,1)(1,0,1,1)(0,1,1,1)$
$1, 4, 16, 56, 184, 592, 1896:$	$(1,1,1,1)$
$2, 5, 15, 48, 155, 499, 1602:$	$(01,0,0,0)(0,01,0,0)(0,0,01,0)(0,0,0,01)$
$2, 6, 20, 65, 208, 665, 2130:$	$(01,0,1,0)(1,0,01,0)(0,01,0,1)(0,1,0,01)$
$2, 7, 21, 64, 201, 641, 2054:$	$(01,1,0,0)(1,01,0,0)(0,0,01,1)(0,0,1,01)$
$2, 7, 21, 65, 207, 664, 2130:$	$(0,01,1,0)(0,1,01,0)(01,0,0,1)(1,0,0,01)$
$2, 7, 25, 83, 267, 854, 2734:$	$(1,01,1,0)(01,1,0,1)(0,1,01,1)(1,0,1,01)$
$2, 8, 26, 82, 260, 830, 2658:$	$(1,1,01,0)(01,0,1,1)(0,01,1,1)(1,1,0,01)$
$2, 8, 26, 83, 266, 853, 2734:$	$(01,1,1,0)(1,01,0,1)(1,0,01,1)(0,1,1,01)$
$2, 8, 30, 102, 332, 1066, 3414:$	$(01,1,1,1)(1,01,1,1)(1,1,01,1)(1,1,1,01)$
$4, 11, 35, 113, 363, 1164, 3732:$	$(01,0,01,0)(0,01,0,01)$
$4, 12, 36, 112, 356, 1140, 3656:$	$(01,01,0,0)(0,0,01,01)$
$4, 12, 36, 113, 362, 1163, 3732:$	$(0,01,01,0)(01,0,0,01)$
$4, 14, 46, 147, 468, 1495, 4788:$	$(1,01,01,0)(0,01,01,1)(01,1,0,01)(01,0,1,01)$
$4, 14, 46, 148, 474, 1518, 4864:$	$(01,01,1,0)(01,01,0,1)(1,0,01,01)(0,1,01,01)$
$4, 15, 47, 147, 467, 1494, 4788:$	$(01,1,01,0)(01,0,01,1)(1,01,0,01)(0,01,1,01)$
$4, 15, 55, 185, 599, 1920, 6148:$	$(01,1,01,1)(1,01,1,01)$
$4, 16, 56, 184, 592, 1896, 6072:$	$(01,01,1,1)(1,1,01,01)$
$4, 16, 56, 185, 598, 1919, 6148:$	$(1,01,01,1)(01,1,1,01)$
$8, 26, 82, 260, 830, 2658, 8520:$	$(01,01,01,0)(01,01,0,01)(01,0,01,01)(0,01,01,01)$
$8, 30, 102, 332, 1066, 3414, 10936:$	$(01,01,01,1)(01,01,1,01)(01,1,01,01)(1,01,01,01)$
$16, 56, 184, 592, 1896, 6072, 19456:$	$(01,01,01,01)$

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