SKEW QUANTUM MURNAGHAN-NAKAYAMA RULE

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ABSTRACT. We extend recent results of Assaf and McNamara on a skew Pieri rule and a skew Murnaghan-Nakayama rule to a more general identity, which gives an elegant expansion of the product of a skew Schur function with a quantum power sum function in terms of skew Schur functions. We give two proofs, one completely bijective in the spirit of Assaf-McNamara's original proof, and one via Lam-Lauve-Sotille's skew Littlewood-Richardson rule. We end with some conjectures for skew rules for Hall-Littlewood polynomials.

1. INTRODUCTION

Schur functions $\{s_\lambda \colon \lambda \text{ a partition}\}$ form the most important basis of the algebra of symmetric functions. They are not only interesting in their own right, but they appear in many areas of mathematics, such as the representation theory of the symmetric group and general linear groups, the cohomology ring of the Grassmanian etc. It is only natural that the formula for the product of two Schur functions, the celebrated Littlewood-Richardson rule, is of paramount importance in the theory. The rule says that the coefficient of s_λ in the product $s_\mu s_\nu$ is the number of certain (easily described) objects; in particular, all these coefficients are non-negative integers. For general λ, μ, ν the exact enumeration of these objects is still a difficult task.

The rule, however, becomes very simple if one of the Schur functions in the product is of a special form, namely, if ν a partition with either one row or one column. The (conjugate) Pieri rule states that in this case, all coefficients are either 0 or 1, and the coefficient is 1 only if we can obtain λ from μ in a certain way. The Pieri rule has many generalizations. Surprisingly, a generalization to skew Schur functions was proved only very recently by Assaf and McNamara in [1] (see §1 in that paper for a sample of other generalizations of the Pieri rule and references). Assaf and McNamara also proved a skew version of another important special case of the Littlewood-Richardson rule, the Murnaghan-Nakayama rule, which gives an expansion of the product of a Schur function with a power sum function.

The authors provide a beautiful combinatorial proof of the skew Pieri rule (and Thomas Lam gives an algebraic proof in the appendix), but they were unable to find a combinatorial proof for the skew Murnaghan-Nakayama rule, and they challenged their readers to find one. Our paper does more than that: it gives an involutive proof of the following generalization of both the skew Pieri rule and the skew Murnaghan-Nakayama rule.

Main Theorem (Skew Quantum Murnaghan-Nakayama Rule) *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum_{j=0}^{r} (-1)^{r+1-j} \sum_{\lambda^+, \mu^-} (-1)^{\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)} q^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} s_{\lambda^+/\mu^-},$$

where the internal sum on the right is over λ^+, μ^- such that λ^+/λ is a broken ribbon of size $r - j$, and μ/μ^- is a broken ribbon of size j.

Here $\widetilde{p}_r$ are certain symmetric functions with the parameter q that specialize to the power sum functions for $q = 1$ and to the Schur function s_r for $q = 0$. See the next section for a precise definition of all terms.

The paper is structured as follows. In Section 2, we give all necessary definitions and background, and state all our results. In Section 3, we describe the sign-reversing involution of Assaf and McNamara that was used to prove their skew Pieri rule. Furthermore, we show a variant of this involution that proves the conjugate skew Pieri rule. Note that this involution is actually much simpler than the one in [1]. In Section 4, we present an extension of these involutions that proves the skew quantum Murnaghan-Nakayama rule. There is quite some work involved to interpret the right-hand side of SQMNR in an appropriate way, but once this is done the involution is just a natural combination of the two involutions in Section 3. In Section 5, we present another proof of SQMNR, via the skew Littlewood-Richardson rule of Lam-Lauve-Sotille [4]; since their result (at the moment) only has an algebraic proof, this proof of SQMNR is not completely combinatorial. In Section 6, we give some conjectured skew Pieri-type rules for Hall-Littlewood polynomials, for which our combinatorial methods seem to fail. We finish with some concluding remarks in Section 7.

2. PRELIMINARIES AND RESULTS

A *partition* λ of n is a sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$; we use the notation $\lambda \vdash n$, $\ell = \ell(\lambda)$ (*length* of λ), $n = |\lambda|$ (*size* of λ), $\lambda_i = 0$ if $i > \ell(\lambda)$. We sometimes write $(\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots)$ if λ_1 is repeated k_1 times, $\lambda_2 < \lambda_1$ is repeated k_2 times etc. We write $m_j(\lambda)$ for the number of parts of λ equal to j. The *conjugate* partition of λ, denoted λ^c, is the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{\lambda_1})$ defined by $\mu_i = \max\{j \colon \lambda_j \geq i\}$. The *Young diagram* $[\lambda]$ of a partition λ is the set $\{(i,j) \colon 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. For partitions λ, μ we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, the *skew Young diagram* $[\lambda/\mu]$ of λ/μ is the set $\{(i,j) \colon 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}$. We denote $|\lambda| - |\mu|$ by $|\lambda/\mu|$. The elements of $[\lambda/\mu]$ are called *cells*. We treat λ and $\lambda/\emptyset$ as identical.

We say that λ/μ is a *horizontal strip* (respectively *vertical strip*) if $[\lambda/\mu]$ is non-empty, contains no 2×1 (respectively 1×2) block, equivalently, if $\lambda_i^c \leq \mu_i^c + 1$ (respectively $\lambda_i \leq \mu_i + 1$) for all i. We say that λ/μ is a *ribbon* if $[\lambda/\mu]$ is non-empty, connected and if it contains no 2×2 block, and that λ/μ is a *broken ribbon* if $[\lambda/\mu]$ contains no 2×2 block, equivalently, if $\lambda_i \leq \mu_{i-1} + 1$ for $i \geq 2$. The Young diagram of a broken ribbon is a disjoint union of $\mathrm{rib}(\lambda/\mu)$ number of ribbons. The *height* $\mathrm{ht}(\lambda/\mu)$ (respectively *width* $\mathrm{wt}(\lambda/\mu)$) of a ribbon is the number of non-empty rows (respectively columns) of $[\lambda/\mu]$, minus 1. The height (respectively width) of a broken ribbon is the sum of heights (respectively widths) of the components. Clearly, λ/μ is a horizontal (respectively vertical) strip if and only if it is a broken ribbon of height (respectively width) 0. Figure 1 shows examples of a horizontal strip, vertical strip, ribbon (with $\mathrm{ht}(\lambda/\mu) = 8$ and $\mathrm{wt}(\lambda/\mu) = 7$) and broken ribbon (with $\mathrm{ht}(\lambda/\mu) = 6$, $\mathrm{wt}(\lambda/\mu) = 6$ and $\mathrm{rib}(\lambda/\mu) = 3$).
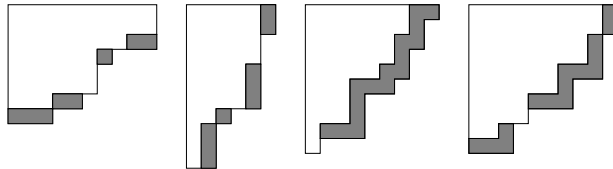

FIGURE 1

A map $T \colon [\lambda/\mu] \to \mathbb{N}$ is called a *skew semistandard Young tableau of shape* λ/μ if, for all i and j, $T(i, j_1) \leq T(i, j_2)$ for $j_1 < j_2$, and $T(i_1, j) < T(i_2, j)$ for $i_1 < i_2$. If T is a skew semistandard Young tableau, we denote by $t_i(T)$ the number of cells that map to i. If T is a bijection into $[n]$, we call T a *standard Young tableau*. Define the *skew Schur function*

$$(1) \qquad\qquad s_{\lambda/\mu} = \sum_T x_1^{t_1(T)} x_2^{t_2(T)} \cdots,$$

where the sum is over all semistandard Young tableaux of shape λ/μ. A skew Schur function is a formal power series in $x_1, x_2, \ldots$, and it is not hard to see that it is a symmetric function. Moreover, the set of

Schur functions $\{s_\lambda \colon \lambda \text{ partition}\}$ is a basis of the space of symmetric functions. For more details, and for some of the amazing properties of Schur functions, see [8, §7].

There are several other bases of the space of symmetric functions. For the purposes of this paper, the most important one is the *power sum basis* $\{p_\lambda \colon \lambda \text{ partition}\}$, defined by

$$p_r = x_1^r + x_2^r + \ldots,$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

Let us also mention the *monomial basis* $\{m_\lambda \colon \lambda \text{ partition}\}$, defined by

$$m_\lambda = \sum x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(\ell)}^{\lambda_\ell},$$

where the sum is over all injective maps $\pi \colon \{1, \ldots, \ell\} \to \mathbb{N}$.

The product of Schur functions can be (uniquely) expressed as a linear combination of Schur functions:

$$s_\lambda s_\mu = \sum c_{\lambda,\mu}^\nu s_\nu.$$

The coefficients $c_{\lambda,\mu}^\nu$ are called *Littlewood-Richardson coefficients* and can be computed using the celebrated *Littlewood-Richardson rule*, see [8, Appendix A1.3]. This rule is quite complicated, but it is very simple if μ has only one row or column. Namely, we have the *Pieri rule*:

$$(2) \qquad\qquad s_\lambda s_r = \sum s_{\lambda^+},$$

where the sum on the right is over all λ^+ such that λ^+/λ is a horizontal strip of size r. Similarly, the *conjugate Pieri rule* says that

$$(3) \qquad\qquad s_\lambda s_{1^r} = \sum s_{\lambda^+},$$

where the sum on the right is over all λ^+ such that λ^+/λ is a vertical strip of size r.

We also have a rule for the product of a Schur function with a power sum symmetric function, the *Murnaghan-Nakayama rule*:

$$(4) \qquad\qquad s_\lambda p_r = \sum (-1)^{\operatorname{ht}(\lambda^+/\lambda)} s_{\lambda^+},$$

where the sum on the right is over all λ^+ such that λ^+/λ is a ribbon of size r. See [8, Theorems 7.15.7 and 7.17.1].

In [1] and [2], Assaf and McNamara found a beautiful extension of both the Pieri rule and the Murnaghan-Nakayama rule.

Theorem 1 (Skew Pieri Rule – SPR) *For any partitions* $\lambda, \mu,\ \mu \subseteq \lambda$, *we have*

$$s_{\lambda/\mu} \cdot s_r = \sum_j (-1)^j \sum s_{\lambda^+/\mu^-},$$

where the inner sum on the right is over all λ^+, μ^- *such that* λ^+/λ *is a horizontal strip of size* $r - j$, *and* μ/μ^- *is a vertical strip of size* j.

The skew Pieri rule has an equivalent conjugate form.

Corollary 2 (Conjugate skew Pieri rule – CSPR) *For any partitions* $\lambda, \mu,\ \mu \subseteq \lambda$, *we have*

$$s_{\lambda/\mu} \cdot s_{1^r} = \sum_j (-1)^j \sum s_{\lambda^+/\mu^-},$$

where the inner sum on the right is over all λ^+, μ^- *such that* λ^+/λ *is a vertical strip of size* $r - j$, *and* μ/μ^- *is a horizontal strip of size* j

CSPR can be proved from SPR via the involution ω on the algebra of symmetric functions, which maps $s_{\lambda/\mu}$ to s_{λ^c/μ^c} and is an algebra homomorphism. See [8, §7.6 and §7.14] for details.

Theorem 3 (Skew Murnaghan-Nakayama Rule – SMNR) *For any partitions λ, μ, $\mu \subseteq \lambda$, we have*

$$s_{\lambda/\mu} \cdot p_r = \sum (-1)^{\mathrm{ht}(\lambda^+/\lambda)} s_{\lambda^+/\mu} - \sum (-1)^{\mathrm{ht}(\mu/\mu^-)} s_{\lambda/\mu^-},$$

where the first (respectively second) sum on the right is over all λ^+ (respectively μ^-) such that λ^+/λ (respectively μ/μ^-) is a ribbon of size r.

EXAMPLE By SPR, we have

$$s_{322/11} \cdot s_2 = s_{522/11} + s_{432/11} + s_{4221/11} + s_{3321/11} + s_{3222/11} - s_{422/1} - s_{332/1} - s_{3221/1} + s_{322},$$
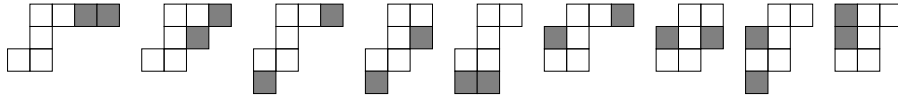
as shown by Figure 2.

FIGURE 2

By CSPR, we have

$$s_{322/11} \cdot s_{11} = s_{432/11} + s_{4221/11} + s_{333/11} + s_{3321/11} + s_{32211/11} - s_{422/1} - s_{332/1} - s_{3221/1},$$
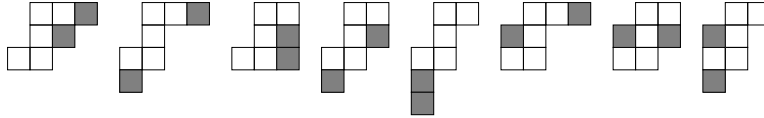
as shown by Figure 3.

FIGURE 3

By SMNR, we have

$$s_{433/22} \cdot p_3 = s_{733/22} - s_{553/22} + s_{4333/22} - s_{43321/22} + s_{433111/22} + s_{433/1},$$
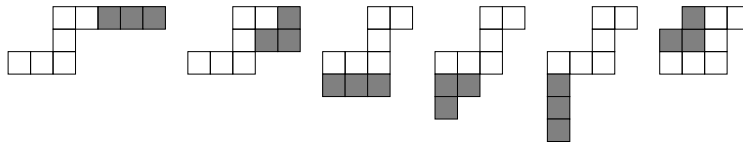
as shown by Figure 4.

FIGURE 4

Note that while the Pieri rule and the Murnaghan-Nakayama rule expand products in terms of a basis, their skew versions give only one possible (but obviously special) expansion in terms of skew Schur functions, which are not a basis of the space of symmetric functions.

Assaf and McNamara provide an elegant bijective proof of their skew Pieri rule (but not of the skew Murnaghan-Nakayama rule; see Section 7). We describe this rule in detail in Section 3 since an extension of it proves our main result.

Define *quantum power sum symmetric functions* by

$$\widetilde{p}_r = \sum_{\tau \vdash r} (-1)^{\ell(\tau)-1} (q-1)^{\ell(\tau)-1} m_\tau,$$

$$\widetilde{p}_\mu = \widetilde{p}_{\mu_1}\widetilde{p}_{\mu_2}\cdots.$$

For example,

$$\widetilde{p}_4 = m_4 - (q-1)m_{31} - (q-1)m_{22} + (q-1)^2 m_{211} - (q-1)^3 m_{1111}$$

and

$$\widetilde{p}_{22} = m_4 - 2(q-1)m_{31} + (q^2 - 2q + 3)m_{22} + 2(q-1)(q-2)m_{211} + 6(q-1)^2 m_{1111}.$$

The functions $\widetilde{p}_\mu$ have connections with representation theory (more precisely, characters of the Hecke algebra of type A; see for example [3, Theorem 6.5.3]). See also [6] and [5, Chapter III.2].

We have

$$\widetilde{p}_r|_{q=1} = m_r = p_r, \qquad \widetilde{p}_r|_{q=0} = \sum_{\tau \vdash r} m_\tau = s_r, \qquad \lim_{q \to \infty} \frac{\widetilde{p}_r}{q^{r-1}} = (-1)^{r-1} m_{1^r} = (-1)^{r-1} s_{1^r}.$$

There exists a natural generalization of the Murnaghan-Nakayama rule, the *quantum Murnaghan-Nakayama rule* (QMNR):

$$s_\lambda \cdot \widetilde{p}_r = (-1)^{r+1} \sum_{\lambda^+} (-1)^{\mathrm{wt}(\lambda^+/\lambda)} q^{\mathrm{ht}(\lambda^+/\lambda)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)-1} s_{\lambda^+},$$

where the internal sum on the right is over λ^+ such that λ^+/λ is a broken ribbon of size r. See for example [3, Theorem 6.5.2] for a slightly different version.

The following is our main result, the skew quantum Murnaghan-Nakayama rule.

Theorem 4 (SQMNR) *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum_{j=0}^r (-1)^{r+1-j} \sum_{\lambda^+,\mu^-} (-1)^{\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)} q^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} s_{\lambda^+/\mu^-},$$

where the internal sum on the right is over λ^+, μ^- such that λ^+/λ is a broken ribbon of size $r - j$, and μ/μ^- is a broken ribbon of size j.

There is another version of the statement that will be slightly more useful for our purposes.

Theorem 5 (SQMNR$'$) *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} s_{\lambda^+/\mu^-},$$

where the sum on the right is over λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

To see that these two versions are equivalent, note that

$$q^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} =$$

$$= (-1)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)+\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} (-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1},$$

which means that the sign of $(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}$ of a term on the right-hand side of SQMNR is

$$(-1)^{r+1-j+\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)+\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)+\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}.$$

If π/σ is a ribbon, we have $\mathrm{wt}(\pi/\sigma) + \mathrm{ht}(\pi/\sigma) + 1 = |\pi/\sigma|$. Therefore if π/σ is a broken ribbon,

$$(5) \qquad\qquad \mathrm{wt}(\pi/\sigma) + \mathrm{ht}(\pi/\sigma) + \mathrm{rib}(\pi/\sigma) = |\pi/\sigma|.$$

That means that the sign above is equal to

$$(-1)^{r+1-j+|\lambda^+/\lambda|+|\mu/\mu^-|-1} = (-1)^{2r+j} = (-1)^{|\mu/\mu^-|}.$$

The main theorem is a generalization of several statements. The following is a sample:

- $q = 0$: a term on the right-hand side of SQMNR$'$ is non-zero if and only if $\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-) = 0$. In this case, λ^+/λ has height 0 (and is a horizontal strip) and μ/μ^- has width 0 (and is a vertical strip). As noted above, $\widetilde{p}_r|_{q=0} = s_r$. SQMNR$'$ specializes to the skew Pieri rule due to Assaf-McNamara [1].
- $q = 1$: a term on the right-hand side of SQMNR$'$ is non-zero if and only if $\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1 = 0$. In this case, one of λ^+/λ and μ/μ^- is empty, and the other one is a ribbon. As noted above, $\widetilde{p}_r|_{q=1} = p_r$. SQMNR$'$ therefore states

$$s_{\lambda/\mu} \cdot p_r = \sum_{\lambda^+}(-1)^{\mathrm{ht}(\lambda^+/\lambda)}s_{\lambda^+/\mu} + \sum_{\mu^-}(-1)^k(-1)^{\mathrm{wt}(\mu/\mu^-)}s_{\lambda/\mu^-} =$$

$$= \sum_{\lambda^+}(-1)^{\mathrm{ht}(\lambda^+/\lambda)}s_{\lambda^+/\mu} - \sum_{\mu^-}(-1)^{\mathrm{ht}(\mu/\mu^-)}s_{\lambda/\mu^-},$$

  where the first sum is over λ^+ so that λ^+/λ is a ribbon, and the second sum is over μ^- so that μ/μ^- is a ribbon. This is the skew Murnaghan-Nakayama rule due to Assaf-McNamara [2].
- $q \to \infty$: divide SQMNR by q^{r-1} and send $q \to \infty$. The limit of the left-hand side is $(-1)^{r-1}s_{\lambda/\mu}s_{1^r}$. The coefficient of s_{λ^+/μ^-} on the right is

$$\lim_{q\to\infty}(-1)^{r+1-j}(-1)^{\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)}\frac{q^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(q-1)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}}{q^{r-1}} =$$

$$= (-1)^{r+1-j}(-1)^{\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)}\lim_{q\to\infty}q^{-(\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-))},$$

  where we used (5). This is non-zero if and only if $\mathrm{wt}(\lambda^+/\lambda) + \mathrm{ht}(\mu/\mu^-) = 0$, i.e. if λ^+/λ is a vertical strip and μ/μ^- is a horizontal strip, and the limit is $(-1)^{r-1}(-1)^j$. SQMNR therefore implies the conjugate skew Pieri rule.
- $\mu = \emptyset$: SQMNR is obviously the quantum Murnaghan-Nakayama rule.
- $\mu = \emptyset$, $q = 0$: this is the classical Pieri rule.
- $\mu = \emptyset$, $q = 1$: this is the classical Murnaghan-Nakayama rule.
- $\mu = \emptyset$, $q \to \infty$: this implies the classical conjugate Pieri rule.
- $\lambda = \mu = \emptyset$: this gives the expansion of quantum power sum functions in the basis of Schur functions. The only Young diagrams of size r that are also broken ribbons are hooks, i.e. diagrams of partitions of the type $(k, 1^{r-k})$ for $1 \le k \le r$. Therefore (as we verify independently in Lemma 10),

$$\widetilde{p}_r = \sum_{k=1}^{r}(-q)^{r-k}s_{k,1^{r-k}}.$$

Define a *broken ribbon tableau of shape λ/μ and type τ* (respectively, *reverse broken ribbon tableau of shape λ/μ and type τ*) as an assignment of positive integers to the cells of $[\lambda/\mu]$ satisfying the following;

- every row and column is weakly increasing (respectively, weakly decreasing);
- the integer i appear τ_i times;
- the set T_i of cells occupied by i forms a broken ribbon.

For a (reverse) broken ribbon tableau T we define $\mathrm{ht}(T) = \sum \mathrm{ht}(T_i)$, $\mathrm{wt}(T) = \sum \mathrm{wt}(T_i)$, $\mathrm{rib}(T) = \sum \mathrm{rib}(T_i)$.

The main theorem implies the following corollary; compare to [8, Theorem 7.17.3].

Corollary 6 *We have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_\tau = \sum_{\substack{\lambda^+ \supseteq \lambda \\ \mu^- \subseteq \mu}}(-1)^{|\mu/\mu^-|}\chi(\lambda^+, \lambda, \mu, \mu^-; \tau)s_{\lambda^+/\mu^-},$$

where

$$\chi(\lambda^+, \lambda, \mu, \mu^-; \tau) = \sum(-q)^{\mathrm{ht}(T')+\mathrm{wt}(T'')}(1-q)^{\mathrm{rib}(T')+\mathrm{rib}(T'')-1}$$

with the sum over all pairs (T', T'') of a broken ribbon tableau and a reverse broken ribbon tableau of shapes λ^+/λ and μ/μ^-, respectively, and types τ' and τ'', respectively, so that $\tau' + \tau'' = \tau$.

3. Proofs of the skew Pieri rule and its conjugate

One of the most important algorithms on semistandard Young tableaux is the Robinson-Schensted *row insertion*. Given a semistandard Young tableau T of shape λ and an integer k, we can *insert k into T* as follows. Define $k_1 = k$. Find the smallest j so that $T_{1j} > k_1$, replace T_{1j} by k_1, and define k_2 to be the previous value of T_{1j}. Then find the smallest j so that $T_{2j} > k_2$, replace T_{2j} by k_2, and define k_3 to be the previous value of T_{2j}. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda'_i+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion of k into T *exits in row i'*. See [8, §7.11] for details.

EXAMPLE Inserting 1 into the tableau on the left of Figure 5 produces the tableau on the right.

FIGURE 5

Now assume we have a *skew* semistandard Young tableau T of some shape λ/μ. We can *insert k into T* for some integer k in almost exactly the same way. Define $k_1 = k$. Find the smallest j, $\mu_1 < j \leq \lambda_1$, so that $T_{1j} > k_1$, replace T_{1j} by k_1, and define k_2 to be the previous value of T_{1j}. Then find the smallest j, $\mu_2 < j \leq \lambda_2$, so that $T_{2j} > k_2$, replace T_{2j} by k_2, and define k_3 to be the previous value of T_{2j}. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda'_i+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion of k into T *exits in row i'*.

There is, however, another natural kind of insertion. Take i_0 so that either $i_0 = 1$ or $\mu_{i_0-1} > \mu_{i_0}$, and take $k_{i_0+1} = T_{i_0,\mu_{i_0}+1}$. We can *insert from row i_0 in T* as follows. Erase the entry $T_{i_0,\mu_{i_0}+1}$. Find the smallest j, $\mu_{i_0+1} < j \leq \lambda_{i_0+1}$, so that $T_{i_0+1,j} > k_{i_0+1}$, replace $T_{i_0+1,j}$ by k_{i_0+1}, and define k_{i_0+2} to be the previous value of $T_{i_0+1,j}$. Then find the smallest j, $\mu_{i_0+2} < j \leq \lambda_{i_0+2}$, so that $T_{i_0+2,j} > k_{i_0+2}$, replace $T_{i_0+2,j}$ by k_{i_0+2}, and define k_{i_0+3} to be the previous value of $T_{i_0+2,j}$. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda_i+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion from row i_0 in T *exits in row i'*.

EXAMPLE In the following figures, we have an insertion of 1 into a tableau, and insertion from row 2 in a tableau.

FIGURE 6

Note that insertion into T is in a way a special case of insertion from a row in T. Indeed, take $\mu_0 = \lambda_1$, $\lambda_0 = \lambda_1 + 1$, and define $T_{0,\lambda_1} = k$. Then insertion from row 0 in the new tableau gives the same result as insertion of k into the original tableau.

Insertion has an inverse operation, *reverse insertion*. Say we are given a semistandard Young tableau T of shape λ/μ. Take i' so that $\lambda_{i'+1} < \lambda_{i'}$. We *reverse insert from row i' in T* as follows. Define $k_{i'-1} = T_{i',\lambda_{i'}}$. Erase the entry $T_{i',\lambda_{i'}}$. Find the largest j, $\mu_{i'-1} < j \leq \lambda_{i'-1}$, so that $T_{i'-1,j} < k_{i'-1}$, replace $T_{i'-1,j}$ by $k_{i'-1}$, and define $k_{i'-2}$ to be the previous value of $T_{i'-1,j}$. Then find the largest j, $\mu_{i'-2} < j \leq \lambda_{i'-2}$, so that $T_{i'-2,j} < k_{i'-2}$, replace $T_{i'-2,j}$ by $k_{i'-2}$, and define $k_{i'-3}$ to be the previous value of $T_{i'-2,j}$. Continue until we have k_{i_0}, where either $i_0 = 0$ or all elements of row i_0 are $\geq k_{i_0}$. If $i_0 = 0$, the result is a pair (S, k), where S is a semistandard Young tableau and $k = k_0$. We call k the *exiting integer*. If $i_0 \geq 1$ and all elements of row i_0 are $\geq k_{i_0}$, define $T_{i_0,\mu_{i_0}} = k_{i_0}$. The result is a semistandard Young tableau S. We say that the reverse insertion from row i' in T *exits in row i_0*.

EXAMPLE In the following figures, we have reverse insertion from rows 2 (which exits in row 0 with exiting integer 2) and 4 (which exits in row 1).

FIGURE 7

In [1], the operations of insertion and reverse insertion are proved to be inverses of one another in the following sense. If the insertion of an integer k into a semistandard Young tableau T exits in row i' and the resulting tableau is S, then the reverse insertion from row i' in S exits in row 0 and the result is (T, k). If the insertion from row i_0 into T exits in row i' and the resulting tableau is S, then the reverse insertion from row i' in S exits in row i_0 and the result is T. Similarly, if the reverse insertion from row i' in T exits in row 0 and the result is (S, k), then the insertion of k into S exits in row i' and the result is T. And if the reverse insertion from row i' in T exits in row $i_0 \geq 1$ and the result is S, then the insertion from row i_0 into S exits in row i' and the result is T.

We will also need the following property of insertion and reverse insertion. The lemma essentially states that insertion paths never cross.

Lemma 7 *Say we are given a semistandard Young tableau T.*
  (a) *If S is obtained by reverse insertion from row i' in T that exits in row $i_0 > 0$, and R is obtained by reverse insertion from row $i'' < i'$ in S that exits in row i_0', then $i_0' < i_0$.*
  (b) *If S is obtained by reverse insertion from row i' in T that exits in row 0 with exiting integer k', and R is obtained by reverse insertion from row $i'' < i'$ in S that exits in row i_0', then $i_0' = 0$ and the reverse insertion exits with exiting integer $k'' > k'$.*
  (c) *If reverse insertion from row i' in T exits in i_0 and insertion from row $i_0' > i_0$ in T exits in i'', then $i'' > i'$.*

Proof. (a) If $i_0 \geq i''$, then $i_0' < i'' \leq i_0$ and the claim follows. Assume $i_0 < i''$. We claim that if the reverse insertion from row i' in T passes through (i, j') and the reverse insertion from row i'' in S passes through (i, j''), then $j'' \geq j'$; in other words, reverse insertion from T lies weakly to the left of the reverse insertion from S. The statement is true for $i = i''$ because in this case, $j'' = \lambda_i$. If the statement holds for i and $j' < j''$, the reverse insertion from row i' in T bumps the entry $T(i, j')$ into row $i-1$; then the reverse insertion from row i'' in S bumps the entry $T(i, j'') \geq T(i, j')$ into a position which cannot be the left of the new position of $T(i, j')$ in row $i-1$. If, on the other hand, $j' = j''$, the reverse insertion from row i' in

T again bumps the entry $T(i, j')$ into row $i - 1$ and the entry $T(i, j')$ is itself replaced by a strictly larger entry. Then reverse insertion from row i'' in S bumps this strictly larger entry into the next row into a position which cannot be the the left of the new position of $T(i, j')$ in row $i - 1$.

This means that the reverse insertion from row i'' in S passes through row i_0 and so it exits in row $< i_0$.

(b) By the reasoning in (a), the reverse insertion from S is weakly to the right of the reverse insertion from T. In particular, reverse insertion from S reaches row 1, and if the exiting integer k' is bumped from position $(1, j')$, then the exiting integer k'' is bumped from $(1, j'')$ for $j'' \geq j'$. In particular, $k'' > k'$.

(c) We claim that if reverse insertion from row i' in T passes through $(i, j',$ where $i_0' \leq i \leq i'$, then insertion from i_0' in T passes through the cell (i, j'') for some $j'' \leq j'$. The statement is true for $i = i_0'$ because in that case, $j'' = \mu_i + 1 \leq j'$. If it holds for i, then the entry from row $i + 1$, say a, that was bumped into row i during the reverse insertion from row i' in T, must be $< a$, and lies in position (i, j') in T. Therefore $T(i, j'') < a$ and cannot be bumped into a position to the right of a in row $i + 1$ in T.

In particular, insertion from row i_0' in T passes through row i', and so the insertion exits in row $i'' > i'$. $\square$

The involution by Assaf and McNamara that proves the skew Pieri rule works as follows. Say we are given a skew shape λ/μ and a semistandard Young tableau T of shape λ^+/μ^-, where λ^+/λ is a horizontal strip and μ/μ^- is a vertical strip. Let v be the empty word. Let $i = \infty$ if $\mu = \mu^-$, and let i be the top row of μ/μ^- otherwise.

While $\lambda^+ \neq \lambda$ and the reverse insertion from row i', the top row of λ^+/λ, in T exits in row 0 and results in (S, k), attach k to the beginning of v, let $T = S$, and let λ^+/μ^- be the shape of the new T (note that $\lambda_{i'}^+$ is decreased by 1 and μ^- remains the same).

If the while loop stops when $\lambda^+ \neq \lambda$ and the reverse insertion from row i' in T exits in row i_0, $0 < i_0 < i$, and results in S, let $T = S$.

If the while loop stops when $\lambda^+ = \lambda$, $\mu \neq \mu^-$, or when $\lambda^+ \neq \lambda$ and the reverse insertion from row i' in T would exit in row i_0, $i_0 \geq i$, insert from row i into T and call the resulting tableau T.

Finish the algorithm by inserting the entries of v from left to right into T. The final result is a semistandard Young tableau, we denote it $\Phi_{\lambda, \mu, \lambda^+, \mu^-}(T)$.

EXAMPLE (Consult Figure 8.) The left drawing shows a skew semistandard Young tableau with $\lambda^+ = 8855432$, $\lambda = 855533$, $\mu = 43222$, $\mu^- = 42111$. The while loop changes v to 2445 and it stops because after four reverse insertions, the next reverse insertion (from row 7) exits in row 1 (see the second drawing). Since this is strictly above the top row of μ/μ^-, i.e. 2, we also perform this reverse insertion from row 7 (see the third drawing). Then we insert the integers 2, 4, 4, 5 and we get the skew semistandard Young tableau pictured on the right, with $\lambda^+ = 8855431$, $\lambda = 855533$, $\mu = 43222$, $\mu^- = 32111$.

```
Drawing 1                Drawing 2                Drawing 3                Drawing 4
      2 4 4 5                  3 5 6 6                1 3 5 6 6              1 2 4 4 5
    1 1 3 5 6 6            1 1 5                    1 2 5                  1 2 3 5 6 6
  2 2 2 5                2 2 2 7                  2 2 4 7                2 2 4 5
  3 3 4 7                3 3 4 8                  3 3 6 8                3 3 6 7
  8 6 8                  5 6                      5 7                    5 7 8
1 6 7                  1 6 7                    1 6 8                  1 6 8
2 8                    2 8                      2                      2
```
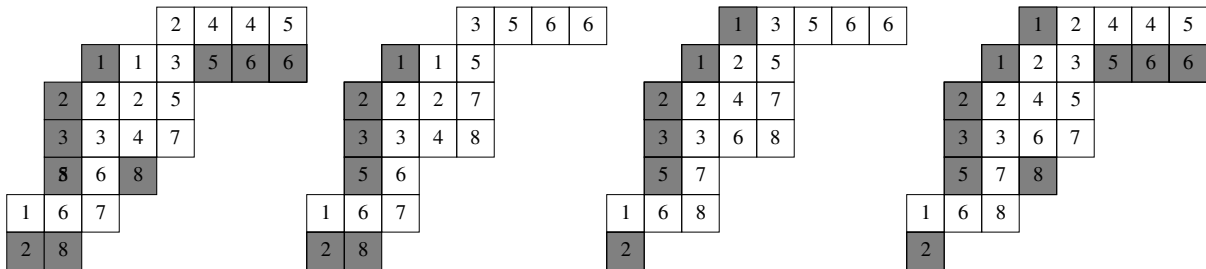
FIGURE 8

(Consult Figure 9.) In the second example, we start with $\lambda^+ = 8855431$, $\lambda = 855533$, $\mu = 43222$, $\mu^- = 42111$, see the left drawing. The while loop again changes v to 2445 and it stops because after four reverse insertions, the next reverse insertion (from row 7) exits in row 5 (see the second drawing). Since

this is not above the top row of μ/μ^-, we do not perform this reverse insertion. Instead, we insert from the top row of μ/μ^-, i.e. 2 (see the third drawing). Then we insert the integers 2, 4, 4 and 5 and we get the skew semistandard Young tableau pictured on the right, with $\lambda^+ = 8855432$, $\lambda = 855533$, $\mu = 43222$, $\mu^- = 43111$.
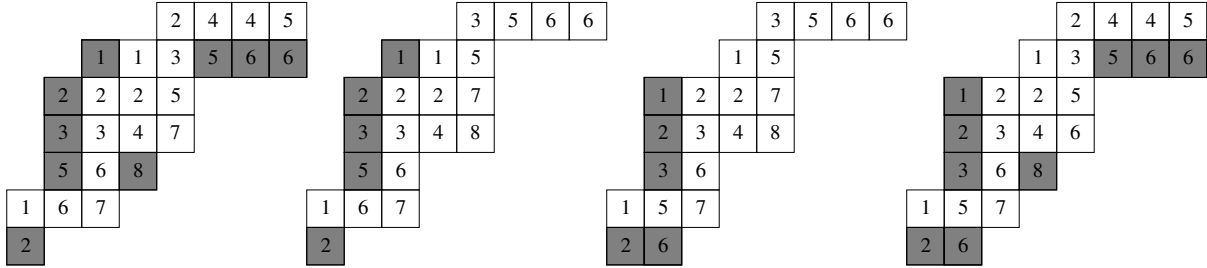

FIGURE 9

(Consult Figure 10.) In the third example, we start with $\lambda^+ = 9855331$, $\lambda = 855533$, $\mu = 43222$, $\mu^- = 43222$, see the left drawing. The while loop changes v to 11245 and it stops because after five reverse insertions, $\lambda^+ = \lambda$ and $\mu = \mu^-$. So we insert the integers 1, 1, 2, 4 and 5 and we get the original skew semistandard Young tableau, pictured on the right.
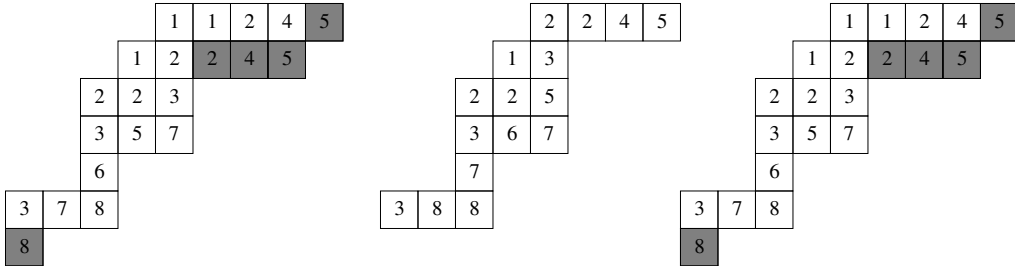

FIGURE 10

It turns out that Φ is an involution, and T is a fixed point if and only if $\mu = \mu^-$ and the while loop stops when $\lambda^+ = \lambda$. Such fixed points are in one-to-one correspondence with pairs (S, v), where S is a semistandard Young tableau of shape λ/μ and v is a weakly increasing word. Indeed, if we stop the algorithm after the while loop, we have exactly such a pair, and given a pair (S, v), we can insert the entries of v from left to right into S to get the corresponding T. Furthermore, if T is not a fixed point, then $|\mu^{--}| = |\mu^-| \pm 1$. It is easy to see that this shows the skew Pieri rule. See [1] for details and a precise proof.

As mentioned in Section 2, the conjugate skew Pieri rule follows from SPR by applying the involution ω on the algebra of symmetric functions. There is, however, an involution in the spirit of Assaf-McNamara that proves CSPR.

Fix λ, μ, r. A term on the right-hand side of CSPR is represented by a semistandard skew Young tableau of shape λ^+/μ^-, where λ^+/λ is a vertical strip, μ/μ^- is a horizontal strip, and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. Such a tableau T is weighted by $(-1)^{|\mu/\mu^-|}$. Let i denote the bottom row of μ/μ^- (unless $\mu = \mu^-$, in which case take $i = 0$). Now reverse insert from row i', the bottom row of λ^+/λ, in T (unless $\lambda^+ = \lambda$). If the reverse insertion exits the diagram in row $\geq i$ (*except* in the case when $\mu = \mu^-$ and the reverse insertion exits in

row 0), call this new diagram $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T)$. See Figure 11, left. If this reverse insertion exits the diagram in row $< i$, or if $\lambda^+ = \lambda$, insert from row i in T and call the result $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T)$. See Figure 11, middle. When $\mu = \mu^-$ and the reverse insertion exits in row 0, take $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T) = T$. See Figure 11, right.

EXAMPLE For the skew semistandard Young tableau on the left in Figure 11, reverse insertion from row 9 (the bottom row of λ^+/λ) exits in row 5, which is weakly below the bottom row of μ/μ^-. Therefore we perform this reverse insertion, and the result is the left picture of Figure 12. For the skew semistandard Young tableau in the middle of Figure 11, reverse insertion from row 9 (the bottom row of λ^+/λ) exits in row 4, which is strictly above the bottom row of μ/μ^-. Therefore we insert from row 5 (the bottom row of μ/μ^-), the result is the middle picture of Figure 12. For the skew semistandard Young tableau on the right of Figure 11, reverse insertion from row 9 (the bottom row of λ^+/λ) exits in row 0. This means that the tableau is a fixed point of Ψ. The right picture in Figure 12 shows the skew semistandard Young tableau that we get if we repeatedly reverse insert from the bottom row of λ^+/λ; the exiting integers are $1, 2, 3, 4, 5, 6$.
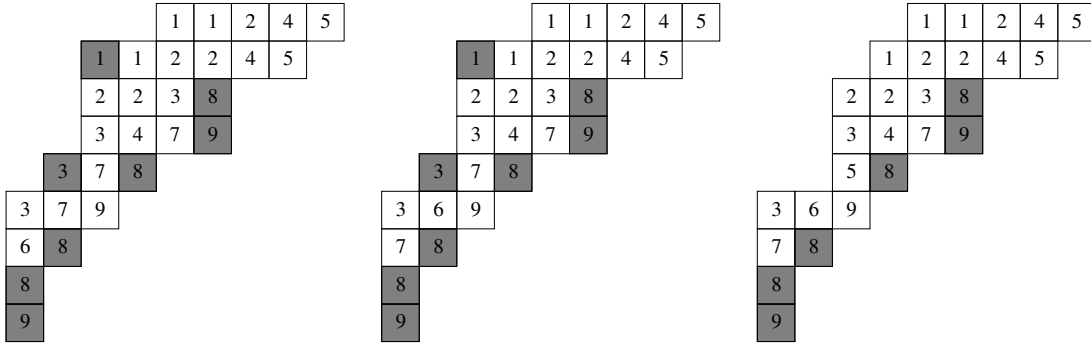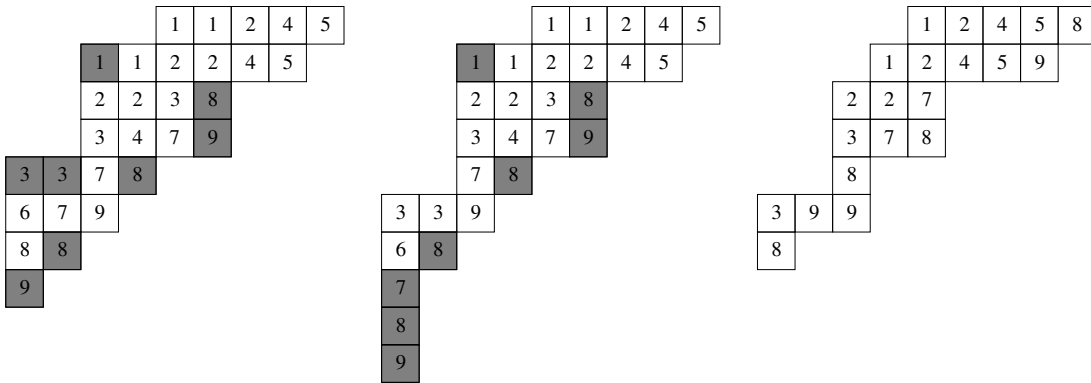

FIGURE 11

FIGURE 12

Proposition 8 *The map $\Psi_{\lambda,\mu,\lambda^+,\mu^-}$ is an involution that is sign-reversing except on fixed points. Furthermore, the fixed points are in a natural bijective correspondence with elements on the left-hand side of CSPR.*

Proof. Say that $\lambda^+ \neq \lambda$ and the reverse insertion from row i', the bottom row of λ^+/λ, exits in row i_0, $0 \neq i_0 \geq i$, where i is the bottom row of μ/μ^-, and results in S of shape λ^{++}/μ^{--}. Recall that in this

case, $\Psi(T) = S$. The partition μ^{--} differs from μ^- only in row i_0, and $\mu_{i_0}^{--} = \mu_{i_0}^- - 1$. Also, λ^{++} differs from λ^+ only in row i', and $\lambda_{i'}^{++} = \lambda_{i'}^+ - 1$. Note that the bottom row of μ/μ^{--} is i_0. If $\lambda^{++} = \lambda$, then $\Psi(S)$ is obtained by inserting from row i_0 in S, which is T (because insertion and reverse insertion are inverse operations). If $\lambda^{++} \neq \lambda$, then the bottom row of λ^{++}/λ is strictly above i'; furthermore, reverse insertion from this row exits in row $< i_0$ by Lemma 7, part (a). So we also obtain $\Psi(S)$ by inserting from row i_0 in S, and we get T.

Now assume that $\lambda^+ \neq \lambda$ and the reverse insertion from row i', the bottom row of λ^+/λ, exits in row $< i$. Then $S = \Psi(T)$ of shape λ^{++}/μ^{--} is the result of inserting from row i in T, assume that this insertion exits in row i''. We know that μ^{--} differs from μ^- only in row i, $\mu_i^{--} = \mu_i^- + 1$, and λ^{++} differs from λ^+ only in row i'', $\lambda_{i''}^{++} = \lambda_{i''}^+ + 1$. By Lemma 7, part (c), $i'' > i'$. That means that when we perform Ψ on S, we reverse insert from row i'' in S. The reverse insertion results in T and exits in row i, which is weakly below the bottom row of μ/μ^{--}, so $\Psi(S) = T$.

If $\lambda^+ = \lambda$, we obtain $S = \Psi(T)$ of shape λ^{++}/μ^{--} by inserting from row i in T, say that the insertion exits in row i'. In S, λ^+/λ has only one cell, which is in row i'. Furthermore, reverse insertion from row i' in S exits in row i, which is weakly below the bottom row of μ/μ^{--}. So the result of this reverse insertion, T, is also $\Psi(S)$.

Finally, assume that T is a fixed point, i.e. that $\mu = \mu^-$ and that the reverse insertion from row i', the bottom row of λ^+/λ, exits in row 0. Call the resulting tableau T_1 (of shape λ^{++}/μ) and the exiting integer k_1. By Lemma 7, part (b), that means that if we again reverse insert from the bottom row of λ_1^+/λ in T_1, the reverse insertion again exits in row 0, and the exiting integer k_2 is strictly greater than k_1. Call the resulting tableau T_2, and continue. After r steps, we have a semistandard Young tableau $S = T_r$ of shape λ/μ, and a strictly decreasing word $w = k_r k_{r-1} \cdots k_1$. Such pairs (S, w) are obviously enumerated by the left-hand side of CSPR. $\qquad\square$

4. A BIJECTIVE PROOF OF THE MAIN THEOREM

The first step of our proof is to interpret the right-hand side of SQMNR$'$ as a weighted sum over some combinatorial objects. The appropriate objects turn out to be skew semistandard Young tableaux with some cells colored gray. To motivate these colorings, observe the following. If we "glue" together a vertical strip and a horizontal strip in such a way that the result is a skew diagram, then this skew diagram cannot have any 2×2 squares. In other words, it is a broken ribbon. This also holds the other way around: if we are given a broken ribbon, we can break it up into a vertical strip and a horizontal strip. See Figure 13 for two examples. Note that the right example is special: the white cells (i.e. the cells of μ and the cells of λ/μ we put in the horizontal strip) form a partition. In other words, the cells of the horizontal strip never have cells of the vertical strip to the left or above them.
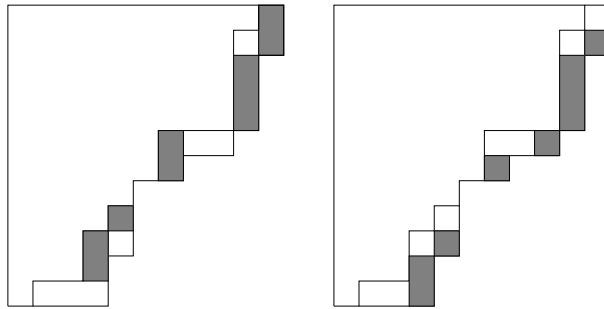

FIGURE 13

Let us multiply both sides of SQMNR$'$ by $1 - q$ and call this statement SQMNR$''$:

$$s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r} (1-q)^{\ell(\tau)} m_\tau \right) = \sum_{\lambda^+, \mu^-} (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda) + \mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)} s_{\lambda^+/\mu^-}.$$

We have fixed λ, μ, r. Say that we are given λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, and a skew semistandard Young tableau T of shape λ^+/μ^-. Our first goal is to break up each of the broken ribbons λ^+/λ and μ/μ^- into a vertical strip and a horizontal strip. More precisely, we wish to choose partitions λ', μ' such that λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips. We weight such a selection with

$$(-1)^{|\mu/\mu^-|} (-q)^{|\lambda^+/\lambda'| + |\mu'/\mu^-|}.$$

We color the cells of λ^+/λ' and μ'/μ^- gray and leave the other cells white. So our requirements are saying that both the gray cells of λ^+/λ and the white cells of μ/μ^- form a vertical strip, and both the white cells of λ^+/λ and the gray cells of μ/μ^- form a horizontal strip; also, the white cells form a diagram of some shape λ'/μ' for $\lambda \subseteq \lambda' \subseteq \lambda^+$, $\mu^- \subseteq \mu' \subseteq \mu$. Furthermore, the weight of such an object is $(-1)^{|\mu/\mu^-|}(-q)^j$, where j is the number of gray cells.

EXAMPLE Figure 14 shows four examples with weights q^{16}, q^{14}, $-q^{13}$ and $-q^{11}$.
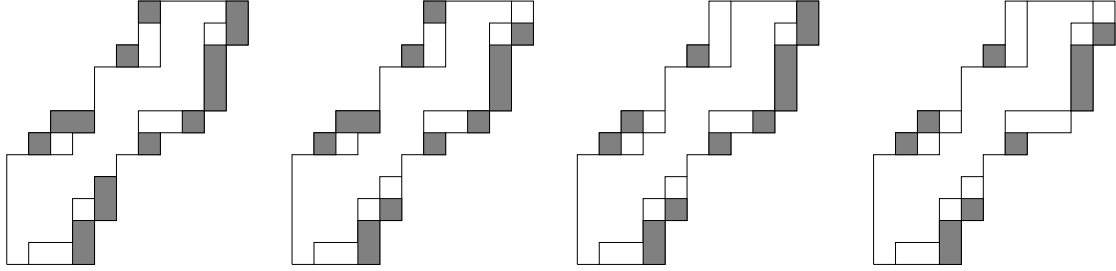

FIGURE 14

We claim that these objects indeed enumerate the right-hand side of SQMNR$'$.

Lemma 9 *For fixed $\lambda, \mu, \lambda^+, \mu^-$, we have*

$$\sum_{\lambda', \mu'} (-1)^{|\mu/\mu^-|} (-q)^{|\lambda^+/\lambda'| + |\mu'/\mu^-|} = (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda) + \mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)},$$

where the sum on the left runs over all λ', μ' such that λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips.

Proof. For each cell of λ^+/λ, we have to decide whether or not to put it in λ'/λ or in λ^+/λ' (i.e. whether to make it white or gray). If a cell in λ^+/λ has a right neighbor in λ^+/λ, it cannot be in λ^+/λ', since its right neighbor would also have to be in λ^+/λ', and this would contradict the requirement that λ^+/λ' is a vertical strip. Similarly, if a cell in λ^+/λ has an upper neighbor in λ^+/λ, it cannot be in λ'/λ, since its upper neighbor would also have to be in λ'/λ, and this would contradict the requirement that λ'/λ is a horizontal strip.

This means that the colors of all the cells in λ^+/λ are determined, except for the top right cell of each ribbon of λ^+/λ, which can be either white or gray.

If a cell in μ/μ^- has a right neighbor in μ/μ^-, it cannot be in μ/μ', since its right neighbor would also have to be in μ/μ', and this would contradict the requirement that μ/μ' is a vertical strip. Similarly, if a cell in μ/μ^- has an upper neighbor in μ/μ^-, it cannot be in μ'/μ^-, since its upper neighbor would also have to be in μ'/μ^-, and this would contradict the requirement that μ'/μ^- is a horizontal strip.

This means that the colors of all the cells in μ/μ^- are determined, except for the top right cell of each ribbon of μ/μ^-, which can be either white or gray.

In other words, we have two choices for each upper right cell of each ribbon of $(\lambda^+/\lambda) \cup (\mu/\mu^-)$. This already means that there are $2^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)}$ terms on the left-hand side.

We have at least $\mathrm{ht}(\lambda^+/\lambda)$ gray cells in λ^+/λ, and at least $\mathrm{wt}(\mu/\mu^-)$ gray cells in μ/μ^-. So the weight of a term on the left-hand side is

$$(-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(-q)^j,$$

where j is the number of cells that are gray by choice, and these choices are made independently. By the binomial theorem,

$$\sum \binom{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)}{j}(-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(-q)^j =$$

$$(-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)},$$

which finishes the proof of the lemma. $\qquad\square$

We have managed to rewrite SQMNR″ as follows:

$$s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r}(1-q)^{\ell(\tau)}m_\tau\right) = \sum_{\lambda^+,\lambda',\mu^-,\mu'}(-1)^{|\mu/\mu^-|}(-q)^{|\lambda^+/\lambda'|+|\mu'/\mu^-|}s_{\lambda^+/\mu^-},$$

where the sum is over partitions $\lambda^+, \lambda', \mu^-, \mu'$ such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips.

Fix λ, μ and r. By 1, a term on the right-hand side of SQMNR″ therefore corresponds to a semistandard Young tableau T with some cells colored white and some cells colored gray, such that the following properties are satisfied:

- the shape of T is λ^+/μ^- for some $\lambda^+ \supseteq \lambda$ and $\mu^- \subseteq \mu$, $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, and λ^+/λ and μ/μ^- are broken ribbons;
- the white cells form a skew diagram λ'/μ' for some partitions λ', μ';
- the white cells in λ^+/λ form a horizontal strip, and the white cells in μ/μ^- form a vertical strip;
- the gray cells are in $(\lambda^+/\lambda) \cup (\mu/\mu^-)$, and they form a vertical strip in λ^+/λ and a horizontal strip in μ/μ^-;

We call such an object a *colored tableau* of shape $(\lambda, \mu, \lambda', \mu', \lambda^+, \mu^-)$. We weight a colored tableau by

$$(-1)^{|\mu/\mu^-|}(-q)^{|\lambda^+/\lambda'|+|\mu'/\mu^-|}.$$

Now perform the involution Ψ on the gray cells of a colored tableau. More specifically, find $\Psi_{\lambda',\mu',\lambda^+,\mu^-}(T)$. Since λ^+/λ' is a vertical strip and μ'/μ^- is a horizontal strip, the map is well defined. One gray cell is removed, and one gray cell is added in the process. The result is a colored tableau T' of shape $(\lambda, \mu, \lambda', \mu', \lambda^{++}, \mu^{--})$ for some λ^{++}, μ^{--}; it has the same white cells as T, the same number of gray cells as T, and with the property that $|\mu/\mu^{--}| = |\mu/\mu^-| \pm 1$ unless $T = T'$ is a fixed point.

This already cancels a large number of terms. The ones that remain correspond to fixed points of $\Psi_{\lambda',\mu',\lambda^+,\mu^-}$. Each such fixed point consists of a semistandard skew Young tableau S of some shape λ'/μ', where λ'/λ is a horizontal strip and μ/μ' is a vertical strip, and of a strictly decreasing word w coming from reverse insertion of λ^+/λ'. Such an object is weighted by $(-1)^{|\mu/\mu'|}(-q)^{|w|}$.

Now apply the Assaf-McNamara involution Φ to the tableau. More specifically, find $\Phi_{\lambda,\mu,\lambda',\mu'}(S)$. The result is a semistandard Young tableaux S' of some shape λ^{+++}/μ^{---} with the property that $|\mu/\mu^{---}| = |\mu/\mu^-| \pm 1$ unless $S = S'$ is a fixed point. This cancels more terms. The ones that remain correspond to fixed points of $\Phi_{\lambda,\mu,\lambda',\mu'}$, together with a strictly decreasing word w and the weight $(-q)^{|w|}$. Each such fixed point consists of a semistandard Young tableau R of shape λ/μ, together with a weakly increasing word v and a strictly decreasing word w. Such an object is weighted by $(-q)^{|w|}$. Furthermore, every such triple (R, v, w) appears as a non-canceling term on the right. Indeed, insert the elements of v into R to

get a semistandard Young tableau S of shape λ'/μ for some partition λ' so that λ'/λ is a horizontal strip; then insert the elements of w into S and color the new cells gray to get a colored tableau T of shape λ^+/μ for some partition λ^+ so that λ^+/λ' is a vertical strip. Then applying Ψ and Φ to T yields (R, v, w).

It remains to enumerate all triples (R, v, w). If we want (v, w) to contain, say, τ_i copies of i, $1 \leq i \leq \ell$, we can choose any j-subset of $\{1, \ldots, \ell\}$ and put the elements in decreasing order to form w, and then put the remaining elements of the multiset $\{1^{\tau_1}, 2^{\tau_2}, \ldots, \ell^{\tau_\ell}\}$ in weakly increasing order to form v. Furthermore, the weight of (R, v, w) for these v and w is $(-q)^j$. That means that the right-hand side of SQMNR$''$ becomes, after cancellations,

$$s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r} (1-q)^{\ell(\tau)} m_\tau \right),$$

which is the left-hand side of SQMNR$''$.

5. A PROOF VIA SKEW LITTLEWOOD-RICHARDSON RULE

It is informative to use Lam-Lauve-Sotille's [4] skew Littlewood-Richardson rule to find another proof of SQMNR. The first lemma is a simple computation that allows us to replace the quantum power sum functions with "hook" Schur functions and should remind the reader of the enumeration of pairs (v, w) for v a weakly increasing word and w a strictly decreasing word at the end of the previous section. The second lemma is technical and states that a certain property is preserved in jeu de taquin slides. And the third lemma sheds some light on connections between jeu de taquin, hooks, and decompositions of broken ribbons into vertical and horizontal strips.

Lemma 10 *For all r, we have*

$$\widetilde{p}_r = \sum_{k=1}^{r} (-q)^{r-k} s_{k,1^{r-k}}.$$

Proof. Let us compute the expansion of the right-hand side in basis m_λ. Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and k, $1 \leq k \leq r$, it is easy to count the number of semistandard Young tableaux of shape $(k, 1^{r-k})$ and type λ: place 1 in cell $(1,1)$, choose the elements to place in the (only) cell of rows $2, \ldots, r-k+1$ in $\binom{\ell-1}{r-k}$ ways and place them in the first column in strictly increasing order, and place the remaining elements in weakly increasing order in the first row. This tells us that the coefficient of m_λ in the right-hand side is

$$\sum_{k=1}^{r} (-q)^{r-k} \binom{\ell-1}{r-k} = (1-q)^{\ell-1},$$

which is also the coefficient of m_λ in $\widetilde{p}_r$. $\square$

For the second lemma, we have to recall the celebrated *backward (respectively, forward) jeu de taquin slide* due to Schützenberger. Say we are given a skew standard Young tableau of shape λ/μ. Let $c = c_0$ be a cell that is not in λ/μ, shares the right or lower edge (respectively, the left or upper edge) with λ/μ, and is such that $\lambda/\mu \cup c$ is a valid skew diagram. Let c_1 be the cell of λ/μ that shares an edge with c_0; if there are two such cells, take the one with the smaller entry (respectively, larger entry). Then move the entry occupying c_1 to c_0, look at the tableau entries below or to the right of c_1 (respectively, above or to the left of c_1), and repeat the same procedure. We continue until we reach the boundary, say in m moves. The new tableau is a standard Young tableau and is called $\mathrm{jdt}_c(T)$. We say that $c_0, c_1, \ldots, c_m$ is the *path* of the slide.

If T is a skew standard Young tableau, we can repeatedly perform backward jeu de taquin slides. The final result S is a standard Young tableau of straight shape, and it is independent of the choices during the execution of the algorithm. We say that T *rectifies* to S. See [8, Appendix A1.2].

Say we are given a standard Young tableau T of shape λ/μ. We say that T has the *k-NE property* if the following statements are true:

NE1 the entry in the last cell of the first non-empty row (i.e. the northeast cell) of λ/μ is k;

NE2 if $i < j < k$, then i appears strictly to the left of j in T;
NE3 if $j > i > k$, then i appears strictly above j in T.

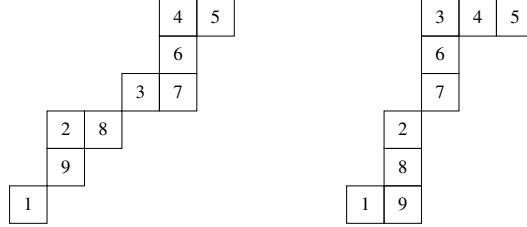The following figure shows some tableaux with 5-NE property.

FIGURE 15

Lemma 11 *If a tableau T has the k-NE property, its shape is a broken ribbon. Furthermore, the k-NE property is preserved in a jeu de taquin slide.*

Proof. For the first statement, assume that there is a 2×2 square in λ/μ, assume it has numbers a, b in the upper row and c, d in the lower row. If $c < k$, then $a < c$ implies $a < k$, and this is a contradiction with property NE2. If $c > k$, then $d > c$ implies $d > k$, and this is contradiction with property NE3. But we cannot have $c = k$ since c is obviously not the northeast cell of T.

Now take the path $c_0, c_1, \ldots, c_m$ of a backward slide in T. We claim that either all c_i are in the same row, or all c_i are in the same column, or $m = 2$, c_1 is below c_0, and c_2 is to the right of c_1.

If they are not in the same row or column, it means that the c_i's take a turn. All three cells involved in the turn (i.e. c_i, c_{i+1}, c_{i+2}, where either c_{i+1} lies to the right of c_i and c_{i+2} lies below c_{i+1}, or c_{i+1} lies below c_i and c_{i+2} lies to the right of c_{i+1}) cannot be in λ/μ, since that would imply that there is a 2×2 block in λ/μ. Therefore the only option is if the entries involved in the turn are c_0, c_1, c_2. Say that c_1 lies to the right of c_0 and c_2 lies below c_1. The fact that we can add c_0 to λ/μ implies that there is a cell c' of λ/μ below c_0, with entry, say, a_3. Say that we have a_1 in c_1 and a_2 in c_2. We have $a_3 < k$ (since $a_3 > k$ would imply $a_2 > a_3 > k$, and this would contradict NE3). We also have $a_1 < a_3$, since otherwise we would be sliding from c' into c_0 rather than from c_1. But then $a_1 < k$ lies to the right of $a_3 < k$, even though $a_1 < a_3$, which contradicts NE2.

That means that c_1 lies below c_0 and c_2 lies to the right of c_1. If $m \geq 3$, there must be cells both above c_2 and c_3 in λ/μ, and this would give a 2×2 block.

Assume first that all c_i are in the same row. Obviously the northeast cell is preserved, so property NE1 holds for the new tableau. Furthermore, since all cells of the tableau stay in the same row, NE3 is preserved. If NE2 is violated in the new tableau, it must mean that there is a cell c' with entry $< k$ in T in the same column as c_0. But this can only happen if c' lies immediately below c_0; by NE2, its entry is less than the entry of c_1, and therefore we would slide from c' into c_0, not from c_1.

If all c_i are in the same column, the proof that the properties NE1, NE2 and NE3 are preserved is completely analogous. So let us assume that we have $m = 2$, c_1 is below c_0, and c_2 is to the right of c_1. There must be a cell c' of λ/μ to the right of c_0, say with entry a_3. Assume we have a_1 in c_1 and a_2 in c_2. Then $a_1 < k$ ($a_1 > k$ would imply $a_2 > k$ and contradict NE3) and $a_2 > k$ ($a_2 < k$ would imply $a_3 < k$ and contradict NE2). So all cells with entries $< k$ stay in the same column, and all cells with entries $> k$ stay in the same row. Therefore NE2 and NE3 are still satisfied, and it is clear that the northeast cell stays in place.

The proof for a forward slide is analogous. This completes the proof of the lemma. □

Lemma 12 *Take r, k, $1 \leq k \leq r$, and let S be the standard Young tableau of shape $(k, 1^{r-k})$ with $1, 2, \ldots, k$ in the first row, and $k+1, k+2, \ldots, r$ in rows $2, 3, \ldots, r-k+1$. Choose a skew shape λ/μ. Then the*

number of standard Young tableaux of shape λ/μ that rectify to S is $\binom{\operatorname{rib}(\lambda/\mu)-1}{k-1-\operatorname{wt}(\lambda/\mu)}$ if λ/μ is a broken ribbon of size r, and 0 otherwise.

Proof. Obviously the number is 0 unless $|\lambda/\mu| = r$.

Note that S has the k-NE property. By Lemma 11, that means that if T of shape λ/μ rectifies to S, T has the k-NE property and its shape λ/μ is a broken ribbon. Furthermore, there is only one non-skew standard Young tableau that has the k-NE property, and that is S.

It remains to assume that λ/μ is a broken ribbon of size r, and to count the number of standard Young tableaux of shape λ/μ that have the k-NE property. Place k in the northeast cell. If a cell in λ/μ has a right neighbor in λ/μ, then the entry has to be less than k (otherwise both this entry and the entry to the right would be greater than k, and this would contradict NE3). Similarly, if a cell in λ/μ has an upper neighbor in λ/μ, then the entry has to be greater than k (otherwise both this entry and the entry above it would be less than k, and this would contradict NE2).

This means that there are at least $\operatorname{wt}(\lambda/\mu)$ elements that are $< k$. We can choose the northeast element of any ribbon except the northeast ribbon and make it $< k$. Since there are $k-1$ elements total that are less than k, we have

$$\binom{\operatorname{rib}(\lambda/\mu)-1}{k-1-\operatorname{wt}(\lambda/\mu)}$$

choices. □

Finally, recall the following result from [4]. For standard Young tableaux T and S, we let $T * S$ be the tableau we get by placing T below and to the left of S. See Figure 16 for an example.
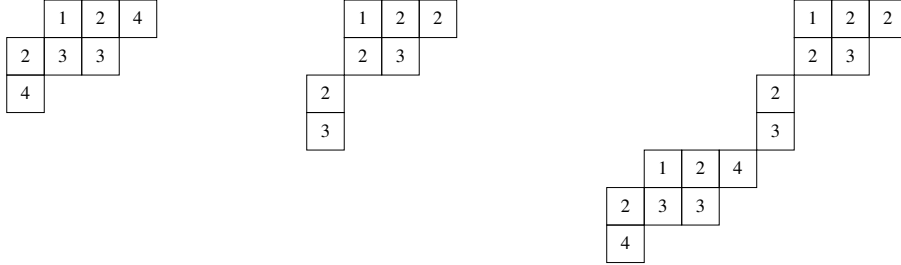

FIGURE 16. Tableaux T, S and $T * S$.

Theorem 13 (Skew Littlewood-Richardson rule – SLRR) *Let λ, μ, σ, τ be partitions and fix a standard Young tableau T of shape σ. Then*

$$s_{\lambda/\mu} s_{\sigma/\tau} = \sum (-1)^{|R^-|} s_{\lambda^+/\mu^-},$$

where the sum is over triples (R^-, R^+, R) of standard Young tableaux of respective shapes $(\mu/\mu^-)^c$, λ^+/λ and τ such that $R^- * R^+ * R$ rectifies to T.

SLRR and the lemmas indeed prove SQMNR as follows. By Lemma 10,

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum_{k=1}^{r} (-q)^{r-k} s_{\lambda/\mu} \cdot s_{k,1^{r-k}}.$$

By SLRR,

$$s_{\lambda/\mu} \cdot s_{k,1^{r-k}} = \sum_{R^-, R^+} (-1)^{|R^-|} s_{\lambda^+/\mu^-},$$

where the sum is over $R^- \in \operatorname{SYT}((\mu/\mu^-)^c)$, $R^+ \in \operatorname{SYT}(\lambda^+/\lambda)$ such that $R^- * R^+$ rectifies to T, where T is the standard Young tableau of shape $(k, 1^{r-k})$ with $1, 2, \ldots, k$ in the first row, and $k+1, k+2, \ldots, r$ in

rows $2, 3, \ldots, r - k + 1$. By Lemma 12, the sum on the right is over λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons, and for such λ^+, μ^-, the coefficient of s_{λ^+/μ^-} is

$$(-1)^{|\mu/\mu^-|}\binom{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-) - 1}{k - 1 - \mathrm{wt}(\lambda^+/\lambda) - \mathrm{ht}(\mu/\mu^-)}.$$

This means that the coefficient of s_{λ^+/μ^-} in $s_{\lambda/\mu} \cdot \widetilde{p}_r$ is

$$(-1)^{|\mu/\mu^-|}\sum_k (-q)^{r-k}\binom{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-) - 1}{k - 1 - \mathrm{wt}(\lambda^+/\lambda) - \mathrm{ht}(\mu/\mu^-)}.$$

Since $r = \mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-) + \mathrm{wt}(\lambda^+/\lambda) + \mathrm{wt}(\mu/\mu^-) + \mathrm{ht}(\lambda^+/\lambda) + \mathrm{ht}(\mu/\mu^-)$, the sum equals

$$(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}\sum_k (-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1-(k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-))}\binom{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}{k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-)}$$

$$= (-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}$$

by the binomial theorem. This is SQMNR$'$.

6. Some conjectures involving Hall-Littlewood polynomials

The quantum power sum functions $\widetilde{p}_r$ are equal to Hall-Littlewood polynomials P_r (with parameter q instead of the usual t), see e.g. [5, page 214]. So while SPR gives the expansion of $s_{\lambda/\mu}s_r$, SQMNR gives the expansion of $s_{\lambda/\mu}P_r$. Of course, the expansion of $P_\lambda P_r$ and $P_\lambda P_{1^r} = P_\lambda s_{1^r}$ in terms of P_{λ^+} are two of the basic results for Hall-Littlewood polynomials (see [5, §III, (3.2) and (3.10)]). The following questions naturally arise. Can we exchange the roles of P and s in SQMNR, i.e. is there a natural expansion of $P_{\lambda/\mu}s_r$ in terms of P_{λ^+/μ^-}? What about $P_{\lambda/\mu}s_{1^r}$? The following conjectures suggest that the answers to these questions are in the affirmative.

Recall the definition of the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}, \text{ where } [i]! = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\ldots+q^{i-1}).$$

For a horizontal strip λ/μ, define

$$\mathrm{hs}(\lambda/\mu) = \prod_{\substack{\lambda_j^c = \mu_j^c + 1 \\ \lambda_{j+1}^c = \mu_{j+1}^c}} (1 - q^{m_j(\lambda)}).$$

For a vertical strip λ/μ, define

$$\mathrm{vs}(\lambda/\mu) = \prod_{j \geq 1}\begin{bmatrix} \lambda_j^c - \lambda_{j+1}^c \\ \lambda_j^c - \mu_j^c \end{bmatrix}_q.$$

For a broken ribbon λ/μ, define

$$\mathrm{br}(\lambda/\mu) = (-q)^{\mathrm{ht}(\lambda/\mu)}(1-q)^{\mathrm{rib}(\lambda/\mu)}.$$

For any skew shape λ/μ, define

$$\mathrm{sk}(\lambda/\mu) = q^{\sum_j \binom{\lambda_j^c - \mu_j^c}{2}}\prod_j \begin{bmatrix} \lambda_j^c - \mu_{j+1}^c \\ m_j(\mu) \end{bmatrix}_q.$$

With this notation, SQMNR$'$ can be expressed as

$$s_{\lambda/\mu} \cdot P_r = \frac{1}{1-q}\sum_{\lambda^+, \mu^-} (-1)^{|\mu/\mu^-|}\,\mathrm{br}(\lambda^+/\lambda)\,\mathrm{br}((\mu/\mu^-)^c)s_{\lambda^+/\mu^-},$$

where the sum on the right is over λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

Conjecture 14 *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$ we have*

$$P_{\lambda/\mu} \cdot s_r = \sum (-1)^{|\mu/\mu^-|} \operatorname{sk}(\lambda^+/\lambda) \operatorname{vs}(\mu/\mu^-) P_{\lambda^+/\mu^-},$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that μ/μ^- is a vertical strip and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

For $\lambda = \mu = \emptyset$, this is identity (2) on page 219 in [5]. The formula seems to be new for $\mu = \emptyset$ and arbitrary λ.

Conjecture 15 *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$ we have*

$$P_{\lambda/\mu} \cdot s_{1^r} = P_{\lambda/\mu} \cdot P_{1^r} = \sum (-1)^{|\mu/\mu^-|} \operatorname{vs}(\lambda^+/\lambda) \operatorname{sk}(\mu/\mu^-) P_{\lambda^+/\mu^-},$$

where the sum on the right is over all $\lambda^+ \supseteq \lambda$, $\mu^- \subseteq \mu$ such that λ^+/λ is a vertical strip and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

For $\mu = \emptyset$, this is [5, §III, (3.2)].

The methods of this paper do *not* seem to work for these conjectures. In other words, the sign-reversing involutions described in Sections 3 and 4 cancel only the constant coefficients on both sides of the conjectured equalities; positive powers of q cancel in some other, mysterious manner.

The author was unable to find a (conjectural) skew version of the Pieri rule for Hall-Littlewood polynomials, i.e. an expansion of $P_{\lambda/\mu} P_r$ in terms of P_{λ^+/μ^-}.

7. Final remarks

7.1. The motivation for this work was the open problem posed by Assaf and McNamara in [2]: to find a combinatorial proof of the skew Murnaghan-Nakayama rule (SMNR). Even though this paper provides a completely bijective proof of the skew *quantum* Murnaghan-Nakayama rule, which obviously specializes to the non-quantum rule, Assaf-McNamara's problem remains open. Indeed, plugging $q = 1$ into SMNR″, which is the identity we proved bijectively, gives 0 on both sides. To get SMNR, we have to divide SMNR″ by $1 - q$ and then set $q = 1$.

One possibility would be to find a bijective proof of SQMNR′ instead. This would mean that one of the northeast corners of ribbons of $(\lambda^+/\lambda) \cup (\mu/\mu^-)$ would have to be colored white (or gray), perhaps the northeast corner of λ^+/λ or the northeast corner of μ/μ^-. We were unable to find such a bijection. Even such a bijection, however, would not be enough to construct a bijection that proves SMNR. Indeed, plugging in $q = 1$ makes many of the skew tableaux weighted with 0, and hence would not appear on the right-hand side of SMNR at all. We would want to avoid such 0-weight objects in the sign-reversing involution.

One possibility would be to construct an involution-principle type of a bijection. Namely, given a skew semistandard Young tableau of shape λ^+/μ, with λ^+/λ a ribbon of size r, we would map it to a tableau of shape λ^{++}/μ^{--}, where λ^{++}/λ is a broken ribbon of size $r - 1$ and $|\mu/\mu^{--}|$ a broken ribbon of size 1, then map this to a tableau of shape λ^{+++}/μ^{---}, where λ^{+++}/λ is a broken ribbon of size $r - 2$ and $|\mu/\mu^{--}|$ a broken ribbon of size 2 etc., until finally reaching a tableau of shape μ/μ^-, where μ/μ^- is a ribbon of size r. We leave this as motivation for further work.

7.2. There is another natural q-version of power sum functions, defined by

$$\bar{p}_r = \sum_{\tau \vdash r} q^{r-\ell(\tau)} (q-1)^{\ell(\tau)-1} m_\tau,$$

$$\bar{p}_\mu = \bar{p}_{\mu_1} \bar{p}_{\mu_2} \cdots .$$

For example,

$$\bar{p}_4 = q^3 m_4 + q^2(q-1)m_{31} + q^2(q-1)m_{22} + q(q-1)^2 m_{211} + (q-1)^3 m_{1111}$$

and

$$\bar{p}_{22} = q^2 m_4 + 2q(q-1)m_{31} + (3q^2 - 2q + 1)m_{22} + 2(q-1)(2q-1)m_{211} + 6(q-1)^2 m_{1111}.$$

We have

$$\bar{p}_r|_{q=1} = m_r = p_r, \qquad \bar{p}_r|_{q=0} = (-1)^{r-1} m_{1^r} = (-1)^{r-1} s_{1^r}, \qquad \lim_{q \to \infty} \frac{\bar{p}_r}{q^{r-1}} = \sum_{\tau \vdash r} m_\tau = s_r.$$

Theorem 16 (SQMNR‴) *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \bar{p}_r = (-1)^{r-1} \sum_{\lambda^+, \mu^-} (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{wt}(\lambda^+/\lambda) + \mathrm{ht}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-) - 1} s_{\lambda^+/\mu^-},$$

where the sum on the right is over λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

For $q = 0$, this is the conjugate skew Pieri rule (multiplied by $(-1)^{r-1}$), for $q = 1$, this is again the skew Murnaghan-Nakayama rule, and if we divide by q^{r-1} and send q to ∞, we get the skew Pieri rule. We could give an involutive proof in the spirit of the proof of SQMNR′, but we can also derive SQMNR‴ from SQMNR′ by observing that $\bar{p}_r = q^r \widetilde{p}_r|_{q=q^{-1}}$.

7.3. Lam-Lauve-Sotille's skew Littlewood-Richardson rule is very general, but the computation of actual coefficients in the expansion, i.e. counting all standard Young tableaux of a given shape that rectify to a given tableau, is complicated in practice. In light of Section 5, our work can be seen as one possible answer to the following question. For what special shapes of $\lambda, \mu, \sigma, \tau$ can we actually compute the coefficients? SQMNR can be interpreted as saying that if $\tau = \emptyset$ and σ is a hook, the coefficients are certain binomial coefficients, while SPR says that the coefficient is ± 1 if $\tau = \emptyset$ and $\sigma = r$.

It would be interesting to find other examples when the coefficients can be computed and yield elegant answers, both for Schur functions and for other Hopf algebras.

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