RESULTS AND CONJECTURES ON THE NUMBER OF STANDARD STRONG MARKED TABLEAUX

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Abstract. Many results involving Schur functions have analogues involving k-Schur functions. Standard strong marked tableaux play a role for k-Schur functions similar to the role standard Young tableaux play for Schur functions. We discuss results and conjectures toward an analogue of the hook-length formula.

1. Introduction

In 1988, Macdonald [Mac95] introduced a new class of polynomials and conjectured that they expand positively in terms of Schur functions. This conjecture, verified in [Hai01], has led to an enormous amount of work, including the development of the k-Schur functions. The k-Schur functions were defined in [LLM03]. Lascoux, Lapointe, and Morse conjectured that they form a basis for a certain subspace of the space of symmetric functions and that the Macdonald polynomials indexed by partitions whose first part is not larger than k expand positively in terms of the k-Schur functions, leading to a refinement of the Macdonald conjecture. The k-Schur functions have since been found to arise in other contexts; for example, as the Schubert cells of the cohomology of affine Grassmannian permutations [Lam06], and they are related to the quantum cohomology of the affine permutations [LM08].

One of the intriguing features of standard Young tableaux is the Frame-Thrall-Robinson hook-length formula, which enumerates them. It has many different proofs and many generalizations, see e.g. [Sta99, Chapter 7], [GNW79], [CFKP11] and the references therein.

In this paper, we partially succeed in finding an analogue of the hook-length formula for standard strong marked tableaux (or starred tableaux for short), which are a natural generalization of standard Young tableaux in the context of k-Schur functions. For a fixed n, the shape of a starred tableau (see Subsection 2.6 for a definition) is necessarily an n-core, a partition for which all hook-lengths are different from n. In [LLMS10], a formula is given for the number of starred tableaux for $n = 3$.

Proposition 1.1 ([LLMS10], Proposition 9.17)**.** *For a 3-core λ, the number of starred tableaux of shape λ equals*

$$\frac{m!}{2^{\lfloor \frac{m}{2} \rfloor}},$$

where m is the number of boxes of λ with hook-length $< n$. □

The number of 2-hooks is $\lfloor \frac{m}{2} \rfloor$. Therefore we can rewrite the result as

$$\frac{m!}{\prod\limits_{\substack{i,j \in \lambda \\ h_{ij} < 3}} h_{ij}}.$$

Note that this is reminiscent of the classical hook-length formula.

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The authors left the enumeration for $n > 3$ as an open problem. The main result (Theorem 3.1) of this paper implies the existence, for each n, of $(n-1)!$ rational numbers which we call *correction factors*. Once the corrections factors have been calculated by enumerating all starred tableaux for certain shapes, the number of starred tableaux of shape λ for any n-core λ can be easily computed. In fact, Theorem 3.1 is a t-analogue of the hook formula. The theorem is "incomplete" in the sense that we were not able to find explicit formulas for the (weighted) correction factors. We have, however, been able to state some of their properties (some conjecturally), the most interesting of these properties being unimodality (Conjecture 3.7).

Another result of interest is a new, alternative description of strong marked covers via simple triangular arrays of integers which we call *residue tables* and *quotient tables* (Theorem 5.2).

The paper is structured as follows. In Section 2, we give the requisite background, notation, definitions, and results. In Section 3, we state the main results and conjectures. In Section 4, we give a proof of the main theorem via quasisymmetric functions. In Section 5, we make the first steps toward an inductive, GNW-style proof of the non-weighted version of the main formula, and discuss how to prove it for small n. The main tool is an alternative description of strong covers directly in terms of bounded partitions (instead of via cores, abacuses or affine permutations). We prove this description in Section 6. In Section 7, we present a (conjectured) stronger statement that would give an inductive proof of the main result. We finish with some remarks and open questions in Section 8.

2. Preliminaries

Here we introduce notation and review some constructions.

2.1. **Partitions.** A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of weakly decreasing positive integers, called the *parts* of λ. The *length* of λ, $\ell(\lambda)$, is the number of parts, and the *size* of λ, $|\lambda|$, is the sum of parts. A *(weak) composition* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of positive (non-negative) integers. The *Young diagram* of a partition λ is the left-justified array of boxes with $\ell(\lambda)$ rows and λ_i boxes in row i. (Note that we are using the English convention for drawing diagrams.) We will often refer to both the partition and the diagram of the partition by λ. We write $\lambda \subseteq \mu$ if the diagram of λ is contained in the diagram of μ, i.e. if $\ell(\lambda) \leq \ell(\mu)$ and $\lambda_i \leq \mu_i$ for $1 \leq i \leq \ell(\lambda)$. If $\lambda \subseteq \mu$, we can define the *skew diagram* μ/λ as the boxes which are in the diagram of μ but not in the diagram of λ. If no two boxes of μ/λ are in the same column (respectively, row), we say that μ/λ is a horizontal (resp., vertical) strip. A subset of the boxes in μ/λ is a *connected component* if for any two boxes there is a sequence of adjacent boxes in μ/λ from one to the other. A connected component of μ/λ is called a *ribbon* if it does not contain any 2×2 block. The *head* of a connected component is the box furthest to the northeast and its *tail* is the box furthest to the southwest.

For $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$, box (i,j) refers to the box in row i, column j of λ. The *conjugate* of λ is the partition λ' whose diagram is obtained by reflecting the diagram of λ about the diagonal. The (i,j)*-hook* of a partition λ consists of the box (i,j) of λ, all the boxes to the right of it in row i, together with all the boxes below it in column j. The *hook-length* h_{ij}^λ is the number of boxes in the (i,j)-hook. The *content* of box (i,j) is $j-i$. When n is clear, for example, when λ an n-core partition, the *residue* of box $(i,j) \in \lambda$ is $j - i \bmod n$.

2.2. **Cores and bounded partitions.** Let n be a positive integer. An *n-core* is a partition λ such that $h_{ij}^\lambda \neq n$ for all $(i,j) \in \lambda$. Core partitions were introduced by Nakayama [Nak41a, Nak41b] to describe when two ordinary irreducible representations of the symmetric group belong to the same block. There is a close connection between $(k+1)$-cores and *k-bounded partitions*, which are partitions whose first part (and hence every part) is $\leq k$. Indeed, in [LM05], a simple bijection between $(k+1)$-cores and k-bounded partitions is presented. Given a $(k+1)$-core λ, let π_i be the number of boxes in row i of λ with hook-length $\leq k$. The resulting $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ is a k-bounded partition, we denote it $\mathfrak{b}(\lambda)$. Conversely, given a k-bounded partition π, move from the last row of π upwards, and in row i, shift the π_i boxes of the diagram of π to the right until their hook-lengths are at most k. The resulting $(k+1)$-core is denoted $\mathfrak{c}(\pi)$.

Example 2.1. On the left-hand side of Figure 1, the hook-lengths of the boxes of the 5-core $\lambda = 953211$ are shown, with the ones that are < 5 in bold. That means that $\mathfrak{b}(\lambda) = 432211$.

14	11	9	7	6	**4**	**3**	**2**	**1**
9	6	**4**	**2**	**1**				
6	**3**	**1**						
4	**1**							
2								
1								

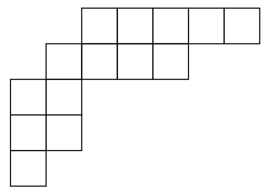


FIGURE 1. Bijections $\mathfrak{b}$ and $\mathfrak{c}$.

The right-hand side shows the construction of $\mathfrak{c}(\pi) = 75221$ for the 6-bounded partition $\pi = 54221$. $\diamond$

Of particular importance are k-bounded partitions π that satisfy $m_i(\pi) \leq k - i$ for all $i = 1, \dots, k$. We call such partitions *k-irreducible partitions*, see [LLM03]. The number of k-irreducible partitions is $k!$.

Note that some confusion exists in the literature as to whether it is better to use k (which appears in connection with, say, bounded partitions and k-Schur functions), n (which appears in connection with, say, cores, affine permutations and abacuses), or, as Lam suggests, to use both k and n and have $n = k + 1$ always. In this paper, we mostly use k, but whenever n appears, it should be construed as $k + 1$.

2.3. **Young tableaux and the hook-length formula.** *Young's lattice* $\mathcal{Y}$ takes as its vertices all integer partitions, and the relation is containment. If λ and μ are partitions, then μ covers λ if and only if $\lambda \subseteq \mu$ and $|\mu| = |\lambda| + 1$. The rank of a partition is given by its size.

A *semistandard Young tableau* T of shape λ is a Young diagram of shape λ whose boxes have been filled with positive integers satisfying the following: the integers must be nondecreasing as we read a row from left to right, and increasing as we read a column from top to bottom. The *weight* of T is the composition $(\alpha_1, \alpha_2, \ldots)$, where α_i is the number of i's in T. The tableau T is a *standard Young tableau* if the entries are $1, \ldots, |\lambda|$ in some order, i.e. if the weight is $(1, \ldots, 1)$. A standard Young tableau of shape λ represents a saturated chain in the interval $[\emptyset, \lambda]$ of the Young's lattice. Let $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$, $\lambda^{(0)} = \emptyset$, $\lambda^{(m)} = \lambda$, be such a chain. Then in the tableau corresponding to this chain, i is the entry in the box added in moving from $\lambda^{(i-1)}$ to $\lambda^{(i)}$.

The Frame-Thrall-Robinson hook-length formula shows how to compute f_λ, the number of standard Young tableaux of shape λ. We have:

$$f_\lambda = \frac{|\lambda|!}{\prod_{i,j \in \lambda} h_{ij}^\lambda}. \tag{2.1}$$

This formula has a well-known weighted version; see [Sta99, Corollary 7.21.5]. For a standard Young tableau T, define a *descent* to be an integer i such that $i + 1$ appears in a lower row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T. Define the *major index* of T as $\mathrm{maj}(T) = \sum_{i \in D(T)} i$, and the polynomial

$$f_\lambda(t) = \sum t^{\mathrm{maj}(T)},$$

where the sum is over all standard Young tableaux of shape λ. Then

$$f_\lambda(t) = \frac{t^{b(\lambda)}(\boldsymbol{|\lambda|})\boldsymbol{!}}{\prod_{i,j\in\lambda}(\boldsymbol{h_{ij}^\lambda})} \tag{2.2}$$

Here $b(\lambda) = \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda'_i}{2}$, $\boldsymbol{(i)} = 1 + t + \ldots + t^{i-1}$ and $\boldsymbol{(i)!} = \boldsymbol{(1)}\cdot\boldsymbol{(2)}\cdots\boldsymbol{(i)}$.

2.4. **Strong marked and starred tableaux.** The *strong n-core poset* $\mathcal{C}_n$ is the subposet of $\mathcal{Y}$ induced by the set of all n-core partitions. That is, its vertices are n-core partitions and $\lambda \le \mu$ in $\mathcal{C}_n$ if $\lambda \subseteq \mu$. The cover relations are trickier to describe in $\mathcal{C}_n$ than in $\mathcal{Y}$.

Proposition 2.2 ([LLMS10], Proposition 9.5)**.** *Suppose $\lambda \le \mu$ in $\mathcal{C}_n$, and let $C_1, \ldots, C_m$ be the connected components of μ/λ. Then μ covers λ (denoted $\lambda \lessdot \mu$) if and only if each C_i is a ribbon, and all the components are translates of each other with heads on consecutive diagonals with the same residue.*

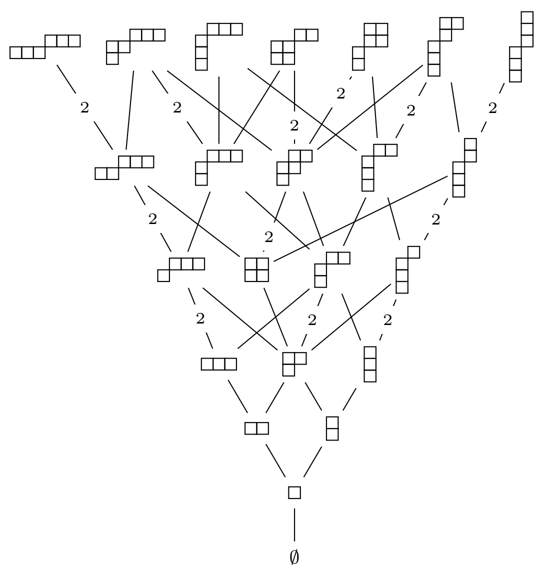


FIGURE 2. The 4-core lattice up to rank 6.

The rank of an n-core is the number of boxes of its diagram with hook-length $< n$. If $\lambda \lessdot \mu$ and μ/λ consists of m ribbons, we say that μ *covers λ in the strong order with multiplicity m*. Figure 2 shows the strong marked covers for 4-cores with rank at most 6. Only multiplicities $\neq 1$ are marked.

A *strong marked cover* is a triple (λ, μ, c) such that $\lambda \lessdot \mu$ and that c is the content of the head of one of the ribbons. We call c the *marking* of the strong marked cover. A *strong marked horizontal strip of size r and shape μ/λ* is a sequence $(\nu^{(i)}, \nu^{(i+1)}, c_i)_{i=0}^{r-1}$ of strong marked covers such that $c_i < c_{i+1}$, $\nu^{(0)} = \lambda$, $\nu^{(r)} = \mu$. If λ is an n-core, a *strong marked tableau* T of shape λ is a sequence of strong marked horizontal strips of shapes $\mu^{(i+1)}/\mu^{(i)}$, $i = 0, \ldots, m-1$, such that $\mu^{(0)} = \emptyset$ and $\mu^{(m)} = \lambda$. The *weight* of T is the composition $(r_1, \ldots, r_m)$, where r_i is the size of the strong marked horizontal strip $\mu^{(i)}/\mu^{(i-1)}$. If all strong marked horizontal strips are of size 1, we call T a *standard strong marked tableau* or a *starred tableau* for short. For a k-bounded partition π (recall that $n = k+1$), denote the number of starred tableaux of shape $\mathfrak{c}(\pi)$ by $F_\pi^{(k)}$.

Example 2.3. Take $k = 3$. Figure 3 represents a strong marked tableau of shape 6311 and weight 421.

Here a star means that the box (necessarily the head of a ribbon) is one whose content is the marking of a strong marked cover. The number, say 1 in box $(2,1)$, tells us which of the strong marked

$$\begin{array}{|c|c|c|c|c|c|}\hline 1_1^* & 1_2^* & 1_3^* & 1_4^* & 2_2^* & 3_1 \\ \hline 1_4 & 2_2 & 3_1 \\ \cline{1-3} 2_1^* \\ \cline{1-1} 3_1^* \\ \cline{1-1} \end{array}$$

FIGURE 3. An example of a strong marked tableau.

horizontal strips the box belongs to. And the index, say 4 in the box $(2,1)$, tells us which of the strong marked covers in the strong marked horizontal strip the box belongs to. So the sequence of strong marked covers in the first strong marked horizontal strip is $(\emptyset, 1, 0), (1, 2, 1), (2, 3, 2), (3, 41, 4)$, in the second strong marked horizontal strip it is $(41, 411, -2), (411, 521, 5)$, and the third strong marked horizontal strip consists of only one strong marked cover $(521, 6311, -3)$.

Standard strong marked tableaux, or *starred tableaux*, represent saturated chains in the strong n-core lattice much as standard Young tableaux do in Young's lattice. The difference is that now more than one component may be added in moving from $\lambda^{(i-1)}$ to $\lambda^{(i)}$; one of those components must be starred.

$$\begin{array}{|c|c|c|}\hline 1^* & 2^* & 3^* \\ \hline 4^* \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1^* & 2^* & 4^* \\ \hline 3^* \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1^* & 2^* & 4 \\ \hline 3^* \\ \cline{1-1} 4^* \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1^* & 3^* & 4^* \\ \hline 2^* \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1^* & 3^* & 4 \\ \hline 2^* \\ \cline{1-1} 4^* \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1^* & 4 & 4^* \\ \hline 2^* \\ \cline{1-1} 3^* \\ \cline{1-1} \end{array}$$

FIGURE 4. All starred tableaux of shape 311.

Figure 4 illustrates $F_{211}^{(3)} = 6$. $\diamond$

If λ is a k-bounded partition that is also a $(k+1)$-core (i.e., if $\lambda_1 + \ell(\lambda) \leq k+1$), then strong marked covers on the interval $[\emptyset, \lambda]$ are equivalent to the covers in the Young lattice, strong marked tableaux of shape λ are equivalent to semistandard Young tableaux of shape λ, and starred tableaux of shape λ are equivalent to standard Young tableaux of shape λ.

As with semistandard Young tableaux, we may *standardize* any strong marked tableau. We standardize the tableau strip by strip by replacing j_i by $(j+i-1)_1$, and then renumbering to avoid repetition. The marks remain with their boxes.

Example 2.4. The standard marked tableau in Figure 3 standardizes to the starred tableau in Figure 5.

$$\begin{array}{|c|c|c|c|c|c|}\hline 1_1^* & 2_1^* & 3_1^* & 4_1^* & 6_1^* & 7_1 \\ \hline 4_1 & 6_1 & 7_1 \\ \cline{1-3} 5_1^* \\ \cline{1-1} 7_1^* \\ \cline{1-1} \end{array}$$

FIGURE 5. Standardization of the tableau in Figure 3.

Note that we can delete the indices 1 without losing any information. $\diamond$

2.5. **Schur functions and fundamental quasisymmetric functions.** For the definition of Λ, the ring of symmetric functions, see [Mac95] or [Sta99]. For a partition λ, define the *monomial symmetric function*

$$m_\lambda = m_\lambda(x_1, x_2, \ldots) = \sum_\alpha x^\alpha,$$

where the sum is over all weak compositions α that are a permutation of λ, and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. For partitions λ and μ of the same size, define the *Kostka number* $K_{\lambda\mu}$ as the number of semistandard Young tableaux of shape λ and weight μ. Define the *Schur function*

$$s_\lambda = \sum K_{\lambda\mu} m_\mu$$

with the sum over all partitions. The Schur functions form the most important basis of Λ and have numerous beautiful properties. See for example [Sta99, Chapter 7] and [Mac95, Chapter 1].

Let m be a positive integer. *Fundamental quasisymmetric functions* may be indexed by m and subsets of $\{1, 2, \ldots, m-1\}$ (for example, [Hag08]) or by compositions of m (for example, [Sta99]). In this paper, we use subsets. For a subset D of $\{1, 2, \ldots, m-1\}$, let

$$Q_{m,D} = \sum_{\substack{i_1 \leq \cdots \leq i_m \\ i_h < i_{h+1} \text{ if } h \in D}} x_{i_1} x_{i_2} \cdots x_{i_m} \tag{2.3}$$

denote the fundamental quasisymmetric function corresponding to m and D.

We have the classical result of Gessel [Ges84] that Schur functions can be expanded in fundamental quasisymmetric functions. To state this, we define a *descent* of a standard Young tableau T to be an integer i such that $i+1$ appears in a lower row of T than i, and define the *descent set* $D(T)$ to be the set of all descents of T. Then we have

$$s_\lambda = \sum_T Q_{|\lambda|, D(T)}. \tag{2.4}$$

Here the sum is over all standard Young tableaux of shape λ.

2.6. ***k*-Schur functions.** There are at least three conjecturally equivalent definitions of k-Schur functions. Here, we give the definition from [LLMS10] via strong marked tableaux. For k-bounded partitions π and τ, define the *k-Kostka number* $K_{\pi\tau}^{(k)}$ as the number of strong marked tableaux of shape $\mathfrak{c}(\pi)$ and content τ. Then we define the *k-Schur function*

$$s_\pi^{(k)} = \sum_\tau K_{\pi\tau}^{(k)} m_\tau, \tag{2.5}$$

where the sum is over all k-bounded partitions τ.

If π is also a $(k+1)$-core, then strong marked tableaux of shape π are equivalent to semistandard Young tableaux of shape π, and therefore in this case $s_\pi^{(k)} = s_\pi$.

The original definition of k-Schur functions was via *atoms* [LLM03], which we will not use here (but see **8.2**). Note that in full generality, the k-Schur functions (in any definition) have a parameter t. In this paper, $t = 1$.

2.7. **Splitting of bounded partitions.** For a k-bounded partition π, denote by $\partial_k(\pi)$ the boxes of $\mathfrak{c}(\pi)$ with hook-length $\leq k$. If $\partial_k(\pi)$ is not connected, we say that π *splits*. Each of the connected components of $\partial_k(\pi)$ is a horizontal translate of $\partial_k(\pi^i)$ for some k-bounded partition π^i. Call $\pi^1, \pi^2, \ldots$ the *components* of π.

Example 2.5. Figure 6 depicts $\partial_5(54433211)$.

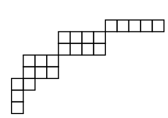


FIGURE 6. Splitting of a k-partition.

It follows that π splits into components $5, 44, 33211$. ◇

Denton [Den12, Theorem 1.1] proved the following.

Theorem 2.6. *Suppose π splits into $\pi^1, \ldots, \pi^m$. Then*

$$s_\pi^{(k)} = \prod_{i=1}^m s_{\pi^i}^{(k)}. \qquad \square$$

3. Main results and conjectures

For a starred tableau T, define the *descent set of* T, $D(T)$, as the set of all i for which the marked box at i is strictly above the marked box at $i+1$. Define the *major index of* T, $\operatorname{maj}(T)$, by $\sum_{i \in D(T)} i$. For a k-bounded partition π, define the polynomial

$$F_\pi^{(k)}(t) = \sum_T t^{\operatorname{maj}(T)}, \tag{3.1}$$

where the sum is over all starred tableaux of shape $\mathfrak{c}(\pi)$. Recall that $F_\pi^{(k)}$ denotes the number of such starred tableaux, i.e. $F_\pi^{(k)} = F_\pi^{(k)}(1)$.

Our main result is the following theorem.

Theorem 3.1. *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1 + 1 \cdot w_1}, (k-1)^{a_2 + 2 \cdot w_2}, \ldots, 1^{a_k + k \cdot w_k} \rangle,$$

for $0 \le a_i < i$. Then

$$F_\pi^{(k)}(t) = \frac{t^{\sum_{i=1}^k w_i \binom{i}{2}(k-i+1)} (|\boldsymbol{\pi}|)!\, F_\sigma^{(k)}(t)}{(|\boldsymbol{\sigma}|)! \prod_{j=1}^k (\boldsymbol{j})^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$.

By plugging in $t = 1$, we get the following.

Corollary 3.2. *Let π be a k-bounded partition, and write*

$$\pi = \langle k^{a_1 + 1 \cdot w_1}, (k-1)^{a_2 + 2 \cdot w_2}, \ldots, 1^{a_k + k \cdot w_k} \rangle,$$

for $0 \le a_i < i$. Then

$$F_\pi^{(k)} = \frac{|\pi|!\, F_\sigma^{(k)}}{|\sigma|! \prod_{j=1}^k j^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}},$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$. $\square$

The theorem (respectively, corollary) implies that in order to compute $F_\pi^{(k)}(t)$ (resp., $F_\pi^{(k)}$) for all k-bounded partitions π, it suffices to compute $F_\sigma^{(k)}(t)$ (resp., $F_\sigma^{(k)}$) only for k-irreducible partitions σ; recall that there are $k!$ such partitions.

We prove the theorem in Section 4.

Example 3.3. The following gives the formulas for $k \le 3$.

(1) For $k = 1$, we have $F_{1^0}^{(1)}(t) = 1$ and therefore

$$F_{1^{w_1}}^{(1)}(t) = \frac{(\boldsymbol{w_1})! \cdot 1}{(\mathbf{0})! \cdot (\mathbf{1})^{w_1}} = (\boldsymbol{w_1})!.$$

This is consistent with [LLMS10, §9.4.1], which states that

$$F_{1^{w_1}}^{(1)} = w_1!.$$

(2) For $k=2$, we have $F^{(2)}_{2^0 1^0}(t)=1$ and $F^{(2)}_{2^0 1^1}(t)=1$. Therefore,

$$F^{(2)}_{2^{w_1}1^{2w_2}}(t)=\frac{t^{w_2}\boldsymbol{(2w_1+2w_2)!}\cdot 1}{\boldsymbol{(0)!}\cdot\boldsymbol{(2)}^{w_1+w_2}}=\frac{t^{w_2}\boldsymbol{(2w_1+2w_2)!}}{\boldsymbol{(2)}^{w_1+w_2}}.$$

$$F^{(2)}_{2^{w_1}1^{1+2w_2}}(t)=\frac{t^{w_2}\boldsymbol{(2w_1+2w_2+1)!}\cdot 1}{\boldsymbol{(0)!}\cdot\boldsymbol{(2)}^{w_1+w_2}}=\frac{t^{w_2}\boldsymbol{(2w_1+2w_2+1)!}}{\boldsymbol{(2)}^{w_1+w_2}}.$$

This is consistent with [LLMS10, Proposition 9.17], which states that

$$F^{(2)}_{2^l 1^{m-2l}}=\frac{m!}{2^{\lfloor m/2\rfloor}}.$$

(3) For $k=3$, we have

$$\begin{array}{lclclclclcl} F^{(3)}_{3^0 2^0 1^0} &=& 1 & \quad F^{(3)}_{3^0 2^0 1^1} &=& 1 & \quad F^{(3)}_{3^0 2^0 1^2} &=& t \\ F^{(3)}_{3^0 2^1 1^0} &=& 1 & \quad F^{(3)}_{3^0 2^1 1^1} &=& t(1+t) & \quad F^{(3)}_{3^0 2^1 1^2} &=& t\left(t^2+1\right)\left(t^2+t+1\right) \end{array}$$

so, among other formulas, we have

$$\begin{aligned} F^{(3)}_{3^{w_1}2^{1+2w_2}1^{1+3w_3}}(t) &= \frac{t^{2w_2+3w_3}\boldsymbol{(3w_1+4w_2+3w_3+3)!}\cdot t(1+t)}{\boldsymbol{(3)!}\cdot\boldsymbol{(2)}^{w_1+2w_2+w_3}\cdot\boldsymbol{(3)}^{w_1+w_2+w_3}} \\ &= \frac{t^{2w_2+3w_3+1}\cdot\boldsymbol{(3w_3+4w_2+3w_1+3)!}}{\boldsymbol{(2)}^{w_1+2w_2+w_3}\cdot\boldsymbol{(3)}^{w_1+w_2+w_3+1}}. \end{aligned}$$

Using a computer, it is easy to obtain formulas for larger k. ◇

We now introduce *weighted correction factors*. For a k-bounded partition π, let $H^{(k)}_\pi(t)=\prod\boldsymbol{(h_{ij})}$, where the product is over all boxes (i,j) of the $(k+1)$-core $\mathfrak{c}(\pi)$ with hook-lengths at most k, and let $H^{(k)}_\pi=H^{(k)}_\pi(1)$ be the product of all hook-lengths $\le k$ of $\mathfrak{c}(\pi)$. Furthermore, if b_j is the number of boxes in the j-column of $\mathfrak{c}(\pi)$ with hook-length at most k, write $b^{(k)}_\pi=\sum_j\binom{b_j}{2}$.

Example 3.4. For the 6-bounded partition $\pi=54211$ from Example 2.1, we have

$$H^{(6)}_\pi(t)=\boldsymbol{(1)}^4\boldsymbol{(2)}^3\boldsymbol{(3)}^2\boldsymbol{(4)}^2\boldsymbol{(5)}\boldsymbol{(6)}^2,\qquad H^{(6)}_\pi=207360$$

and $b^{(6)}_\pi=2\binom{3}{2}+3\binom{2}{2}+2\binom{1}{2}=9$. ◇

By introducing weighted correction factors $C^{(k)}_\sigma(t)$ for a k-irreducible partition σ, we can, by Theorem 3.1, express $F^{(k)}_\pi(t)$ (for all k-bounded partitions π) in another way which is reminiscent of the classical hook-length formula. More precisely, define a rational function $C^{(k)}_\sigma(t)$ so that

$$F^{(k)}_\sigma(t)=\frac{t^{b^{(k)}_\sigma}\boldsymbol{(|\sigma|)!}C^{(k)}_\sigma(t)}{H^{(k)}_\sigma(t)}. \tag{3.2}$$

Note that this implies, in the notation of Theorem 3.1, that

$$F^{(k)}_\pi(t)=\frac{t^{b^{(k)}_\sigma+\sum_{i=1}^k w_i\binom{i}{2}(k+1-i)}\boldsymbol{(|\pi|)!}\,C^{(k)}_\sigma(t)}{H_\sigma(t)\cdot\prod_{j=1}^k\boldsymbol{(j)}^{\sum_{i=1}^k w_i\min\{i,j,k+1-i,k+1-j\}}}.$$

The *correction factor* $C^{(k)}_\sigma$ is defined as $C^{(k)}_\sigma(1)$.

For $k\le 3$, all weighted correction factors are 1. For $k=4$, all but four of the 24 weighted correction factors – for 4-bounded partitions 2211, 321, 3211 and 32211 – are 1, and the ones different from 1 are

$$\frac{1+2t+t^2+t^3}{\boldsymbol{(2)}\boldsymbol{(3)}},\ \frac{1+t+2t^2+t^3}{\boldsymbol{(2)}\boldsymbol{(3)}},\ \frac{1+2t+2t^2+2t^3+t^4}{\boldsymbol{(3)}^2},\ \frac{1+t+3t^2+t^3+t^4}{\boldsymbol{(3)}^2},$$

respectively.

We state some results and conjectures about the weighted correction factors.

Proposition 3.5. *The weighted correction factors are multiplicative in the following sense. If a k-irreducible partition σ splits into $\sigma^1, \sigma^2, \ldots, \sigma^m$, then $C_\sigma(t) = \prod_{i=1}^m C_{\sigma^i}(t)$.*

We prove the proposition in Section 4.

Conjecture 3.6. *For a k-irreducible partition σ, the weighted correction factor is 1 if and only if σ splits into $\sigma^1, \sigma^2, \ldots, \sigma^l$, where each σ^i is a k-bounded partition that is also a $(k+1)$-core.*

The "if" direction is easy: if a k-bounded partition σ is also a $(k+1)$-core, then strong covers on the interval $[0, \sigma]$ are precisely the regular covers in the Young lattice, the starred tableaux of shape σ are standard Young tableaux of shape σ, and the major index of a starred tableau of shape σ is the classical major index for standard Young tableaux; the fact that the weighted correction factor is 1 then follows from the classical weighted version of the hook-length formula (2.2).

The most interesting conjecture about the weighted correction factors is the following. Recall that a sequence $(\alpha_i)_i$ is *unimodal* if there exists I so that $\alpha_i \leq \alpha_{i+1}$ for $i < I$ and $\alpha_i \geq \alpha_{i+1}$ for $i \geq I$, and a *unimodal polynomial* is a polynomial whose sequence of coefficients is unimodal.

Conjecture 3.7. *For a k-irreducible partition σ, we can write*

$$1 - C_\sigma(t) = \frac{P_1(t)}{P_2(t)},$$

where $P_1(t)$ is a unimodal polynomial with non-negative integer coefficients and $P_2(t)$ is a polynomial of the form $\prod_{i=1}^{k-1} (\boldsymbol{j})^{w_j}$ for some non-negative integers w_j.
In particular, we have $0 < C_\sigma \leq 1$ for all σ.

Example 3.8. The table in the Appendix gives $1 - C_\sigma(t)$ in the required form for 5-bounded partitions with correction factor $\neq 1$. Note that indeed all numerators are unimodal (and almost symmetric), and the factors in the denominators are $\mathbf{(2)}$, $\mathbf{(3)}$ and $\mathbf{(4)}$. ◇

4. Proof via quasisymmetric functions

In this section, we prove Theorem 3.1. Let π be a k-bounded partition, and write

$$\pi = \langle k^{a_1 + 1 \cdot w_1}, (k-1)^{a_2 + 2 \cdot w_2}, \ldots, 1^{a_k + k \cdot w_k} \rangle,$$

for $0 \leq a_i < i$. Our goal is to prove that

$$F_\pi^{(k)}(t) = \frac{t^{\sum_{i=1}^k w_i \binom{i}{2}(k-i+1)} \boldsymbol{(|\pi|)!}\, F_\sigma^{(k)}(t)}{\boldsymbol{(|\sigma|)!} \prod_{j=1}^k \boldsymbol{(j)}^{\sum_{i=1}^k w_i \min\{i,j,k+1-i,k+1-j\}}}, \tag{4.1}$$

where $\sigma = \langle k^{a_1}, (k-1)^{a_2}, \ldots, 1^{a_k} \rangle$.

The theorem comes from the expansion of k-Schur functions into fundamental quasisymmetric functions and a result from [Lam08, LM07].

Standardization of semistandard Young tableaux (roughly) explains the expansion of Schur functions as a sum of fundamental quasisymmetric functions over standard Young tableaux. In the same way, standardization of strong marked tableaux explains the expansion of k-Schur functions as a sum of fundamental quasisymmetric functions, now over starred tableaux.

Conjecture 9.11 of [LLMS10] writes the k-Schur functions as a sum of monomials; namely for the k-bounded partition π

$$s_\pi^{(k)} = \sum_T x^{\operatorname{wt} T}, \tag{4.2}$$

where the summation runs over strong marked tableaux T of shape given by the n-core $\mathfrak{c}(\pi)$, and $\operatorname{wt} T$ denotes the weight of T. We take this as our definition of k-Schur functions; see (2.5).

In a starred tableau, we may replace the label $i+1$ with the label i if and only if the content of the box labeled with $(i+1)^*$ is greater than the content of the box labeled with i^*. This is because the contents of the starred boxes in the i-th component must be increasing in a strong marked tableau. The integer i is a *descent* of the starred tableau T if the box labeled i^* is above that labeled $(i+1)^*$; in other words, if the content of the box labeled with $(i+1)^*$ is less than the content of the box labeled with i^*. The strong marked tableaux which standardize to a given tableau T are the ones which respect T's descents. We may therefore write

$$s_\pi^{(k)} = \sum_T Q_{|\pi|,D(T)}, \tag{4.3}$$

where π is a k-bounded partition and the sum is over starred tableaux of shape $\mathfrak{c}(\pi)$. See [AB12, Equation 3.4].

Let us see how to use that to prove (4.1). We assume that $w_1 = \ldots = w_{i-1} = w_{i+1} = \ldots = w_k = 0$ and $w_i = 1$; the general case is almost exactly the same.

A *k-rectangle* $R(i)$ is a partition of the form (i^{k+1-i}), where $1 \leq i \leq k$. Lapointe, Lascoux, and Morse [LLM03, equation (1.25), for the atom definition], Lam [Lam08, Corollary 8.4], and Lapointe and Morse [LM07, Theorem 40] all showed that if τ is a k-bounded partition, then

$$s^{(k)}_{\tau\cup R(i)} = s_{R(i)} s^{(k)}_\tau \tag{4.4}$$

where $\tau \cup R(i)$ is the partition whose parts are the parts of τ together with $k-i+1$ parts of size i, and we have used the fact that the k-Schur function of a k-rectangle is the same as its Schur function.

Recall that a partition with no more than i parts equal to $k-i$, as with σ above, is called k-irreducible. Equation (4.4) shows that all k-Schur functions can be built up by multiplying the k-Schur function for a k-irreducible partition by the Schur functions for rectangles.

In particular,

$$s^{(k)}_\pi = (s_{R(k)})^{w_1} \cdots (s_{R(1)})^{w_k} s^{(k)}_\sigma, \tag{4.5}$$

where π and σ are the k-bounded partitions above.

Under our assumptions, when $w_i = 1$ and all other $w_1, \ldots, w_k$ are 0, this becomes

$$s^{(k)}_\pi = s_{R(k+1-i)} s^{(k)}_\sigma. \tag{4.6}$$

Using (4.3), expand both k-Schur functions in terms of the fundamental quasisymmetric functions.

$$\sum_{\substack{T\colon T \text{ is a} \\ \text{starred tableau} \\ \text{of shape } \mathfrak{c}(\pi)}} Q_{D(T)} = s_{R(k+1-i)} \times \sum_{\substack{T\colon T \text{ is a} \\ \text{starred tableau} \\ \text{of shape } \mathfrak{c}(\sigma)}} Q_{D(T)}. \tag{4.7}$$

Recall that *stable principal specialization* of a symmetric function is the evaluation at $1, t, t^2, \ldots$, see [Sta99, §7.8]. We consider the stable principal specializations of the functions in (4.7).

First we calculate $s_{R(k+1-i)}(1,t,t^2,\ldots)$. By Corollary 7.21.3 in [Sta99],

$$s_{R(k+1-i)}(1,t,t^2,\ldots) = \frac{t^{\binom{i}{2}(k+1-i)}}{(1-t)^{i(k+1-i)} \prod_{(i,j)\in R(k+1-i)} (\boldsymbol{h_{ij}})}, \tag{4.8}$$

where we used the fact that $R(k+1-i)'$ has $k+1-i$ parts equal to i.

The hook-lengths in a rectangle are arranged in diagonal stripes and there are $\min\{i,j,k+1-i,k+1-j\}$ boxes in $R(k+1-i)$ with hook length j, as illustrated in Figure 7.

Equation (4.8) becomes

$$s_{R(k+1-i)}(1,t,t^2,\ldots) = \frac{t^{\binom{i}{2}(k+1-i)}}{(1-t)^{i(k+1-i)} \prod_{j=1}^k (\boldsymbol{j})^{\min\{i,j,k+1-i,k+1-j\}}}, \tag{4.9}$$

k	$\cdots$					j	$\cdots$	i
					$\iddots$			$\vdots$
$\vdots$				j				3
			j					2
	$\cdots$	j					$\cdots$	1

k	$\cdots$		$\cdots$	i
				$\vdots$
$\vdots$				j
			$\iddots$	$\vdots$
		j	$\cdots$	1

FIGURE 7. Hook-lengths in a rectangle.

We have that $\mathfrak{c}(\sigma)$ is a $(k+1)$-core of rank $|\sigma|$ and any starred tableau T of shape $\mathfrak{c}(\sigma)$ will have its descent set contained in $\{1, 2, \ldots, |\sigma|-1\}$. We can then write

$$Q_{D(T)}(1, t, t^2, \ldots) = \frac{t^{\mathrm{comaj}(T)}}{(|\boldsymbol{\sigma}|)!(1-t)^{|\sigma|}} \tag{4.10}$$

by Corollary 7.19.10 of [Sta99], where $\mathrm{comaj}(T) = \sum_{i \in D(T)}(|\sigma| - i)$. Similarly, when T is a starred tableau of shape $\mathfrak{c}(\pi)$,

$$Q_{D(T)}(1, t, t^2, \ldots) = \frac{t^{\mathrm{comaj}(T)}}{(|\boldsymbol{\pi}|)!(1-t)^{|\sigma|+i(k+1-i)}}. \tag{4.11}$$

Now combine (4.10) with (4.3). The numerators of the stable principal specializations for $s_\sigma^{(k)}$ and $s_\pi^{(k)}$ count their respective starred tableaux by comaj. Since k-Schur functions are symmetric, we can use [Sta99, Proposition 7.19.2] and turn this into counting by maj. We can therefore write

$$s_\sigma^{(k)} = \frac{F_\sigma^{(k)}(t)}{(|\boldsymbol{\sigma}|)!(1-t)^{|\sigma|}} \quad \text{and} \quad s_\pi^{(k)} = \frac{F_\pi^{(k)}(t)}{(|\boldsymbol{\pi}|)!(1-t)^{|\sigma|+i(k+1-i)}}. \tag{4.12}$$

Together with (4.9) and (4.6), (4.12) gives the desired result. $\square$

Proposition 3.5 easily follows as well. Indeed, if σ splits into $\sigma^1, \ldots, \sigma^m$, then, by Theorem 2.6,

$$s_\sigma^{(k)} = \prod_{i=1}^m s_{\sigma^i}^{(k)}.$$

By (4.12),

$$\frac{F_\sigma^{(k)}(t)}{(|\boldsymbol{\sigma}|)!(1-t)^{|\sigma|}} = \prod_{i=1}^m \frac{F_{\sigma^i}^{(k)}(t)}{(|\boldsymbol{\sigma^i}|)!(1-t)^{|\sigma^i|}}$$

If we use (3.2) and the fact that $\sum_{i=1}^m |\sigma^i| = |\sigma|$, we obtain

$$\frac{t^{b_\sigma^{(k)}} C_\sigma(t)}{H_\sigma^{(k)}(t)} = \prod_{i=1}^m \frac{t^{b_{\sigma^i}^{(k)}} C_{\sigma^i}(t)}{H_{\sigma^i}^{(k)}(t)}.$$

Since we clearly have $b_\sigma^{(k)} = \sum_{i=1}^m b_{\sigma^i}^{(k)}$ and $H_\sigma^{(k)}(t) = \prod_{i=1}^m H_{\sigma^i}^{(k)}(t)$, the proposition follows. $\square$

5. Proof by induction for small k

The proof in Section 4 closely follows one of the possible proofs of the classical (non-weighted and weighted) hook-length formula, see e.g. [Sta99, §7.21]. Note, however, that the truly elegant proofs (for example, the celebrated probabilistic proof due to Greene, Nijenhuis and Wilf [GNW79]) are via induction. In this and the next two sections, we show the first steps toward such a proof.

In the process, we present a new description of strong marked covers in terms of bounded partitions (previous descriptions included cores, affine permutations and abacuses). See the definition of residue and quotient tables below, and Theorem 5.2.

We identify a bounded partition $\pi = \langle k^{p_1}, (k-1)^{p_2}, \ldots, 1^{p_k} \rangle$ with the sequence $p = (p_1, \ldots, p_k)$. Given i, j, m, $0 \le m < i \le j \le k$, define $p^{i,j,m}$ as follows.

$$p_h^{i,i,m} = \begin{cases} p_h + m & \text{if } h = i-1 \\ p_h - 2m - 1 & \text{if } h = i \\ p_h + m + 1 & \text{if } h = i+1 \\ p_h & \text{otherwise.} \end{cases}, \qquad p_h^{i,j,m} = \begin{cases} p_h + m & \text{if } h = i-1 \\ p_h - m & \text{if } h = i \\ p_h - m - 1 & \text{if } h = j \\ p_h + m + 1 & \text{if } h = j+1 \\ p_h & \text{otherwise} \end{cases} \quad \text{if } i < j.$$

In other words, to get $p^{i,j,m}$ from p, add m copies of $k+2-i$, remove m copies of $k+1-i$, remove $m+1$ copies of $k+1-j$, and add $m+1$ copies of $k-j$. (If $j = k$, then we are adding $m+1$ copies of $k-j = 0$, which does not change the partition. If $i = 1$, we have $m = 0$, so adding m copies of $k+2-i = k+1$ also does not change the partition.) To put it another way: to get $p^{i,j,m}$ from p, increase the first m copies of $k+1-i$ by 1, and decrease the last $m+1$ copies of $k+1-j$ by 1. See Example 5.3.

Define upper-triangular arrays $\mathcal{R} = (r_{ij})_{1 \leq i \leq j \leq k}$, $\mathcal{Q} = (q_{ij})_{1 \leq i \leq j \leq k}$ by

- $r_{jj} = p_j \bmod j$, $r_{ij} = (p_i + r_{i+1,j}) \bmod i$ for $i < j$,
- $q_{jj} = p_j \operatorname{div} j$, $q_{ij} = (p_i + r_{i+1,j}) \operatorname{div} i$ for $j < i$.

We call $\mathcal{R}$ the *residue table* and $\mathcal{Q}$ the *quotient table.*

Example 5.1. Take $k = 4$ and $p = (1, 3, 2, 5)$. Then the residue and quotient tables are given by

$$\begin{array}{|cccc|} 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 2 & 0 \\ & & & 1 \end{array} \qquad \begin{array}{|cccc|} 1 & 2 & 2 & 2 \\ & 1 & 2 & 1 \\ & & 0 & 1 \\ & & & 1 \end{array}$$

It is easy to reconstruct p from the diagonals of $\mathcal{R}$ and $\mathcal{Q}$: $p_1 = 0 + 1 \cdot 1$, $p_2 = 1 + 1 \cdot 2$, $p_3 = 2 + 0 \cdot 3$, $p_4 = 1 + 1 \cdot 4$. ◇

It turns out that the residue and quotient tables determine strong marked covers (and probably other important relations as well, see **8.5**).

Theorem 5.2. *Take $p = (p_1, \ldots, p_k)$ and $1 \leq i \leq j \leq k$. If $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$, then p covers $p^{i,j,r_{ij}}$ in the strong order with multiplicity $q_{ij} + \ldots + q_{jj}$. Furthermore, these are precisely all strong covers. In particular, an element of the $(k+1)$-core lattice covers at most $\binom{k+1}{2}$ elements.*

Example 5.3. Take $k = 4$ and $p = (1, 3, 2, 5)$ as before. Let us underline the entries r_{ij} in the residue table $\mathcal{R}$ for which $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$.

$$\begin{array}{|cccc|} \underline{0} & \underline{0} & \underline{0} & 0 \\ & \underline{1} & \underline{1} & 1 \\ & & \underline{2} & \underline{0} \\ & & & \underline{1} \end{array}$$

By Theorem 5.2, p covers the (exactly) following elements in the strong order:

- $p^{1,1,0} = (0, 4, 2, 5)$ with multiplicity 1,
- $p^{1,2,0} = (1, 2, 3, 5)$ with multiplicity $2 + 1 = 3$,
- $p^{2,2,1} = (2, 0, 4, 5)$ with multiplicity 1,
- $p^{1,3,0} = (1, 3, 1, 6)$ with multiplicity $2 + 2 + 0 = 4$,
- $p^{2,3,1} = (2, 2, 0, 7)$ with multiplicity $2 + 0 = 2$,
- $p^{3,3,2} = (1, 5, -3, 8)$ with multiplicity 0,
- $p^{3,4,0} = (1, 3, 2, 4)$ with multiplicity $1 + 1 = 2$, and
- $p^{4,4,1} = (1, 3, 3, 2)$ with multiplicity 1.

Note that while $(1, 5, -3, 8)$ does not represent a valid partition, the multiplicity of the cover is 0, so we can ignore this cover relation. ◇

For a k-bounded partition π, we clearly have

$$F_\pi^{(k)} = \sum_\tau m_{\tau\pi} F_\tau^{(k)},$$

where the sum is over all k-bounded τ that are covered by π, and $m_{\tau\pi}$ is the multiplicity of the cover. Therefore the theorem can be used to prove Corollary 3.2 for small values of k by induction. First, we need the following corollary.

Corollary 5.4. *Let $p = (p_1, \ldots, p_k)$, $p_i < i$, with corresponding residue and quotient tables $\mathcal{R}$ and $\mathcal{Q}$. Assume that for $1 \le i \le j \le k$, we have $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$. For $s_i \in \mathbb{N}$, write $s = (s_1, 2s_2, \ldots, ks_k)$. Then $p + s$ covers $p^{i,j,r_{ij}} + s$ with multiplicity $q_{ij} + \ldots + q_{jj} + s_i + \ldots + s_j$.*

The corollary implies that in order to prove Corollary 3.2, all we have to do is check $k!$ equalities. Let us illustrate that with an example.

Example 5.5. The 4-bounded partition $p = (w_1, 1 + 2w_2, 2 + 3w_3, 1 + 4w_4)$ covers:

- $(w_1 - 1, 2 + 2w_2, 2 + 3w_3, 1 + 4w_4)$ with multiplicity w_1
- $(w_1, 2w_2, 3 + 3w_3, 1 + 4w_4)$ with multiplicity $1 + w_1 + w_2$
- $(1 + w_1, 2w_2 - 2, 4 + 3w_3, 1 + 4w_4)$ with multiplicity w_2
- $(w_1, 1 + 2w_2, 1 + 3w_3, 2 + 4w_4)$ with multiplicity $2 + w_1 + w_2 + w_3$
- $(1 + w_1, 2w_2, 3w_3, 3 + 4w_4)$ with multiplicity $1 + w_2 + w_3$
- $(w_1, 3 + 2w_2, -3 + 3w_3, 4 + 4w_4)$ with multiplicity w_3
- $(w_1, 1 + 2w_2, 2 + 3w_3, 4w_4)$ with multiplicity $1 + w_3 + w_4$
- $(w_1, 1 + 2w_2, 3 + 3w_3, -2 + 4w_4)$ with multiplicity w_4

We reconstruct the previous example by taking $w_1 = 1, w_2 = 1, w_3 = 0, w_4 = 1$.
By the cover relations,

$$\begin{aligned}
&F^{(4)}_{4^{w_1}3^{1+2w_2}2^{2+3w_3}1^{1+4w_4}} \\
&\quad = w_1 \cdot F^{(4)}_{4^{w_1-1}3^{2+2w_2}2^{2+3w_3}1^{1+4w_4}} \\
&\quad + (1 + w_1 + w_2) \cdot F^{(4)}_{4^{w_1}3^{2w_2}2^{3+3w_3}1^{1+4w_4}} \\
&\quad + w_2 \cdot F^{(4)}_{4^{1+w_1}3^{2w_2-2}2^{4+3w_3}1^{1+4w_4}} \\
&\quad + (2 + w_1 + w_2 + w_3) \cdot F^{(4)}_{4^{w_1}3^{1+2w_2}2^{1+3w_3}1^{2+4w_4}} \\
&\quad + (1 + w_2 + w_3) \cdot F^{(4)}_{4^{1+w_1}3^{2w_2}2^{3w_3}1^{3+4w_4}} \\
&\quad + w_3 \cdot F^{(4)}_{4^{w_1}3^{3+2w_2}2^{-3+3w_3}1^{4+4w_4}} \\
&\quad + (1 + w_3 + w_4) \cdot F^{(4)}_{4^{w_1}3^{1+2w_2}2^{2+3w_3}1^{4w_4}} \\
&\quad + w_4 \cdot F^{(4)}_{4^{w_1}3^{1+2w_2}2^{3+3w_3}1^{-2+4w_4}}
\end{aligned}$$

Assume that Theorem 3.2 holds for all partitions with size less than p. Let us show the computations for the first term on the right-hand side only. By Theorem 3.2 for $F^{(4)}_{4^{w_1-1}3^{2+2w_2}2^{2+3w_3}1^{1+4w_4}}$ (which has size 1 less than $F^{(4)}_{4^{w_1}3^{1+2w_2}2^{2+3w_3}1^{1+4w_4}}$),

$$\begin{aligned}
&F^{(4)}_{4^{w_1-1}3^{2+2w_2}2^{2+3w_3}1^{1+4w_4}} = F^{(4)}_{4^{0+(w_1-1)}3^{2(1+w_2)}2^{2+3w_3}1^{1+4w_4}} = \\
&\frac{(4(w_1-1) + 3(2+2w_2) + 2(2+3w_3) + 1(1+4w_4))! F^{(4)}_{4^0 3^0 2^2 1^1}}{5! \cdot 2^{(w_1-1)+2(w_2+1)+2w_3+w_4} 3^{(w_1-1)+2(w_2+1)+2w_3+w_4} 4^{(w_1-1)+(w_2+1)+w_3+w_4}} = \\
&\frac{(4w_1 + 6w_2 + 6w_3 + 4w_4 + 7)! \cdot 5}{5! \cdot 2 \cdot 3 \cdot 2^{w_1+2w_2+2w_3+w_4} 3^{w_1+2w_2+2w_3+w_4} 4^{w_1+w_2+w_3+w_4}}
\end{aligned}$$

After seven more similar calculations, we get

$$\begin{aligned}
F^{(4)}_{4^{w_1}3^{1+2w_2}2^{2+3w_3}1^{1+4w_4}} &= \frac{(4w_1+6w_2+6w_3+4w_4+7)!}{144\cdot 2^{w_1+2w_2+2w_3+w_4}3^{w_1+2w_2+2w_3+w_4}4^{w_1+w_2+w_3+w_4}}\times \\
&\quad \big(w_1+(1+w_1+w_2)+2w_2+2(2+w_1+w_2+w_3)+ \\
&\quad (1+w_2+w_3)+w_3+2(1+w_3+w_4)+2w_4\big) \\
&= \frac{(4w_1+6w_2+6w_3+4w_4+8)!}{144\cdot 2^{w_1+2w_2+2w_3+w_4}3^{w_1+2w_2+2w_3+w_4}4^{w_1+w_2+w_3+w_4}} \\
&= \frac{(4w_1+6w_2+6w_3+4w_4+8)!F^{(4)}_{4^0 3^1 2^2 1^1}}{8!\cdot 2^{w_1+2w_2+2w_3+w_4}3^{w_1+2w_2+2w_3+w_4}4^{w_1+w_2+w_3+w_4}}
\end{aligned}$$

This completes the calculation for $\sigma = 4^0 3^1 2^2 1^1$. In order to prove the statement for $k = 4$, we would need to do 24 such calculations. $\diamond$

The authors did all such calculations with a computer small k ($k \le 8$).

Of course, one would want a proof for general k, preferably one in the (probabilistic) spirit of the Greene-Nijenhuis-Wilf proof [GNW79]. It seems likely that before one could find such a proof, an explicit formula for the correction factors would have to be known.

6. Description of strong covers

In this section, we prove Theorem 5.2. The proof is via the known description of strong covers in terms of *abacuses*. In particular, we show that the residue table counts inversions in certain permutations.

We know that k-bounded partitions, and k-bounded multiplicity vectors are in an obvious bijective correspondence with each other. They inherit the cover relations from the strong Bruhat order on n-core partitions and this is what we mean, for example, when we say one k-bounded multiplicity vector covers another.

Lapointe, Morse, Lam, and Shimozono [LLMS10] assign each core partition an *offset sequence* and then describe covers in the core lattice in terms of the offset sequence. We prove Theorem 5.2 by explaining the connection between the residue and quotient tables and offset vectors, so their construction is reviewed here.

The edge sequence of a core partition is equivalent to its beta sequence (see Van Leeuven [vL99]) and we use the beta sequence/abacus diagram to describe the offset sequence. Let γ be an n-core partition with ℓ parts, let $r = \ell \bmod n$, and let $\beta_1 > \beta_2 > \ldots > \beta_\ell$ be the first column hook-lengths of γ. Then the *abacus* of γ has n runners, labelled by 1 to n and we place bead i, for i from 1 to ℓ, on runner $(\beta_i - r) \bmod n + 1$.

The offset sequence $d = (d_1, \ldots, d_n)$ is defined by

$$d_i = \begin{cases} \text{the number of beads on runner } i \text{ minus } q & \text{if } 1 \le i < (-r \bmod n) + 1 \\ \text{the number of beads on runner } i \text{ minus } (q+1) & \text{otherwise.} \end{cases}$$

The extended offset sequence $(d_i)_{i\in\mathbb{Z}}$ of an n-core is defined by $d_{i+jn} = d_i - j$.

We will need the following description of strong covers found in [LLMS10]. Let $d = (d_1, \ldots, d_n)$ be an offset sequence. Then the offset sequence $(d_1, \ldots, d_a, \ldots, d_b, \ldots, d_n)$ covers the offset sequence $(d_1, \ldots, d'_a, \ldots, d'_b, \ldots, d_n)$ if and only if either

(1) $d_a < d_b$ and for all $a < k < b$, $d_k \notin [d_a, d_b]$,

or

(2) $d_b < d_a - 1$ and for all $b < k \le n$, $d_k \notin [d_b, d_a - 1]$ and for $1 \le k < a$, $d_k - 1 \notin [d_b, d_a - 1]$.

In both cases, we say that this is an (a, b) cover. In the first case, $d'_a = d_b$ and the multiplicity is $d_b - d_a$. In the second case, $d'_a = d_b + 1$ and $d'_b = d_a - 1$ and the multiplicity is $d_a - 1 - d_b$.

In Algorithm 6.1, we convert the k-bounded multiplicity sequence $p = (p_1, \ldots, p_k)$ to the offset sequence of the n-core corresponding to the k-bounded partition, via the abacus. This produces the same abacus as produced by the β numbers of the corresponding $(k+1)$-core. Right after that, in Algorithm 6.2 based on Algorithm 6.1, we will describe how to assign a permutation π_p to p.

Algorithm 6.1. *Let ℓ be the number of parts; that is, $\sum p_i$, and set r to be $\ell \bmod n$ and q to be $\ell \operatorname{div} n$. We start with an empty abacus of n runners, labelled by $1, 2, \ldots, n$ from left to right. For step 0, we mark runner $(-r \bmod n) + 1$ as done; all others are free. For step 1, distribute p_k beads on the abacus, one on each runner, in order, always skipping runner $(-r \bmod n) + 1$ and cycling back to runner 1 when necessary. The runner following the runner where the last bead was placed is now marked as done and out of play. Suppose step i has been completed and $p_k + p_{k-1} + \ldots + p_{k-i+1}$ beads have been distributed. Again, the next available runner following the runner where the last bead was placed is now out of play. For step $i+1$, start distributing the next p_{k-i} beads on the next available runner after that one, always skipping runners marked as done. This continues until all ℓ beads have been distributed.*

The components $d_i, 1 \le i \le n$, of the offset vector can be read from the abacus. In particular,

$$d_i = \begin{cases} \text{the number of beads on runner } i \text{ minus } q & \text{if } 1 \le i < (-r \bmod n) + 1 \\ \text{the number of beads on runner } i \text{ minus } (q+1) & \text{otherwise.} \end{cases}$$

Algorithm 6.2. *The permutation π_p, which we now define, keeps track of the order in which runners are marked as done, from last to first. Thus $\pi_p(n) = (-r \bmod n) + 1$, since runner $(-r \bmod n) + 1$ was the first marked as done. $\pi_p(n-1)$ records the second runner marked as done and so on. The value of $\pi_p(i)$ is determined after step $n - i$, where p_i beads have been placed.*

By its definition, the permutation π_p has the property that it orders the offset sequence

$$d_{\pi_p(1)} \ge d_{\pi_p(1)} \ge \cdots \ge d_{\pi_p(n)}, \tag{6.1}$$

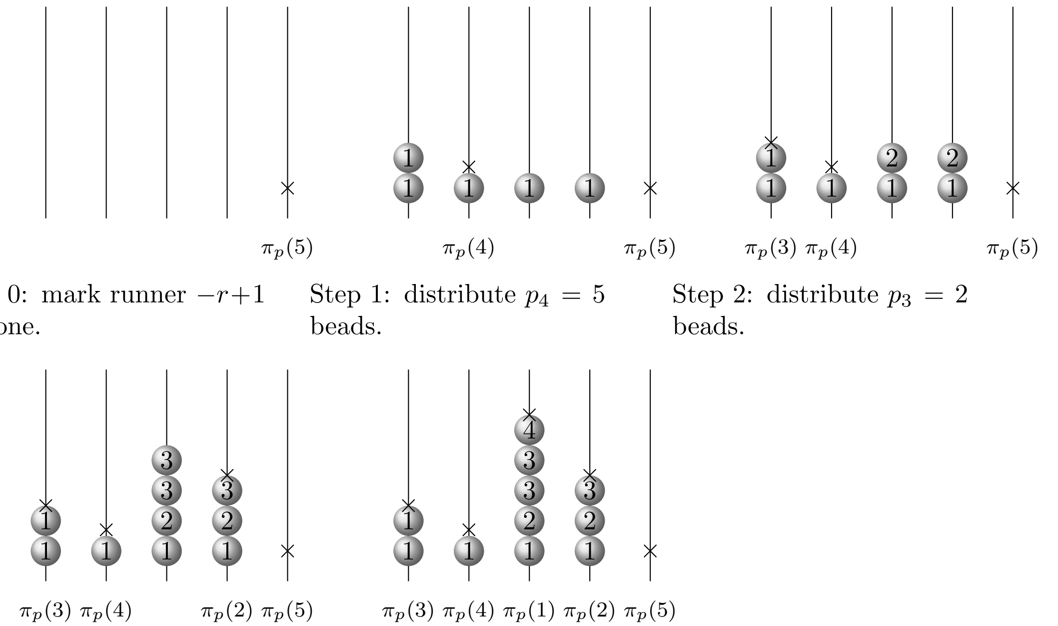
subject to the condition that if $i < j$ and $d_i = d_j$, then $\pi_p^{-1}(j) < \pi_p^{-1}(i)$.

Example 6.3. Let us take the 4-bounded partition from Example 5.1, $p = (1, 3, 2, 5)$. The offset sequence for the partition in Example 5.1 is $d = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 3 & 1 & -3 \end{pmatrix}$.

The corresponding permutation π_p is $(3, 4, 1, 2, 5)$. This partition/offset sequence/multiplicity sequence covers 7 partitions/offset sequences/multiplicity sequences. Below we list them and the corresponding pairs in the offset sequence.

$p^{i,j,r_{ij}}$	$\pi_p(j+1)$	$\pi_p(i)$	offset sequence cover	multiplicity
$p^{1,1,0}$	4	3	$\left(\begin{smallmatrix} 3 & 4 \\ 3 & 1 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 3 & 4 \\ 2 & 2 \end{smallmatrix}\right)$	1
$p^{1,2,0}$	1	3	$\left(\begin{smallmatrix} 1 & 3 \\ 0 & 3 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 1 & 3 \\ 3 & 0 \end{smallmatrix}\right)$	3
$p^{2,2,1}$	1	4	$\left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 1 & 4 \\ 1 & 0 \end{smallmatrix}\right)$	1
$p^{1,3,0}$	2	3	$\left(\begin{smallmatrix} 2 & 3 \\ -1 & 3 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 2 & 3 \\ 3 & -1 \end{smallmatrix}\right)$	4
$p^{2,3,1}$	2	4	$\left(\begin{smallmatrix} 2 & 4 \\ -1 & 1 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 2 & 4 \\ 1 & -1 \end{smallmatrix}\right)$	2
$p^{3,4,0}$	5	1	$\left(\begin{smallmatrix} 1 & 5 \\ 0 & -3 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 1 & 5 \\ -2 & -1 \end{smallmatrix}\right)$	2
$p^{4,4,1}$	5	2	$\left(\begin{smallmatrix} 2 & 5 \\ -1 & -3 \end{smallmatrix}\right) \gtrdot \left(\begin{smallmatrix} 2 & 5 \\ -2 & -2 \end{smallmatrix}\right)$	1

The construction of the abacus is presented in Figure 8; here $n = 5$, $\ell = 11$, $r = 1$, $q = 2$, and $-r+1 \bmod n = 5$. ◇



Step 0: mark runner $-r+1$ as done.

Step 1: distribute $p_4 = 5$ beads.

Step 2: distribute $p_3 = 2$ beads.

Step 3: distribute $p_2 = 3$ beads.

Step 4: distribute $p_1 = 1$ beads.

FIGURE 8. Construction of the abacus corresponding to $(1, 3, 2, 5)$.

We use the following notation, because it helps us avoid many case statements. Let $\pi \in \mathfrak{S}_n$ and let $1 \le i < k \le n$

$$[\pi(i)]_k = \begin{cases} \pi(i) & \text{if } \pi(k) < \pi(i) \\ \pi(i) + n & \text{otherwise} \end{cases}$$

One important fact about π_p which is easy to see from the algorithm to build d and the definition of π_p is

$$p_i \bmod i + |\{k : i + i < k \le n \text{ and } \pi_p(i+1) < [\pi_p(k)]_{i+1} < [\pi_p(i)]_{i+1}\}| = [\pi_p(i)]_{i+1} - \pi_p(i+1). \tag{6.2}$$

Proof of Theorem 5.2. We claim that every entry r_{ij} in the residue table with the property that $r_{ij} < r_{i+1,j}, \ldots, r_{jj}$ and $q_{ij} + \cdots + q_{jj} > 0$ corresponds to a multiplicity vector $p^{i,j,r_{ij}}$ which p covers. More precisely, a k-bounded partition (i, j)-cover corresponds to the offset sequence cover given by $(\min(\pi_p(j+1), \pi_p(i)), \max(\pi_p(j+1), \pi_p(i)))$. We will refer to this as a $\{\pi_p(j+1), \pi_p(i)\}$ cover for the offset vector. In this subsection, we explain the connection between these two descriptions of covers. We need to explain why p covers $p^{i,j,r_{ij}}$ with multiplicity $q_{ij} + q_{i+1,j} + \cdots + q_{jj}$ if and only if $\{\pi_p(j+1), \pi_p(i)\}$ is a cover with multiplicity $[d_{\pi(i)}]_{j+1} - d_{\pi(j+1)}$.

Here is the plan. In Subsection 6.1, we define a statistic ρ_{ij} for the permutation π_p. We show that the offset vector d has a $\{\pi_p(j+1), \pi_p(i)\}$ cover if and only if $\rho_{ij} < \rho_{kj}$ for $j \le k \le i+1$. Then in Subsection 6.2 we show that $\rho_{ij} = r_{ij}$. We discuss the multiplicities of the cover in Subsection 6.3. Finally, in Subsection 6.4, we verify the form of the multiplicity vector given in Theorem 5.2.

6.1. **The statistic ρ_{ij}.** This subsection defines the statistic ρ_{ij} for $\pi \in \mathfrak{S}_n$ and makes no mention of offset sequences, k-bounded partitions, etc. Let $\pi \in \mathfrak{S}_n$. The set $\mathcal{L}_{ij} = \mathcal{L}_{ij}(\pi)$ is defined for $i < j$ by

$$\mathcal{L}_{ij} = \{m : 1 \le m \le i-1 \text{ and } \pi(j+1) < [\pi(m)]_{j+1} < [\pi(i)]_{j+1}\}.$$

Now we can define ρ_{ij} as $|\mathcal{L}_{ij}|$.

Example 6.4. We continue with the same example. We have $\pi = (3,4,1,2,5) \in \mathfrak{S}_5$, so that

$$\begin{aligned}
\mathcal{L}_{44} &= \{m\colon 1 \le m \le 3 \text{ and } 5 < [\pi(m)]_5 < 7\} = \{3\},\\
\mathcal{L}_{34} &= \{m\colon 1 \le m \le 2 \text{ and } 5 < [\pi(m)]_5 < 6\} = \{\},\\
\mathcal{L}_{24} &= \{m\colon 1 \le m \le 1 \text{ and } 5 < [\pi(m)]_5 < 9\} = \{1\},\\
\mathcal{L}_{14} &= \{m\colon 1 \le m \le 1 \text{ and } 5 < [\pi(m)]_5 < 8\} = \{\},\\
\mathcal{L}_{33} &= \{m\colon 1 \le m \le 2 \text{ and } 2 < [\pi(m)]_4 < 6\} = \{1,2\},\\
\mathcal{L}_{23} &= \{m\colon 1 \le m \le 1 \text{ and } 2 < [\pi(m)]_4 < 4\} = \{1\},\\
\mathcal{L}_{13} &= \{m\colon 1 \le m \le 0 \text{ and } 2 < [\pi(m)]_4 < 3\} = \{\},\\
\mathcal{L}_{22} &= \{m\colon 1 \le m \le 1 \text{ and } 1 < [\pi(m)]_3 < 4\} = \{1\},\\
\mathcal{L}_{12} &= \{m\colon 1 \le m \le 0 \text{ and } 1 < [\pi(m)]_3 < 3\} = \{\}, \text{ and}\\
\mathcal{L}_{11} &= \{m\colon 1 \le m \le 0 \text{ and } 4 < [\pi(m)]_3 < 8\} = \{\}.
\end{aligned}$$

Thus, we have

$$\begin{array}{cccc}
\rho_{11} = 0 & \rho_{12} = 0 & \rho_{13} = 0 & \rho_{14} = 0\\
 & \rho_{22} = 1 & \rho_{23} = 1 & \rho_{24} = 1\\
 & & \rho_{33} = 2 & \rho_{34} = 0\\
 & & & \rho_{44} = 1
\end{array}$$

6.2. **Offset sequence covers and ρ_{ij}.** Fix a multiplicity sequence p and its permutation π_p. Suppose $i < k < j+1$. By (6.1), we know $d_{\pi_p(i)} \le d_{\pi_p(k)} \le d_{\pi_p(j+1)}$. Therefore, $d_{\pi_p(k)}$ causes the failure of $\{\pi_p(j+1), \pi_p(i)\}$ as an offset sequence cover if and only if $\pi_p(j+1) < [\pi_p(k)]_{j+1} < [\pi_p(i)]_{j+1}$. This means we will be done with the second step of our plan when we have proved the following lemma.

Lemma 6.5. *Let $1 \le i < j \le n$ and $\pi \in \mathfrak{S}_n$. We have $\rho_{ij} < \rho_{kj}$ for all k such that $i < k \le j$ if and only if $[\pi(k)]_{j+1} \notin [\pi(j+1), [\pi(i)]_{j+1}]$ for all k such that $i < k \le j$.*

Proof. This is a statement about permutations and has nothing to do with the relationship to the multiplicity vector of a k-bounded partition.
Suppose $i < k_0 \le j$ and $\pi(j+1) < [\pi(k_0)]_{j+1} < [\pi(i)]_{j+1}$, so that $\{\pi(j+1), \pi(i)\}$ is not an offset sequence cover. We must show that there is an m such that $i < m \le j$ and $\rho_{ij} \ge \rho_{mj}$. Let m be the least integer such that $i < m \le k_0$ and $[\pi(m)]_{j+1} < [\pi(i)]_{j+1}$. Such an m exists, since k_0 exists. Then

$$[\pi(m-1)]_{j+1}, [\pi(m-2)]_{j+1}, \ldots, [\pi(i+1)]_{j+1} > [\pi(i)]_{j+1} > [\pi(m)]. \tag{6.3}$$

If $x \in \mathcal{L}_{mj}$, then $[\pi(x)]_{j+1} < [\pi(m)]_{j=1}$ and $1 \le x < m$ by definition of $\mathcal{L}_{mj}$. By (6.3), $x \notin \{i+1, i+2, \ldots, m-1\}$, forcing $1 \le x < i$. Therefore, $x \in \mathcal{L}_{ij}$ and $\rho_{ij} \ge \rho_{mj}$.
Conversely, suppose no such k_0 exists. That is, for all $i < k \le j$, we have $[\pi(k)]_{j+1} > [\pi(i)]_{j+1}$. Then $i \in \mathcal{L}_{kj}$ for each k, whereas $i \notin \mathcal{L}_{ij}$. Additionally, suppose $m \in \mathcal{L}_{ij}$. Then $m < i < k$ and $[\pi(m)]_{j+1} < [\pi(i)]_{j+1} < [\pi(k)]_{j+1}$, so that $\mathcal{L}_{ij} \subset \mathcal{L}_{kj}$ and $\rho_{ij} < \rho_{kj}$ for all $i < k \le j$. □

We use induction on $j - i$ to show that $r_{ij} = \rho_{ij}$. We combine the fact that

$$\begin{aligned}
[\pi_p(i)]_{i+1} - \pi_p(i+1) &= |\{k : 1 \le k < i \text{ and } \pi_p(i+1) < [\pi_p(k)]_{i+1} < [\pi_p(i)_{i+1}\}|\\
&\quad + |\{k : i+i < k \le n \text{ and } \pi_p(i+1) < [\pi_p(k)]_{i+1} < [\pi_p(i)_{i+1}\}|\\
&= \rho_{ij} + |\{k : i+i < k \le n \text{ and } \pi_p(i+1) < [\pi_p(k)]_{i+1} < [\pi_p(i)_{i+1}\}|
\end{aligned}$$

with (6.2) to obtain $r_{ii} = \rho_{ii} = p_i \bmod i$.

We cover the induction step by brute force. There are six cases to consider, given by the relative ordering of $\pi_p(j+1)$, $\pi_p(i+1)$, $\pi_p(i)$. We discuss two of these cases; the others are similar.

Case 1: $\pi_p(j+1) < \pi_p(i+1) < \pi_p(i)$. Here $\mathcal{L}_{ij}$ is the disjoint union of $\mathcal{L}_{i+1,j}$ and $\{k : 1 \le k < i \text{ and } \pi_p(i+1) < \pi_p(k) < \pi_p(i)\}$. The size of the first set is $\rho_{i+1,j}$ and the size of the second is $\rho_{ii} = p_i \bmod i$, so the size of their union is $\rho_{i+1,j} + p_i \bmod i = (\rho_{i+1,j} + p_i) \bmod i$.

Case 2: $\pi_p(j+1) < \pi_p(i) < \pi_p(i+1)$. Let $A = \{k : 1 \le k < i \text{ and } \pi_p(i) < \pi_p(k) < \pi_p(i+1)\}$. Then $\mathcal{L}_{ij} = \mathcal{L}_{i+1,j} \setminus A$, where $A \subset \mathcal{L}_{i+1,j}$. Then since $|\mathcal{L}_{i+1,j}|$ is $\rho_{i+1,j}$ and $|A| = -p_i \bmod i$, we have

$$\rho_{ij} = |\mathcal{L}_{i+1,j}| - |A| = \rho_{i+1,j} - (-p_i \bmod i) = (\rho_{i+1,j} + p_i) \bmod i.$$

6.3. **Cover multiplicity.** Suppose we have a $\{\pi_p(i), \pi_p(j+1)\}$ cover for the offset vector. The multiplicity of this cover is $[d_{\pi_p(i)}]_{j+1} - d_{\pi_p(i)}$, which is $d_{\pi_p(i)} - d_{\pi_p(j+1)}$ if $\pi_p(j+1) < \pi_p(i)$ and is $d_{\pi_p(i)} - 1 - d_{\pi_p(j+1)}$ if $\pi_p(j+1) > \pi_p(i)$. We have to show $[d_{\pi_p(i)}]_{j+1} - d_{\pi_p(i)} = \sum_{h=i}^{j} q_{hj}$.

On the one hand, we have that

$$d_{\pi_p(i)} - d_{\pi_p(i+1)} = \begin{cases} q_{ii} & \text{if } \pi_p(i) > \pi_p(i+1) \\ q_{ii} + 1 & \text{if } \pi_p(i) < \pi_p(i+1). \end{cases}$$

Thus,

$$d_{\pi_p(i)} - d_{\pi_p(j+1)} = \sum_{h=i}^{j} q_{hh} + |\{m : i \le m \le j \text{ and } \pi_p(m) < \pi_p(m+1)\}|.$$

Let $A_{ij} = |\{m : i \le m \le j \text{ and } \pi_p(m) < \pi_p(m+1)\}|$ and note that

$$A_{ij} = \begin{cases} A_{i+1,j} & \text{if } \pi_p(i) > \pi_p(i+1) \\ A_{i+1,j} + 1 & \text{if } \pi_p(i) < \pi_p(i+1), \end{cases}$$

where

$$A_{ii} = 0 \text{ if } \pi_p(i+1) < \pi_p(i) \text{ and } 1 \text{ otherwise.} \tag{6.4}$$

On the other hand,

$$q_{hj} = \begin{cases} q_{hh} & \text{if } r_{hh} + r_{h+1,j} < h \\ q_{hh} + 1 & \text{if } r_{hh} + r_{h+1,j} \ge h. \end{cases}$$

Therefore, we must show that

$$A_{ij} = \begin{cases} |\{m : i \le m \le j-1 \text{ and } r_{mm} + r_{m+1,j} \ge m\}| & \text{if } \pi_p(j+1) < \pi_p(i) \\ |\{m : i \le m \le j-1 \text{ and } r_{mm} + r_{m+1,j} \ge m\}| + 1 & \text{if } \pi_p(j+1) > \pi_p(i) \end{cases}$$

Let $B_{ij} = |\{m : i \le m \le j-1 \text{ and } r_{mm} + r_{m+1,j} \ge m\}|$, so our goal is to show

$$A_{ij} = \begin{cases} B_{ij} & \text{if } \pi_p(j+1) < \pi_p(i) \\ B_{ij} + 1 & \text{if } \pi_p(j+1) > \pi_p(i) \end{cases} \tag{6.5}$$

We note the simple recursion for B_{ij}:

$$B_{ij} = \begin{cases} B_{i+1,j} & \text{if } r_{ii} + r_{i+1,j} < i \\ B_{i+1,j} + 1 & \text{if } r_{ii} + r_{i+1,j} \ge i, \end{cases}$$

where

$$B_{ii} = 0. \tag{6.6}$$

We will need the following claim, and then the proof will follow by induction and consideration of cases.

Lemma 6.6. *We have that*

$$r_{ii} + r_{i+1,j} \ge i \textit{ if and only if } [\pi_p(i)]_{j+1} \in [\pi_p(j+1), [\pi_p(i+1)]_{j+1}].$$

Proof. Consider the algorithm to construct the permutation π_p (and the offset sequence) from k-bounded multiplicity vector $p = (p_1, \dots, p_k)$. Suppose $\pi_p(i+1)$ has been determined. Place $p_i = q_{ii} \cdot i + r_{ii}$ beads on the remaining i runners, starting at the first free runner after the one labelled by $\pi_p(i+1)$. The last r_{ii} determine $\pi_p(i)$, and we deposit these starting again at the first free runner after the one labelled by $\pi_p(i+1)$. These beads will first be placed on runners in the complement of $[\pi_p(j+1), [\pi_p(i+1)]_{j=1}]$, and then in $[\pi_p(j+1), [\pi_p(i+1)]_{j+1}]$ if there are enough beads. There are $i - r_{i+1,j}$ free runners in the complement of $[\pi_p(j+1), [\pi_p(i+1)]]$. Therefore, we have $r_{ii} \geq i - r_{i+1,j}$ if and only if $[\pi_p(i)]_{j+1} \in [\pi_p(j+1), [\pi_p(i+1)]_{j+1}]$. □

The base step of the induction follows from (6.4) and (6.6). Now assume that (6.5) holds for $i+1$ and j. Again, we have six cases, based on the relative positions of $\pi_p(i)$, $\pi_p(i+1)$, and $\pi_p(j+1)$, and again we prove only two of them.

Case 1: $\pi_p(i) < \pi_p(i+1) < \pi_p(j+1)$. By induction, we have $A_{i+1,j} = B_{i+1,j} + 1$ and we want $A_{ij} = B_{ij} + 1$. Since $\pi_p(i) < \pi_p(i+1)$, we have $A_{ij} = A_{i+1,j} + 1$ and since $[\pi_p(i)]_{j+1} \in [\pi_p(j+1), [\pi_p(i+1)]_{j+1}]$, we have $r_{ii} + r_{i+1,j} \geq i$, so that by Claim 6.6, $B_{ij} = B_{i+1,j} + 1$.

Then $A_{ij} = A_{i+1,j} + 1 = (B_{i+1,j} + 1) + 1 = B_{ij} + 1$.

Case 2: $\pi_p(i+1) < \pi_p(i) < \pi_p(j+1)$. By induction, we have $A_{i+1,j} = B_{i+1,j} + 1$ and again we want $A_{ij} = B_{ij} + 1$. Since $\pi_p(i+1) < \pi_p(i)$, we have $A_{ij} = A_{i+1,j}$ and since $[\pi_p(i)]_{j+1} \notin [\pi_p(j+1), [\pi_p(i+1)]_{j+1}]$, we have $r_{ii} + r_{i+1,j} < i$, so $B_{ij} = B_{i+1,j}$.

Then $A_{ij} = A_{i+1,j} = (B_{i+1,j} + 1) = B_{ij} + 1$.

6.4. **k-bounded multiplicity vector.** Suppose the k-bounded multiplicity vector p corresponds to the offset vector d and that d is a $\{\pi_p(i), \pi_p(j+1)\}$ cover, where $i \leq j$. Call the offset vector d covers $\tilde{d}$. We have to show that $\tilde{d}$ correponds to the k-bounded multiplicity vector $p^{i,j,r_{ij}}$, which will call $\tilde{p}$ in this section.

By [LLMS10], we

$$\tilde{d} = \begin{cases} (d_1, \dots, d_{\pi_p(i)}, \dots, d_{\pi_p(j+1)}, \dots, d_n) & \text{if } d = (d_1, \dots, d_{\pi_p(j+1)}, \dots, d_{\pi_p(i)}, \dots, d_n) \\ (d_1, \dots, d_{\pi_p(j+1)} + 11, \dots, d_{\pi_p(i)} - 1, \dots, d_n) & \text{if } d = (d_1, \dots, d_{\pi_p(i)}, \dots, d_{\pi_p(j+1)}, \dots, d_n). \end{cases}$$

We consider how we would have to change p so that in Algorithm 6.1 we produce $\tilde{d}$ instead of d. We assume $r = 0$, $i < j$, and $\pi_p(j+1) < \pi_p(i)$; otherwise, the arithmetic muddies the water too much. Between runners $\pi_p(j+1)$ and $\pi_p(i)$, there are r_{ij} runners whose labels are $\pi_p(m)$ for $1 \leq m < i$. Since d covers $\tilde{d}$, there are no runners between runners $\pi_p(j+1)$ and $\pi_p(i)$ whose labels are $\pi_p(m)$ for $i < m < j+1$. That is, when Algorithm 6.1 is used on p, at all steps from step $n-(j+1)$ to step $n-i$, there are the same number of free runners between runners $\pi_p(j+1)$ and $\pi_p(i)$: r_{ij}.

At step $n-(j+1)$, we start at the first free runner after $\pi_p(j+2)$. Up to now, the abacus for p and $\tilde{p}$ are the same. Suppose we distribute $\tilde{p}_{j+1} = t_{j+1} + r_{ij} + 1$ instead of t_{j+1} beads. Then one bead is for runner $\pi_P(j+1)$ and we end at $\pi_p(i)$, so that that $d_{\pi_{\tilde{p}}(j+1)} = d_{\pi_p(i)}$. Next we distribute $\pi_{\tilde{p}}(j) = \pi_p(j) - r_{ij} - 1$ beads, landing back at runner $\pi_p(j) = \pi_{\tilde{p}}(j)$.

Similarly, at step $n-i$, we start at the first free runner after $\pi_p(i+1) = \pi_{\tilde{p}}(i+1)$ and distribute $t_{\tilde{p}} = t_p - r_{ij}$, landing at $\pi_p(j+1)$ instead of $\pi_(i)$. For the next step, we compensate by distributing $t_{\tilde{p}} = t_p + r_{ij}$ beads, forcing $\pi_p(i-1) = \pi_{\tilde{p}}(i-1)$ and realigning d with $\tilde{d}$.

This process is reversible; that is, if $p^{i,j,m}$ is a covered by p, then we must have $m = r_{ij}$.

This completes the proof of Theorem 5.2.

7. Induction for the weighted version

It is clear that an inductive proof of the weighted version, Theorem 3.1, is necessarily more complicated. Indeed, it is not clear what the recursive formula for $F_\pi^{(k)}(t)$ would be: if we add a strong marked cover to a starred tableau T to get T', the major index of T' does not depend only on the marked cover and the major index of T, but also on the the last cover of T.

It follows that in order to prove Theorem 3.1 by induction (for small k), we have to state a stronger statement. The following conjecture postulates that it is possible to compute $F_{\pi,h}^{(k)}(t) = \sum_T t^{\mathrm{maj}(T)}$, where the sum is over all standard strong marked tableaux T of shape $\mathfrak{c}(\pi)$ for which the marked box with the largest entry is in row h.

Conjecture 7.1. *For each k, there exist rational functions $p_{\sigma,I,J}^{(k)}(t)$ for a k-irreducible partition σ, $1 \leq I \leq J \leq k$ such that*

$$\sum_{I=1}^{J} p_{\sigma,I,J}(t) = (\boldsymbol{J(k+1-J)}) + (t^{J(k+1-J)} - 1) \sum_{j=1}^{J-1} \sum_{i=1}^{m_{k+1-j}(\sigma)} p_{\sigma,i,j}(t)$$

$$\sum_{J=1}^{k} \sum_{I=1}^{m_{k+1-J}(\sigma)} p_{\sigma,I,J}(t) = (\boldsymbol{|\sigma|})$$

and such that the following holds. For a k-bounded partition π,

$$\pi = \langle k^{p_1}, (k-1)^{p_2}, \ldots, 1^{p_k} \rangle,$$

write $p_i = a_i + iw_i$, where $0 \leq a_i < i$. For h, $1 \leq h \leq \ell(\pi)$, find (the unique) J, $1 \leq J \leq k$, so that $p_1 + \ldots + p_{J-1} < h \leq p_1 + \ldots + p_J$, and write $h = p_1 + \ldots + p_{J-1} + I + Jm$, where $1 \leq I \leq J$ and $m \geq 0$. Then

$$\frac{F_{\pi,h}^{(k)}(t)}{F_\pi^{(k)}(t)} = \frac{t^{\sum_{i=1}^{J-1} i(k+1-i)w_i + mJ(k+1-J)}\, p_{\sigma,I,J}^{(k)}(t)}{(\boldsymbol{|\pi|})}.$$

In other words,

$$F_{\pi,h}^{(k)}(t) = \frac{t^{\sum_{i=1}^{k} w_i \binom{i}{2}(k-i+1) + \sum_{i=1}^{J-1} i(k+1-i)w_i + mJ(k+1-J)} (\boldsymbol{|\pi|-1})!\, F_\sigma^{(k)}(t) p_{\sigma,I,J}^{(k)}(t)}{(\boldsymbol{|\sigma|})! \prod_{j=1}^{k} (\boldsymbol{j})^{\sum_{i=1}^{k} w_i \min\{i,j,k+1-i,k+1-j\}}}.$$

The conjecture is essentially saying that each possible row h get a predictable, "fair" share of starred tableaux. This is quite suprising; it does not seem to hold for standard Young tableaux.

Example 7.2. The following tables give the rational functions $p_{\sigma,I,J}^{(k)}(t)$ for $k = 2, 3$ (in row I and column J of the table for σ).

$2^0 1^0$	$\mathbf{(2)}$	0	$2^0 1^1$	$\mathbf{(2)}$	1
		$\mathbf{(2)}$			t

$3^0 2^0 1^0$	$\mathbf{(3)}$	0	0	$3^0 2^0 1^1$	$\mathbf{(3)}$	1	1
		$\mathbf{(4)}$	0			$t\mathbf{(3)}$	0
			$\mathbf{(3)}$				$t\mathbf{(2)}$
$3^0 2^0 1^2$	$\mathbf{(3)}$	0	0	$3^0 2^1 1^1$	$\mathbf{(3)}$	$\mathbf{(2)}$	0
		$\mathbf{(4)}$	$\mathbf{(2)}$			$t^2\mathbf{(2)}$	0
			t^2				$t^2\mathbf{(3)}$
$3^0 2^1 1^1$	$\mathbf{(3)}$	$\frac{\mathbf{(3)}}{\mathbf{(2)}}$	$\frac{t\mathbf{(3)}}{\mathbf{(2)}}$	$3^0 2^1 1^2$	$\mathbf{(3)}$	$\mathbf{(2)}$	$\frac{t^2\mathbf{(2)}}{\mathbf{(3)}}$
		$\frac{t(1+t+2t^2+t^3)}{\mathbf{(2)}}$	0			$t^2\mathbf{(2)}$	$\frac{t^3\mathbf{(2)}^2}{\mathbf{(3)}}$
			$\frac{t^3\mathbf{(3)}}{\mathbf{(2)}}$				t^4

Of course, $p_{\sigma,I,J}^{(k)}(t)$ get progressively more complicated for higher k. ◇

It follows immediately from the definition of the major index that

$$F^{(k)}_{\pi,h}(t) = \sum_{i=1}^{h-1} q^{|\pi|-1} \sum_{\tau} F^{(k)}_{\tau,i}(t) + \sum_{i=h}^{\ell(\pi)} \sum_{\tau} F^{(k)}_{\tau,i}(t),$$

where both inner sums are over k-bounded partitions τ of size $|\pi|-1$ that are covered by π and so that one of the ribbons of $\mathfrak{c}(\pi)/\mathfrak{c}(\tau)$ has its head in row h (note that this can be described explicitly). Therefore it remains to check that the formula for $F^{(k)}_{\pi,h}(t)$ from the conjecture satisfies the same recursion.

8. Final remarks

8.1. There are also notions of *weak horizontal strips* and *weak tableaux*. For n-cores λ and μ, $\lambda \subseteq \mu$, we say that μ/λ is a weak horizontal strip if $\mathfrak{b}(\mu)/\mathfrak{b}(\lambda)$ is a horizontal strip and $\mathfrak{b}(\mu')/\mathfrak{b}(\lambda')$ is a vertical strip. If in addition $|\mathfrak{b}(\mu)| = |\mathfrak{b}(\lambda)|+1$, we say that μ *covers* λ *in the weak order*. A weak tableau of shape λ is a sequence of weak horizontal strips $\mu^{(i+1)}/\mu^{(i)}$, $i=0,\ldots,m-1$, such that $\mu^{(0)}=\emptyset$ and $\mu^{(m)}=\lambda$. Define $f^{(k)}_\pi$ to be the number of weak tableaux of shape $\mathfrak{c}(\pi)$. In [LLMS10], it was proved that $f^{(2)}_{2^{w_1}1^{2w_2}} = f^{(2)}_{2^{w_1}1^{1+2w_2}} = \frac{(w_1+w_2)!}{w_1!w_2!}$.

It is not hard to prove by induction that

$$\begin{aligned}
f^{(3)}_{3^{w_1}2^{2w_2}1^{3w_3}} &= \tfrac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3-1)!(2w_1+2w_2+2w_3)!}{w_1!w_2!w_3!(w_1+w_2+w_3-1)!(2w_1+2w_2)!(2w_2+2w_3)!}\\
f^{(3)}_{3^{w_1}2^{2w_2}1^{1+3w_3}} &= \tfrac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3)!(2w_1+2w_2+2w_3)!}{w_1!w_2!w_3!(w_1+w_2+w_3)!(2w_1+2w_2)!(2w_2+2w_3)!}\\
f^{(3)}_{3^{w_1}2^{2w_2}1^{2+3w_3}} &= \tfrac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3)!(2w_1+2w_2+2w_3+1)!}{w_1!w_2!w_3!(w_1+w_2+w_3)!(2w_1+2w_2)!2w_2+2w_3+1)!}\\
f^{(3)}_{3^{w_1}2^{1+2w_2}1^{3w_3}} &= \tfrac{2^{2w_2}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3)!(2w_1+2w_2+2w_3+1)!}{w_1!w_2!w_3!(w_1+w_2+w_3)!(2w_1+2w_2+1)!(2w_2+2w_3)!}\\
f^{(3)}_{3^{w_1}2^{1+2w_2}1^{1+3w_3}} &= \tfrac{2^{2w_2+1}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3+1)!(2w_1+2w_2+2w_3+1)!}{w_1!w_2!w_3!(w_1+w_2+w_3)!(2w_1+2w_2+1)!(2w_2+2w_3+1)!}\\
f^{(3)}_{3^{w_1}2^{1+2w_2}1^{2+3w_3}} &= \tfrac{2^{2w_2+1}(w_1+w_2)!(w_2+w_3)!(w_1+2w_2+w_3+1)!(2w_1+2w_2+2w_3+1)!}{w_1!w_2!w_3!(w_1+w_2+w_3)!(2w_1+2w_2+1)!(2w_2+2w_3+1)!}
\end{aligned}$$

We were unable to find formulas for $k \geq 4$, but it seems unlikely that simple formulas exist. For example, the simplest recurrence relation that $g(i,j) = f^{(4)}_{2^{3i}1^{4j}}$ seem to satisfy is

$$a(i,j)g(i,j) + b(i,j)g(i,j+1) - c(i,j)g(i+1,j) = 0,$$

where

$$\begin{aligned}
a(i,j) =& \tfrac{(3i+2j+1)(3i+2j+2)\left(63i^2+111ij+108i+36j^2+89j+45\right)}{3}\\
b(i,j) =& \tfrac{5(j+1)\left(216i^3+432i^2j+432i^2+219ij^2+513ij+264i+36j^3+125j^2+139j+48\right)}{3}\\
c(i,j) =& (i+1)(3i+1)(3i+2)(9i+12j+11)
\end{aligned}$$

8.2. Our work has led us to consider (weighted) correction factors. They seem to be mysterious objects that deserve further study. The unimodality conjecture (Conjecture 3.7) is certainly intriguing and could hint that the factors have some geometric meaning.

Let us give another perspective on these factors. Since k-Schur functions are symmetric, they can be expanded in terms of Schur functions; in fact, the original definition (conjecturally equivalent to our definition) of k-Schur functions via atoms gives precisely such an expansion. For example,

$$s^{(4)}_{2211} = s_{2211} + s_{321}.$$

Take the stable principle specialization (i.e., evaluate at $1, t, t^2, \ldots$) and multiply by $\mathbf{(6)!}(1-t)^6$. By (4.12) and [Sta99, Proposition 7.19.11], we have

$$F^{(4)}_{2211}(t) = f_{2211}(t) + f_{321}(t).$$

Then, by (3.2) and [Sta99, Corollary 7.21.5],

$$\frac{q^4 C_{2211}(t)}{\mathbf{(2)(3)(4)}} = \frac{q^7}{\mathbf{(2)}^2\mathbf{(4)(5)}} + \frac{q^4}{\mathbf{(3)}^2\mathbf{(5)}}$$

and so

$$C_{2211}(t) = \mathbf{(2)(3)(4)}\left(\frac{t^3}{\mathbf{(2)}^2\mathbf{(4)(5)}} + \frac{1}{\mathbf{(3)}^2\mathbf{(5)}}\right) = \frac{1+2t+t^2+t^3}{\mathbf{(2)(3)}}.$$

8.3. There is also a formula for the principal specialization of s_λ of order i (i.e. evaluation at $1, t, \ldots, t^{i-1}$, see e.g. [Sta99, Theorem 7.21.2]), in which both hook-lengths and contents of boxes appear. By imitating **8.2**, we can get rational functions (which depend on i) which converge to the weighted correction factors as $i \to \infty$. These rational functions also seem interesting and worthy of further study.

8.4. As we already mentioned, it would be preferable to prove Corollary 3.2 by induction, as in Section 5, but for a general k and in a way that would make apparent the meaning of hook-lengths and correction factors (the ideal being a variant of the probabilistic proof from [GNW79]). It seems likely that one would need to know a formula for the correction factors before such a proof would be feasible.

8.5. We showed (in Theorem 5.2) how to interpret the residue and quotient table to find strong covers. We feel that residue (and quotient) tables could prove important in other aspects of the k-Schur function theory. In the follow-up paper [Kon], the following results are presented:

Description of k-conjugates: Take a k-irreducible partition π. Then the number of parts of $\pi^{(k)}$ equal to i is the number of elements in row $i+1$ that are strictly greater than the element immediately above them. (A similar description exists for general k-bounded partitions.)

Description of weak covers: Identify a k-bounded partition $\pi = \langle k^{p_1}, \ldots, 1^{p_k}\rangle$ with $p = (p_1, \ldots, p_k)$, and write $\varepsilon_i = (0, \ldots, 1, -1, \ldots, 0)$ (with 1 in position i and -1 in position $i+1$). It is obvious that $p + \varepsilon_i$ covers p in the Young lattice if and only if $p_{i+1} > 0$. Denote the residue table of p by $\mathcal{R}$. Then $p + \varepsilon_i$ covers p in the weak order if and only if $r_{i+1,i+1}, \ldots, r_{i+1,k} > 0$.

Description of weak strips: More generally, for a set $S \subseteq \{1, \ldots, k\}$, denote by π^S the partition whose corresponding sequence is $p + \sum_{i \in S} \varepsilon_i$. Then π^S/π is a weak strip if and only if $r_{i+1,j} > 0$ for all $i \in S$, $j \notin S$, $i \leq j$.

Description of LLMS insertion: In [LLMS10], a variant of the Robinson-Schensted insertion for weak and strong marked tableaux is presented. The procedure has very important implications, but is extremely complicated. It simplifies slightly when specialized to standard tableaux [LLMS10, §10.4]. The following description of case X hints that a description in terms of residue tables could be possible. Note that case X occurs when all three known corners of a square in the growth diagram are the same, say π, and the number within in the square is 1. Then the unknown corner of the square is $\pi + \varepsilon_i$, where i is the unique index for which $r_{i+1,j} > 0$ for $i+1 \leq j \leq k$ and $r_{ji} < j-1$ for $2 \leq j \leq i$.

9. The appendix: Correction factors for $k = 5$

$2^2 1^3$ $\frac{t^3}{\mathbf{(2)(4)}}$	$2^3 1^2$ $\frac{t^2(1+t)}{\mathbf{(3)}^2}$	$2^3 1^3$ $\frac{t^2}{\mathbf{(4)}}$	$3^1 2^1 1^2$ $\frac{t^2}{\mathbf{(3)}^2}$
$3^1 2^1 1^3$ $\frac{t^3}{\mathbf{(3)(4)}}$	$3^1 2^2 1^1$ $\frac{t}{\mathbf{(4)}}$	$3^1 2^2 1^2$ $\frac{t^2(1+2t+2t^2+2t^3+2t^4+t^5)}{\mathbf{(3)(4)}}$	$3^1 2^2 1^3$ $\frac{t^2(1+2t+2t^2+t^3+t^4)}{\mathbf{(2)(4)}^2}$
$3^1 2^3 1^2$ $\frac{t^2(2+2t+2t^2+t^3)}{\mathbf{(3)}^3}$	$3^1 2^3 1^3$ $\frac{t(1+t+t^2+t^3+t^4)}{\mathbf{(4)}^2}$	$3^2 1^2$ $\frac{t^2}{\mathbf{(4)}}$	$3^2 2^1 1^1$ $\frac{t(1+t+t^2)}{\mathbf{(2)(4)}}$
$3^2 2^1 1^2$ $\frac{t(1+t)}{\mathbf{(3)}^2}$	$3^2 2^1 1^3$ $\frac{t^2(1+t)}{\mathbf{(3)(4)}}$	$3^2 2^2$ $\frac{t^2\mathbf{(2)}}{\mathbf{(3)}^2}$	$3^2 2^2 1^1$ $\frac{t}{\mathbf{(3)}}$
$3^2 2^2 1^2$ $\frac{t^2(2+3t+3t^2+2t^3+t^4)}{\mathbf{(3)}^2\mathbf{(4)}}$	$3^2 2^2 1^3$ $\frac{\beta_1(t)}{\mathbf{(2)}^2\mathbf{(4)}^3}$	$3^2 2^2 1^4$ $\frac{t^2(1+t)}{\mathbf{(3)}^2}$	$3^2 2^3 1^2$ $\frac{t(1+2t+2t^2+2t^3+t^4)}{\mathbf{(3)}^3}$
$3^2 2^3 1^3$ $\frac{t(1+2t+3t^2+3t^3+2t^4)}{\mathbf{(2)(3)(4)}}$	$4^1 2^1 1^1$ $\frac{t}{\mathbf{(2)(4)}}$	$4^1 2^1 1^2$ $\frac{t^2}{\mathbf{(3)(4)}}$	$4^1 2^1 1^3$ $\frac{t^3}{\mathbf{(4)}^2}$
$4^1 2^2 1^3$ $\frac{t(1+t+t^2+t^3+t^4+t^5)}{\mathbf{(2)(4)}^2}$	$4^1 2^3 1^2$ $\frac{t^2(2+2t+2t^2+2t^3+t^4)}{\mathbf{(4)}}$	$4^1 2^3 1^3$ $\frac{t^2}{\mathbf{(4)}}$	$4^1 3^1 1^1$ $\frac{t(1+t)}{\mathbf{(3)}^2}$
$4^1 3^1 1^2$ $\frac{t^2(1+t)}{\mathbf{(3)(4)}}$	$4^1 3^1 2^1$ $\frac{t}{\mathbf{(4)}}$	$4^1 3^1 2^1 1^1$ $\frac{t(1+t+2t^2+2t^3+t^4)}{\mathbf{(2)(4)}^2}$	$4^1 3^1 2^1 1^2$ $\frac{t(1+2t+2t^2+2t^3)}{\mathbf{(3)}^3}$
$4^1 3^1 2^1 1^3$ $\frac{t(1+2t+2t^2+2t^3+2t^4)}{\mathbf{(3)}^2\mathbf{(4)}}$	$4^1 3^1 2^1 1^4$ $\frac{t}{\mathbf{(4)}}$	$4^1 3^1 2^2$ $\frac{t^2(1+t)}{\mathbf{(3)(4)}}$	$4^1 3^1 2^2 1^1$ $\frac{t(1+2t+3t^2+3t^3+2t^4)}{\mathbf{(3)}^2\mathbf{(4)}}$
$4^1 3^1 2^2 1^2$ $\frac{t^2(2+3t+2t^2)}{\mathbf{(3)}^3}$	$4^1 3^1 2^2 1^3$ $\frac{t(1+t+t^2)}{\mathbf{(2)(4)}}$	$4^1 3^1 2^2 1^4$ $\frac{t^2(1+t)}{\mathbf{(3)(4)}}$	$4^1 3^1 2^3 1^2$ $\frac{t(1+3t+3t^2+3t^3+t^4)}{\mathbf{(3)}^3}$
$4^1 3^1 2^3 1^3$ $\frac{\beta_2(t)}{\mathbf{(3)}^2\mathbf{(4)}^2}$	$4^1 3^2 1^2$ $\frac{t(1+t+t^2+t^3+t^4)}{\mathbf{(4)}^2}$	$4^1 3^2 2^1 1^1$ $\frac{\beta_3(t)}{\mathbf{(2)}^2\mathbf{(4)}^3}$	$4^1 3^2 2^1 1^2$ $\frac{t(1+t)}{\mathbf{(3)}^2}$
$4^1 3^2 2^1 1^3$ $\frac{t^2(1+t)}{\mathbf{(3)(4)}}$	$4^1 3^2 2^2$ $\frac{t(1+2t+2t^2+2t^3+t^4)}{\mathbf{(3)}^3}$	$4^1 3^2 2^2 1^1$ $\frac{t(2+3t+3t^2+2t^3+t^4)}{\mathbf{(2)(3)(4)}}$	$4^1 3^2 2^2 1^2$ $\frac{\beta_4(t)}{\mathbf{(3)}^2\mathbf{(4)}^2}$
$4^1 3^2 2^2 1^3$ $\frac{\beta_5(t)}{\mathbf{(2)}^2\mathbf{(4)}^4}$	$4^1 3^2 2^2 1^4$ $\frac{t(1+2t+2t^2+2t^3+t^4)}{\mathbf{(3)}^3}$	$4^1 3^2 2^3 1^2$ $\frac{t(1+2t+2t^2+2t^3+t^4)}{\mathbf{(3)}^3}$	$4^1 3^2 2^3 1^3$ $\frac{t(2+3t+5t^2+5t^3+5t^4+3t^5+2t^6)}{\mathbf{(3)(4)}^2}$

where:

$$\beta_1(t) = 1 + 4t + 9t^2 + 15t^3 + 19t^4 + 20t^5 + 16t^6 + 10t^7 + 5t^8 + 2t^9$$
$$\beta_2(t) = 1 + 3t + 7t^2 + 11t^3 + 13t^4 + 11t^5 + 8t^6 + 4t^7 + t^8$$
$$\beta_3(t) = 2 + 5t + 10t^2 + 16t^3 + 20t^4 + 19t^5 + 15t^6 + 9t^7 + 4t^8 + t^9$$
$$\beta_4(t) = 1 + 4t + 8t^2 + 11t^3 + 13t^4 + 11t^5 + 7t^6 + 3t^7 + t^8$$
$$\beta_5(t) = 2 + 7t + 18t^2 + 36t^3 + 58t^4 + 75t^5 + 82t^6 + 75t^7 + 58t^8 + 36t^9 + 18t^{10} + 7t^{11} + 2t^{12}$$

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