NON-COMMUTATIVE EXTENSIONS OF THE MACMAHON MASTER THEOREM

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Abstract. We present several non-commutative extensions of the MacMahon Master Theorem, further extending the results of Cartier-Foata and Garoufalidis-Lê-Zeilberger. The proofs are combinatorial and new even in the classical cases. We also give applications to the β-extension and Krattenthaler-Schlosser's q-analogue.

Introduction

The MacMahon Master Theorem is one of the jewels in enumerative combinatorics, and it is as famous and useful as it is mysterious. Most recently, a new type of algebraic generalization was proposed in [GLZ] and was further studied in [FH1, FH2, FH3, HL]. In this paper we present further generalizations of the MacMahon Master Theorem and several other related results. While our generalizations are algebraic in statement, the heart of our proofs is completely bijective, unifying all generalizations. In fact, we give a new bijective proof of the (usual) MacMahon Master Theorem, modulo some elementary linear algebra. Our approach seems to be robust enough to allow further generalizations in this direction.

Let us begin with a brief outline of the history of the subject. The Master Theorem was discovered in 1915 by Percy MacMahon in his landmark two-volume *"Combinatory Analysis"*, where he called it "a Master Theorem in the Theory of Partitions" [MM, page 98]. Much later, in the early sixties, the real power of Master Theorem was discovered, especially as a simple tool for proving binomial identities (see [GJ]). The proof of the MacMahon Master Theorem using Lagrange inversion is now standard, and the result is often viewed in the analytic context [Go, GJ].

An algebraic approach to MacMahon Master Theorem goes back to Foata's thesis [F1], parts of which were later expanded in [CF] (see also [L]). The idea was to view the theorem as a result on "words" over a (partially commutative) alphabet, so one can prove it and generalize it by means of simple combinatorial and algebraic considerations. This approach became highly influential and led to a number of new related results (see e.g. [K, Mi, V, Z]).

While the Master Theorem continued to be extended in several directions (see [FZ, KS]), the "right" q- and non-commutative analogues of the results evaded discovery until recently. This was in sharp contrast with the Lagrange inversion, whose q- and non-commutative analogues were understood fairly well [Ga, GaR, Ge, GS, Kr, PPR, Si]. Unfortunately, no reasonable generalizations of the Master Theorem followed from these results.

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An important breakthrough was made by Garoufalidis, Lê and Zeilberger (GLZ), who introduced a new type of q-analogue, with a puzzling algebraic statement and a technical proof [GLZ]. In a series of papers, Foata and Han first modified and extended the Cartier-Foata combinatorial approach to work in this algebraic setting, obtaining a new (involutive) proof of the GLZ-theorem [FH1]. Then they developed a beautiful "$1 = $ q" principle which gives perhaps the most elegant explanation of the results [FH2]. They also analyze a number of specializations in [FH3]. Most recently, Hai and Lorenz gave an interesting algebraic proof of the GLZ-theorem, opening yet another direction for exploration (see Section 13).

This paper presents a number of generalizations of the MacMahon Master Theorem in the style of Cartier-Foata and Garoufalidis-Lê-Zeilberger. Our approach is bijective and is new even in the classical cases, where it is easier to understand. This is reflected in the structure of the paper: we present generalizations one by one, gradually moving from well known results to new ones. The paper is largely self-contained and no background is assumed.

We begin with basic definitions, notations and statements of the main results in Section 1. The proof of the (usual) MacMahon Master Theorem is given in Section 2. While the proof here is elementary, it is the basis for our approach. A straightforward extension to the Cartier-Foata case is given in Section 3. The right-quantum case is presented in Section 4. This is a special case of the GLZ-theorem, when $q = 1$. Then we give a q-analogue of the Cartier-Foata case (Section 5), and the GLZ-theorem (Section 6). The subsequent results are our own and can be summarized as follows:

- The Cartier-Foata (q_{ij})-analogue (Section 7).
- The right-quantum (q_{ij})-analogue (Section 8).
- The super-analogue (Section 9).
- The β-extension (Section 10).

The (q_{ij})-analogues are our main result; one of them specializes to the GLZ-theorem when all $q_{ij} = q$. The super-analogue is a direct extension of the classical MacMahon Master Theorem to commuting and anti-commuting variables. Having been overlooked in previous investigations, it is a special case of the (q_{ij})-analogue, with some $q_{ij} = 1$ and others $= -1$. Our final extension is somewhat tangential to the main direction, but is similar in philosophy. We show that our proof of the MacMahon Master Theorem can be easily modified to give a non-commutative generalization of the so called β-extension, due to Foata and Zeilberger [FZ].

In Section 11 we present one additional observation on the subject. In [KS], Krattenthaler and Schlosser obtained an intriguing q-analogue of the MacMahon Master Theorem, a result which on the surface does not seem to fit the above scheme. We prove that in fact it follows from the classical Cartier-Foata generalization.

As the reader shall see, an important technical part of our proof is converting the results we obtain into traditional form. This is basic linear algebra in the classical case, but in non-commutative cases the corresponding determinant identities are either less known or new. For the sake of completeness, we present concise proofs of all of them in Section 12. We conclude the paper with final remarks and open problems.

1. Basic definitions, notations and main results

1.1. **Classical Master Theorem.** We begin by stating the Master Theorem in the classical form:

Theorem 1.1. (MacMahon Master Theorem) *Let $A = (a_{ij})_{m\times m}$ be a complex matrix, and let $x_1, \ldots, x_m$ be a set of variables. Denote by $G(k_1, \ldots, k_m)$ the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in*

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}. \tag{1.1}$$

Let $t_1, \ldots, t_m$ be another set of variables, and $T = (\delta_{ij} t_i)_{m\times m}$. Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m)\ t_1^{k_1} \cdots t_m^{k_m} \ = \ \frac{1}{\det(I - TA)}, \tag{1.2}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$.

By taking $t_1 = \ldots = t_m = 1$ we get

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m) \ = \ \frac{1}{\det(I - A)}\,, \tag{1.3}$$

whenever both sides of the equation are well defined, for example when all a_{ij} are formal variables. Moreover, replacing a_{ij} in (1.3) with $a_{ij}\, t_i$ shows that (1.3) is actually equivalent to (1.2). We will use this observation throughout the paper.

1.2. **Non-commuting variables.** Consider the following algebraic setting. Denote by $\mathcal{A}$ the algebra (over $\mathbb{C}$) of formal power series with non-commuting variables a_{ij}, $1 \le i, j \le m$. Elements of $\mathcal{A}$ are infinite linear combinations of words in variables a_{ij} (with coefficients in $\mathbb{C}$). In most cases we will take elements of $\mathcal{A}$ modulo some ideal $\mathcal{I}$ generated by a finite number of relations. For example, if $\mathcal{I}$ is generated by $a_{ij}a_{kl} = a_{kl}a_{ij}$ for all i, j, k, l, then $\mathcal{A}/\mathcal{I}$ is the symmetric algebra (the free commutative algebra with m^2 variables a_{ij}, $1 \le i, j \le m$).

Throughout the paper we assume that $x_1, \ldots, x_m$ commute with a_{ij}, and that x_i and x_j commute up to some nonzero complex weight, i.e. that

$$x_j x_i \ = \ q_{ij}\, x_i x_j\,, \quad \text{for all} \quad i < j$$

with $q_{ij} \in \mathbb{C}$, $q_{ij} \ne 0$. We can then expand the expression

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i}, \tag{1.4}$$

move all x_i's to the right and order them. Along the way, we will exchange pairs of variables x_i and x_j, producing a product of q_{ij}'s. We can then extract the coefficient at $x_1^{k_1} \cdots x_m^{k_m}$. As before, we will denote this coefficient by $G(k_1, \ldots, k_m)$. Each such coefficient will be a finite sum of products of a monomial in q_{ij}'s, $1 \le i < j \le m$, and a word $a_{i_1 j_1} \ldots a_{i_\ell j_\ell}$, such that $i_1 \le \ldots \le i_\ell$, the number of variables $a_{i,*}$ is equal to k_i, and the number of variables $a_{*,j}$ is equal to k_j.

To make sense of the right-hand side of (1.3) in the non-commutative case, we need to generalize the determinant. Throughout the paper the (non-commutative) determinant will be given by the formula

$$\det(B) \,=\, \sum_{\sigma\in S_m} w(\sigma)\, b_{\sigma_1 1}\cdots b_{\sigma_m m}\,, \tag{1.5}$$

where $\sigma = (\sigma_1,\ldots,\sigma_m)$ is a permutation and $w(\sigma)$ is a certain constant weight of σ. Of course, $w(\sigma) = (-1)^{\operatorname{inv}(\sigma)}$ is the usual case, where $\operatorname{inv}(\sigma)$ is the number of inversions in σ.

Now, in all cases we consider the weight of the identity permutation will be equal to 1: $w(1,\ldots,m) = 1$. Substituting $B = I - A$ in (1.5), this gives us

$$\frac{1}{\det(I-A)} \,=\, \frac{1}{1-\Sigma} \,=\, 1 \,+\, \Sigma \,+\, \Sigma^2 \,+\, \ldots\,,$$

where Σ is a certain finite sum of words in a_{ij} and both left and right inverse of $\det(I-A)$ are equal to the infinite sum on the right. From now on, whenever justified, we will always use the fraction notation as above in non-commutative situations.

In summary, we just showed that both

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) \quad \text{and} \quad \frac{1}{\det(I-A)}$$

are well-defined elements of $\mathcal{A}$. The generalizations of the Master Theorem we present in this paper will state that these two expressions are equal modulo a certain ideal $\mathcal{I}$. In the classical case, the MacMahon Master Theorem gives that for the ideal $\mathcal{I}_{\text{comm}}$ generated by $a_{ij}a_{kl} = a_{kl}a_{ij}$, for all $1 \le i, j, k, l \le m$.

1.3. **Main theorem.** Fix complex numbers $q_{ij} \neq 0$, where $1 \le i < j \le m$. Suppose the variables $x_1,\ldots,x_m$ are $\mathbf{q}$-commuting:

$$x_j\, x_i \,=\, q_{ij}\, x_i\, x_j\,, \quad \text{for all} \;\; i<j, \tag{1.6}$$

and that they commute with all a_{ij}. Suppose also that the variables a_{ij} $\mathbf{q}$-commute within columns:

$$a_{jk}\, a_{ik} \,=\, q_{ij}\, a_{ik}\, a_{jk}\,, \quad \text{for all} \;\; i<j, \tag{1.7}$$

and in addition satisfy the following quadratic equations:

$$a_{jk}\, a_{il} \,-\, q_{ij}\, a_{ik}\, a_{jl} \,+\, q_{kl}\, a_{jl}\, a_{ik} \,-\, q_{kl}\, q_{ij}\, a_{il}\, a_{jk} \,=0\,, \quad \text{for all} \;\; i<j, \;\; k<l. \tag{1.8}$$

We call $A = (a_{ij})$ with entries satisfying (1.7) and (1.8) a *right-quantum* $\mathbf{q}$*-matrix*.

For a matrix $B = (b_{ij})_{m\times m}$, define the $\mathbf{q}$*-determinant* by

$$\det_{\mathbf{q}}(b_{ij}) = \sum_{\sigma} w(\sigma)\, b_{\sigma_1 1}\cdots b_{\sigma_m m}, \tag{1.9}$$

where

$$w(\sigma) = \prod_{i<j,\;\sigma_i>\sigma_j} (-q_{\sigma_j\sigma_i})^{-1}.$$

Theorem 1.2. *Let $A = (a_{ij})_{m\times m}$ be a right-quantum $\mathbf{q}$-matrix. Denote the coefficient of $x_1^{k_1}\cdots x_m^{k_m}$ in*

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i}.$$

by $G(k_1,\ldots,k_m)$. Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) = \frac{1}{\det_{\mathbf{q}}(I-A)}, \tag{1.10}$$

where the summation is over all nonnegative integer vectors $(k_1,\ldots,k_m)$.

Theorem 1.2 is the ultimate extension of the classical MacMahon Master Theorem. Our proof of the theorem uses a number of technical improvements which become apparent in special cases. While the proof is given in Section 8, it is based on all previous sections.

2. A combinatorial proof of the MacMahon Master Theorem

2.1. **Determinant as a product.** Let $B = (b_{ij})$ be an invertible $m\times m$ matrix over $\mathbb{C}$. Denote by B^{11} the matrix B without the first row and the first column, by $B^{12,12}$ the matrix B without the first two rows and the first two columns, etc. For the entries of the inverse matrix we have:

$$\big(B^{-1}\big)_{11} = \frac{\det B^{11}}{\det B}. \tag{2.1}$$

Substituting $B = I - A$ and iterating (2.1), we obtain:

$$\begin{aligned}
&\left(\frac{1}{I-A}\right)_{11}\left(\frac{1}{I-A^{11}}\right)_{22}\left(\frac{1}{I-A^{12,12}}\right)_{33}\cdots\frac{1}{1-a_{mm}}\\
&\quad= \frac{\det(I-A^{11})}{\det(I-A)}\cdot\frac{\det(I-A^{12,12})}{\det(I-A^{11})}\cdot\frac{\det(I-A^{123,123})}{\det(I-A^{12,12})}\cdots\frac{1}{1-a_{mm}}\\
&\quad= \frac{1}{\det(I-A)},
\end{aligned}$$

provided that all minors are invertible. Now let a_{ij} be commuting variables as in Subsection 1.1. We obtain that the right-hand side of equation (1.3) is the product of entries in the inverses of matrices, and we need to prove the following identity:

$$\sum G(k_1,\ldots,k_m) = \left(\frac{1}{I-A}\right)_{11}\left(\frac{1}{I-A^{11}}\right)_{22}\left(\frac{1}{I-A^{12,12}}\right)_{33}\cdots\frac{1}{1-a_{mm}}. \tag{2.2}$$

Since $(I-A)^{-1} = I + A + A^2 + \ldots$, we get a combinatorial interpretation of the (11)-entry:

$$\left(\frac{1}{I-A}\right)_{11} = \sum a_{1j_1}a_{j_1j_2}\cdots a_{j_\ell 1}, \tag{2.3}$$

where the summation is over all finite sequences $(j_1,\ldots,j_\ell)$, where $j_r\in\{1,\ldots,m\}$, $1\le r\le \ell$. A combinatorial interpretation of the other product terms is analogous. Recall that we already have a combinatorial interpretation of $G(k_1,\ldots,k_m)$ as a summation of words. Therefore, we have reduced the Master Theorem to an equality between two summations of words (1.3), where all the summands have a positive sign. To finish the proof we construct an explicit bijection between the families of words corresponding to both sides.

2.2. **The bijection.** Throughout the paper we consider *lattice steps* of the form $(x, i) \to (x+1, j)$ for some $x, i, j \in \mathbb{Z}$, $1 \le i, j \le m$. We think of x being drawn along x-axis, increasing from left to right, and refer to i and j as the *starting height* and *ending height*, respectively.

From here on, we represent the step $(x, i) \to (x+1, j)$ by the variable a_{ij}. Similarly, we represent a finite sequence of steps by a word in the alphabet $\{a_{ij}\}$, $1 \le i, j \le m$, i.e. by an element of algebra $\mathcal{A}$. If each step in a sequence starts at the ending point of the previous step, we call such a sequence a *lattice path*.

Define a *balanced sequence* (*b-sequence*) to be a finite sequence of steps

$$\alpha = \big\{(0, i_1) \to (1, j_1)\,,\ (1, i_2) \to (2, j_2)\,,\ \ldots\,,\ (\ell - 1, i_\ell) \to (\ell, j_\ell)\big\}\,, \tag{2.4}$$

such that the number of steps starting at height i is equal to the number of steps ending at height i, for all i. We denote this number by k_i, and call $(k_1, \ldots, k_m)$ the *type* of the b-sequence. Clearly, the total number of steps in the path $\ell = k_1 + \ldots + k_m$.

Define an *ordered sequence* (*o-sequence*) to be a b-sequence where the steps starting at smaller height always precede steps starting at larger heights. In other words, an o-sequence of type $(k_1, \ldots, k_m)$ is a sequence of k_1 steps starting at height 1, then k_2 steps starting at height 2, etc., so that k_i steps end at height i. Denote by $\mathbf{O}(k_1, \ldots, k_m)$ the set of all o-sequences of type $(k_1, \ldots, k_m)$.

Now consider a lattice path from $(0, 1)$ to $(x_1, 1)$ that never goes below $y = 1$ or above $y = m$, then a lattice path from $(x_1, 2)$ to $(x_2, 2)$ that never goes below $y = 2$ or above $y = m$, etc.; in the end, take a straight path from (x_{m-1}, m) to (x_m, m). We will call this a *path sequence* (*p-sequence*). Observe that every p-sequence is also a b-sequence. Denote by $\mathbf{P}(k_1, \ldots, k_m)$ the set of all p-sequences of type $(k_1, \ldots, k_m)$.

Example 2.1. Figure 1 presents the o-sequence associated with the word

$$a_{13}a_{11}a_{12}a_{13}a_{22}a_{23}a_{22}a_{21}a_{23}a_{22}a_{23}a_{32}a_{31}a_{31}a_{33}a_{32}a_{32}a_{33}a_{33}$$

and the p-sequence associated with

$$a_{13}a_{32}a_{22}a_{23}a_{31}a_{11}a_{12}a_{22}a_{21}a_{13}a_{31}a_{23}a_{33}a_{32}a_{22}a_{23}a_{32}a_{33}a_{33}.$$

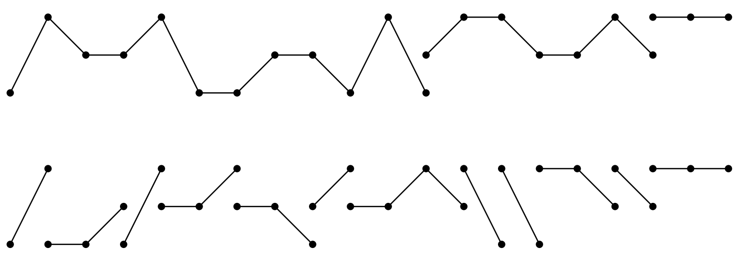

FIGURE 1. An o-sequence and a p-sequence of type $(4, 7, 8)$

We are ready now to establish a connection between balanced sequences and the equation (2.2). First, observe that choosing a term of

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}$$

means choosing a term $a_{1*}x_*$ k_1 times, then choosing a term $a_{2*}x_*$ k_2 times, etc., and then multiplying all these terms. In other words, each term on the left-hand side of (2.2) corresponds to an o-sequence in $\mathbf{O}(k_1, \ldots, k_m)$ for a unique vector $(k_1, \ldots, k_m)$. Similarly, by (2.3), a term on the right-hand side of (2.2) corresponds to a p-sequence, i.e. to an element of $\mathbf{P}(k_1, \ldots, k_m)$ for a unique vector $(k_1, \ldots, k_m)$.

Let us define a bijection

$$\varphi : \mathbf{O}(k_1, \ldots, k_m) \longrightarrow \mathbf{P}(k_1, \ldots, k_m)$$

with the property that the word $\varphi(\alpha)$ is a rearrangement of the word α, for every o-sequence α.

Take an o-sequence α, and let $[0, x]$ be the maximal interval on which it is part of a p-sequence, i.e. the maximal interval $[0, x]$ on which the o-sequence has the property that if a step ends at level i, and the following step starts at level $j > i$, the o-sequence stays on or above height j afterwards. Let i be the height at x. Choose the step $(x', i) \to (x'+1, i')$ in the o-sequence that is the first to the right of x that starts at level i (such a step exists because an o-sequence is a balanced sequence). Continue switching this step with the one to the left until it becomes the step $(x, i) \to (x+1, i')$. The new object is part of a p-sequence at least on the interval $[0, x+1]$. Continuing this procedure we get a p-sequence $\varphi(\alpha)$.

For example, for the o-sequence given in Figure 1 we have $x = 1$ and $i = 3$. The step we choose then is $(12, 3) \to (13, 1)$, i.e. $x' = 12$.

Lemma 2.2. *The map* $\varphi : \mathbf{O}(k_1, \ldots, k_m) \to \mathbf{P}(k_1, \ldots, k_m)$ *constructed above is a bijection.*

Proof. Since the above procedure never switches two steps that begin at the same height, there is exactly one o-sequence that maps into a given b-sequence: take all steps starting at height 1 in the b-sequence in the order they appear, then all the steps starting at height 2 in the p-sequence in the order they appear, etc. Clearly, this map preserves the type of a b-sequence. □

Example 2.3. Figure 2 shows the switches for an o-sequence of type $(3, 1, 1)$, and the p-sequence in Figure 1 is the result of applying this procedure to the o-sequence in the same figure (we need 33 switches).

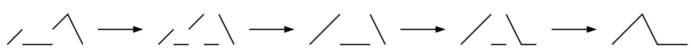

FIGURE 2. Transforming an o-sequence into a p-sequence.

In summary, Lemma 2.2 establishes the desired bijection between two sides of equation (2.2). This completes the proof of the theorem. □

2.3. **Refining the bijection.** Although we already established the MacMahon Master Theorem, in the next two subsections we will refine and then elaborate on the proof. This will be useful when we consider various generalizations and modifications of the theorem.

First, let us define *q-sequences* to be the b-sequences we get in the transformation of an o-sequence into a p-sequence with the above procedure (including the o-sequence and the p-sequence). Examples of q-sequences can be seen in Figure 2, where an o-sequence is transformed into a p-sequence via the intermediate q-sequences.

Formally, a q-sequence is a b-sequence with the following properties: it is part of a p-sequence on some interval $[0, x]$ (and this part ends at some height i); the rest of the sequence has non-decreasing starting heights, with the exception of the first step to the right of x that starts at height i, which can come before some steps starting at lower levels. For a q-sequence α, denote by $\psi(\alpha)$ the q-sequence we get by performing the switch defined above; for a p-sequence α (where no more switches are needed), $\psi(\alpha) = \alpha$. By construction, map ψ always switches steps that start on different heights.

For a balanced sequence (2.4), define the *rank* r as follows:

$$r := \left|\{(s,t) \ : \ i_s > i_t, \, 1 \leq s < t \leq \ell\}\right|.$$

Clearly, o-sequences are exactly the balanced sequences of rank 0. Note also that the map ψ defined above increases by 1 the rank of sequences that are not p-sequences.

Write $\mathbf{Q}_n(k_1, \ldots, k_m)$ for the union of two sets of b-sequences of type $(k_1, \ldots, k_m)$: the set of all q-sequences with rank n and the set of p-sequences with rank $< n$; in particular, $\mathbf{O}(k_1, \ldots, k_m) = \mathbf{Q}_0(k_1, \ldots, k_m)$ and $\mathbf{P}(k_1, \ldots, k_m) = \mathbf{Q}_N(k_1, \ldots, k_m)$ for N large enough (say, $N \geq \binom{\ell}{2}$ will work).

Lemma 2.4. *The map* $\psi : \mathbf{Q}_n(k_1, \ldots, k_m) \to \mathbf{Q}_{n+1}(k_1, \ldots, k_m)$ *is a bijection for all* n.

Proof. A q-sequence of rank n which is not a p-sequence is mapped into a q-sequence of rank $n+1$, and ψ is the identity map on p-sequences. This proves that ψ is indeed a map from $\mathbf{Q}_n(k_1, \ldots, k_m)$ to $\mathbf{Q}_{n+1}(k_1, \ldots, k_m)$. It is easy to see that ψ is injective and surjective. □

The lemma gives another proof that $\varphi = \psi^N : \mathbf{O}(k_1, \ldots, k_m) \to \mathbf{P}(k_1, \ldots, k_m)$ is a bijection. This is the crucial observation which will be used repeatedly in the later sections.

Let us emphasize the importance of bijections ψ and φ in the language of ideals. Obviously we have $\psi(\alpha) = \alpha$ modulo $\mathcal{I}_{\text{comm}}$ for every q-sequence α. Consequently, $\varphi(\alpha) = \alpha$ modulo $\mathcal{I}_{\text{comm}}$ for every o-sequence, and we have

$$\sum \varphi(\alpha) = \sum \alpha \mod \mathcal{I}_{\text{comm}},$$

where the sum is over all o-sequences α. From above, this can be viewed as a restatement of the MacMahon Master Theorem 1.1.

2.4. **Meditation on the proof.** The proof we presented above splits into two (unequal) parts: combinatorial and linear algebraic. The combinatorial part (the construction of the bijection φ) is the heart of the proof and will give analogues of (2.2) in non-commutative cases as well. While it is fair to view the equation (2.2) as the "right" generalization of the Master Theorem, it is preferable if the right-hand side is the inverse of some version of the determinant, for both aesthetic and traditional reasons. This is also how our Main Theorem 1.2 is stated.

The linear algebraic part, essentially the equation (2.1), is trivial in the commutative (classical) case. The non-commutative analogues we consider are much less trivial, but largely known. In the most general case considered in the Main Theorem the formula follows easily from the results of Manin on quantum determinants [M2, M3] and advanced technical results of Etingof and Retakh who proved (2.1) for quantum determinants [ER] in a more general setting (see further details in Section 13).

To avoid referring the technicalities to other people's work and deriving these basic linear algebra facts from much more general results, we include our own proofs of the analogues of (2.1). These proofs are moved to Section 12 and we try to keep them as concise and elementary as possible.

3. The Cartier-Foata case

In this section, we will assume that the variables $x_1, \ldots, x_m$ commute with each other and with all a_{ij}, and that

$$a_{ij}\, a_{kl} \;=\; a_{kl}\, a_{ij} \quad \text{for all} \;\; i \neq k\,. \tag{3.1}$$

The matrix $A = (a_{ij})$ which satisfies the conditions above is called a *Cartier-Foata matrix.*

For any matrix $B = (b_{ij})_{m\times m}$ (with non-commutative entries) define the *Cartier-Foata determinant*:

$$\det B \;=\; \sum_{\sigma\in S_m} (-1)^{\mathrm{inv}(\sigma)}\, b_{\sigma_1 1} \cdots b_{\sigma_m m}\,.$$

Note that the order of terms in the product is important in general, though not for a Cartier-Foata matrix.

Theorem 3.1 (Cartier-Foata)**.** *Let $A = (a_{ij})_{m\times m}$ be a Cartier-Foata matrix. Denote by $G(k_1, \ldots, k_r)$ the coefficient of $x_1^{k_1}\cdots x_m^{k_m}$ in the product*

$$\overrightarrow{\prod_{i=1..m}} \, \bigl(a_{i1}x_1 + \ldots + a_{im}x_m\bigr)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m) \;=\; \frac{1}{\det(I-A)}\,, \tag{3.2}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$, and $\det(\cdot)$ is the Cartier-Foata determinant.

Clearly, Theorem 3.1 is a generalization of the MacMahon Master Theorem 1.1. Let us show that our proof of the Master Theorem easily extends to this case. We start with the following well known technical result (see e.g. [F2]).

Proposition 3.2. *If* $A = (a_{ij})_{m\times m}$ *is a Cartier-Foata matrix, then*

$$\left(\frac{1}{I-A}\right)_{11} \, = \, \frac{1}{\det(I-A)} \cdot \det\left(I - A^{11}\right),$$

where $\det(\cdot)$ *is the Cartier-Foata determinant.*

For completeness, we include a straightforward proof of the proposition in Section 12.

Proof of Theorem 3.1. Denote by $\mathcal{I}_{\rm cf}$ the ideal generated by relations $a_{ij}a_{kl} = a_{kl}a_{ij}$ for all $1 \le i,j,k,l \le m$, with $i \ne k$. Observe that the terms of the left-hand side of (3.2) correspond to o-sequences. Similarly, by Proposition 3.2 and equation (2.3), the terms on the right-hand side correspond to p-sequences. Therefore, to prove the theorem it suffices to show that

$$\sum \alpha \, = \, \sum \varphi(\alpha) \mod \mathcal{I}_{\rm cf}\,, \tag{3.3}$$

where the sum is over all o-sequences of a fixed type $(k_1,\ldots,k_m)$.

As mentioned earlier, all switches we used in the construction of ψ involve steps starting at different heights. This means that for a q-sequence α, we have

$$\psi(\alpha) \, = \, \alpha \mod \mathcal{I}_{\rm cf}\,,$$

which implies (3.3). This completes the proof of the theorem. □

4. The right-quantum case

In this section, we will assume that the variables $x_1,\ldots,x_m$ commute with each other and with all a_{ij}, and that we have

$$a_{jk}\,a_{ik} \; = \; a_{ik}\,a_{jk}, \tag{4.1}$$

$$a_{ik}\,a_{jl} \,-\, a_{jk}\,a_{il} \; = \; a_{jl}\,a_{ik} \,-\, a_{il}\,a_{jk}, \tag{4.2}$$

for all $1 \le i,j,k,l \le m$. We call $A = (a_{ij})_{m\times m}$ whose entries satisfy these relations a *right-quantum matrix.*

Note that a Cartier-Foata matrix is a right-quantum matrix. The following result is an important special case of the GLZ-theorem (Theorem 6) and a generalization of Theorem 3.1.

Theorem 4.1. *Let* $A = (a_{ij})_{m\times m}$ *be a right-quantum matrix. Denote by* $G(k_1,\ldots,k_r)$ *the coefficient of* $x_1^{k_1}\cdots x_m^{k_m}$ *in the product*

$$\overrightarrow{\prod_{i=1..m}} \left(a_{i1}x_1 + \ldots + a_{im}x_m\right)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) \, = \, \frac{1}{\det(I-A)}\,, \tag{4.3}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$*, and* $\det(\cdot)$ *is the Cartier-Foata determinant.*

Let us show that our proof of the Master Theorem extends to this case as well, with some minor modifications. We start with the following technical result generalizing Proposition 3.2.

Proposition 4.2. *If* $A = (a_{ij})$ *is a right-quantum matrix, then*

$$\left(\frac{1}{I-A}\right)_{11} = \frac{1}{\det(I-A)} \cdot \det\left(I - A^{11}\right).$$

For completeness, we include a proof of the proposition in Section 12.

Proof of Theorem 4.1. Denote by $\mathcal{I}_{\mathrm{rq}}$ the ideal of $\mathcal{A}$ generated by the relations (4.1) and (4.2). As before, the proposition implies that the right-hand side of (4.3) enumerates all p-sequences, and it is again obvious that the left-hand side of (4.3) enumerates all o-sequences. Note that it is no longer true that for an o-sequence α, $\varphi(\alpha) = \alpha$ modulo $\mathcal{I}_{\mathrm{rq}}$. However, it suffices to prove that

$$\sum \varphi(\alpha) = \sum \alpha \mod \mathcal{I}_{\mathrm{rq}}, \tag{4.4}$$

where the sum goes over all o-sequences $\alpha \in \mathbf{O}(k_1, \ldots, k_m)$. We show this by making switches in the construction of φ *simultaneously*.

Take a q-sequence α. If α is a p-sequence, then $\psi(\alpha) = \alpha$. Otherwise, assume that $(x-1, i) \to (x, k)$ and $(x, j) \to (x+1, l)$ are the steps to be switched in order to get $\psi(\alpha)$. If $k = l$, then $\psi(\alpha) = \alpha$ modulo $\mathcal{I}_{\mathrm{rq}}$ by (4.1). Otherwise, denote by β the sequence we get by replacing these two steps with $(x-1, i) \to (x, l)$ and $(x, j) \to (x+1, k)$. The crucial observation is that β is also a q-sequence, and that its rank is equal to the rank of α. Furthermore, $\alpha + \beta = \psi(\alpha) + \psi(\beta) \mod \mathcal{I}_{\mathrm{rq}}$ because of (4.2). This implies that $\sum \psi(\alpha) = \sum \alpha \mod \mathcal{I}_{\mathrm{rq}}$ with the sum over all sequences in $\mathbf{Q}_n(k_1, \ldots, k_m)$. From here we obtain (4.4) and conclude the proof of the theorem. $\square$

Example 4.3. Figure 3 provides a graphical illustration for $k_1 = 3$, $k_2 = 1$, $k_3 = 1$; here p-sequences are drawn in bold, an arrow from a q-sequence α of rank n to a q-sequence of rank $n+1$ α' means that $\alpha' = \psi(\alpha)$ and $\alpha' = \alpha \mod \mathcal{I}_{\mathrm{rq}}$, and arrows from q-sequences α, β of rank n to q-sequences α', β' of rank $n+1$ whose intersection is marked by a dot mean that $\alpha' = \psi(\alpha)$, $\beta' = \psi(\beta)$, and $\alpha' + \beta' = \alpha + \beta \mod \mathcal{I}_{\mathrm{rq}}$.

5. The Cartier-Foata q-case

In this section, we assume that variables $x_1, \ldots, x_m$ satisfy

$$x_j x_i = q\, x_i x_j \quad \text{for} \quad i < j, \tag{5.1}$$

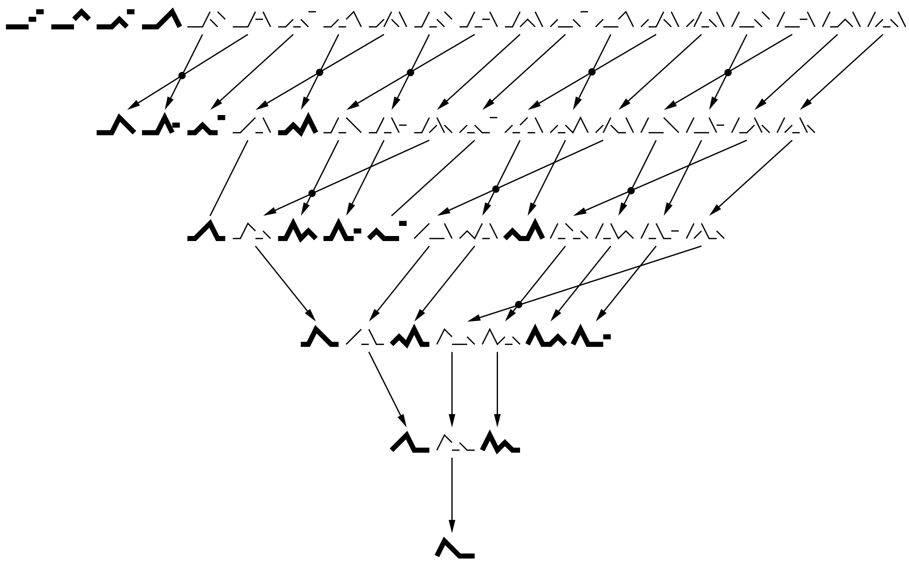

FIGURE 3. Transforming o-sequences into p-sequences via a series of simultaneous switches.

where $q \in \mathbb{C}$, $q \neq 0$, is a fixed complex number. Suppose also that $x_1, \ldots, x_m$ they commute with all a_{ij} and that we have:

$$a_{jl}\, a_{ik} \;=\; a_{ik}\, a_{jl} \quad \text{for} \;\; i < j,\; k < l\,, \tag{5.2}$$

$$a_{jl}\, a_{ik} \;=\; q^2\, a_{ik}\, a_{jl}\,, \quad \text{for} \;\; i < j,\; k > l\,, \tag{5.3}$$

$$a_{jk}\, a_{ik} \;=\; q\, a_{ik}\, a_{jk}\,, \quad \text{for} \;\; i < j\,. \tag{5.4}$$

Let us call such a matrix $A = (a_{ij})$ a *Cartier-Foata q-matrix.* As the name suggests, when $q = 1$ the Cartier-Foata q-matrix becomes a Cartier-Foata matrix.

For a matrix $B = (b_{ij})_{m\times m}$ with non-commutative entries, define a *quantum determinant* (*q-determinant*) by the following formula:

$$\det_q B \;=\; \sum_{\sigma\in S_m} (-q)^{-\,\mathrm{inv}(\sigma)}\, b_{\sigma_1 1} \cdots b_{\sigma_m m}$$

The following result is another important special case of the GLZ-theorem and a generalization of the Cartier-Foata Theorem 3.1.

Theorem 5.1. *Let $A = (a_{ij})_{m\times m}$ be a Cartier-Foata q-matrix. Denote by $G(k_1, \ldots, k_r)$ the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in*

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m) \;=\; \frac{1}{\det_q(I - A)}\,, \tag{5.5}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$.

The proof of the theorem is a weighted analogue of the proof of Theorem 3.1. The main technical difference is essentially bookkeeping of the powers of q which appear after switching the letters a_{ij} (equivalently, the lattice steps in the q-sequences). We begin with some helpful notation which will be used throughout the remainder of the paper.

We abbreviate the product $a_{\lambda_1\mu_1}\cdots a_{\lambda_n\mu_n}$ to $a_{\lambda,\mu}$ for $\lambda = \lambda_1\cdots\lambda_n$ and $\mu = \mu_1\cdots\mu_n$, where λ and μ are regarded as words in the alphabet $\{1,\dots,m\}$. For any such word $\nu = \nu_1\cdots\nu_n$, define the *set of inversions*

$$\mathcal{I}(\nu) \,=\, \{(i,j)\colon i<j, \nu_i>\nu_j\},$$

and let $\operatorname{inv}\nu = |\mathcal{I}(\nu)|$.

Proof of Theorem 5.1. Denote by $\mathcal{I}_{q-\mathrm{cf}}$ the ideal of $\mathcal{A}$ generated by relations (5.2) – (5.4). When we expand the product

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1+\ldots+a_{im}x_m\big)^{k_i},$$

move the x_i's to the right and order them, the coefficient at $a_{\lambda,\mu}$ is $q^{\operatorname{inv}\mu}$. This means that $\sum G(k_1,\dots,k_m)$ is a weighted sum of o-sequences, with an o-sequence $a_{\lambda,\mu}$ weighted by $q^{\operatorname{inv}\mu} = q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}$.

Choose a q-sequence $\alpha = a_{\lambda,\mu}$ and let $\psi(\alpha) = a_{\lambda',\mu'}$. Assume that the switch we perform is between steps $(x-1,i)\to(x,k)$ and $(x,j)\to(x+1,l)$; write $\lambda=\lambda_1ij\lambda_2$, $\mu=\mu_1kl\mu_2$, $\lambda'=\lambda_1ji\lambda_2$, $\mu'=\mu_1lk\mu_2$. If $i<j$ and $k<l$, we have $\operatorname{inv}\lambda' = \operatorname{inv}\lambda+1$, $\operatorname{inv}\mu'=\operatorname{inv}\mu+1$. By (5.2), $\psi(\alpha)=\alpha$ modulo $\mathcal{I}_{q-\mathrm{cf}}$ and

$$q^{\operatorname{inv}\mu'-\operatorname{inv}\lambda'}\psi(\alpha) = q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}\alpha \mod \mathcal{I}_{q-\mathrm{cf}}\,. \tag{5.6}$$

Similarly, if $i<j$ and $k>l$, we have $\operatorname{inv}\lambda'=\operatorname{inv}\lambda+1$, $\operatorname{inv}\mu'=\operatorname{inv}\mu-1$. By (5.3), we have $\psi(\alpha)=q^2\alpha$ modulo $\mathcal{I}_{q-\mathrm{cf}}$, which implies equation (5.6). If $i<j$ and $k=l$, we have $\operatorname{inv}\lambda'=\operatorname{inv}\lambda+1$, $\operatorname{inv}\mu'=\operatorname{inv}\mu$. By (5.4), we have $\psi(\alpha)=q\alpha$ modulo $\mathcal{I}_{q-\mathrm{cf}}$, which implies (5.6) again. Other cases are analogous.

Iterating equation (5.6), we conclude that if $\alpha=a_{\lambda,\mu}$ is an o-sequence and $\varphi(\alpha) = a_{\lambda',\mu'}$ is the corresponding p-sequence, then

$$q^{\operatorname{inv}\mu'-\operatorname{inv}\lambda'}\varphi(\alpha) \,=\, q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}\,\alpha \mod \mathcal{I}_{q-\mathrm{cf}}\,.$$

Therefore,

$$\sum_{(k_1,\dots,k_m)} G(k_1,\dots,k_m) \,=\, \sum q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}\,\alpha \mod \mathcal{I}_{q-\mathrm{cf}}\,, \tag{5.7}$$

where the sum on the right-hand side goes over all p-sequences $\alpha = a_{\lambda,\mu}$.

Let us call a p-sequence *primitive* if it starts at some height y and stays strictly above y until the last step (when it returns to y). For example, the p-sequence in Figure 4 is a product of four primitive p-sequences. For a primitive p-sequence $a_{\lambda,\mu}$ of length ℓ, $\operatorname{inv}\mu-\operatorname{inv}\lambda=\ell-1$, and for an arbitrary p-sequence $a_{\lambda,\mu}$ of length ℓ that decomposes into n primitive p-sequences, $\operatorname{inv}\mu-\operatorname{inv}\lambda=\ell-n$.

Consider a matrix

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ qa_{21} & qa_{22} & \cdots & qa_{2m} \\ qa_{31} & qa_{32} & \cdots & qa_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ qa_{m1} & qa_{m2} & \cdots & qa_{mm} \end{pmatrix}. \tag{5.8}$$

Clearly $(\widetilde{A}^\ell)_{11}$ enumerates paths starting and ending at height 1 weighted by $q^{\ell-n}$, where n is the number of steps starting at height 1. At this point we need the following generalization of Proposition 3.2.

Proposition 5.2. *If $A = (a_{ij})_{m\times m}$ is a Cartier-Foata q-matrix, then*

$$\left(\frac{1}{I-\widetilde{A}}\right)_{11} = \frac{1}{\det_q(I-A)} \cdot \det_q\left(I - A^{11}\right).$$

The proposition implies that the right-hand side of (5.5) in the theorem enumerates all p-sequences, with $\alpha = a_{\lambda,\mu}$ weighted by $q^{\operatorname{inv}\mu - \operatorname{inv}\lambda}$. The equation (5.7) above shows that this is also the left-hand side of (5.5). This completes the proof of the theorem. ☐

Example 5.3. For the p-sequence

$$\alpha = a_{13}a_{32}a_{24}a_{43}a_{31}a_{11}a_{22}a_{34}a_{44}a_{43}$$

shown in Figure 4, we have

$$\operatorname{inv}(1324312344) = 0+3+1+4+2+0+0+0+0+0 = 10$$

and

$$\operatorname{inv}(3243112443) = 4+2+5+3+0+0+0+1+1+0 = 16.$$

Therefore, the p-sequence α is weighted by q^6.

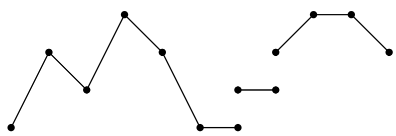

Figure 4. A p-sequence with weight q^6.

6. The right-quantum q-case

As in the previous section, we assume that variables $x_1, \ldots, x_m$ satisfy

$$x_j x_i = q x_i x_j \quad \text{for } i < j, \tag{6.1}$$

where $q \in \mathbb{C}$, $q \neq 0$ is a fixed complex number. Suppose also that $x_1, \ldots, x_m$ commute with all a_{ij} and that we have:

$$a_{jk} a_{ik} = q a_{ik} a_{jk} \quad \text{for all } i < j, \tag{6.2}$$

$$a_{ik} a_{jl} - q^{-1} a_{jk} a_{il} = a_{jl} a_{ik} - q a_{il} a_{jk} \quad \text{for all } i < j, k < l. \tag{6.3}$$

We call such matrix $A = (a_{ij})$ the *right quantum q-matrix.* It is easy to see that when $q = 1$ we get a right quantum matrix defined in Section 4. In a different direction, every Cartier-Foata q-matrix is also a right quantum q-matrix. The following result of Garoufalidis, Lê and Zeilberger [GLZ] generalizes Theorems 4.1 and 5.1.

Theorem 6.1 (GLZ-theorem)**.** *Let $A = (a_{ij})_{m\times m}$ be a right quantum q-matrix. Denote by $G(k_1,\ldots,k_r)$ the coefficient of $x_1^{k_1}\cdots x_m^{k_m}$ in*

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) \,=\, \frac{1}{\det_q(I-A)}\,, \tag{6.4}$$

where the summation is over all nonnegative integer vectors $(k_1,\ldots,k_m)$.

The proof of the theorem is almost identical to the one given in the previous section, with some modifications similar to those in the proof of Theorem 4.1.

Proof of Theorem 6.1. Denote by $\mathcal{I}_{q-\mathrm{rq}}$ the ideal of $\mathcal{A}$ generated by relations (6.2) and (6.3). Now, when we expand the product

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i},$$

move the x_i's to the right and order them, the coefficient at $a_{\lambda,\mu}$ is $q^{\operatorname{inv}\mu}$. Therefore, $\sum G(k_1,\ldots,k_m)$ is a weighted sum of o-sequences, with an o-sequence $a_{\lambda,\mu}$ weighted by $q^{\operatorname{inv}\mu} = q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}$. Similar arguments as before, now using (6.2) and (6.3) instead of (5.2) – (5.4), show that

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) \,=\, \sum q^{\operatorname{inv}\mu-\operatorname{inv}\lambda}\, a_{\lambda,\mu} \mod \mathcal{I}_{q-\mathrm{rq}}\,, \tag{6.5}$$

where the sum on the right-hand side is over all p-sequences $\alpha = a_{\lambda,\mu}$. The following proposition generalizes Propositions 4.2 and 5.2.

Proposition 6.2. *If $A = (a_{ij})_{m\times m}$ is a right quantum q-matrix, then*

$$\left(\frac{1}{I-\widetilde{A}}\right)_{11} \,=\, \frac{1}{\det_q(I-A)}\,\cdot\,\det_q\big(I-A^{11}\big)\,,$$

where $\widetilde{A}$ is defined by (5.8)*.*

The proposition is proved in Section 12. Now Theorem 6.1 follows from the proposition and equation (6.5). $\square$

7. The Cartier-Foata q_{ij}-case

We can extend the results of the previous sections to the multiparameter case. Assume that variables $x_1, \ldots, x_m$ satisfy

$$x_j x_i \, = \, q_{ij} \, x_i x_j \quad \text{for} \;\; i < j \, , \tag{7.1}$$

where $q_{ij} \in \mathbb{C}$, $q_{ij} \neq 0$ are fixed complex numbers, $1 \leq i < j \leq m$. Suppose also that $x_1, \ldots, x_m$ commute with all a_{ij} and that we have:

$$q_{kl} \, a_{jl} \, a_{ik} \;=\; q_{ij} \, a_{ik} \, a_{jl} \quad \text{for} \;\; i < j, k < l \, , \tag{7.2}$$

$$a_{jl} \, a_{ik} \;=\; q_{ij} \, q_{lk} \, a_{ik} \, a_{jl} \quad \text{for} \;\; i < j, k > l \, , \tag{7.3}$$

$$a_{jk} \, a_{ik} \;=\; q_{ij} \, a_{ik} \, a_{jk} \, , \quad \text{for} \;\; i < j \, . \tag{7.4}$$

We call $A = (a_{ij})_{m \times m}$ whose entries satisfy these relations a *Cartier-Foata* $\mathbf{q}$*-matrix.* When all $q_{ij} = q$ we obtain a Cartier-Foata q-matrix. Thus the following result is a generalization of Theorem 5.1 and is a corollary of our Main Theorem 1.2.

Theorem 7.1. *Assume that* $A = (a_{ij})_{m \times m}$ *is a Cartier-Foata* $\mathbf{q}$*-matrix. Denote by* $G(k_1, \ldots, k_r)$ *the coefficient of* $x_1^{k_1} \cdots x_m^{k_m}$ *in*

$$\overrightarrow{\prod_{i=1..m}} \left(a_{i1} x_1 + \ldots + a_{im} x_m \right)^{k_i}.$$

Then

$$\sum_{(k_1, \ldots, k_m)} G(k_1, \ldots, k_m) \, = \, \frac{1}{\det_{\mathbf{q}}(I - A)} \, , \tag{7.5}$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$ *and* $\det_{\mathbf{q}}(\cdot)$ *is the* $\mathbf{q}$*-determinant defined by* (1.9)*.*

Remark 7.2. If we define $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for $i < j$, we can write the conditions (7.2) – (7.4) more concisely as

$$q_{kl} \, a_{jl} \, a_{ik} \, = \, q_{ij} \, a_{ik} \, a_{jl}, \tag{7.6}$$

for all i, j, k, l, and $i \neq j$.

Let us note also that the definition of $\mathbf{q}$-determinant $\det_{\mathbf{q}}(B)$ for the minors of B has to be adapted as follows. The weights q_{ij} always correspond to indices i, j of the entries b_{ij}, not the column and row numbers. For example,

$$\det_{\mathbf{q}} \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \, = \, a_{22} \, a_{33} \, - \, q_{23}^{-1} \, a_{32} \, a_{23}.$$

We can repeat the proof of Theorem 5.1 almost verbatim. This only requires a more careful "bookkeeping" as we need to keep track of the set of inversions, not just its cardinality (the number of inversions).

Proof of Theorem 7.1. Denote by $\mathcal{I}_{\mathbf{q}-\mathrm{cf}}$ the ideal of $\mathcal{A}$ generated by the relations (7.2) – (7.4). When we expand the product

$$\overrightarrow{\prod_{i=1..m}} \bigl(a_{i1}x_1 + \ldots + a_{im}x_m\bigr)^{k_i},$$

move the x_i's to the right and order them, the coefficient at $a_{\lambda,\mu}$ is equal to

$$\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i}\,.$$

This means that $\sum G(k_1,\ldots,k_m)$ is a weighted sum of o-sequences, with an o-sequence $a_{\lambda,\mu}$ weighted by

$$\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} = \prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in\mathcal{I}(\lambda)} q^{-1}_{\lambda_j\lambda_i}.$$

Now, the equation (7.6) implies that for every o-sequence $\alpha = a_{\lambda,\mu}$ and $\varphi(\alpha) = a_{\lambda',\mu'}$, we have

$$\left(\prod_{(i,j)\in\mathcal{I}(\mu')} q_{\mu'_j\mu'_i} \prod_{(i,j)\in\mathcal{I}(\lambda')} q^{-1}_{\lambda'_j\lambda'_i}\right)\varphi(\alpha) = \left(\prod_{(i,j)\in\mathcal{I}(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in\mathcal{I}(\lambda)} q^{-1}_{\lambda_j\lambda_i}\right)\alpha \mod \mathcal{I}_{\mathbf{q}-\mathrm{cf}}.$$

On the other hand, for a primitive p-sequence $a_{\lambda,\mu}$ starting and ending at 1 we have:

$$\prod_{(i,j)\in I(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in I(\lambda)} q^{-1}_{\lambda_j\lambda_i} = q_{1\mu_1}q_{1\mu_2}\cdots q_{1\mu_{n-1}}.$$

This shows that all weighted p-sequences starting and ending at 1 are enumerated by

$$(7.7)\qquad \left(\frac{1}{I-\widetilde{A}}\right)_{11}, \quad \text{where} \quad \widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ q_{12}a_{21} & q_{12}a_{22} & \cdots & q_{12}a_{2m} \\ q_{13}a_{31} & q_{13}a_{32} & \cdots & q_{13}a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1m}a_{m1} & q_{1m}a_{m2} & \cdots & q_{1m}a_{mm} \end{pmatrix}.$$

We need the following generalization of Proposition 5.2.

Proposition 7.3. *If $A = (a_{ij})_{m\times m}$ is a Cartier-Foata $\mathbf{q}$-matrix, then*

$$\left(\frac{1}{I-\widetilde{A}}\right)_{11} = \frac{1}{\det_{\mathbf{q}}(I-A)} \cdot \det_{\mathbf{q}}\bigl(I - A^{11}\bigr)\,.$$

The proposition is proved in Section 12. From here, by the same logic as in the proofs above we obtain the result. $\square$

8. The right-quantum q_{ij}-case (proof of Main Theorem 1.2)

First, by taking $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for $j < i$, we can assume (1.6) holds for all $1 \le i, j \le m$. Now equations (1.7) and (1.8) can be more succinctly written as

$$a_{ik}\, a_{jl} \,-\, q_{ij}^{-1}\, a_{jk}\, a_{il} \;=\; q_{kl}\big(q_{ij}^{-1}\, a_{jl}\, a_{ik} \,-\, a_{il}\, a_{jk}\big) \tag{8.1}$$

for all i, j, k, l, such that $i \ne j$. Note that in this form equation (8.1) is a direct generalization of equation (6.3) on one hand, with the q_{ij}'s arranged as in equations (7.2)–(7.4) on the other hand.

We also need the following (straightforward) generalization of Propositions 6.2 and 7.3.

Proposition 8.1. *If $A = (a_{ij})_{m\times m}$ is a right-quantum $\mathbf{q}$-matrix, then*

$$\left(\frac{1}{I - \widetilde{A}}\right)_{11} = \frac{1}{\det_{\mathbf{q}}(I - A)} \cdot \det_{\mathbf{q}}\left(I - A^{11}\right),$$

where $\widetilde{A}$ is given in (7.7)*.*

The proof of the proposition is in Section 12. From here, the proof of the Main Theorem follows verbatim the proof of Theorem 7.1. We omit the details. $\square$

9. The super-case

In this section we present an especially interesting corollary of Theorem 7.1.

Fix a vector $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_2^m$ and write $\hat{\imath}$ for γ_i. If $\hat{\imath} = 0$, index i is called *even*, otherwise it is called *odd*. We will assume that the variables $x_1, \ldots, x_m$ satisfy

$$x_j\, x_i \;=\; (-1)^{\hat{\imath}\hat{\jmath}}\, x_i x_j \quad \text{for } \ i \ne j \tag{9.1}$$

In other words, variables x_i and x_j commute unless they are both odd: $\gamma_i = \gamma_j = 1$, in which case they anti-commute. As before, suppose $x_1, \ldots, x_m$ commute with all a_{ij}'s, and that we have

$$a_{ik}\, a_{jk} \;=\; (-1)^{\hat{\imath}\hat{\jmath}}\, a_{jk}\, a_{ik}, \qquad \text{for all } \ i \ne j, \tag{9.2}$$

$$a_{ik}\, a_{jl} \;=\; (-1)^{\hat{\imath}\hat{\jmath} + \hat{k}\hat{l}}\, a_{jl}\, a_{ik}, \quad \text{for all } \ i \ne j, \ k \ne l. \tag{9.3}$$

We call $A = (a_{ij})$ as above a *Cartier-Foata super-matrix.* Clearly, when $\gamma = (0, \ldots, 0)$, we get the (usual) Cartier-Foata matrix (see Section 3).

For a permutation σ of $\{1, \ldots, m\}$, we will denote by $\operatorname{oinv}(\sigma)$ the number of *odd inversions*, i.e. the number of pairs (i, j) with $\hat{\imath} = \hat{\jmath} = 1$, $i < j$, $\pi(i) > \pi(j)$. For a matrix $B = (b_{ij})_{m\times m}$ define its *super-determinant* as

$$\operatorname{sdet} B = \sum_{\sigma \in S_m} (-1)^{\operatorname{inv}(\sigma) - \operatorname{oinv}(\sigma)}\, b_{\sigma_1 1} \cdots b_{\sigma_m m}.$$

Theorem 9.1. (Super Master Theorem) *Let $A = (a_{ij})_{m\times m}$ be a Cartier-Foata super-matrix, and let $x_1, \ldots, x_m$ be as above. Denote by $G(k_1, \ldots, k_r)$ the coefficient of $x_1^{k_1} \cdots x_m^{k_m}$ in*

$$\overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{k_i}.$$

Then

$$\sum_{(k_1,\ldots,k_m)} G(k_1,\ldots,k_m) = \frac{1}{\operatorname{sdet}(I-A)}\,, \tag{9.4}$$

where the summation is over all nonnegative integer vectors $(k_1,\ldots,k_m)$.

Proof. This is a special case of Theorem 7.1 for $q_{ij} = (-1)^{\hat{\imath}\hat{\jmath}}$. It is easy to see that $\det_{\mathbf{q}}(B) = \operatorname{sdet}(B)$ for all B, by definition of even and odd inversions. The rest is a straightforward verification. □

In conclusion, let us note that when $\gamma = (1,\ldots,1)$ we get a Cartier-Foata q-matrix with $q=-1$. Interestingly, here sdet becomes a permanent.

10. The β-extension

In this section we first present an extension of MacMahon Master Theorem due to Foata and Zeilberger, and then show how to generalize it to a non-commutative setting.

First, assume that a_{ij} are commutative variables and let $\beta \in \mathbb{N}$ be a non-negative integer. For $\mathbf{k} = (k_1,\ldots,k_m)$, let $\Sigma(\mathbf{k})$ denote the set of all permutations of the set

$$\{(1,1),\ldots,(1,k_1),(2,1),\ldots,(2,k_2),\ldots,(m,1),\ldots,(m,k_m)\}.$$

For a permutation $\pi \in \Sigma(\mathbf{k})$, we define $\pi_{ij} := i'$ whenever $\pi(i,j) = (i',j')$. Define the *weight* $v(\pi)$ by a word

$$v(\pi) = \overrightarrow{\prod_{i=1..m}}\ \overrightarrow{\prod_{j=1..k_i}}\ a_{i,\pi_{ij}}$$

and the *β-weight* $v_\beta(\pi)$ by a product

$$v_\beta(\pi) = \beta^{\operatorname{cyc}\pi}\, v(\pi)\,,$$

where $\operatorname{cyc}\pi$ is the number of cycles of the permutation π. For example, if

$$\pi = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (2,1) & (3,1) \\ (2,1) & (1,2) & (1,1) & (3,1) & (1,3) \end{pmatrix} \in \Sigma(3,1,1)\,,$$

then $v(\pi) = a_{12}a_{11}a_{11}a_{23}a_{31}$ and $v_\beta(\pi) = \beta^2\, a_{12}a_{11}a_{11}a_{23}a_{31}$.

By definition, the word $v(\pi)$ is always an o-sequence of type $(k_1,\ldots,k_m)$. Note now that the word $\alpha \in \mathbf{O}(k_1,\ldots,k_m)$ does not determine the permutation π uniquely, since the second coordinate j' in $(i',j') = \pi(i,j)$ can take any value between 1 and $k_{i'}$. From here it follows that there are exactly $k_1!\cdots k_m!$ permutations $\pi \in \Sigma(\mathbf{k})$ corresponding to a given o-sequence $\alpha \in \mathbf{O}(k_1,\ldots,k_m)$.

Now, the (usual) MacMahon Master Theorem can be restated as

$$\sum_{\mathbf{k}=(k_1,\ldots,k_m)} \frac{1}{k_1!\cdots k_m!} \sum_{\pi\in\Sigma(\mathbf{k})} v(\pi) \;=\; \frac{1}{\det(I-A)}\,, \tag{MMT}$$

where the summation is over all non-negative integer vectors $\mathbf{k} = (k_1, \ldots, k_m)$. Foata and Zeilberger proved in [FZ] the following extension of (MMT):

$$\text{(FZ)} \qquad \sum_{\mathbf{k}=(k_1,\ldots,k_m)} \frac{1}{k_1! \cdots k_m!} \sum_{\pi \in \Sigma(\mathbf{k})} v_\beta(\pi) = \left(\frac{1}{\det(I - A)} \right)^\beta .$$

Note that the right-hand side of (FZ) is well defined for all complex values β, but we will avoid this generalization for simplicity.

Now, in the spirit of Subsection 1.2 one can ask whether (FZ) can be extended to a non-commutative setting. As it turns out, this is quite straightforward given the structure of our bijection φ. As an illustration, we will work in the setting of Section 4.

Theorem 10.1. *Let $A = (a_{ij})_{m \times m}$ be a right quantum matrix and assume that the variables $x_1, \ldots, x_m$ commute with each other and with all a_{ij}, $1 \le i, j \le m$. Then, in the above notation, we have:*

$$\text{(10.1)} \qquad \sum_{\mathbf{k}=(k_1,\ldots,k_m)} \frac{1}{k_1! \cdots k_m!} \sum_{\pi \in \Sigma(\mathbf{k})} v_\beta(\pi) = \left(\frac{1}{\det(I - A)} \right)^\beta ,$$

where the summation is over all non-negative integer vectors $\mathbf{k} = (k_1, \ldots, k_m)$ and $\det(\cdot)$ *is the Cartier-Foata determinant.*

Proof. We prove the theorem by reduction to Foata-Zeilberger's identity (FZ) and our previous results. First, by Theorem 4.1, every term on the right-hand side of equation (10.1) is a concatenation of β o-sequences. Using bijection φ as in the proof of Theorem 4.1, we conclude that the sum of all concatenations of β o-sequences is equal to a weighted sum of all o-sequences modulo the ideal $\mathcal{I}_{\text{rq}}$. In other words, $(\det(I - A))^{-\beta}$ is a weighted sum of words $v(\pi)/(k_1! \cdots k_m!)$, for $\pi \in \Sigma(\mathbf{k})$, and the coefficients are equal to the number of concatenations of β o-sequences that are transformed into the given p-sequence. Therefore, the coefficients must be the same as in the commutative case. Now Foata-Zeilberger's equation (FZ) immediately implies the theorem. □

Example 10.2. Figure 5 illustrates the term $(a_{13}a_{22}a_{31})(a_{11}a_{12}a_{23}a_{31})(a_{23}a_{32})$ in $(\det(I - A))^{-3}$.

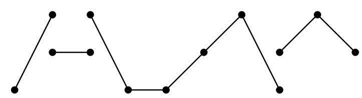

FIGURE 5. Concatenation of three o-sequences of lengths 3, 4 and 2.

For $\beta = 2$, Figure 6 shows all $(\beta^3 + \beta^2)/2 = 6$ pairs of o-sequences whose concatenation gives the term $a_{11}a_{13}a_{22}a_{31}$ in $(\det(I - A))^{-\beta}$.

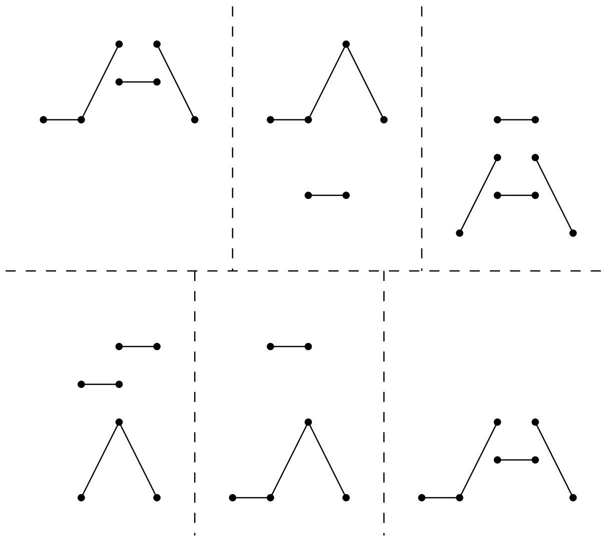

FIGURE 6. Pairs of o-sequences whose concatenation give $a_{11}a_{13}a_{22}a_{31}$ after shuffling.

11. Krattenthaler-Schlosser's q-analogue

In the context of multidimensional q-series an interesting q-analogue of MacMahon Master Theorem was obtained in [KS, Theorem 9.2]. In this section we place the result in our non-commutative framework and quickly deduce it from Theorem 3.1.

We start with some basic definitions and notations. Let z_i, b_{ij}, $1 \le i, j \le m$, be commutative variables, and let $q_1, \ldots, q_m \in \mathbb{C}$ be fixed complex numbers. Denote by $\mathcal{E}_i$ the q_i-shift operator

$$\mathcal{E}_i : \mathbb{C}[z_1, \ldots, z_m] \longrightarrow \mathbb{C}[z_1, \ldots, z_m]$$

that replaces each occurrence of z_i by $q_i\, z_i$. We assume that $\mathcal{E}_r$ commutes with b_{ij}, for all $1 \le i, j, r \le m$. For a nonnegative integer vector $\mathbf{k} = (k_1, \ldots, k_m)$, denote by $[\mathbf{z}^\mathbf{k}]\, F$ the coefficient of $z_1^{k_1} \cdots z_m^{k_m}$ in the series F. Denote by $\mathbf{1}$ the constant polynomial 1. Finally, let

$$(a; q)_k \;=\; (1 - a)(a - a\, q) \cdots (1 - a\, q^{k-1}).$$

Theorem 11.1 (Krattenthaler-Schlosser)**.** *Let $A = (a_{ij})_{m\times m}$, where*

$$a_{ij} \;=\; z_i\, \delta_{ij} - z_i\, b_{ij}\, \mathcal{E}_i\,, \quad \textit{for all} \;\; 1 \le i, j \le m.$$

Then, for non-negative integer vector $\mathbf{k} = (k_1, \ldots, k_m)$ we have:

$$[\mathbf{z}^\mathbf{0}] \prod_{i=1}^{m} \left(\sum_{j=1}^{m} b_{ij} z_j / z_i; q_i \right)_{k_i} \;=\; [\mathbf{z}^\mathbf{k}] \left(\frac{1}{\det(I - A)} \cdot \mathbf{1} \right), \tag{11.1}$$

where $\det(\cdot)$ *is the Cartier-Foata determinant.*

Note that the right-hand side of (11.1) is non-commutative and (as stated) does not contain q_i's, while the left-hand side contains only commutative variables and q_i's. It is not immediately obvious and was shown in [KS] that the theorem reduces to the MacMahon Master Theorem. Here we give a new proof of the result.

Proof of Theorem 11.1. Think of variables z_i and b_{ij} as operators acting on polynomials by multiplication. Then a matrix entry a_{ij} is an operator as well. Note that multiplication by z_i and the operator $\mathcal{E}_j$ commute for $i \neq j$. This implies that the equation (3.1) holds, i.e. that A is a Cartier-Foata matrix. Let $x_1, \ldots, x_m$ be formal variables that commute with each other and with a_{ij}'s. By Theorem 3.1, for the operator on the right-hand side of (11.1) we have:

$$\frac{1}{\det(I-A)} \, = \sum_{\mathbf{r}=(r_1,\ldots,r_m)} G(r_1,\ldots,r_m),$$

where

$$G(r_1,\ldots,r_m) \, = \, [\mathbf{x}^{\mathbf{r}}] \overrightarrow{\prod_{i=1..m}} \big(a_{i1}x_1 + \ldots + a_{im}x_m\big)^{r_i}.$$

Recall that $a_{ij} = z_i\,(\delta_{ij} - b_{ij}\,\mathcal{E}_i)$. Now observe that every coefficient $G(r_1,\ldots,r_m)\cdot \mathbf{1}$ is equal to $\mathbf{z}^{\mathbf{r}}$ times a polynomial in b_{ij} and q_i. Therefore, the right-hand side of (11.1) is equal to

$$[\mathbf{z}^{\mathbf{k}}] \left(\frac{1}{\det(I-A)}\cdot \mathbf{1}\right) \, = \, [\mathbf{z}^{\mathbf{k}}] \left(\sum_{\mathbf{r}} G(r_1,\ldots,r_m)\cdot \mathbf{1}\right) \, = \, [\mathbf{z}^{\mathbf{k}}] \, \big(G(k_1,\ldots,k_m)\cdot \mathbf{1}\big).$$

This is, of course, a sum of $[\mathbf{z}^{\mathbf{k}}](\alpha\cdot\mathbf{1})$ over all o-sequences α of type $\mathbf{k}$. Define

$$c^k_{ij} = z_i\delta_{ij} - z_i b_{ij} q_i^{k-1} \quad \text{and} \quad d^k_{ij} = z_j\delta_{ij} - z_j b_{ij} q_i^{k-1}.$$

It is easy to prove by induction that

$$a_{i\lambda_1}a_{i\lambda_2}\cdots a_{i\lambda_\ell}\cdot\mathbf{1} \, = \, c^\ell_{i\lambda_1}c^{\ell-1}_{i\lambda_2}\cdots c^1_{i\lambda_\ell}.$$

Therefore, for every o-sequence

$$\alpha \, = \, a_{1\lambda^1_1}a_{1\lambda^1_2}\cdots a_{1\lambda^1_{k_1}}\,a_{2\lambda^2_1}a_{2\lambda^2_2}\cdots a_{2\lambda^2_{k_2}}\,\cdots a_{m\lambda^m_1}a_{m\lambda^m_2}\cdots a_{m\lambda^m_{k_m}} \tag{11.2}$$

we have:

$$\begin{aligned}\alpha\cdot\mathbf{1} \, &= \, c^{k_1}_{1\lambda^1_1}c^{k_1-1}_{1\lambda^1_2}\cdots\, c^1_{1\lambda^1_{k_1}}\; c^{k_2}_{2\lambda^2_1}c^{k_2-1}_{2\lambda^2_2}\cdots\, c^1_{2\lambda^2_{k_2}}\;\cdots\; c^{k_m}_{m\lambda^m_1}\,c^{k_m-1}_{m\lambda^m_2}\cdots\, c^1_{m\lambda^m_{k_m}}\\ &= \, d^{k_1}_{1\lambda^1_1}d^{k_1-1}_{1\lambda^1_2}\cdots\, d^1_{1\lambda^1_{k_1}}\; d^{k_2}_{2\lambda^2_1}d^{k_2-1}_{2\lambda^2_2}\cdots\, d^1_{2\lambda^2_{k_2}}\;\cdots\; d^{k_m}_{m\lambda^m_1}d^{k_m-1}_{m\lambda^m_2}\cdots\, d^1_{m\lambda^m_{k_m}},\end{aligned}$$

where the second equality holds because α is a balanced sequence. On the other hand,

$$[\mathbf{z}^{\mathbf{0}}]\prod_{i=1}^m\left(\sum_{j=1}^m b_{ij}z_j/z_i;q_i\right)_{k_i} \, = \, [\mathbf{z}^{\mathbf{k}}]\prod_{i=1}^m\prod_{j=1}^{k_i}(d^j_{i1}+\ldots+d^j_{im})$$

is equal to the sum of

$$[\mathbf{z}^{\mathbf{k}}]\left(d^{k_1}_{1\lambda^1_1}d^{k_1-1}_{1\lambda^1_2}\cdots\, d^1_{1\lambda^1_{k_1}}\,d^{k_2}_{2\lambda^2_1}d^{k_2-1}_{2\lambda^2_2}\cdots\, d^1_{2\lambda^2_{k_2}}\,\cdots\, d^{k_m}_{m\lambda^m_1}d^{k_m-1}_{m\lambda^m_2}\cdots\, d^1_{m\lambda^m_{k_m}}\right)$$

over all o-sequences α of form (11.2). This completes the proof. □

12. Proofs of linear algebra propositions

12.1. **Proof of Proposition 3.2.** The proof imitates the standard linear algebra proof in the commutative case. We start with the following easy result.

Lemma 12.1. *Let* $B = (b_{ij})_{m \times m}$.

(1) *If* B *satisfies* (3.1) *and if* B' *denotes the matrix we get by interchanging adjacent columns of* B*, then* $\det B' = -\det B$.
(2) *If* B *satisfies* (3.1) *and has two columns equal, then* $\det B = 0$.
(3) *If* B^{ij} *denotes the matrix obtained from* B *by deleting the* i*-th row and the* j*-th column, then*

$$\det B = \sum_{i=1}^{m} (-1)^{m+i} (\det B^{im})\, b_{im}.$$

The proof of the lemma is completely straightforward. Now take $B = I - A$ and recall that B is invertible. The j-th coordinate of the matrix product

$$(\det B^{11}, -\det B^{21}, \ldots, (-1)^m B^{m1}) \cdot B$$

is $\sum_{i=1}^{m} (-1)^i \det B^{i1} b_{ij}$. Since B satisfies (3.1), this is equal to $\det B \cdot \delta_{1j}$ by the lemma. But then

$$(\det B^{11}, -\det B^{21}, \ldots, (-1)^m B^{m1}) = \det B \cdot (1, 0, \ldots, 0) \cdot B^{-1}$$

and

$$(B^{-1})_{11} = (\det B)^{-1} \cdot \det B^{11}.$$ □

12.2. **Proof of Proposition 4.2.** Follow the same scheme as in the previous subsection. The following is a well-known result (see e.g. [GLZ, Lemmas 2.3 and 2.4] or [FH2, Properties 5 and 6]).

Lemma 12.2. *Let* $B = (b_{ij})_{m \times m}$.

(1) *If* B *satisfies* (4.2) *and if* B' *denotes the matrix we get by interchanging adjacent columns of* B*, then* $\det B' = -\det B$.
(2) *If* B *satisfies* (4.2) *and has two columns equal, then* $\det B = 0$.
(3) *If* B^{ij} *denotes the matrix obtained from* B *by deleting the* i*-th row and the* j*-th column, then*

$$\det B = \sum_{i=1}^{m} (-1)^{m+i} (\det B^{im})\, b_{im}.$$ □

The rest follows verbatim the previous argument.

12.3. **Proof of Proposition 6.2.** Foata and Han introduced ([FH1, Section 3]) the so-called "$1 = q$ principle" to derive identities in the algebra $\mathcal{A}/\mathcal{I}_{q-\mathrm{rq}}$ from those in the algebra $\mathcal{A}/\mathcal{I}_{\mathrm{rq}}$.

Lemma 12.3. ("$1 = q$ principle") *Let $\phi\colon \mathcal{A} \to \mathcal{A}$ denote the linear map induced by*

$$\phi\left(a_{\lambda,\mu}\right) = q^{\operatorname{inv}\mu - \operatorname{inv}\lambda} a_{\lambda,\mu}.$$

Then:

(a) *ϕ maps $\mathcal{I}_{\mathrm{rq}}$ into $\mathcal{I}_{q-\mathrm{rq}}$*

(b) *Call $a_{\lambda,\mu}$ a* circuit *if λ is a rearrangement of μ (i.e. if λ and μ contain the same letters with the same multiplicities). Then $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for α, β linear combinations of circuits.*

We include the proof of the lemma since we need to generalize it later on.

Proof. (a) It suffices to prove the claim for elements of the form

$$\alpha = a_{\lambda,\mu}(a_{ik}a_{jk} - a_{jk}a_{ik})a_{\lambda',\mu'}$$

and

$$\beta = a_{\lambda,\mu}(a_{ik}a_{jl} - a_{jk}a_{il} - a_{jl}a_{ik} + a_{il}a_{jk})a_{\lambda',\mu'}$$

with $i < j$ (and $k < l$). Note that the sets of inversions of the words $\lambda ij\lambda'$ and $\lambda ij\lambda'$ differ only in the inversion (i, j). Therefore $\phi(\alpha)$ is a multiple of

$$a_{ik}a_{jk} - q^{-1}a_{jk}a_{ik}.$$

For the β the proof is analogous.

(b) It suffices to prove the claim for α, β circuits, i.e. $\alpha = a_{\lambda,\mu}$ with λ a rearrangement of μ and $\beta = a_{\lambda',\mu'}$ with λ' a rearrangement of μ'. The set of inversions of $\lambda\lambda'$ consists of the inversions of λ, the inversions of λ', and the pairs (i, j) where $\lambda_i > \mu_j$. Similarly, the set of inversions of $\mu\mu'$ consists of the inversions of μ, the inversions of μ', and the pairs (i, j) where $\lambda'_i > \mu'_j$. Since λ is a rearrangement of μ and λ' is a rearrangement of μ', $\operatorname{inv}(\mu\mu') - \operatorname{inv}(\lambda\lambda') = (\operatorname{inv}\mu - \operatorname{inv}\lambda) + (\operatorname{inv}\mu' - \operatorname{inv}\lambda')$, which concludes the proof. $\square$

By Proposition 4.2, we have:

$$\det(I - A) \cdot \left((I - A)^{-1}\right)_{11} - \det\left(I - A^{11}\right) \in \mathcal{I}_{\mathrm{rq}}\,.$$

It is clear that

$$\phi(\det(I - A)) = \phi\left(\sum(-1)^{|J|}\det A_J\right) = \sum(-1)^{|J|}\det\nolimits_q A_J = \det\nolimits_q(I - A),$$

where the sums go over all subsets $J \subseteq \{1, \ldots, m\}$. Similarly,

$$\phi\left(\left((I - A)^{-1}\right)_{11}\right) = \left((I - \widetilde{A})^{-1}\right)_{11}.$$

Now the result follows from Lemma 12.3. $\square$

12.4. **Proofs of Propositions 5.2, 7.3 and 8.1.** The result can be derived from Propositions 3.2 and 4.2 by a simple extension of the "$1 = q$ principle".

Lemma 12.4. ("$1 = q_{ij}$ principle") *Call an element $\sum_{i\in\mathcal{J}} c_i a_{\lambda_i,\mu_i}$ of $\mathcal{A}$* balanced *if for any $i, j \in \mathcal{J}$, λ_i is a reshuffle of λ_j and μ_i is a reshuffle of μ_j.*
Let $\phi\colon \mathcal{A} \to \mathcal{A}$ denote the linear map induced by

$$\phi\left(a_{\lambda,\mu}\right) = \left(\prod_{(i,j)\in I(\mu)} q_{\mu_j\mu_i} \prod_{(i,j)\in I(\lambda)} q^{-1}_{\lambda_j\lambda_i}\right) a_{\lambda,\mu}.$$

Choose a set $\mathcal{S}$ with balanced elements, denote by $\mathcal{I}$ the ideal generated by $\mathcal{S}$, and by $\mathcal{I}_{\mathbf{q}}$ the ideal generated by $\phi(\mathcal{S})$. Then

(a) *ϕ maps $\mathcal{I}$ into $\mathcal{I}_{\mathbf{q}}$,*
(b) *$\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for α, β linear combinations of circuits.*

The proof of lemma follows verbatim the proof of Lemma 12.3. Propositions 5.2 and 7.3 follow from Proposition 3.2, and Proposition 8.1 follows from Proposition 4.2. We omit the details. □

13. Final remarks

13.1. A connection between Cartier-Foata free partially-commutative monoids and Koszul duality was established by Kobayashi [K] and can be stated as follows. Let G be a graph on $[n] = \{1, \ldots, n\}$. Consider a quadratic algebra $\mathcal{A}_G$ over $\mathbb{C}$ with variables $x_1, \ldots, x_n$ and relations $x_i x_j = x_j x_i$ for every edge $(i, j) \in G$, $i \neq j$, and $x_i^2 = i$ if there is a loop at i. It was shown by Fröberg in full generality that $\mathcal{A}_G$ is Koszul, and the Koszul dual algebra $\mathcal{A}_G^!$ has a related combinatorial structure (see [Fr]). This generalizes the classical case of a complete graph $G = K_n$, where $\mathcal{A}_G$ is a symmetric and $\mathcal{A}_G^!$ is an exterior algebra. We refer to [PP] for a extensive recent survey on quadratic algebras and Koszul duality.

Now, Kobayashi observed that one can view the Cartier-Foata Möbius inversion theorem for the partially commutative monoid corresponding to a graph G (see [CF]) as a statement about Hilbert series:

$$A_G(t) \cdot A_G^!(t) = 1 \tag{13.1}$$

where $A(t) = \sum_i \dim \mathcal{A}^i t^i$ for a graded algebra $\mathcal{A} = \oplus \mathcal{A}^i$. In effect, Kobayashi gives an explicit construction of the Koszul complex for $\mathcal{A}_G$ by using Cartier-Foata's involution [K].

Most recently, Hai and Lorenz made a related observation, by showing that one can view the Master Theorem as an identity of the same type as (13.1) but for the characters rather than dimensions [HL]. This allowed them to give an algebraic proof of the Garoufalidis-Lê-Zeilberger theorem. In fact, they present a general framework to obtain versions of the Master Theorem for other Koszul algerbras (which are necessarily quadratic) and a (quantum) group acting on it.

13.2. From our presentation, one may assume that the choice of a (q_{ij})-analogue was a lucky guess or a carefully chosen deformation designed to make the technical lemmas work. This was not our motivation, of course. These quadratic algebras are well known generalizations of the classical quantum groups of type A (see [M1, M2, M3]). They were introduced and extensively studied by Manin, who also proved their Koszulity and defined the corresponding (generalized) quantum determinants.

While our proof is combinatorial, we are confident that the Hai-Lorenz approach will work in the (q_{ij})-case as well. While we do not plan to further investigate this connection, we hope the reader find it of interest to pursue that direction.

13.3. For matrices over general rings, the elements of the inverse matrices are called *quasi-determinants* [GeR] (see also [GGRW]). They were introduced by Gelfand and Retakh, who showed that in various special cases these quasi-determinants are the ratios of two (generalized) determinants. In particular, in the context of non-commutative determinants they established Propositions 3.2, 6.2 and a (slightly weaker) corresponding result for the super-analogue.

In a more general context, Etingof and Retakh showed the analogue of this result for all twisted quantum groups [ER]. Although they do not explicitly say so, we believe one can probably deduce our most general Proposition 8.1 from [ER] and the above mentioned Manin's papers. Interestingly, it follows from [ER] and our work that the (non-commutative) determinants of minors considered in this paper always commute with each other. We do not need this observation for our telescoping argument.

Let us mention here that the inverse matrix $(I - A)^{-1}$ appears in the same context as in this paper in the study of quasi-determinants [GGRW] and the non-commutative Lagrange inversion [PPR].

13.4. The relations for variables in our super-analogue are somewhat different from those studied in the literature (see e.g. [M3]). Note also that our super-determinant is different from the *Berezinian* [B] (see also [GGRW, M1]). We are somewhat puzzled by this and hope to obtain the "real" super-analogue in the future.

13.5. The relations studied in this paper always lead to quadratic algebras. While the deep reason lies in the Koszul duality, the fact that Koszulity can be extended to non-quadratic algebras is suggestive [Be]. The first such effort is made in [EP] where an unusual algebraic extension of MacMahon Master Theorem is obtained.

13.6. While we do not state the most general result combining both β-extension and (q_{ij})-analogue, both the statement and the proof follow verbatim the presentation in Section 10. Similarly, the results easily extend to all complex values $\beta \in \mathbb{C}$.

Let us mention here that the original β-extension of the Master Theorem (given in [FZ]) follows easily from the β-extension of the Lagrange inversion [Ze]. In fact, the proof of the latter is bijective.

13.7. In the previous papers [FH1, FH2, FH3, GLZ] the authors used $Boz(\cdot)$ and $Fer(\cdot)$ notation for the left- and the right-hand side of (1.3). While the implied connection is not unjustified, it might be misleading when the results are generalized.

Indeed, in view of Koszul duality connection (see Subsection 13.1 above) the algebras can be interchanged, while giving the same result with notions of *Bozon* and *Fermion* summations switched. On the other hand, we should point out that in the most interesting cases the *Fermion* summation is finite, which makes it special from combinatorial point of view.

13.8. The Krattenthaler-Schlosser's q-analogue (Theorem 11.1) is essentially a byproduct of the author's work on q-series. It was pointed out to us by Michael Schlosser that the Cartier-Foata matrices routinely appear in the context of "matrix inversions" for q-series (see [KS, Sc]). It would be interesting to see if our extensions (such as Cartier-Foata q_{ij}-case in Section 7) can can be used to obtain new results, or give new proofs of existing results.

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