THE WEIGHTED HOOK LENGTH FORMULA III: SHIFTED TABLEAUX

MATJAŽ KONVALINKA*

ABSTRACT. Recently, a simple proof of the hook length formula was given via the branching rule. In this paper, we extend the results to shifted tableaux. We give a bijective proof of the branching rule for the hook lengths for shifted tableaux; present variants of this rule, including weighted versions; and make the first tentative steps toward a bijective proof of the hook length formula for d-complete posets.

1. INTRODUCTION

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$, be a partition of n, $\lambda \vdash n$, and let $[\lambda] = \{(i,j) \in \mathbb{Z}^2 \colon 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$ be the corresponding *Young diagram*. The *conjugate partition* λ' is defined by $\lambda'_j = \max\{i : \lambda_i \geq j\}$. The *hook* $H_{\mathbf{z}} = H_{ij} \subseteq [\lambda]$ is the set of squares weakly to the right and below of $\mathbf{z} = (i,j) \in [\lambda]$, and the *hook length* $h_{\mathbf{z}} = h_{ij} = |H_{\mathbf{z}}| = \lambda_i + \lambda'_j - i - j + 1$ is the size of the hook. The *punctured hook* $\overline{H}_{\mathbf{z}} \subseteq [\lambda]$ is the set $H_{\mathbf{z}} \setminus \{\mathbf{z}\}$. See Figure 1, left and center drawing.

We write $[n]$ for $\{1, \ldots, n\}$, $\mathcal{P}(A)$ for the power set of A, and $A \sqcup B$ for the disjoint union of (not necessarily disjoint) sets A and B. Furthermore, for squares (i,j) and (i',j') of $[\lambda]$, write $(i,j) \geq (i',j')$ if $i \leq i'$ and $j \leq j'$.

A *standard Young tableau* of shape λ is a bijective map $f : [\lambda] \to [n]$, such that $f(\mathbf{z}) < f(\mathbf{z}')$ whenever $\mathbf{z} \geq \mathbf{z}'$ and $\mathbf{z} \neq \mathbf{z}'$. See Figure 1, right drawing. We denote the number of standard Young tableaux of shape λ by f_λ. The remarkable hook length formula states that if λ is a partition of n, then

$$f_\lambda = \frac{n!}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}}.$$

For example, for $\lambda = (3,2,2) \vdash 7$, the hook length formula gives

$$f_{322} = \frac{7!}{5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 21.$$

This gives a short formula for dimensions of irreducible representations of the symmetric group, and is a fundamental result in algebraic combinatorics. The formula was discovered by Frame, Robinson and Thrall in [FRT54] based on earlier resultsof Young, Frobenius and Thrall. Since then, it has been reproved, generalized and extended in several different ways, and applications have been found in a number of fields ranging from algebraic geometry to probability, and from group theory to the analysis of algorithms.

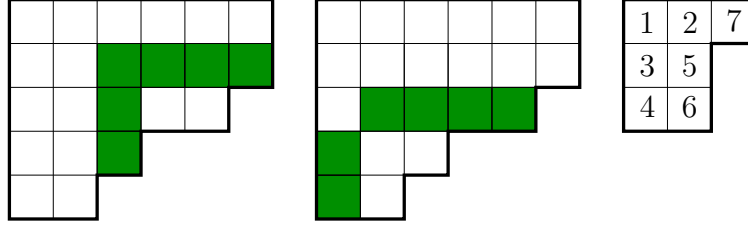*Department of Mathematics, Vanderbilt University, Nashville; matjaz.konvalinka@vanderbilt.edu.

FIGURE 1. Young diagram $[\lambda]$, $\lambda = 66532$, hook H_{23} with hook length $h_{23} = 6$ (left), punctured hook $\overline{H}_{31}$ (center); a standard Young tableau of shape 322 (right).

One way to prove the hook length formula is by induction on n. Namely, it is obvious that in a standard Young tableau, n must be in one of the *corners*, squares (i, j) of $[\lambda]$ satisfying $(i + 1, j), (i, j + 1) \notin [\lambda]$. Therefore

$$f_\lambda = \sum_{\mathbf{c} \in \mathcal{C}[\lambda]} f_{\lambda - \mathbf{c}},$$

where $\mathcal{C}[\lambda]$ is the set of all corners of λ, and $\lambda - \mathbf{c}$ is the partition whose diagram is $[\lambda] \setminus \{\mathbf{c}\}$.

That means that in order to prove the hook length formula, we have to prove that $F_\lambda = n! / \prod h_{\mathbf{z}}$ satisfy the same recursion. It is easy to see that this is equivalent to the following *branching rule for the hook lengths*:

$$(1) \qquad n \cdot \prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda]}} (h_{ij} - 1) = \sum_{(r,s) \in \mathcal{C}[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda] \\ i \neq r, j \neq s}} (h_{ij} - 1) \right] \prod_{i=1}^{r-1} h_{is} \prod_{j=1}^{s-1} h_{rj}.$$

In [CKP], a weighted generalization of this formula was presented, with a simple bijective proof.

Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell > 0$, is a partition of n *with distinct parts*, and let $[\lambda]^* = \{(i, j) \in \mathbb{Z}^2 \colon 1 \leq i \leq \ell, i \leq j \leq \lambda_i + i - 1\}$ be the corresponding *shifted Young diagram*. The *hook* $H_{\mathbf{z}}^* = H_{ij}^* \subseteq [\lambda]^*$ is the set of squares weakly to the right and below of $\mathbf{z} = (i, j) \in [\lambda]$, and in row $j + 1$, and the *hook length* $h_{\mathbf{z}}^* = h_{ij}^* = |H_{\mathbf{z}}^*|$ is the size of the hook. It is left as an exercise for the reader to check that $h_{ij}^* = \lambda_i + \lambda_{j+1}$ if $j < \ell(\lambda)$ and $h_{ij}^* = \lambda_i + \max\{k \colon \lambda_k \geq j + 1 - k \geq 1\} - j$ if $j \geq \ell(\lambda)$. The *punctured hook* $\overline{H}_{\mathbf{z}}^* \subseteq [\lambda]^*$ is the set $H_{\mathbf{z}}^* \setminus \{\mathbf{z}\}$. See Figure 2, left and center drawing. For squares (i, j) and (i', j') of $[\lambda]^*$, write $(i, j) \geq (i', j')$ if $i \leq i'$ and $j \leq j'$.

A *standard shifted Young tableau* of shape λ is a bijective map $f : [\lambda]^* \to [n]$, such that $f(\mathbf{z}) < f(\mathbf{z}')$ whenever $\mathbf{z} \geq \mathbf{z}'$ and $\mathbf{z} \neq \mathbf{z}'$. See Figure 2, right drawing. We denote the number of standard shifted Young tableaux of shape λ by f_λ^*. The shifted hook length formula states that if λ is a partition of n with distinct parts, then

$$f_\lambda^* = \frac{n!}{\prod_{\mathbf{z} \in [\lambda]} h_{\mathbf{z}}^*}.$$

For example, for $\lambda = (5, 3, 2) \vdash 10$, the hook length formula gives

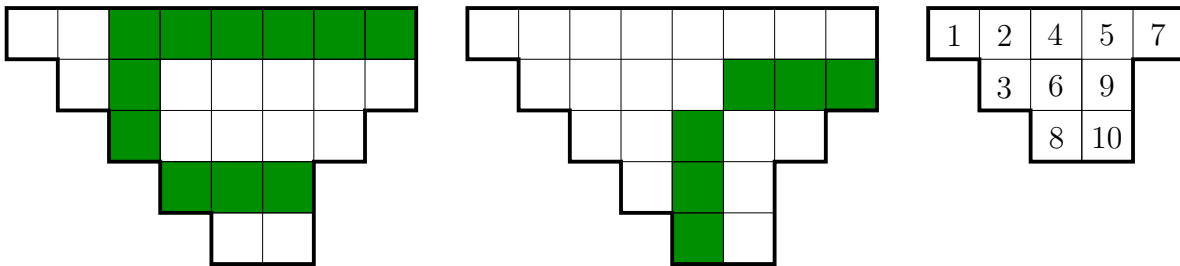$$f^*_{532} = \frac{10!}{8 \cdot 7 \cdot 5 \cdot 4 \cdot 1 \cdot 5 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 54.$$

FIGURE 2. Shifted Young diagram $[\lambda]$, $\lambda = 87532$, a hook H^*_{13} with hook length $h^*_{13} = 11$ (left), a punctured hook $\overline{H}^*_{25}$ (center); a standard shifted Young tableau of shape 532 (right).

Again, one way to prove the shifted hook length formula is by induction on n. Namely, it is obvious that in a standard shifted Young tableau, n must be in one of the *shifted corners*, squares (i, j) of $[\lambda]^*$ satisfying $(i+1, j), (i, j+1) \notin [\lambda]^*$. Therefore

$$f^*_\lambda = \sum_{\mathbf{c} \in \mathcal{C}^*[\lambda]} f^*_{\lambda - \mathbf{c}},$$

where $\mathcal{C}^*[\lambda]$ is the set of all shifted corners of λ, and $\lambda - \mathbf{c}$ is the partition whose shifted diagram is $[\lambda]^* \setminus \{\mathbf{c}\}$.

That means that in order to prove the shifted hook length formula, we have to prove that $F^*_\lambda = n! / \prod h^*_{\mathbf{z}}$ satisfy the same recursion. It is easy to see that this is equivalent to the following *shifted branching rule for the hook lengths*:

$$(2) \quad n \cdot \prod_{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda]} (h^*_{ij} - 1) = \sum_{(r,s) \in \mathcal{C}^*[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda] \\ i \neq r, j \neq r-1, s}} (h^*_{ij} - 1) \right] \prod_{i=1}^{r-1} h^*_{is} \prod_{j=1}^{s-1} h^*_{rj} \prod_{i=1}^{r-1} h^*_{i, r-1}.$$

Interestingly, the shifted hook length formula was discovered before the non-shifted one [Thr52]. Shortly after the famous Greene-Nijenhuis-Wilf's probabilistic proof of the ordinary hook length formula [GNW79], Sagan [Sag79] extended the argument to the shifted case. The proof, however, is rather technical. In 1995, Krattenthaler [Kra95] provided a bijective proof. While short, it is very involved, as it needs a variant of Hillman-Grassl algorithm, a bijection that comes from Stanley's (P, ω)-partition theory, and the involution principle of Garsia and Milne. A few years later, Fischer [Fis] gave the most direct proof of the formula, in the spirit of Novelli-Pak-Stoyanovskii's bijective proof of the ordinary hook-length formula. At almost 50 pages in length, the proof is very involved. Bandlow [Ban08] gave a short proof via interpolation.

The main result of this paper is a bijective proof of the branching formula for the hook lengths for shifted tableaux, which is trivially equivalent to the hook length formula. While

the proof still cannot be described as simple, it seems that it is the most intuitive of the known proofs. Once the proof in the non-shifted case is understood in a proper context, the bijection for shifted tableaux has only two (main) extra features: a description of a certain map s, described by (S1)–(S5) on page 13, and the process of "flipping the snake" of Lemma 1 with a half-page proof.

This paper is organized as follows. Section 2 presents the non-weighted bijection from [CKP] in a new way, to make it easier to adapt to the shifted case. Section 3 discusses the basic features of the bijective proof of (2), and Section 4 gives all the requisite details. Sections 3 and 4 thus give the first completely bijective proof of the shifted branching rule. In Section 5, we discuss the above-mentioned paper by Sagan [Sag79]. Section 6 gives variants of the formulas in the spirit of [CKP, §2], including weighted formulas. In Section 7, we make the first step to extending the bijective proof to d-complete posets. We conclude with final remarks in Section 8.

2. BIJECTION IN THE NON-SHIFTED CASE

In this section, we essentially describe the bijection that was used in [CKP] to prove the weighted branching rule for hook lengths (see also [Zei84]). The reader is advised to read Subsection 2.2 of that paper first, however, as the description that follows is much more abstract. This way, it is more easily adaptable to the shifted case.

Let us give an interpretation for each side of equality (1). The left-hand side counts all pairs $(\mathbf{s}, F)$, where $\mathbf{s} \in [\lambda]$ and F is a map from $[\lambda] \setminus \mathcal{C}[\lambda]$ to $[\lambda]$ so that $F(\mathbf{z})$ is in the punctured hook of $\mathbf{z}$ for every $\mathbf{z} \in [\lambda] \setminus \mathcal{C}[\lambda]$. We also write $F_{\mathbf{z}} = F_{i,j}$ for $F(\mathbf{z})$ when $\mathbf{z} = (i, j)$. We think of the map F as an arrangement of labels; we label the square (i, j) by $F_{i,j} = (k, l)$ or sometimes just kl. The left drawing in Figure 3 shows an example of such a pair $(\mathbf{s}, F)$; the square $\mathbf{s}$ is drawn in green. Denote the set of all such pairs by $\mathcal{F}$.

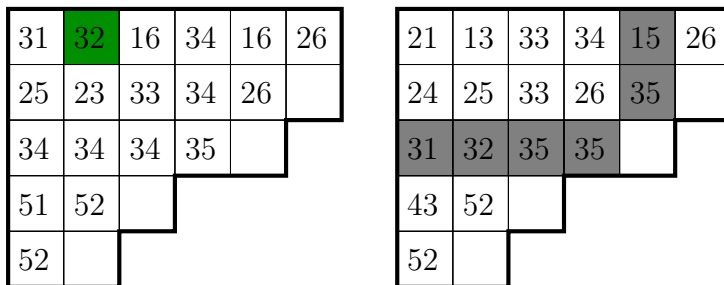

FIGURE 3. A representative of the left-hand side and the right-hand side of (1) for $\lambda = 66532$.

Similarly, the right-hand side of (1) counts the number of pairs $(\mathbf{c}, G)$, where $\mathbf{c} \in \mathcal{C}[\lambda]$ and G is a map from $[\lambda] \setminus \mathcal{C}[\lambda]$ to $[\lambda]$ satisfying the following:

- if $\mathbf{c}$ lies in the hook of $\mathbf{z}$, then $G(\mathbf{z})$ lies in the hook of $\mathbf{z}$;
- otherwise, $G(\mathbf{z})$ lies in the punctured hook of $\mathbf{z}$.

Note that if $\mathbf{c} = (r, s)$, then $\mathbf{c}$ lies in the hook of $\mathbf{z} = (i, j)$ if and only if $i = r$ or $j = s$. We think of the map G as an arrangement of labels. The right drawing in Figure 3 shows an example of such a pair $(\mathbf{c}, G)$; the row r and the column s without the corner (r, s) are shaded. Denote the set of all G such that $(\mathbf{c}, G)$ satisfies the above properties by $\mathcal{G}_{\mathbf{c}}$, and the disjoint union $\bigsqcup_{\mathbf{c}} \mathcal{G}_{\mathbf{c}}$ by $\mathcal{G}$. We also write $G_{\mathbf{z}} = G_{i,j}$ for $G(\mathbf{z})$ when $\mathbf{z} = (i, j)$.

Some squares $\mathbf{z}$ in the shaded row and column may satisfy $G_{\mathbf{z}} = \mathbf{z}$. Let us define sets A, B as follows:

$$A = A(G) = \{i \colon G_{i,s} = (i, s)\}$$
$$B = B(G) = \{j \colon G_{r,j} = (r, j)\}$$

We think of the set A as representing column s without the corner (r, s), and of B as representing row r without the corner (r, s), with a dot in all squares satisfying $G_{\mathbf{z}} = \mathbf{z}$. Figure 4 shows the shaded row and column for the example from the right-hand side of Figure 3.
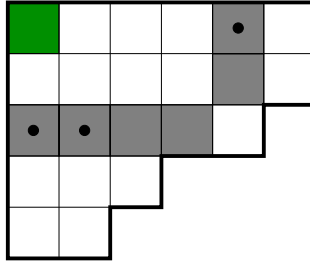

FIGURE 4. Shaded row and column and sets A and B for the example on the right-hand side of Figure 3, and $s(A, B)$.

Furthermore, denote by $\mathcal{G}_{\mathbf{c}}^{k}$ the set of all $G \in \mathcal{G}_{\mathbf{c}}$ for which exactly k of the squares in $[\lambda] \setminus \mathcal{C}[\lambda]$ satisfy $G_{\mathbf{z}} = \mathbf{z}$ (note that such $\mathbf{z}$ must be in the shaded row or column); in other words, $|A| + |B| = k$.

We construct two maps, $s = s_{\mathbf{c}}$ and $e = e_{\mathbf{c}}$. The map $s \colon \mathcal{P}([r - 1]) \times \mathcal{P}([s - 1]) \to [\lambda]$ ("starting square") is defined by

$$s(A, B) = (\min(A \cup \{r\}), \min(B \cup \{s\})).$$

We can extend it to a function $s \colon \mathcal{G}_{\mathbf{c}} \to [\lambda]$ by $s(G) = s(A(G), B(G))$. See Figure 4; there, $s(A, B)$ is green.

Now let us define the map $e(G) \colon \mathcal{G}_{\mathbf{c}} \to \mathcal{G}_{\mathbf{c}}$ ("erasing a dot"). If $G \in \mathcal{G}_{\mathbf{c}}^{0}$, then $e(G) = G$. Otherwise, the arrangement $e(G)$ differs from G in only one or two squares. The first case is when $s(G)$ is a shaded square; this happens if $A(G) = \emptyset$ or $B(G) = \emptyset$. If $A = A(G) = \emptyset$ and $B = B(G) = \{j, \ldots\}$, then $s(G) = (r, j)$; define $A' = A = \emptyset$, $B' = B \setminus \{j\}$ and $e(G)_{r,j} = s(A', B')$. If $A = A(G) = \{i, \ldots\}$ and $B = B(G) = \emptyset$, then $s(G) = (i, s)$; define $A' = A \setminus \{i\}$, $B' = B = \emptyset$ and $e(G)_{i,s} = s(A', B')$. In each case, we have $A(e(G)) = A'$, $B(e(G)) = B'$. See Figure 7.

The second case is when $s(G) = (i, j)$ for $i < r$ and $j < s$; this happens when $A = A(G)$ and $B = B(G)$ are both non-empty. The crucial observation is that the punctured hook of (i, j), which consists of squares (i, k) for $j < k \leq \lambda_i$ and (k, j) for $i < k \leq \lambda'_j$, is in a natural correspondence with $\overline{H}_{is} \sqcup \overline{H}_{rj}$. Indeed, squares of the form (i, k) for $s < k \leq \lambda_i$ are also in the punctured hook of (i, s), and squares of the form (k, j) for $i < k \leq r$ are in a natural correspondence with squares of the form (k, s) for $i < k \leq r$, which are in the punctured hook of (i, s). On the other hand, squares of the form (i, k) for $j < k \leq s$ are in a natural correspondence with squares of the form (r, k) for $j < k \leq s$, which are in the punctured hook of (r, j), and squares of the form (k, j) for $r < k \leq \lambda'_j$ are also in the punctured hook of (r, j). See Figure 5.
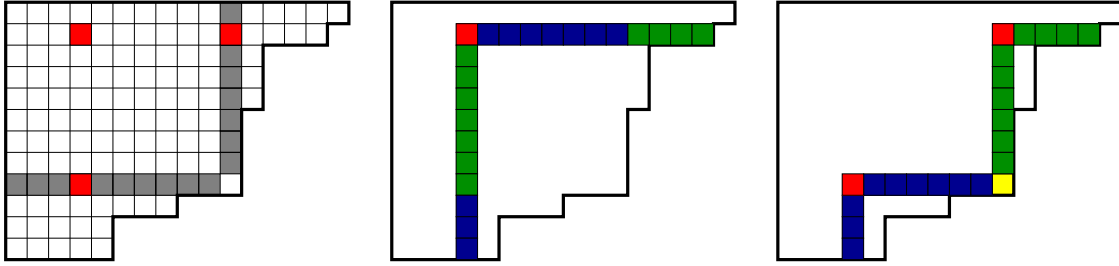

FIGURE 5. The punctured hook of a square is the disjoint union of the punctured hooks of projections onto shaded row and column. The yellow square is in the punctured hook of both projections.

That means that by slight abuse of notation, we can say that $G_{i,j} \in \overline{H}_{is} \sqcup \overline{H}_{rj}$. If $G_{i,j} \in \overline{H}_{is}$, define $e(G)_{i,s} = G_{i,j}$ and $e(G)_{i,j} = s(A', B')$, where $A' = A \setminus \{i\}$ and $B' = B$. If $G_{i,j} \in \overline{H}_{rj}$, define $e(G)_{r,j} = G_{i,j}$ and $e(G)_{i,j} = s(A', B')$, where $A' = A$ and $B' = B \setminus \{j\}$. In each case, we have $A(e(G)) = A'$, $B(e(G)) = B'$. See Figure 6.
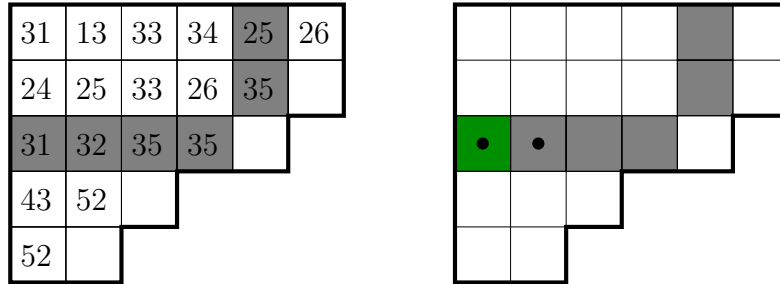

FIGURE 6. Arrangement $G' = e(G)$ and corresponding A', B', $s(A', B')$ for G from Figure 3.

We claim that the maps s and e have the following three properties:

(P1) if $G \in \mathcal{G}^0_{\mathbf{c}}$, then $s(G) = \mathbf{c}$ and $e(G) = G$, and if $G \in \mathcal{G}^k_{\mathbf{c}}$ for $k \geq 1$, then $e(G) \in \mathcal{G}^{k-1}_{\mathbf{c}}$;

(P2) if $G \in \mathcal{G}^k_{\mathbf{c}}$ for $k \geq 1$, then $e(G)_{s(G)} = s(e(G))$; furthermore, if $\mathbf{z} \not\leq s(G)$, then $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$;
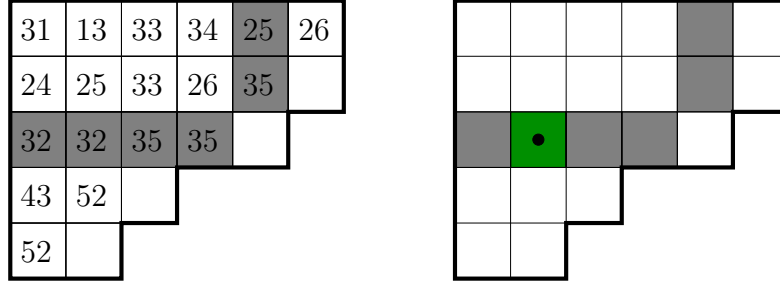
FIGURE 7. Arrangement $G'' = e(G')$ and corresponding A', B', $s(A', B')$ for G' from Figure 6.

(P3) given $G' \in \mathcal{G}_{\mathbf{c}}^k$ and $\mathbf{z}$ for which $G'_{\mathbf{z}} = s(G')$, there is exactly one $G \in \mathcal{G}_{\mathbf{c}}^{k+1}$ satisfying $s(G) = \mathbf{z}$ and $e(G) = G'$.

Indeed, the first statement of (P1) is clear from the definition of s and the second statement is part of the definition of e. The third statement of (P1) is immediately obvious by inspection of all cases: we always leave one of the sets A, B intact, and remove one element from the other. Part (P2) also follows by construction. For example, for $A = \{i, i', \ldots\}$, $B = \{j, \ldots\}$, $G_{ij} \in \overline{H}_{is}$, $A(e(G)) = \{i', \ldots\} = A'$, $B(e(G)) = \{j, \ldots\} = B'$ and $s(e(G)) = (i', j)$. By definition, $e(G)_{s(G)} = e(G)_{ij} = (i', j) = s(e(G))$. Also, $e(G)$ differs from G only in (i, j) and (i, s), and so $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ if $\mathbf{z} \not\leq s(G)$. Other cases are very similar. That leaves only (P3). Assume that $A(G') = \{i', \ldots\}$, $B(G') = \{j', \ldots\}$, $s(G') = (i', j')$, $\mathbf{z} = (i, j')$ for $i < i'$, $G'_{i,j'} = s(G') = (i', j')$. It is easy to see that G defined by $G_{i,j'} = G'_{i,s}$, $G_{i,s} = (i, s)$, $G_{\mathbf{z}} = G'_{\mathbf{z}}$ for $\mathbf{z} \neq (i, j'), (i, s)$ satisfies $s(G) = \mathbf{z}$ and $e(G) = G'$, and that this is the only G with these properties. Other cases are similar.

Observe that by (P1), the sequence $G, e(G), e^2(G), \ldots$ eventually becomes constant, say $\phi(G)$, for every G. More specifically, if $G \in \mathcal{G}_{\mathbf{c}}^k$, then $\phi(G) = e^k(G)$. Let us first prove a simple consequences of (P2). Take $G \in \mathcal{G}_{\mathbf{c}}^k$ for $k \geq 1$. Since $s(e(G))$ is in the punctured hook of $s(G)$, we have $s(G) \not\leq s(e(G))$ and by the last statement in (P2), $e^2(G)_{s(G)} = e(G)_{s(G)} = s(e(G))$. Similarly, $s(G) \not\leq s(e^2(G))$, and so $e^3(G)_{s(G)} = e^2(G)_{s(G)} = s(e(G))$. By induction, we have

$$\phi(G)_{s(G)} = s(e(G)). \tag{3}$$

We claim that $\Phi \colon G \mapsto (s(G), \phi(G))$ is a bijection between $\mathcal{G}$ and $\mathcal{F}$. To see that, note first that an element $(\mathbf{z}, F)$ of $\mathcal{F}$ naturally gives a "hook walk" $\mathbf{z} \to F(\mathbf{z}) \to F^2(\mathbf{z}) \to F^3(\mathbf{z}) \to \ldots \to \mathbf{c}$ for some $\mathbf{c} \in \mathcal{C}[\lambda]$. For $\mathbf{c} \in \mathcal{C}[\lambda]$ and $k \geq 0$, denote by $\mathcal{F}_{\mathbf{c}}^k$ the set of all $(\mathbf{z}, F)$ for which $F^k(\mathbf{z}) = \mathbf{c}$. We prove by induction on k that given $(\mathbf{z}, F) \in \mathcal{F}_{\mathbf{c}}^k$, $\Phi(G) = (\mathbf{z}, F)$ for exactly one $G \in \mathcal{G}$, and that this G is an element of $\mathcal{G}_{\mathbf{c}}^k$.

If $(\mathbf{z}, F) \in \mathcal{F}_{\mathbf{c}}^0$ (i.e., if $\mathbf{z} = \mathbf{c}$), then $\Phi(G) = (\mathbf{c}, F)$ if and only if $G = F \in \mathcal{G}_{\mathbf{c}}^0$. Now assume we have proved the statement for $0, \ldots, k$, and take $(\mathbf{z}, F) \in \mathcal{F}_{\mathbf{c}}^{k+1}$. Because $(F(\mathbf{z}), F) \in \mathcal{F}_{\mathbf{c}}^k$, we have $(F(\mathbf{z}), F) = \Phi(G') = (s(G'), \phi(G')) = (s(G'), e^k(G'))$ for (exactly one) $G' \in \mathcal{G}$, and we have $G' \in \mathcal{G}_{\mathbf{c}}^k$. Since $G'_{\mathbf{z}} = e(G')_{\mathbf{z}} = e^2(G')_{\mathbf{z}} = \ldots = F_{\mathbf{z}} = s(G')$ by an argument similar to the proof of (3), there is (exactly one) $G \in \mathcal{G}_{\mathbf{c}}^{k+1}$ satisfying $s(G) = \mathbf{z}$

and $e(G) = G'$, by (P3). Obviously, $\Phi(G) = (s(G), \phi(G)) = (\mathbf{z}, \phi(e(G'))) = (\mathbf{z}, \phi(G')) = (\mathbf{z}, F)$. If $\Phi(G'') = (\mathbf{z}, F)$ for some $G'' \in \mathcal{G}$, then $s(G'') = \mathbf{z}$ and $\phi(G'') = F$. We have $\Phi(e(G'')) = (s(e(G'')), \phi(e(G''))) = (s(e(G'')), \phi(G'')) = (\phi(G'')_{s(G'')}, \phi(G''))$ by equation (3), and hence $\Phi(e(G'')) = (F(\mathbf{z}), F) = \Phi(G')$. By induction, $e(G'') = G'$. Since both $s(G'') = s(G) = \mathbf{z}$ and $e(G'') = e(G) = G'$, we have $G'' = G$ by (P3).

The message of this section is the following. The crux of the bijective proof of the branching formula is to find the maps s (start) and e (erase). Properties (P1), (P2) and (P3) are easy corollaries of the construction, and we can then prove the bijectivity of the map $\mathcal{G} \to \mathcal{F}$, $G \mapsto (s(G), e(e(\cdots(e(G))\cdots)))$, via a completely abstract proof. The branching formula for the hook lengths, equation (1), follows because the right-hand side enumerates $\mathcal{G}$, and the left-hand side enumerates $\mathcal{F}$.

3. Basic features of the bijection

The descriptions of the left-hand and right-hand sides of equation (2) are very similar. The left-hand side counts the number of pairs $(\mathbf{s}, F)$, where $\mathbf{s} \in [\lambda]^*$ and F is a map from $[\lambda]^* \setminus \mathcal{C}^*[\lambda]$ to $[\lambda]^*$ so that $F(\mathbf{z})$ is in the punctured hook of $\mathbf{z}$ for every $\mathbf{z} \in [\lambda]^* \setminus \mathcal{C}^*[\lambda]$. We think of the map F as an arrangement of labels. The left drawing in Figure 8 shows an example of such a pair $(\mathbf{s}, F)$; the square $\mathbf{s}$ is green. Denote the set of all such pairs by $\mathcal{F}$. We also write $F_{\mathbf{z}} = F_{i,j}$ for $F(\mathbf{z})$ when $\mathbf{z} = (i, j)$.
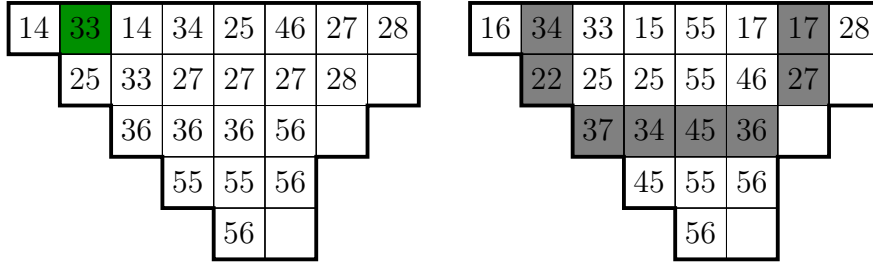

FIGURE 8. A representative of the left-hand side and the right-hand side of (2) for $\lambda = 87532$.

Similarly, the left-hand side of (2) counts the number of pairs $(\mathbf{c}, G)$, where $\mathbf{c} \in \mathcal{C}^*[\lambda]$ and G is a map from $[\lambda]^* \setminus \mathcal{C}^*[\lambda]$ to $[\lambda]^*$ satisfying the following:

- if $\mathbf{c}$ lies in the hook of $\mathbf{z}$, then $G(\mathbf{z})$ lies in the hook of $\mathbf{z}$;
- otherwise, $G(\mathbf{z})$ lies in the punctured hook of $\mathbf{z}$.

Note that if $\mathbf{c} = (r, s)$, then $\mathbf{c}$ lies in the hook of $\mathbf{z} = (i, j)$ if and only if $i = r$ or $j = r - 1$ or $j = s$. The right drawing in Figure 8 shows an example of such a pair $(\mathbf{c}, G)$; the row r without the corners (r, s) and the columns $r - 1$ and s without the corner (r, s) are shaded. Denote the set of all G such that $(\mathbf{c}, G)$ satisfies the above properties by $\mathcal{G}_{\mathbf{c}}$, and the disjoint union $\bigsqcup_{\mathbf{c}} \mathcal{G}_{\mathbf{c}}$ by $\mathcal{G}$. We also write $G_{\mathbf{z}} = G_{i,j}$ for $G(\mathbf{z})$ when $\mathbf{z} = (i, j)$. We think of the map G as an arrangement of labels.

Some squares $\mathbf{z}$ in shaded row and columns may satisfy $G_{\mathbf{z}} = \mathbf{z}$. Let us define sets A, B, C as follows:

$$A = A(G) = \{i \colon G_{i,s} = (i,s)\}$$
$$B = B(G) = \{j \colon G_{r,j} = (r,j)\}$$
$$C = C(G) = \{i \colon G_{i,r-1} = (i, r-1)\}$$

We think of sets A, B and C as representing column s without (r,s), row r without (r,s), and column $r-1$ respectively, with a dot in all squares satisfying $G_{\mathbf{z}} = \mathbf{z}$. Figure 9 shows shaded row and columns for the example from the right-hand side of Figure 8. The square $s(A, B, C)$, which will be defined in the next section, is green.
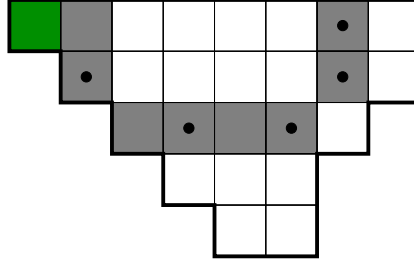

FIGURE 9. Shaded row and columns and sets A, B and C for the example on the right-hand side of Figure 8, and $s(A, B, C)$.

Furthermore, denote by $\mathcal{G}_{\mathbf{c}}^k$ the set of all $G \in \mathcal{G}_{\mathbf{c}}$ for which exactly k of the squares in $[\lambda]^* \setminus \mathcal{C}^*[\lambda]$ satisfy $G_{\mathbf{z}} = \mathbf{z}$ (note that such $\mathbf{z}$ must be in shaded row and columns); in other words, $|A| + |B| + |C| = k$.

Before we continue, let us introduce some terminology relating to the shaded columns (the shaded row will not be relevant for these purposes). If C is empty, we say that the shaded columns form a *right stick*. If A is empty and C is non-empty, we say that the shaded columns form a *left stick*. If $A = C$ and $|A| = |C| \geq 1$, we say that the shaded columns form a *block*. If A and C are non-empty and $\min A = \max C$, we say that the shaded columns form a *right snake*. If A and C are non-empty and $\max A = \min C$, we say that the shaded columns form a *left snake*. The origin of these names should be clear from examples in Figure 10. Formations such as snake-dot, stick-dot, snake-block, snake-block-dot should be self-explanatory; see Figure 10 for examples. We will often say that the shaded columns *start* with a certain formation, for example, that the shaded columns start with a snake-dot. This means that if we erase dots in rows from some row i onward, the resulting columns have the specified form. See Figure 10 for examples.

Like in the non-shifted case, we will construct two maps, $s = s_{\mathbf{c}}$ and $e = e_{\mathbf{c}}$. The map $s \colon \mathcal{P}([r-1]) \times \mathcal{P}([r, \ldots, s-1]) \times \mathcal{P}([r-1]) \to [\lambda]$ ("starting square"), which we extend to $s \colon \mathcal{G}_{\mathbf{c}} \to [\lambda]$ by $s(G) = s(A(G), B(G), C(G))$, and $e \colon \mathcal{G}_{\mathbf{c}} \to \mathcal{G}_{\mathbf{c}}$ ("erase a dot"). Again, they will have the properties (P1), (P2) and (P3) from Section 2. By the same argument as in the previous section, this implies that $G \mapsto (s(G), \phi(G))$, $\phi(G) = e(e(\cdots (e(G)) \cdots))$, is a bijection from $\mathcal{G}$ to $\mathcal{F}$, and proves (2).
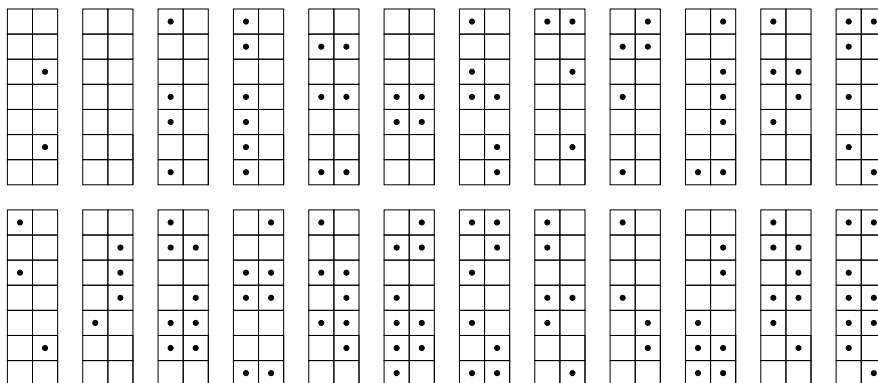
FIGURE 10. In this figure, we have two examples of each of the follow-
ing: the shaded columns form a right stick, left stick, block, right snake,
left snake, snake-dot (top row), stick-dot, snake-block, snake-block-dot; the
shaded columns start with a snake-dot, stick-dot, snake-block-dot (bottom
row).

While this all seems almost exactly the same as in the non-shifted case, there are a few
important differences. First, the map s is fairly complicated; we have to split the possible
configurations of the sets A, B and C into five separate cases, with the rule for finding the
starting column fairly unintuitive. Second, in certain cases it is not immediately obvious
which of the dots in the shaded columns should be erased. Note that a square (i, j) with
$j < r - 1$ has two "projections" in column $r - 1$ and two in column s, see Figure 11.
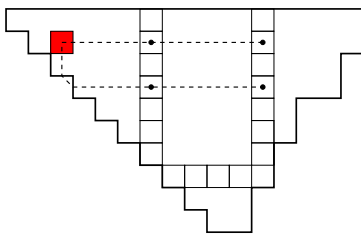

FIGURE 11. Two projections of (i, j) in column $r - 1$, $(i, r - 1)$ and $(j +
1, r - 1)$, and two projections in columns s, (i, s) and $(j + 1, s)$.

On a related note, while the equality $h_{i,j} - 1 = (h_{i,s} - 1) + (h_{r,j} - 1)$ translates nicely into
equalities $h_{i,j}^* - 1 = (h_{i,s}^* - 1) + (h_{j+1,r-1}^* - 1) = (h_{i,r-1}^* - 1) + (h_{j+1,s}^* - 1)$, it is far less
obvious how to decompose the punctured hook of (i, j) into punctured hooks of (i, s) and
$(j + 1, r - 1)$ (or $(i, r - 1)$ and $(j + 1, s)$). And the final, and perhaps the most intriguing,
difference is that in most cases when the shaded columns start with a snake, the map e
does not just "erase the dot", but also "flips the snake". See Figure 12 for some possible
effects of e.

These issues are addressed in the next section. But first, we describe how to change the
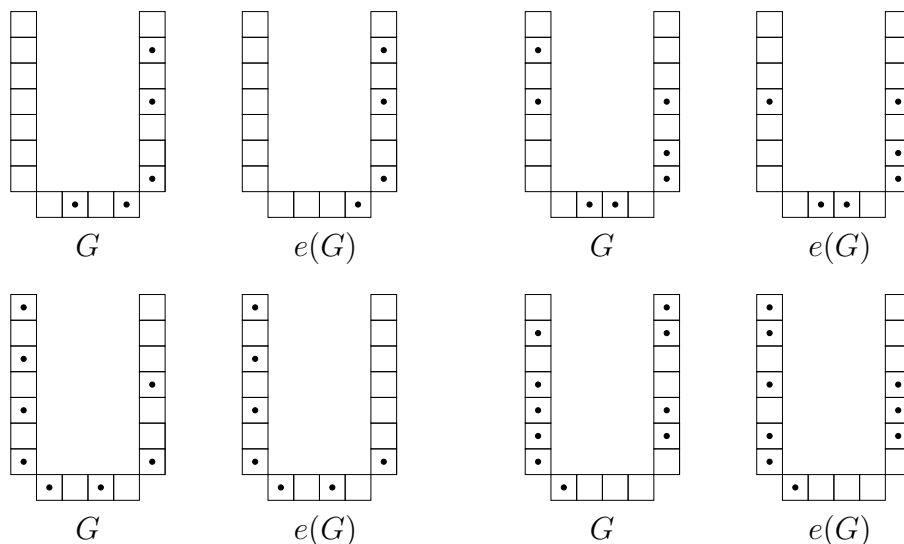direction of a snake.

FIGURE 12. Four possible pairs of $A(G)$, $B(G)$, $C(G)$ and $A(e(G))$, $B(e(G))$, $C(e(G))$. Note that in the last example, the right snake becomes a left snake.

We need some new notation. Pick a set $I = \{i_1, i_2, \ldots, i_m\} \subseteq [r-1]$. For $1 \leq k \leq m$, denote by $\mathcal{L}_{I.k}$ the set of all arrangements $G \in \mathcal{G}$ for which $i_1, \ldots, i_k \in A(G)$, $i_{k+1}, \ldots, i_m \notin A(G)$, $i_1, \ldots, i_{k-1} \notin C(G)$, $i_k, \ldots, i_m \in C(G)$, and by $\mathcal{L}_I$ the set $\bigcup_k L_{I,k}$. For $1 \leq k \leq m$, denote by $\mathcal{R}_{I.k}$ the set of all arrangements $G \in \mathcal{G}$ for which $i_1, \ldots, i_k \in C(G)$, $i_{k+1}, \ldots, i_m \notin C(G)$, $i_1, \ldots, i_{k-1} \notin A(G)$, $i_k, \ldots, i_m \in A(G)$, and by $\mathcal{R}_I$ the set $\bigcup_k R_{I,k}$.

The set $\mathcal{L}_I$ (respectively, $\mathcal{R}_I$) contains all arrangements so that the dots in rows I form a left snake (respectively, right snake). Figure 13 shows two examples.
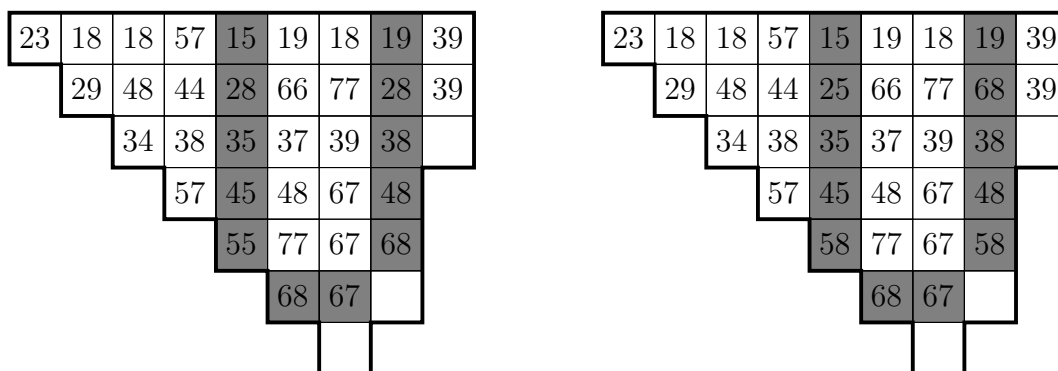

FIGURE 13. An element of $\mathcal{L}_I$ and $\mathcal{R}_I$ for $\lambda = 9875431$, $\mathbf{c} = (6, 8)$, $I = \{2, 3, 5\}$.

Lemma 1 (Flipping the snake) *There is a bijection* $\Psi_I \colon \mathcal{L}_I \to \mathcal{R}_I$ *with the property that* $G_{\mathbf{z}} = \Psi_I(G)_{\mathbf{z}}$ *if* $\mathbf{z} \notin I \times \{r-1, s\}$.

Proof. We construct the map Ψ_I inductively. If $|I| = 0$ or $|I| = 1$, the sets $\mathcal{L}_I$ and $\mathcal{R}_I$ are equal and we can take Ψ_I to be the identity map. Now take $I = \{i_1, i_2, \ldots, i_m\}$, $m \geq 2$, $I' = I \setminus \{i_1\} = \{i_2, \ldots, i_m\}$, $G \in \mathcal{L}_{I.k}$, and look at $G_{i_1,r-1}$.

First take $k = 1$. Then $G_{i_1,r-1} = (i_1, r-1)$, $G_{i_1,s} = (i_1, s)$, $G_{i_2,r-1} = (i_2, r-1)$, $G_{i_2,s} \in \overline{H}_{i_2,s}^*$. A square (i, s), $i > i_2$, in $\overline{H}_{i_2,s}^*$ is also in the punctured hook of (i_1, s), and a square (i_2, j), $j > s$, of $\overline{H}_{i_2,s}^*$ is in a natural correspondence with the square (i_1, j), which is in the punctured hook of (i_1, s). By slight abuse of notation, we can therefore define $G'_{i_1,s} = G_{i_2,s}$; we also define $G'_{i_2,s} = (i_2, s)$ and $G'_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq (i_1, s), (i_2, s)$.

Now note that G' is an element of $\mathcal{L}_{I'}$, and therefore we can take $G'' = \Psi_{I'}(G') \in \mathcal{R}_{\mathcal{I}'}$. By assumption, $\Phi_{I'}$ does not change values in row i_1, so $G''_{i_1,r} = G'_{i_1,r} = G_{i_1,r} = (i_1, r)$ and $G''_{i_1,s} = G'_{i_1,s} \in \overline{H}_{i_1,s}^*$. That means that $G'' \in \mathcal{R}_I$, and we take $\Phi_I(G) = G''$.

Now assume $k > 1$. If $G_{i_1,r-1}$ is of the form (i_1, j) for $j > s$, then it is also in the punctured hook of (i_1, s), and if it is of the form $(i, r-1)$ for $i \leq i_l$, then (i, s) is in the punctured hook of (i_1, s). By slight abuse of notation, we can then take $G'_{i_1,s} = G_{i_1,r-1}$, $G'_{i_1,r-1} = (i_1, r-1)$, $G'_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq (i_1, r-1), (i_1, s)$. Since $G' \in \mathcal{L}_{I'}$, we can take $G'' = \Phi_{I'}(G')$, and since $G'' \in \mathcal{R}_I$, we define $\Phi_I(G) = G''$.

Otherwise, $G_{i_1,r-1}$ is in the punctured hook of $(i_k, r-1)$ (where we identify squares (i_1, j) and (i_k, j) for $r-1 < j \leq s$). Note that if $k < m$, the square $G_{i_{k+1},s}$ is in the punctured hook of (i_1, s) (where we identify squares (i_1, j) and (i_{k+1}, j) for $s < j$). Define $G'_{i_1,r-1} = (i_1, r-1)$, $G'_{i_1,s} = G_{i_{k+1},s}$ (if $k < m$), $G'_{i_1,s} = G_{i_1,s} = (i_1, s)$ (if $k = m$), $G'_{i_k,r-1} = G_{i_1,r-1}$, $G'_{i_{k+1},s} = (i_{k+1}, s)$ (if $k < m$), $G'_{\mathbf{z}} = G_{\mathbf{z}}$ for all other $\mathbf{z}$. If $k = m$, then $G' \in \mathcal{R}_I$, take $G'' = G'$; otherwise, $G' \in \mathcal{L}_{I'}$, so we can define $G'' = \Phi_{I'}(G')$. Since $G'' \in \mathcal{R}_I$ in either case, we define $\Phi_I(G) = G''$.

It is easy to construct the analogous map $\mathcal{R}_I \to \mathcal{L}_I$, and to prove that such maps are inverses. The details are left to the reader. $\square$

EXAMPLE The arrangement on the right-hand side of Figure 13 is the image of the arrangement on the left-hand side.

The following figure shows four further examples. Each block shows shaded columns for G, G' and G''. In the first example, we have $G \in \mathcal{L}_{I,1}$. In the second and third example, we have $G \in \mathcal{L}_{I,l}$ with $1 < k < m$; G' can have different A and C, depending on $G_{i_1,s}$. The last example shows what happens if $G \in \mathcal{L}_{I,m}$ and $G_{i_1,r-1}$ is in the punctured hook of $(i_m, r-1)$.
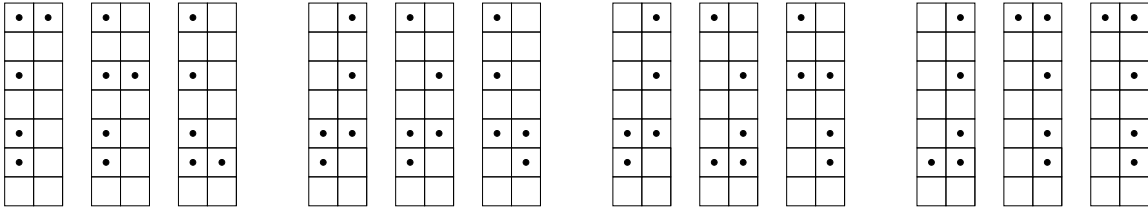

FIGURE 14. Left snake, intermediate step, right snake, for four examples, with $r = 8$ and $I = \{1, 3, 5, 6\}$.

4. Detailed description of the maps s and e

In this section, we describe in full detail the two maps mentioned in Section 3, s and e. While we call the map e "erasing the dot", its effect on the shaded row and columns is, as mentioned above, a bit more complicated.

Choosing the starting square. We are given $G \in \mathcal{G}_{\mathbf{c}}$, and we want to construct the square $\mathbf{z} = (i, j) \in [\lambda]^*$. For the row, the rule is simple and is a straightforward generalization of the rule in the non-shifted case.

(S0) i is the row coordinate of the upper-most dot in the shaded columns; if both A and C are empty, take $i = r$. In other words, $i = \min(A \cup C \cup \{r\})$.

For the column j, there are several different rules depending on the formation of the shaded columns. The motivation for these rules is as follows. We want to think of $s(A, B, C)$ as the starting square of the hook walk that corresponds to the arrangement G. If C is empty, we are essentially in the situation of the non-shifted case, and $s(A, B, C)$ is defined accordingly. If the shaded columns form a left stick or a right snake and B is empty, then the dots determine a hook walk, which, of course, starts in column $r - 1$. In other cases, we start from the top of the shaded columns, and try to see for how long the dots form a hook walk, i.e., a snake (either left or right). Once we hit an obstruction (like a block or a dot in the wrong column), we take the column corresponding to the row where the obstruction occurs, or to the last snake row.

More precisely, we have exactly one of the following options:

(S1) the shaded columns form a right stick; in this case, j is the column coordinate of the left-most dot in the shaded row or, if B is empty, $j = s$;

(S2) the shaded columns form a left stick or a right snake; in this case, $j = r - 1$;

(S3) the shaded columns start with a stick-dot; in this case $j = k - 1$, where k is the row coordinate of the dot;

(S4) the shaded columns start with a block of length ≥ 2; in this case $j = k - 1$, where k is the row coordinate of the second block row;

(S5) the shaded columns start with a snake and do not form a right snake; in this case, count the number of true statements in the following list:

 • the shaded columns start with a left snake;
 • the shaded columns start with a snake-odd block;
 • the shaded columns start with a snake-left dot or snake-block-left dot.

If this number is odd, let $j = k - 1$, where k is the row coordinate of the last snake row. If this number is even, the shaded columns must start with a snake-block; let $j = k - 1$, where k is the row coordinate of the first block row.

An alternative description of (S5) is as follows. If the shaded columns form a left snake or a left snake snake-even block or a right snake-odd block or start with a snake-dot or a left snake-even block-right dot or a left snake-odd block-left dot or a right snake-even block-left dot or a right snake-odd block-right dot, let $j = k - 1$, where k is the row coordinate of the last snake row. If the shaded columns form a left snake-odd block or right snake-even block or start with a left snake-even block-left dot or a left snake-odd block-right dot or a right snake-even block-right dot or a right snake-odd block-left dot, let $j = k - 1$, where

k is the row coordinate of the first block row. And if the shaded columns start with a snake-dot, let $j = k - 1$, where k is the row coordinate of the first block row.

Figure 15 should illuminate these rules. Clearly, (S0) and (S1) imply the first statement of (P1).
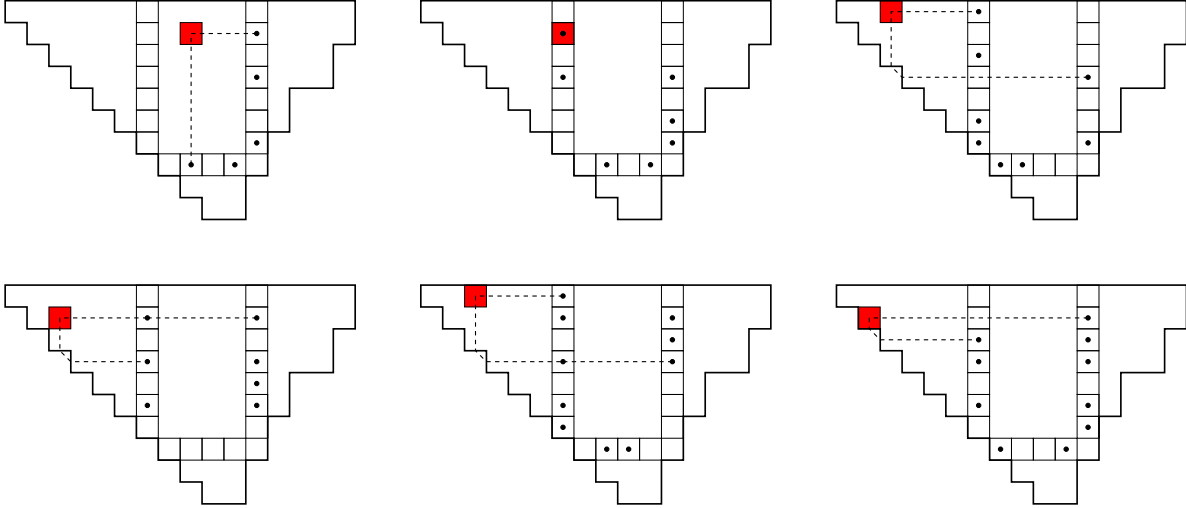

FIGURE 15. The starting square when the shaded columns form a right stick, left snake, or start with a stick-dot (top), block, snake-block-dot – two examples (bottom).

In all cases, we could give these rules in terms of the sets A and C (sets A and B in (S1)). For example, we could state (S3) as follows: if $C \neq \emptyset$ and $A \cap [\min C] \neq \emptyset$ and $\max(A \cap [\min C]) < \min C$, then $j = \min C - 1$, and if $A \neq \emptyset$ and $C \cap [\min A] \neq \emptyset$ and $\max(C \cap [\min A]) < \min A$, then $j = \min A - 1$. Such statements seem to be, however, much less intuitive than the descriptions above.

Erasing the dot. In this subsection, we construct the map e, which acts as identity on $\mathcal{G}_\mathbf{c}^0$, and which maps $\mathcal{G}_\mathbf{c}^k$ into $\mathcal{G}_\mathbf{c}^{k-1}$ for $k \geq 1$; in other words, if there is at least one dot in the shaded columns and row, then $e(G)$ should have one fewer dot in these columns and row. Furthermore, s and e should satisfy properties (P1), (P2) and (P3).

So pick $G \in \mathcal{G}_\mathbf{c} \setminus \mathcal{G}_\mathbf{c}^0$. There are three possibilities. First, $s(G) = (i, j)$ can be one of the shaded squares (i.e., $i = r$ or $j = r - 1$ or $j = s$); this happens in case (S1) if either $A = \emptyset$ or $B = \emptyset$, and in case (S2), and $e(G)$ differs from G in exactly one square. Second, $s(G)$ can be a non-shaded square that appears to the right of the column C (i.e., we have $i < r$ and $r - 1 < j < s$); this happens in case (S1) if both A and B are non-empty, and $e(G)$ differs from G in exactly two squares. And third, $s(G)$ can be a non-shaded square to the left of the column C (i.e., $j < r - 1$); this happens in cases (S3), (S4) and (S5). Now, $e(G)$ differs from G in at least two squares.

In the first case, $e(G)$ differs from G only at $s(G)$. If $B = C = \emptyset$ and $A = \{i, \ldots\}$, then $s(G) = (i, s)$ and $G_{i,s} = (i, s)$; define $A' = A \setminus \{i\}$, $B' = B$, $C' = C$, and $e(G)_{i,s} =$

$s(A', B', C')$. If $A = C = \emptyset$ and $B = \{j, \ldots\}$, then $s(G) = (r, j)$ and $G_{r,j} = (r, j)$, define $A' = A$, $B' = B \setminus \{j\}$, $C' = C$, and $e(G)_{r,j} = s(A', B', C')$. And otherwise (if A and $C = \{i, \ldots\}$ form a left stick or a right snake), we have $s(G) = (i, r-1)$ and $G_{i,r-1} = (i, r-1)$; define $A' = A$, $B' = B$, $C' = C \setminus \{i\}$, and $e(G)_{i,r-1} = s(A', B', C')$. Then define $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq s(G)$. In all these cases, $A(e(G)) = A'$, $B(e(G)) = B'$, $C(e(G)) = C'$ and $e(G)_{s(G)} = s(e(G))$.

The second case, when $A = \{i, \ldots\}$, $B = \{j, \ldots\}$, $C = \emptyset$, is almost exactly the same as in the non-shifted case. The square $G_{s(G)}$ is in the punctured hook of $s(G) = (i, j)$. This punctured hook is in a natural correspondence with $\overline{H}_{is}^* \sqcup \overline{H}_{rj}^*$. See Figure 16. So by slight abuse of notation, we define $e(G)$ as follows. If $G_{i,j} \in \overline{H}_{is}^*$, define $e(G)_{i,s} = G_{i,j}$, and if $G_{s(G)} \in \overline{H}_{rj}^*$, define $e(G)_{r,j} = G_{i,j}$. In the first case, also define $A' = A \setminus \{i\}$, $B' = B$, $C' = C$, and in the second case, define $A' = A$, $B' = B \setminus \{j\}$, $C' = C$. Then take $e(G)_{i,j} = s(A', B', C')$, $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq s(G)$. In both cases, we have $A(e(G)) = A'$, $B(e(G)) = B'$, $C(e(G)) = C'$ and $e(G)_{s(G)} = s(e(G))$.
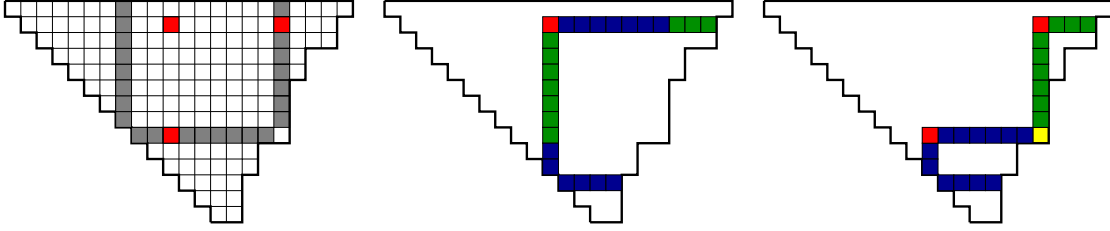

FIGURE 16. The punctured hook of a square is the disjoint union of the punctured hooks of projections onto shaded row and column. The yellow square is in the punctured hook of both projections.

As mentioned before, the third case splits into three possible subcases. the shaded columns either start with a stick-dot, or with a block of length ≥ 2, or start with a snake and do not form a right snake. Again, it is important to decompose the punctured hook of $s(G) = (i, j)$, where $j < r - 1$. The important difference now is that we have two possible decompositions; we can either decompose it as $\overline{H}_{is}^* \sqcup \overline{H}_{j+1,r-1}^*$, or as $\overline{H}_{i,r-1}^* \sqcup \overline{H}_{j+1,s}^*$. See Figure 17.

In the cases (S3), (S4) and (S5), we therefore describe the following:

- which decomposition, $\overline{H}_{is}^* \sqcup \overline{H}_{j+1,r-1}^*$ or $\overline{H}_{i,r-1}^* \sqcup \overline{H}_{j+1,s}^*$, we use;
- whether or not we have to use Lemma 1 to "flip the snake" (this can only happen in case (S5)).

If we use decomposition $\overline{H}_{is}^* \sqcup \overline{H}_{j+1,r-1}^*$ and $G_{i,j} \in \overline{H}_{is}^*$, then define $e(G)_{i,s} = G_{i,j}$, $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq (i, s), (i, j)$, $A' = A \setminus \{i\}$, $B' = B$, $C' = C$. If $G_{i,j} \in \overline{H}_{j+1,r-1}^*$, then define $e(G)_{j+1,r-1} = G_{i,j}$, $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq (j+1, r-1), (i, j)$, $A' = A$, $B' = B$, $C' = C \setminus \{j+1\}$. Similarly, if we use decomposition $\overline{H}_{i,r-1}^* \sqcup \overline{H}_{j+1,s}^*$ and $G_{i,j} \in \overline{H}_{i,r-1}^*$, then define $e(G)_{i,r-1} = G_{i,j}$, $e(G)_{\mathbf{z}} = G_{\mathbf{z}}$ for $\mathbf{z} \neq (i, r-1), (i, j)$, $A' = A$, $B' = B$, $C' = C \setminus \{i\}$.
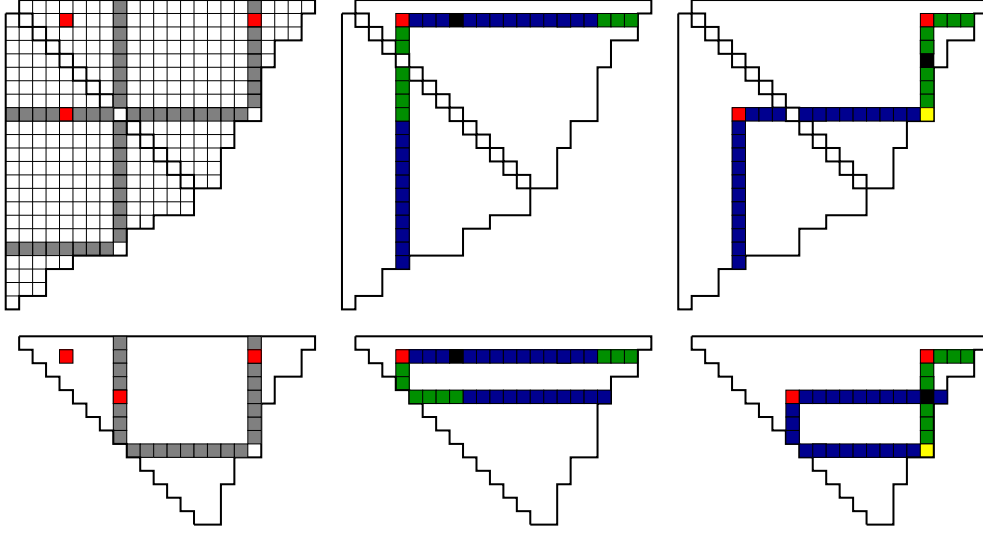
FIGURE 17. The punctured hook of a square is the disjoint union of the punctured hooks of projections onto shaded columns. The decomposition is shown on the bottom, and the top is a graphic explanation of why this decomposition works. The yellow square is in the punctured hook of both projections, as is the black square in the top right picture.

And if $G_{i,j} \in \overline{H}^*_{j+1,s}$, then define $e(G)_{j+1,s} = G_{i,j}$, $e(G)_\mathbf{z} = G_\mathbf{z}$ for $\mathbf{z} \neq (j+1,s), (i,j)$, $A' = A \setminus \{j+1\}$, $B' = B$, $C' = C$.

As we will see, in cases (S3), (S4) and in one subcase of (S5), $s(A', B', C')$ is always in the punctured hook of $s(G) = s(A, B, C)$. In this case, define $e(G)_{i,j} = s(A', B', C')$.

In most subcases of (S5), $s(A', B', C')$ is not in the punctured hook of $s(G) = s(A, B, C)$. In this case, we will use Lemma 1 to get A'', B'' and C''' with $|A''|+|B''|+|C'''| = |A'|+|B'|+|C'|$ and so that $s(A'', B'', C''')$ is in the punctured hook of $s(G) = s(A, B, C)$. In this case, define $e(G)_{i,j} = s(A'', B'', C''')$.

We can only use decomposition $\overline{H}^*_{is} \sqcup \overline{H}^*_{j+1,r-1}$ if $i \in A$ and $j+1 \in C$, and we can only use decomposition $\overline{H}^*_{i,r-1} \sqcup \overline{H}^*_{j+1,s}$ if $i \in C$ and $j+1 \in A$.

Accordingly, for (S3), we have only one choice for the decomposition: if the shaded columns start with a right stick-left dot, use decomposition $\overline{H}^*_{is} \sqcup \overline{H}^*_{j+1,r-1}$, and if they start with a left stick-right dot, use decomposition $\overline{H}^*_{i,r-1} \sqcup \overline{H}^*_{j+1,s}$. In both cases $s(A', B', C')$ is in the punctured hook of $s(A, B, C)$: let us prove this only when we have a right stick-left dot. Suppose first that $A = \{i, i', \dots, \}$, $C = \{j+1, j'+1, \dots\}$ for $i < j+1$. If $G_{i,j} \in \overline{H}^*_{is}$, then $A' = \{i', \dots\}$ and $C' = \{j+1, j'+1, \dots\}$ also form a stick-dot, and $s(A', B', C') = (i', j)$, which is in the punctured hook of (i, j). If $G_{i,j} \in \overline{H}^*_{j+1,r-1}$, then the starting row for A', B' and C' is still i, but the starting column is at least j'; thus $s(A', B', C')$ is in the punctured hook of (i, j). Now suppose that $A = \{i, i', \dots, \}$, $C = \{j+1, j'+1, \dots\}$ for $i' > j+1$. If $G_{i,j} \in \overline{H}^*_{is}$, then the first coordinate of $s(A', B', C')$ for $A' = \{i', \dots\}$ and

$C' = \{j + 1, j' + 1, \ldots\}$ is $j + 1$, and all of row $j + 1$ is in the punctured hook of (i, j). And if $G_{i,j} \in \overline{H}^*_{j+1,r-1}$, then the starting row for A', B' and C' is still i, but the starting column is at least j'; thus $s(A', B', C')$ is in the punctured hook of (i, j). The case when $|A| = 1$ or $|C| = 1$ is easy.

Take case (S4). The shaded columns either form a block or start with a block-dot. If they form an even block, or start with an even block-right dot or odd block-left dot, use decomposition $\overline{H}^*_{is} \sqcup \overline{H}^*_{j+1,r-1}$. If they form an odd block, or start with an odd block-right dot or even block-left dot, use decomposition $\overline{H}^*_{i,r-1} \sqcup \overline{H}^*_{j+1,s}$. It is easy to see that in every case, $s(A', B', C')$ is in the punctured hook of (i, j). For example, if the shaded columns start with an even block-right dot where the block has length > 2, $A = \{i, j+1, j'+1, \ldots\}$ and $C = \{i, j+1, j'+1, \ldots\}$, then erasing i from A or $j+1$ from C produces a formation that starts with right snake-even block-right dot; by (S5), $s(A', B', C') = (i, j')$, which is in the punctured hook of (i, j). All other cases are tackled in a similar way.

That leaves only the case (S5). The rule is as follows. If the shaded columns start with a left snake, use decomposition $\overline{H}^*_{is} \sqcup \overline{H}^*_{j+1,r-1}$, and if they start with a right snake, use decomposition $\overline{H}^*_{i,r-1} \sqcup \overline{H}^*_{j+1,s}$. Erasing i from A or C always produces $s(A', B', C')$ which is in the punctured hook of (i, j), as can be seen by an inspection of all possible cases. However, erasing $j + 1$ from A or C has this property only if the shaded columns form a left stick-even block or right stick-odd block, or if they start with a left stick-even block-right dot, left stick-odd block-left dot, right stick-even block-left dot or right stick-odd block-right dot; see top row of Figure 18. Otherwise, erasing $j + 1$ from A or C produces A', C' with $s(A', B', C') = (i, j')$ for $j' \leq j$, which is *not* in the punctured hook of (i, j); see bottom row of Figure 18.
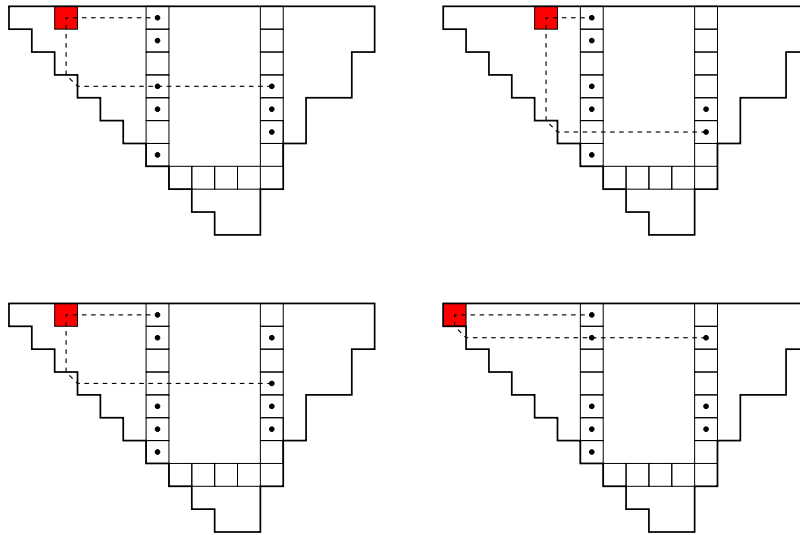

FIGURE 18. In case (S5), deleting a dot in row $j + 1$ can either produce A', C' with $s(A', B, C')$ in the punctured hook of $s(A, B, C)$ (top) or not (bottom).

In all these cases, however, changing a left snake to a right snake or vice versa gives $A'', B'' = B, C'''$ so that $s(A'', B'', C''') = (i, j')$ with $j' > j$. So Lemma 1 is the remaining ingredient of the construction of e. If we are in the position in case (S5) when erasing $j + 1$ from A or C does not produce $A', B' = B, C'$ with $s(A', B', C')$ in the punctured hook of $s(A, B, C)$, apply Ψ_I or Ψ_I^{-1} to G first, where $I = (A \cup C) \cap [j]$, and erase $j + 1$ from the resulting A' (if the original shaded columns started with a right snake) or C' (if the original shaded columns started with a left snake) to get A'' and C'. This way, we get $s(A'', B'', C''')$ in the punctured hook of $s(A, B, C)$, and we are done.

The property (P1) is clear, and the property (P2) follows by construction. Constructing the inverse implicit in (P3) is straightforward, if cumbersome, and is left as an exercise for the reader.

5. Comparison with Sagan's proof

Apart from [CKP] and [Kon], the major source of motivation and inspiration for this paper was [Sag79]. Indeed, some of the notation (such as sets A, B and C) was preserved on purpose. Though not explicitly stated, the map s is hidden in the definition of the operators N, R, L, T and the proof of Lemma 6. On the other hand, our snake-flipping (Lemma 1) is equivalent to Lemma 7. There are, however, several advantages to our bijective approach. We mention a few in the following paragraphs.

An obvious and minor difference is that in [Sag79], the case $C = \emptyset$ is treated separately, in a section of its own, while we are able to place this case in a wider framework, with a special rule for s and e. The second difference is that with our approach, there is no need to study "basic trials" (i.e. the case $B = \emptyset$) any differently than other cases.

The complicated part of the proof is, of course, when $C \neq \emptyset$. Sagan offers an ad hoc construction for the case when $s(A, \emptyset, C) = (r - 2, r - 2)$, and the simultaneous reverse induction in the proof for $s(A, \emptyset, C) = (i, j)$ when $j < r - 2$ is very tricky. Our proof, on the other hand, offers a unified description of how to select the dot to erase.

Furthermore, we feel that the proof of the snake-flipping lemma is much more intuitive than [Sag79, proof of Lemma 7]. It also sheds light on what the "counterexample" from [Sag79, §4] means: that the hook walk by itself does not determine the projection onto shaded row and columns; we also need to pick random elements of the punctured hook on (some of) the squares in shaded columns. More abstractly, it says that snake-flipping is necessary; that we cannot hope to always be able to erase a dot from the shaded row and columns and hope for a new configuration with a starting square in the punctured hook of the starting square of the original configuration.

Also, the bijective approach lends itself very naturally to the weighted formulas, and various other variants mentioned in Sections 6 and 8.

6. Variants and weighted formulas

The true power of a bijection lies in its robustness. In this section, we specialize and slightly adapt the bijection to get variants and generalizations of (2).

Theorem 2 *For a partition λ of n with distinct parts, we have the following equalities:*

$$n \cdot \prod_{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]} (h_{ij}^* - 1) = \sum_{(r,s)\in\mathcal{C}^*[\lambda]} \prod_{\substack{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]\\ i\neq r, j\neq r-1,s}} (h_{ij}^* - 1) \prod_{i=1}^{r-1} h_{is}^* \prod_{j=r}^{s-1} h_{rj}^* \prod_{i=1}^{r-1} h_{i,r-1}^*$$

$$\lambda_1 \cdot \prod_{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]} (h_{ij}^* - 1) = \sum_{(r,s)\in\mathcal{C}^*[\lambda]} (h_{1,r-1}^* + h_{1s}^* - 1) \prod_{\substack{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]\\ i\neq r, j\neq r-1,s}} (h_{ij}^* - 1) \prod_{i=2}^{r-1} h_{is}^* \prod_{j=r}^{s-1} h_{rj}^* \prod_{i=2}^{r-1} h_{i,r-1}^*$$

$$\prod_{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]} (h_{ij}^* - 1) = \sum_{(r,s)\in\mathcal{C}^*[\lambda]} \left(h_{1s}^* h_{2,r-1}^* + (h_{1,r-1}^* - 1)(h_{2s}^* - 1)\right) \prod_{\substack{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]\\ i\neq r, j\neq r-1,s}} (h_{ij}^* - 1) \prod_{i=3}^{r-1} h_{is}^* \prod_{j=\max\{r,2\}}^{s-1} h_{rj}^* \prod_{i=3}^{r-1} h_{i,r-1}^*$$

Here $h_{\mathbf{z}}^* = 1$ if $\mathbf{z} \notin [\lambda]^*$.

Proof. The first equality is just (2). To get the second equality, note that the left-hand side enumerates all pairs $(\mathbf{z}, F)$, where $\mathbf{z}$ is in the first row and $F \in \mathcal{F}$. So we have to prove that for every $\mathbf{c} = (r,s) \in \mathcal{C}^*[\lambda]$, the term on the right-hand side corresponding to (r,s) enumerates all arrangements $G \in \mathcal{G}_{\mathbf{c}}$ and $s(G) = (1,j)$ for some j, $1 \leq j \leq \lambda_1$. If $r = 1$, this is true for all $\mathcal{G}_{\mathbf{c}}$, and there are

$$\prod_{\substack{(i,j)\in[\lambda]\backslash\mathcal{C}[\lambda]\\ i\neq 1}} (h_{ij}^* - 1) \prod_{j=1}^{s-1} h_{1j}^* = \left(h_{1,r-1}^* + h_{1s}^* - 1\right) \prod_{\substack{(i,j)\in[\lambda]\backslash\mathcal{C}[\lambda]\\ i\neq r, j\neq r-1,s}} (h_{ij}^* - 1) \prod_{i=2}^{r-1} h_{is}^* \prod_{j=r}^{s-1} h_{rj}^* \prod_{i=2}^{r-1} h_{i,r-1}^*$$

such arrangements, where we are using the fact that $h_{10}^* = 1$ by definition and $h_{1s}^* = 1$. If $r > 1$, G satisfies $s(G) = (1,j)$ if and only if $1 \in A(G) \cup C(G)$, by (S0). We can therefore put any labels in $(1, r-1)$ and $(1,s)$ except $(1, r-1)$ and $(1,s)$ simultaneously. That means that we have $h_{1,r-1}^* + h_{1,s}^* - 1$ possible labels in these squares. It follows that there are

$$\left(h_{1,r-1}^* + h_{1s}^* - 1\right) \prod_{\substack{(i,j)\in[\lambda]\backslash\mathcal{C}[\lambda]\\ i\neq r, j\neq r-1,s}} (h_{ij}^* - 1) \prod_{i=2}^{r-1} h_{is}^* \prod_{j=r}^{s-1} h_{rj}^* \prod_{i=2}^{r-1} h_{i,r-1}^*$$

such arrangements, as required.

The last proof is a bit more involved. We want to see that a term on the right-hand side enumerates all G so that $s(G) = (1,1)$. If $r = 1$, this is true if and only if $G_{1,1} = (1,1)$, and there are

$$\prod_{\substack{(i,j)\in[\lambda]\backslash\mathcal{C}[\lambda]\\ i\neq 1}} (h_{ij}^* - 1) \prod_{j=2}^{s-1} h_{1j}^*$$

such arrangements. If $r = 2$, we also have $s(G) = (1,1)$ if and only if $G_{1,1} = (1,1)$, and there are

$$\prod_{\substack{(i,j)\in[\lambda]\backslash\mathcal{C}[\lambda]\\ i\neq 2, j\neq 1,s}} (h_{ij}^* - 1) \left[h_{1s}^* \prod_{j=2}^{s-1} h_{2j}^* \right].$$

If $r \geq 3$, we claim the following. Given labels $G_{\mathbf{z}}$ for $\mathbf{z} \notin \{(1, r-1), (1, s), (2, r-1), (2, r)\}$, there are $h_{1s}^* h_{2,r-1}^* + (h_{1,r-1}^* - 1)(h_{2s}^* - 1)$ ways to add labels in $(1, r-1), (1, s), (2, r-1), (2, r)$ so that the resulting G satisfies $s(G) = (1, 1)$.

Consider the rows from 3 to $r - 1$ of the shaded columns. If they are empty or form an even block or start with an even block-right dot or odd block-left dot, a careful application of rules (S0)–(S5) shows that the top two rows can form any of the combinations shown on the top of Figure 19.
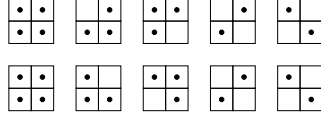

FIGURE 19. Possible top two rows.

There are

$$1 + (h_{1,r-1}^* - 1) + (h_{2s}^* - 1) + (h_{1,r-1}^* - 1)(h_{2s}^* - 1) + (h_{1s}^* - 1)(h_{2,r-1}^* - 1)$$

such labels. If, on the other hand, the rows from 3 to $r - 1$ of the shaded columns form an odd block or start with an odd block-right dot or even block-left dot, the top two rows can form any of the combinations on the bottom drawing in Figure 19, and there are

$$1 + (h_{1s}^* - 1) + (h_{2,r-1}^* - 1) + (h_{1,r-1}^* - 1)(h_{2s}^* - 1) + (h_{1s}^* - 1)(h_{2,r-1}^* - 1)$$

possible labels. Using the formula $(h_{1s}^* - 1) + (h_{2,r-1}^* - 1) = (h_{1,r-1}^* - 1) + (h_{2s}^* - 1)$ and simplifying, we get that both these expressions equal $h_{1s}^* h_{2,r-1}^* + (h_{1,r-1}^* - 1)(h_{2s}^* - 1)$. There are therefore

$$\left(h_{1s}^* h_{2,r-1}^* + (h_{1,r-1}^* - 1)(h_{2s}^* - 1)\right) \prod_{\substack{(i,j) \in [\lambda] \setminus \mathcal{C}[\lambda] \\ i \neq r, j \neq r-1, s}} (h_{ij}^* - 1) \prod_{i=3}^{r-1} h_{is}^* \prod_{j=\max\{r,2\}}^{s-1} h_{rj}^* \prod_{i=3}^{r-1} h_{i,r-1}^*$$

such arrangements, and it is easy to check that this formula is compatible with formulas for $r = 1$ and $r = 2$ obtained above. $\qquad \square$

While weighted formulas were at the very center of [CKP] and [Kon], we mention them here only in passing. The reason is that the formulas we were able to obtain do not seem to simplify the bijection, unlike in the non-shifted case; in fact, because monomials have coefficients other than 1, the bijection becomes more unwieldy. For a discussion of whether or not these weighted formulas are satisfactory and whether a better generalization might exist, see Section 8.

The following figure shows how we weight the punctured hook of a square in the shifted diagram.

More precisely, we define

$$\overline{\mathcal{H}}_{ij}^* = \begin{cases} (2\ell(\lambda) - 2 - i - j)\, x + 2y_1 + \dots + 2y_{\lambda_{j+1} + j - \ell(\lambda) + 1} + y_{\lambda_{j+1} + j - \ell(\lambda) + 2} + \dots + y_{\lambda_i + i - \ell(\lambda)} : j < \ell(\lambda) \\ (\max\{k : \lambda_k \geq j + 1 - k \geq 1\} - i)\, x + y_{j+2-\ell(\lambda)} + \dots + y_{\lambda_i + i - \ell(\lambda)} : j \geq \ell(\lambda) \end{cases}$$
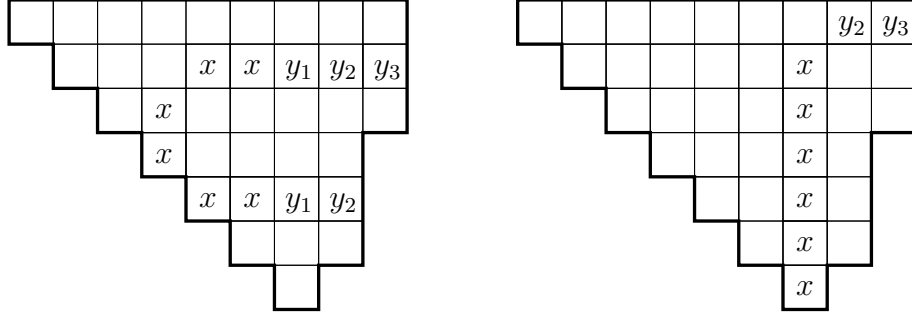
for $(i, j) \in [\lambda]^*$.

FIGURE 20. Weighted punctured hooks of $(2,4)$ and $(1,7)$ in $\lambda = 9875431$ are $6x + 2y_1 + 2y_2 + y_3$ and $6x + y_1 + y_2$, respectively.

Theorem 3 *Let $x, y_1, \ldots, y_{\lambda_1 + 1 - \ell(\lambda)}$ be some commutative variables. For a partition λ of n with distinct parts, we have the following polynomial equalities:*

$$\left[\sum_{(p,q) \in [\lambda]^*} x\, y_{q-\ell(\lambda)+1} \right] \left[\prod_{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda]} \overline{\mathcal{H}}_{ij}^* \right] = \sum_{(r,s) \in \mathcal{C}^*[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda] \\ i \neq r, j \neq r-1, s}} \overline{\mathcal{H}}_{ij}^* \right]$$

$$\times \prod_{i=1}^{r} (\overline{\mathcal{H}}_{is}^* + x) \prod_{j=r}^{\ell(\lambda)-1} (\overline{\mathcal{H}}_{rj}^* + x) \prod_{j=\ell(\lambda)}^{s} (\overline{\mathcal{H}}_{rj}^* + y_{j-\ell(\lambda)+1}) \prod_{i=1}^{r-1} (\overline{\mathcal{H}}_{i,r-1}^* + x)$$

$$\left[(\ell(\lambda) - 1)x + \sum_{j=1}^{\lambda_1 + 1 - \ell(\lambda)} y_j \right] \left[\prod_{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda]} \overline{\mathcal{H}}_{ij}^* \right] = \sum_{(r,s) \in \mathcal{C}^*[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda] \\ i \neq r, j \neq r-1, s}} \overline{\mathcal{H}}_{ij}^* \right] \left[\left\{ \begin{array}{l} \overline{\mathcal{H}}_{1,r-1}^* + \overline{\mathcal{H}}_{1s}^* + x : r \geq 2 \\ 1 : r=1 \end{array} \right. \right]$$

$$\times \prod_{i=2}^{r} (\overline{\mathcal{H}}_{is}^* + x) \prod_{j=r}^{\ell(\lambda)-1} (\overline{\mathcal{H}}_{rj}^* + x) \prod_{j=\ell(\lambda)}^{s} (\overline{\mathcal{H}}_{rj}^* + y_{j-\ell(\lambda)+1}) \prod_{i=2}^{r-1} (\overline{\mathcal{H}}_{i,r-1}^* + x)$$

$$\left[\prod_{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda]} \overline{\mathcal{H}}_{ij}^* \right] = \sum_{(r,s) \in \mathcal{C}^*[\lambda]} \left[\prod_{\substack{(i,j) \in [\lambda]^* \setminus \mathcal{C}^*[\lambda] \\ i \neq r, j \neq r-1, s}} \overline{\mathcal{H}}_{ij}^* \right] \left[\left\{ \begin{array}{l} (\overline{\mathcal{H}}_{1s}^* + x)(\overline{\mathcal{H}}_{2,r-1}^* + x) + \overline{\mathcal{H}}_{1,r-1}^* \overline{\mathcal{H}}_{2s}^* : r \geq 3 \\ \overline{\mathcal{H}}_{1s}^* + x : r=2 \\ 1 : r=1 \end{array} \right. \right]$$

$$\times \prod_{i=3}^{r} (\overline{\mathcal{H}}_{is}^* + x) \prod_{j=\max\{r,2\}}^{\ell(\lambda)-1} (\overline{\mathcal{H}}_{rj}^* + x) \prod_{j=\max\{\ell(\lambda),2\}}^{s} (\overline{\mathcal{H}}_{rj}^* + y_{j-\ell(\lambda)+1}) \prod_{i=3}^{r-1} (\overline{\mathcal{H}}_{i,r-1}^* + x)$$

We omit the (rather tedious) proof; it involves noting that given an arrangement $F \in \mathcal{F}$ or $G \in \mathcal{G}$, we have to replace the square $\mathbf{z}'$ in the hook of $\mathbf{z}$ by a variable x or y_j, using Figure 20 or the definition of $\overline{\mathcal{H}}_{ij}^*$ above. Then we have to prove that the product of labels in G is equal to the product of labels in $\Phi(G)$.

The details are left as an exercise for the reader.

7. TOWARD d-COMPLETE POSETS

An interestion generalization of diagrams and shifted diagrams are d-complete posets of Proctor and Peterson. Let us start with a brief outline of their definition and main features. See [Pro99] for details.

For $k \geq 3$, take a chain of $2k - 3$ elements, and expand the middle element to two incomparable elements. Call the resulting poset a *double-tailed diamond poset* (for $k = 3$, we call it a *diamond*), and denote it by $d_k(1)$. Figure 21 shows the Hasse diagram of $d_6(1)$ (rotated by 90°). An interval in a poset is called a d_k-interval if it is isomorphic to $d_k(1)$. A d_3^--interval in a poset P consists of three elements w, x, y, so that both x and y cover w. For $k \geq 4$, an interval in P is called d_k^--interval if it is isomorphic to $d_k(1) \setminus \{t\}$, where t is the maximal element of $d_k(1)$.
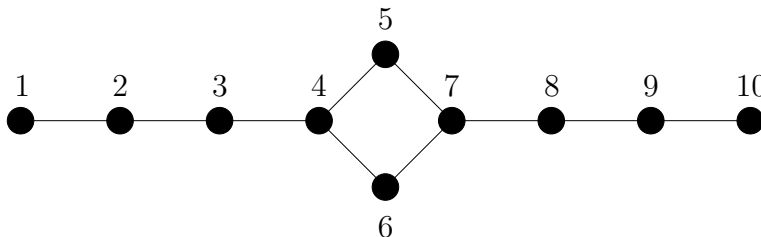

FIGURE 21. The poset $d_5(1)$.

The essential property of d-complete posets is that there are we can "complete" every d_k^--interval. Furthermore, d_k-intervals do not intersect. More precisely, we call a poset $(P, \leq)$ *d-complete* if it has the following properties:

(D1) if x and y cover a third element w, there must exist a fourth element z which covers each of x and y;

(D2) if $\{x, y, z, w\}$, $w < x, y < z$, is a diamond in P, then z covers only x and y in P;

(D3) no two elements x and y can cover each of two other elements w and w';

(D4) if $[w, y]$ is a d_k^--interval, $k \geq 4$, there exists z which covers y and such that $[w, z]$ is a d_k-interval;

(D5) if $[w, z]$ is a d_k-interval, $k \geq 4$, then z covers only one element in P;

(D6) if $[w, y]$ is a d_k^--interval, $k \geq 4$, and x is the (unique) element covering w in this interval, then x is the unique element covering w in P.

The *hook length* of an element z of a d-complete poset P is defined recursively as follows. If z is not an element of a d_k-interval for any $k \geq 3$, then the hook length h_z^P is $|\{w \in P : w \leq z\}|$. If there exists w so that $[w, z]$ is a d_k-interval, and x, y are the incomparable elements in $[w, z]$, then define $h_z^P = h_x^P + h_y^P - h_w^P$. We usually write h_z for h_z^P if the choice of the poset is clear.

We can also define the *hook* H_z of an element z in a d-complete poset P so that $H_z \subseteq \{w \in P : w \leq z\}$, $z \in H_z$ and $|H_z| = h_z$. One possible definition goes as follows. Assume without loss of generality that P is connected. Then it is easy to see that P has a maximal element z_0. Pick z and $w \leq z$. It can be proved that $h_z^{(w, z_0]}$ either equals $h_z^{[w, z_0]}$ or $h_z^{[w, z_0]} - 1$

(note that the posets $(w, z_0]$ and $[w, z_0]$ are also d-complete). We say that $w \in H_z$ if and only if $h_z^{(w,z_0]} = h_z^{[w,z_0]} - 1$. As usual, we write $\overline{H}_z$ for $H_z \setminus \{z\}$.

One of the main properties of d-complete posets is that the hook length formula is still valid. If f_P is the number of bijections $g \colon P \to [n]$ satisfying $g(z) \le g(w)$ for $z \ge w$, then

$$f_P = \frac{n!}{\prod_{z \in P} h_z}.$$

Denote the set of all minimal elements of P by $\min P$. By induction, the hook length formula is equivalent to the *branching rule for d-complete posets*:

$$(4) \qquad n \cdot \prod_{z \in P \setminus \min P} (h_z - 1) \; = \; \sum_{c \in \min P} \prod_{\substack{z \in P \setminus \min P \\ c \notin H_z}} (h_z - 1) \prod_{\substack{z \in P \setminus \min P \\ c \in H_z}} h_z.$$

In analogy with our framework for non-shifted and shifted diagrams, define $\mathcal{F}$ as the set of all pairs (z, F), where $z \in P$ and $F \colon P \setminus \min P \to P$ satisfies $F(z) \in \overline{H}_z$ for all z. Furthermore, define $\mathcal{G}$ as the set of all pairs (c, G), where $c \in \min P$ and $G \colon P \setminus \min P \to P$ satisfies $G(z) \in \overline{H}_z$ for all z, $c \notin H_z$, $G(z) \in H_z$ for all z, $c \in H_z$. For $c \in \min P$ and $k \ge 0$, denote by $\mathcal{G}_c$ the set of all G so that $(c, G) \in \mathcal{G}$ and by $\mathcal{G}_c^k$ the set of all $G \in \mathcal{G}_c$ with $|\{z \in P \colon G(z) = z\}| = k$. Sometimes we write F_z and G_z instead of $F(z)$ and $G(z)$. For a d-complete poset P and $c \in \min P$, denote by $A_c = A_c(G)$ the set $\{z \in P \colon c \in \overline{H}_z\}$. A proof of the following conjecture would provide a bijective proof of the hook length formula for d-complete posets.

Conjecture *For a d-complete poset P and a minimal element c, there exist maps $s = s_c \colon \mathcal{P}(A_c) \to P$, which can be extended to $s \colon \mathcal{G}_c \to P$ by $s(G) = s(A_c(G))$, and $e = e_c \colon \mathcal{G}_c \to \mathcal{G}_c$ satisfying the following properties:*
- (P1) *if $G \in \mathcal{G}_c^0$, then $s(G) = c$ and $e(G) = G$, and if $G \in \mathcal{G}_c^k$ for $k \ge 1$, then $e(G) \in \mathcal{G}_c^{k-1}$;*
- (P2) *if $G \in \mathcal{G}_c^k$ for $k \ge 1$, then $e(G)_{s(G)} = s(e(G))$; furthermore, if $z \not\ge s(G)$, then $e(G)_z = G_z$;*
- (P3) *given $G' \in \mathcal{G}_c^k$ and z for which $G'_z = s(G')$, there is exactly one $G \in \mathcal{G}_c^{k+1}$ satisfying $s(G) = z$ and $e(G) = G'$.*

The following proposition presents some evidence for the conjecture. For the definition of a *slant sum*, see [Pro99, page 67].

Proposition 4 *Let P, P_1, P_2 be d-complete posets.*
- (a) *The conjecture holds for P if P is the diagram of a partition.*
- (b) *The conjecture holds for P if P is the shifted diagram of a partition.*
- (c) *The conjecture holds for P if P is a double-tailed diamond poset.*
- (d) *If the conjecture holds for P_1 and P_2, it also holds for their disjoint union.*
- (e) *If the conjecture holds for P_1 and P_2, it also holds for their slant sum.*

Sketch of proof: We showed (a) in Section 2 and (b) in Sections 3 and 4.
Take $P = d_k(1)$ for $k \ge 3$ (for $k = 3$, this is the diagram of the partition 22, and for $k = 4$, this is the shifted diagram of the partition 321). Label the elements by

$x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$ in such a way that $x_1 > x_2 > \ldots > x_{k-1}, y_{k-1} > y_{k-2} > \ldots > y_1$. The only minimal element is $c = y_1$, and $A_c = P \setminus \{x_1, y_1\}$. It is also important to note that for all $j = 2, \ldots, k-1$, we have an identification of $\overline{H}_{x_1}$ and $\overline{H}_{x_j} \sqcup \overline{H}_{y_j}$, for example

$$
\begin{array}{llllllllllll}
\overline{H}_{x_1}: & x_2 & \cdots & x_j & x_{j+1} & \cdots & x_d & y_d & \cdots & y_{j+1} & y_j & \cdots & y_2 \\
\overline{H}_{x_j}: & & & & x_{j+1} & \cdots & x_d & y_d & \cdots & y_{j+1} & y_{j-1} & \cdots & y_1 \\
\overline{H}_{y_j}: & y_1 & \cdots & y_{j-1} \\
\end{array}
$$

For $A \subseteq A_c$, define

$$
s(A) = \left\{
\begin{array}{lll}
y_1 & : & A = \emptyset \\
\max A & : & A \neq \emptyset, \min\{i\colon x_i \in A\} \neq \max\{i\colon y_i \in A\} \\
x_1 & : & \min\{i\colon x_i \in A\} = \max\{i\colon y_i \in A\}
\end{array}
\right. .
$$

Furthermore, if $A = \emptyset$, define $e(G) = G$; if $A \neq \emptyset$ and $\min\{i\colon x_i \in A\} \neq \max\{i\colon y_i \in A\}$, define $A' = A \setminus \{\max A\}$, $e(G)_{\max A} = s(A')$ and $e(G)_z = G_z$ for $z \neq \max A$. Finally, if $\min\{i\colon x_i \in A\} = \max\{i\colon y_i \in A\} = j$, we have $G_{x_1} \in \overline{H}_{x_j} \sqcup \overline{H}_{y_j}$. If $G_{x_1} \in \overline{H}_{x_j}$, define $A' = A \setminus \{x_j\}$, $e(G)_{x_1} = s(A')$, $e(G)_{x_j} = G_{x_1}$, $e(G)_z = G_z$ for $z \neq x_1, x_j$; and if $G_{x_1} \in \overline{H}_{y_j}$, define $A' = A \setminus \{y_j\}$, $e(G)_{x_1} = s(A')$, $e(G)_{y_j} = G_{x_1}$, $e(G)_z = G_z$ for $z \neq x_1, y_j$. It can be checked that such s and e satisfy (P1), (P2), and (P3). This proves (c).

Suppose P is a disjoint union of d-complete posets P_1 and P_2 with corresponding maps s_1, e_1, s_2, e_2. If $c \in P_i$, then $\{z \in P\colon G(z) = z\} \subseteq P_i$, and therefore we should take

$$
s(G) = s_i(G|_{P_i}), \qquad e(G)_z = \left\{
\begin{array}{lll}
e_i(G|_{P_i})_z & : & z \in P_i \\
G_z & : & z \notin P_i
\end{array}
\right. .
$$

Again, it can be checked that such s and e satisfy (P1), (P2), and (P3), and this proves (d).

Suppose P is a slant sum of d-complete posets P_1 and P_2 with corresponding maps s_1, e_1, s_2, e_2. If $c \in P_1$, or if $c \in P_2$ and $\{z \in P\colon G(z) = z\} \subseteq P_2$, the same construction as in (d) works. If, however, $c \in P_2$ and $\{z \in P\colon G(z) = z\} \not\subseteq P_2$, we have to take

$$
s(G) = \max\{z \in P\colon G(z) = z\}, \qquad e(G)_z = \left\{
\begin{array}{lll}
G_z & : & z \neq s(G) \\
s(A') & : & z = s(G)
\end{array}
\right. ,
$$

where $A' = A \setminus \{s(G)\}$. It can be checked that such s and e satisfy (P1), (P2), and (P3). This proves (e) and finishes the proof of the proposition. $\square$

Proctor [Pro99] gave a classification of irreducible d-complete posets. So the proof of the conjecture would be complete once we found maps s_c and e_c for all minimal elements of all other classes of irreducible d-complete posets. While some of the cases are pretty straightforward, the map s is likely to be extremely complicated for, say, the "bat" and its unique minimam element.

8. Final remarks

8.1. Our weighted formulas are slightly disappointing, since all the "height" variables $x_1, \ldots, x_{\ell(\lambda)}$ have merged into one variable, which also appears in columns $1, \ldots, \ell(\lambda) - 1$. There are certain indications, however, that this is the best possible generalization (the author would be thrilled to be proved wrong though); let us mention two. First, the

weighted punctured hooks $\overline{\mathcal{H}}_{ij}$ should satisfy the formula $\overline{\mathcal{H}}_{ij}^* = \overline{\mathcal{H}}_{i,r-1}^* + \overline{\mathcal{H}}_{j+1,s}^* = \overline{\mathcal{H}}_{i,s}^* + \overline{\mathcal{H}}_{j+1,r-1}^*$, and it should be obvious from Figure 17 that this is not easily obtainable unless many of the variables are equal. Also, for the staircase shape partition $(k, k-1, \ldots, 1)$, the "natural" weighted generalization of n on the left-hand side of (2) is $\sum_{i \leq j} x_i y_j$, which should be factored into linear terms on the right-hand side, and this is obviously impossible. Our weighted version, in which both sides are just integer multiples of a power of x, obviously avoids this issue.

8.2. There is another variant of (2) obtained by finding all formations of the shaded columns that satisfy $s(A, B, C) = (1, m)$ for $1 \leq m \leq \lambda_1$:

$$\prod_{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]} \left(h_{ij}^* - 1\right) = \sum_{(r,s)\in\mathcal{C}^*[\lambda]} H_{\lambda,r,s,m} \prod_{\substack{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda] \\ i\neq r, j\neq r-1, s}} (h_{ij}^* - 1) \prod_{i=m+2}^{r-1} h_{is}^* \prod_{j=\max\{r,m+1\}}^{s-1} h_{rj}^* \prod_{i=m+2}^{r-1} h_{i,r-1}^*,$$

where $H_{\lambda,r,s,m}$ is

$$\prod_{i=1}^{r-1}(h_{i,r-1}^*-1) \prod_{j=r}^{m-1}(h_{r,j}^*-1) \prod_{i=2}^{r-1}(h_{i,s}^*-1) \qquad \text{if } 1 \leq r \leq m$$

$$\sum_{\substack{I\subseteq[m],k\in I \\ \min I=1}} \prod_{i\notin I \text{ or } i>k} (h_{i,r-1}^*-1) \prod_{i\notin I \text{ or } i<k} (h_{i,s}^*-1) + \prod_{i=2}^{r-1} h_{i,r-1}^* \prod_{i=1}^{r-1}(h_{i,s}^*-1) \qquad \text{if } r = m+1$$

$$\sum_{\substack{I\subseteq[m],k\in I \\ \min I=1}} \prod_{i\notin I \text{ or } i>k} (h_{i,r-1}^*-1) \prod_{i\notin I \text{ or } i<k} (h_{i,s}^*-1) + \sum_{\substack{I\subseteq[m+1],k\in I \\ \min I=1,\max I=m+1}} \prod_{i\notin I \text{ or } i>k} (h_{i,r-1}^*-1) \prod_{i\notin I \text{ or } i<k} (h_{i,s}^*-1) \ +$$

$$(h_{m+1,r-1}^*-1)\prod_{i=2}^{m} h_{i,r-1}^* \prod_{i=1}^{m}(h_{i,s}^*-1) + (h_{m+1,s}^*-1)\prod_{i=2}^{m} h_{i,s}^* \prod_{i=1}^{m}(h_{i,r-1}^*-1) \qquad \text{if } r \geq m+2$$

For $m = 1$, this is, of course, the third equality of Theorem 2, and the proof for a general m is very similar. Just let us note that

$$\sum_{\substack{I\subseteq[m],k\in I \\ \min I=1}} \prod_{i\neq I \text{ or } i>k} (h_{i,r-1}^*-1) \prod_{i\neq I \text{ or } i<k}$$

counts the number of ways in which we can choose labels in $[1, m] \times \{r-1, s\}$ so that the shaded columns up to row r form a left snake that starts in row 1.

A weighted version is also possible, but is omitted. While this formula seems to be too complicated to be of real interest, we should mention that it indeed gives $\lambda_1 - 1$ formulas that are not equivalent to the formulas in Theorem 2. In the non-shifted case, finding all G satisfying $s(G) = (1, m)$ would yield formulas that are equivalent to the case $m = 1$ for the partition $(\lambda_1 - m, \lambda_2 - m, \ldots)$.

We can similarly enumerate all G so that $s(G) = (i, m)$ for some fixed m. The resulting formula, which we omit, has $m \cdot \prod_{(i,j)\in[\lambda]^*\backslash\mathcal{C}^*[\lambda]}(h_{ij}^* - 1)$ on the left-hand side, and on the right, the sticks and the snakes need not start in row 1.

8.3. We were not able to find "complementary" formulas in the spirit of [Kon], and we leave them as an open problem. The idea, following [Kon, §6], should be to look at the "complement" of the given shifted tableau (see Figure 22), whose corners are the "outer" corners of the original partition; to write out the formulas from Theorem 2 or Theorem 3 for this complementary partition; to cancel out the terms that the left-hand side and the right-hand side have in common; and then, for each square $\mathbf{z}^*$ appearing in such reduced formula, to find a square $\mathbf{z}$ of the original shifted diagram with satisfying $h_{\mathbf{z}} = h_{\mathbf{z}^*}$ or $h_{\mathbf{z}} + 1 = h_{\mathbf{z}^*} - 1$; and to finally change this "reduced" complementary formula into a formula that involves products over all squares. The author was not able to follow through with this approach.
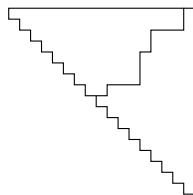

FIGURE 22. Complementary shifted partition.

8.4. Hook walk proofs of Theorem 3 should be possible as a generalization of Sagan's proof [Sag79], or probably in a more intuitive and less technical way, using maps s and e as guiding lights. We leave this as an excercise for the (very determined) reader. Such proofs might also work for the (hypothetical) complementary formulas.

8.5. It was mentioned in [Kon, §4] that variants of the branching rule give new recursions for f_λ, see [Kon, Corollary 5]. This is, of course, also true for the shifted case. For example, the second equality of Theorem 2 has the following corollary:

$$\lambda_1 f_\lambda^* = \sum_{\mathbf{c}=(r,s)\in\mathcal{C}^*[\lambda]} n \left(\frac{1}{\lambda_1+r-s} + \frac{1}{\lambda-r+s+1} - \frac{1}{(\lambda_1+r-s)(\lambda-r+s+1)} \right) f_{\lambda-\mathbf{c}}^*.$$

Again, it would be interesting to know if this identity has a combinatorial meaning.

Acknowledgements

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