Kronecker Bases for Linear Matrix Equations, with Application to Two-Parameter Eigenvalue Problems

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Abstract

The general solutions of the homogeneous matrix equation

$$AXC^T - BXD^T = 0$$

and the system of the matrix equations

$$AX + BY = 0, \quad XC^T + YD^T = 0$$

are described in terms of Kronecker canonical forms, i.e., in terms of Kronecker invariants and Kronecker bases, for pairs of matrices (A, B) and (C, D). A canonical form for a pair of commuting matrices (E, F) such that $E^2 = F^2 = EF = 0$ is discussed. These results are applied to construct a canonical basis for the second root subspace of a two-parameter eigenvalue problem. The corresponding relations for canonical invariants are given.

1 Introduction

In multiparameter spectral theory we consider a system of multiparameter pencils

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} V_{ij}\lambda_j - V_{i0}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where V_{ij} are $n_i \times n_i$ matrices over complex numbers and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are parameters. We remark that when $n = 1$ then (1) is a one-parameter pencil. It is well known that there is a canonical form for such pencils under equivalence of pairs of matrices. This

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Short title : Linear Matrix Equations with Application is the Kronecker canonical form [28] (see [12, 17, 19] for modern versions). In this paper the Kronecker canonical form is applied in two different ways to study two-parameter systems, i.e. systems of form (1) with $n = 2$.

Our approach is based on Atkinson's theory [3]. He associated with (1) an n-tuple of commuting matrices $\{\Gamma_j\}_{j=1}^n$, acting on $\mathbb{C}^N$, $N = \prod_{i=1}^n n_i$. One of the problems considered then is to give bases for joint root subspaces of matrices Γ_j in terms of the matrices V_{ij}. This has been achieved using coalgebraic techniques in [20]. However, the description obtained in [20] shows that in general complex calculations are required in order to compute these bases. An interesting problem now is to find classes of multiparameter systems (1) for which the computation of bases for joint root subspaces of Γ_j is simplified. Some of these are already known in the literature, e.g. [4, 15, 26, 27]. (We discuss the literature, including that on applications, later in the section.) One of simplest cases where to look for motives that would give further algorithms to build bases for root subspaces is to consider two-parameter systems. We do so, and hence we assume hereafter that $n = 2$.

We construct a canonical basis for the second root subspace

$$\mathcal{N} = \ker(\Gamma_1 - \lambda_1 I)^2 \cap \ker[(\Gamma_1 - \lambda_1 I)(\Gamma_2 - \lambda_2 I)] \cap \ker(\Gamma_2 - \lambda_2 I)^2. \tag{2}$$

Bases for the second root subspace for the general multiparameter systems are given in [6]. The main contribution of our paper is that in two-parameter setting we can construct these bases canonically. We remark that in the literature there are known classes of multiparameter systems for which in general the joint eigenvectors for Γ_1 and Γ_2 do not span root subspaces but the second root vectors do span them (cf. [6, §8], [7, §5] and [15, Thm. 5.7]). Thus to find a basis for the second root subspace is not only interesting as an intermediate step towards a construction of a basis for the entire root subspace but for certain cases it already gives a complete basis.

The first application of the Kronecker canonical form in our paper is to describe a canonical form for a pair of commuting matrices A_1 and A_2 that satisfy $A_1^2 = A_1 A_2 = A_2^2 = 0$. Note that the restrictions of $\Gamma_1 - \lambda_1 I$ and $\Gamma_2 - \lambda_2 I$ to the second root subspace $\mathcal{N}$ satisfy these relations. We remark that this is a special case of a canonical form given by Gel'fand and Ponomarev in [18, Ch. II].

The main result of [6] is that a basis for the second root subspace $\mathcal{N}$ is constructed from a basis for the (joint) eigenspace

$$\mathcal{M} = \ker(\Gamma_1 - \lambda_1 I) \cap \ker(\Gamma_2 - \lambda_2 I) \tag{3}$$

and a basis for the kernel of a special matrix, which yields when applied to the two-parameter case a system of matrix equations

$$AX - BY = 0, \quad XC^T - YD^T = 0 \tag{4}$$

(see §2 for details). This system is related to the matrix equation

$$AXC^T - BXD^T = 0. \tag{5}$$

Namely, it is easy to observe that if a pair of matrices (X, Y) solves (4) then both X and Y solve (5). We use Kronecker canonical form for pairs of matrices (A, B) and (C, D)

to construct canonical bases for the space of solutions of (5) and (4). This is our second application of the Kronecker canonical form. We also associate with the construction of bases a set of canonical invariants. These yield when applied to the two-parameter system a basis in which the restrictions of $\Gamma_1 - \lambda_1 I$ and $\Gamma_2 - \lambda_2 I$ to the second root subspace $\mathcal{N}$ are in the canonical form mentioned in the previous paragraph. We believe that the calculations in terms of invariants will yield when translated into appropriate algebraic setting towards better understanding of the main result of [20], and subsequently enable us to find simpler algorithms to calculate bases for joint root subspaces.

The study of two-parameter eigenvalue problems has a long history. Originally they were studied for boundary value problems. Then V_{ij} act in infinite dimensional Hilbert spaces, V_{i0} are differential operators and V_{ij}, $j \neq 0$, are multiplication operators. The description of bases for root subspaces then yields various completeness and expansion results. The cases when the eigenvectors are complete were studied already at the turn of the century. For instance, Dixon [13], Camp [8], and also Hilbert [24] studied expansions of functions in terms of eigenfunctions of a pair of two-parameter differential equations of Sturm-Liouville type. Later Pell [31] studied a two-parameter system of integral equations of Fredholm type. In the 1950s Cordes [10, 11] developed an abstract Hilbert space setting for a special class of two-parameter eigenvalue problems (cf. also [29]). Among recent publications we find work of Almamedov, Aslanov and Isaev [1], Binding and Browne [5], Faierman [15], and many others. Besides the applications to the boundary value problems [2] two-parameter eigenvalue problems (as well as multiparameter ones) occur in various applications, for instance, to the linearized bifurcation models [22], to the inverse eigenvalue problems [21], and to the linearizations of polynomials in two (or more) variables [16].

The matrix equation

$$AXD^T - BXC^T = E \qquad (6)$$

has been studied for a long period of time as well (see [33, 34, 36]). Various applications of these matrix equations are known in the literature, and they motivated several authors to study them, e.g. [9, 14, 23]. The idea to use the Kronecker canonical forms of pairs of matrices (A, B) and (C, D) in order to study the matrix equation (6) was brought forward by Mitra [30] and Rózsa in [32]. Since we could not find in the literature a description of the general solutions of the matrix equations (4) and (5) we first describe these solutions. The methods involved are very standard so we do not give detailed proofs. We remark that a detailed discussion of the general solutions of the matrix equation (6) in terms of the Kronecker canonical forms is given in the internal research paper [25].

We conclude the introduction with some words on the setup of our paper. In §2 we recall Atkinson's construction [3] for two-parameter systems. We also state the two-parameter versions of the main results of [6] and illustrate them with an example. To unify the treatment of various cases our definitions of the Kronecker chains and associated invariants differ slightly from the standard ones. We give the definitions and explain the differences in §3. There we also discuss a canonical form for a pair of commuting matrices (A_1, A_2) such that $A_1^2 = A_1 A_2 = A_2^2 = 0$. Canonical bases, with corresponding invariants, for the spaces of solutions of (5) and (4) are given in §4 and §5, respectively. We apply these bases to construct a canonical basis of the second root subspace for two-parameter systems in §6. We also discuss relations between the invariants and give two examples.

2 Two-parameter Eigenvalue Problems

In order to construct a pair of commuting matrices $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)$ associated with a two-parameter system

$$\mathbf{W} = \{W_i(\boldsymbol{\lambda}) = V_{i1}\lambda_1 + V_{i2}\lambda_2 - V_{i0};\ i = 1, 2\} \tag{7}$$

a regularity assumption is needed. We assume that $\Delta_0 = V_{11} \otimes V_{22} - V_{12} \otimes V_{21}$ is an invertible matrix. Here Δ_0 is acting on $\mathbb{C}^{n_1 n_2}$ ($\equiv \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$). Further we define matrices $\Delta_1 = V_{10} \otimes V_{22} - V_{12} \otimes V_{20}$ and $\Delta_2 = V_{11} \otimes V_{20} - V_{10} \otimes V_{21}$. Atkinson [3, Thm. 6.7.2] showed that matrices $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1, 2$ commute. We call the pair $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)$ the associated system of the system (7).

A pair of complex numbers (λ_1, λ_2) is called an eigenvalue of $\mathbf{\Gamma}$ if the eigenspace $\mathcal{M}$, defined by (3), is nonzero. A pair of complex numbers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is called an eigenvalue of $\mathbf{W}$ if $\ker W_i(\boldsymbol{\lambda}) \neq \{0\}$ for $i = 1, 2$. Atkinson [3, Thm. 6.8.1] showed that the spectra $\sigma(\mathbf{W})$ of a multiparameter system $\mathbf{W}$ and $\sigma(\mathbf{\Gamma})$ of its associated system $\mathbf{\Gamma}$ coincide.

From now we assume that $\boldsymbol{\lambda} \in \mathbb{C}^2$ is an eigenvalue of $\mathbf{W}$. Suppose that $q_i = \dim\ker W_i(\boldsymbol{\lambda})$ and that the columns of matrices X_{i0} and $Y_{i0} \in \mathbb{C}^{n_i \times q_i}$ form bases for the kernels of $W_i(\boldsymbol{\lambda})$ and $W_i(\boldsymbol{\lambda})^*$, respectively. Then we associate with $\boldsymbol{\lambda}$ four matrices

$$V_{ij}^{\lambda} = Y_{i0}^* V_{ij} X_{i0} \left(\in \mathbb{C}^{q_i \times q_i}\right), \quad i, j = 1, 2. \tag{8}$$

We identify the tensor product space $\mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2}$ with the vector space of $q_1 \times q_2$ complex matrices via the isomorphism $\Xi : \mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2} \longrightarrow \mathbb{C}^{q_1 \times q_2}$ defined by

$$\Xi : \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{q_1} \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{q_2} \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_{q_2} \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_{q_2} \\ \vdots & \vdots & & \vdots \\ x_{q_1} y_1 & x_{q_1} y_2 & \cdots & x_{q_1} y_{q_2} \end{bmatrix}$$

for decomposable tensors and extended by linearity. Hereafter we assume this identification. Thus we view

$$\Delta_0^{\lambda} = V_{11}^{\lambda} \otimes V_{22}^{\lambda} - V_{12}^{\lambda} \otimes V_{21}^{\lambda}$$

and

$$\mathcal{D}_0^{\lambda} = \begin{bmatrix} V_{11}^{\lambda} \otimes I & V_{12}^{\lambda} \otimes I \\ I \otimes V_{21}^{\lambda} & I \otimes V_{22}^{\lambda} \end{bmatrix}$$

as linear transformations acting on $\mathbb{C}^{q_1 \times q_2}$ and $\mathbb{C}^{q_1 \times q_2} \oplus \mathbb{C}^{q_1 \times q_2}$ respectively. For instance, if $X \in \mathbb{C}^{q_1 \times q_2}$ then $\Delta_0^{\lambda}(X) = V_{11}^{\lambda} X \left(V_{22}^{\lambda}\right)^T - V_{12}^{\lambda} X \left(V_{21}^{\lambda}\right)^T$.

For every positive integer n we write $\underline{n} = \{1, 2, \ldots, n\}$ and $\underline{0} = \emptyset$. The set of pairs of indices $\underline{q_1} \times \underline{q_2}$ is denoted by $\mathbf{Q}$, and we write $\mathbf{k} = (k_1, k_2)$ for an element $\mathbf{k}$ in $\mathbf{Q}$.

We denote the columns of the matrix X_{i0} by x_{i0}^{k}, $k \in \underline{q_i}$. By [3, Thm. 6.9.1] it follows that $\mathcal{B}_0 = \left\{ z_0^{\mathbf{k}} = x_{10}^{k_1} \otimes x_{20}^{k_2},\ \mathbf{k} \in \mathbf{Q} \right\}$ is a basis for the eigenspace $\mathcal{M}$, and hence $d_0 := \dim \mathcal{M} = q_1 q_2$. We complete the set $\mathcal{B}_0$ to a basis $\mathcal{B} = \mathcal{B}_0 \cup \left\{ z_1^l,\ l \in \underline{d_1} \right\}$ for the second root subspace $\mathcal{N}$ (see (2)). Here $d_1 = \dim \mathcal{N} - d_0$.

Let $A_i = (\Gamma_i - \lambda_i I)\,|_{\mathcal{N}},\ i = 1, 2$. In the basis $\mathcal{B}$ we have that

$$A_i = \begin{bmatrix} 0 & B_i \\ 0 & 0 \end{bmatrix}, \tag{9}$$

where $B_i = \left[b_i^{\mathbf{k}l} \right]_{\mathbf{k} \in \mathbf{Q}, l=1}^{d_1}$ is a matrix of sizes $d_0 \times d_1$. Here we assume that $\mathbf{Q}$ is ordered lexicographically. By (9) we have that

$$(\Gamma_i - \lambda_i I)\, z_1^l = \sum_{\mathbf{k} \in \mathbf{Q}} b_i^{\mathbf{k}l} z_0^{\mathbf{k}} \tag{10}$$

for $i = 1, 2$ and $l \in \underline{d_1}$. We write $\mathbf{B}_i^l = \left[b_i^{\mathbf{k}l} \right]_{k_1=1, k_2=1}^{q_1 \quad q_2}$ and

$$\mathbf{B}^l = \begin{bmatrix} \mathbf{B}_1^l \\ \mathbf{B}_2^l \end{bmatrix}. \tag{11}$$

Note that $\mathbf{B}_i^l$ is the l-th column of B_i written as a matrix.

Theorems 1 and 2 that follow describe the general form of a second root vector, i.e. an element in $\mathcal{N} \backslash \mathcal{M}$, and a basis for $\mathcal{N}$. They are adapted two-parameter versions of Theorems 6.2 and 6.3 in [6].

Theorem 1 *A vector z is in $\mathcal{N} \backslash \mathcal{M}$ if and only if there exist $\left(b_1^{\mathbf{k}}, b_2^{\mathbf{k}} \right) \in \mathbb{C}^2$, $\mathbf{k} \in \mathbf{Q}$, not all 0, and vectors $x_{11}^{k_2} \in \mathbb{C}^{n_1}$, $k_2 \in \underline{q_2}$ and $x_{21}^{k_1} \in \mathbb{C}^{n_2}$, $k_1 \in \underline{q_1}$ such that*

$$\sum_{k_1=1}^{q_1} \left(V_{11} b_1^{\mathbf{k}} + V_{12} b_2^{\mathbf{k}} \right) x_{10}^{k_1} + W_1(\boldsymbol{\lambda})\, x_{11}^{k_2} = 0, \tag{12}$$

$$\sum_{k_2=1}^{q_2} \left(V_{21} b_1^{\mathbf{k}} + V_{22} b_2^{\mathbf{k}} \right) x_{20}^{k_2} + W_2(\boldsymbol{\lambda})\, x_{21}^{k_1} = 0, \tag{13}$$

and

$$z = \sum_{k_1=1}^{q_1} x_{10}^{k_1} \otimes x_{21}^{k_1} + \sum_{k_2=1}^{q_2} x_{11}^{k_2} \otimes x_{20}^{k_2}. \tag{14}$$

Then it follows that

$$(\Gamma_i - \lambda_i I)\, z = \sum_{\mathbf{k} \in \mathbf{Q}} b_i^{\mathbf{k}} z_0^{\mathbf{k}} \tag{15}$$

for $i = 1, 2$.

We recall that columns of the matrix Y_{i0} form a basis for the kernel of $W_i(\boldsymbol{\lambda})^*$. So if we multiply the relation (12) by Y_{10}^* on the left-hand side we get that

$$Y_{10}^* \sum_{k_1=1}^{q_1} \left(V_{11} b_1^{(k_1,k_2)} + V_{12} b_2^{(k_1,k_2)}\right) x_{10}^{k_1} = 0, \quad k_2 \in \underline{q_2} \tag{16}$$

When written in the matrix form the above equalities turn to

$$V_{11}^{\lambda} \mathbf{B}_1 + V_{12}^{\lambda} \mathbf{B}_2 = 0. \tag{17}$$

Here $\mathbf{B}_i = \left[b_i^{(k_1,k_2)}\right]_{k_1=1,k_2=1}^{q_1 \quad q_2}$. In a similar way we get from (13) the matrix equality

$$\mathbf{B}_1 \left(V_{21}^{\lambda}\right)^T + \mathbf{B}_2 \left(V_{22}^{\lambda}\right)^T = 0. \tag{18}$$

Hence it follows that $\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \in \ker \mathcal{D}_0^{\lambda}$ (via the isomorphism Ξ). The system of equations (17) and (18) for $\mathbf{B}_1$ and $\mathbf{B}_2$ is a system of type (4) that is mentioned in the introduction, and is the main motivation for our study. Furthermore, we remark that both $\mathbf{B}_1$ and $\mathbf{B}_2$ solve the matrix equation

$$V_{11}^{\lambda} X \left(V_{22}^{\lambda}\right)^T - V_{12}^{\lambda} X \left(V_{21}^{\lambda}\right)^T = 0,$$

i.e., they are elements of $\ker \Delta_0^{\lambda}$.

The converse part of Theorem 1 says that we can construct a vector $z \in \mathcal{N}/\mathcal{M}$ for every element in the kernel of $\mathcal{D}_0^{\lambda}$ by (14). To find vectors $x_{11}^{k_2}$ note that (17) implies (16), and then vectors $x_{11}^{k_2}$ exist because

$$\sum_{k_1=1}^{q_1} \left(V_{11} b_1^{(k_1,k_2)} + V_{12} b_2^{(k_1,k_2)}\right) x_{10}^{k_1} \in \left(\ker W_1(\boldsymbol{\lambda})^*\right)^{\perp} \left(= \operatorname{im} W_1(\boldsymbol{\lambda})\right).$$

We find vectors $x_{21}^{k_1}$ analogously. Theorem 2 tells that these constructions can be extended to obtain a basis for $\ker \mathcal{D}_0^{\lambda}$ from a basis for $\mathcal{N}/\mathcal{M}$, and conversely.

Theorem 2 *Suppose that* $\left\{\mathbf{B}^l,\ l \in \underline{d_1}\right\}$ *is a basis for the kernel of* $\mathcal{D}_0^{\lambda}$*, and that for each* l *a vector* z_1^l *is associated with* $\mathbf{B}^l$ *as in* (14)*. Then* $\mathcal{B}_0 \cup \left\{z_1^l,\ l \in \underline{d_1}\right\}$ *is a basis for* $\mathcal{N}$*.*

Conversely, suppose that $\mathbf{B}^l$, $l \in \underline{d_1}$ *are given as in* (11)*. Then they form a basis for the kernel of* $\mathcal{D}_0^{\lambda}$*.*

We illustrate Theorems 1 and 2 with an example.

Example 3 Consider the two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix}.$$

Evidently the matrices V_{10} and V_{20} are singular. So $\boldsymbol{\lambda}_0 = (0,0) \in \sigma(\mathbf{W})$ and we have $q_1 = 1$ and $q_2 = 2$. Hence $d_0 = 2$. We choose

$$X_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; Y_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; X_{20} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } Y_{20} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then it follows that vectors

$$z_0^{11} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } z_0^{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for $\mathcal{M}$ and we have

$$V_{11}^{\lambda_0} = V_{12}^{\lambda_0} = [0], \; V_{21}^{\lambda_0} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \text{ and } V_{22}^{\lambda_0} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{19}$$

The space $\mathbb{C} \otimes \mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$ and we identify the direct sum $\mathbb{C}^2 \oplus \mathbb{C}^2$ with $\mathbb{C}^4$. Then

$$\mathcal{D}_0^{\lambda_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Because the matrix $\mathcal{D}_0^{\lambda_0}$ has rank 1 and $d_1 = \dim\ker \mathcal{D}_0^{\lambda_0}$ it follows that $d_1 = 3$. Then we choose

$$\mathbf{B}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \; \mathbf{B}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{B}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

as a basis for $\ker \mathcal{D}_0^{\lambda}$. To construct a vector z_1^1 corresponding to $\mathbf{B}^1$ we need to find vectors x_{11}^{11}, x_{11}^{21} and x_{21}^{11} such that

$$V_{11}x_{10} - V_{10}x_{11}^{11} = 0, \; -2V_{12}x_{10} - V_{10}x_{11}^{21} = 0 \text{ and } V_{21}x_{20}^1 - 2V_{22}x_{20}^2 - V_{20}x_{21}^{11} = 0.$$

Here note that $W_i(\boldsymbol{\lambda}_0) = -V_{i0}$. A possible choice is

$$x_{11}^{11} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \; x_{11}^{21} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \text{ and } x_{21}^{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly we find vectors

$$x_{11}^{12} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \; x_{11}^{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{21}^{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that correspond to $\mathbf{B}^2$, and vectors

$$x_{11}^{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; x_{11}^{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } x_{21}^{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that correspond to $\mathbf{B}^3$. Then we have

$$z_1^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \; z_1^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } z_1^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

and $\{z_0^1, z_0^2;\ z_1^1, z_1^2, z_1^3\}$ is a basis for $\mathcal{N}$. □

By means of the Kronecker bases developed in the next three sections, we shall be able to construct a basis for the above example in a canonical way - see Example 10.

3 Kronecker canonical form

First we introduce some special matrices needed in the construction of the Kronecker canonical form. The $p \times p$ identity matrix is denoted by I_p. The $q \times q$ *Jordan matrix with eigenvalue* α is

$$J_q(\alpha) = \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \alpha \end{bmatrix}.$$

We also define the $p \times (p+1)$ matrices

$$F_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } G_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The pairs of building blocks of the *Kronecker canonical form*for a pair of matrices are of three different types : (L,p), (M,p) and $(J(\alpha),q)$ where $p\geq 0$, $q\geq 1$ and $\alpha\in\mathbb{C}\cup\{\infty\}$. The building blocks of type (L,p) are of sizes $p\times(p+1)$, the building blocks of type (M,p) are of sizes $(p+1)\times p$ and the building blocks of type $(J(\alpha),q)$ are of sizes $q\times q$. Here the blocks of types $(L,0)$ and $(M,0)$ which are of 'sizes' 0×1 and 1×0, respectively, correspond to a column of 0's and a row of 0's, respectively, in the Kronecker canonical form. For $p\geq 1$ the types with the corresponding pairs of building blocks are :

$$\text{type } (L,p) \text{ with } (F_p,\ G_p),$$

$$\text{type } (M,p) \text{ with } \left(F_p^T,\ G_p^T\right),$$

$$\text{type } (J(\alpha),p) \text{ with } \begin{cases} (I,\ J_p(\alpha)) & \text{if } \alpha\in\mathbb{C},\\ (J_p(0),\ I) & \text{if } \alpha=\infty. \end{cases}$$

The theorem of Kronecker (cf. [17, p. 37] or [19, Thm. A.7.3]) states that every pair of $m\times n$ complex matrices (A,B) is equivalent to a pair of matrices in block diagonal form with diagonal blocks of types (L,p), (M,p) and $(J(\alpha),q)$. We call this block diagonal form the *Kronecker canonical form of a pair* (A,B).We call the collection

$$\mathcal{I}=\{(L,l_1),\ldots,(L,l_{p_L});(M,m_1),\ldots,(M,m_{p_M});(J(\alpha_1),j_1),\ldots,(J(\alpha_{p_J}),j_{p_J});\}$$

of the types of the diagonal blocks the *set of invariants*of a pair (A,B). The elements of the set $\mathcal{I}$ are called the *invariants*It is a consequence of the theorem of Kronecker that two pairs of $m\times n$ matrices (A,B) and (C,D) are equivalent if and only if they have the same sets of invariants. See [17, Thm. 5, p. 40] or [19, Cor. A.7.4]. Note that in our discussion we view the initial $u\times v$ block of zeros in [19, Thm. A.7.3] (in [17, p.39, exp. (34)] this is the initial $h\times g$ block of zeros) as a collection of u blocks of type $(L,0)$ and v blocks of type $(M,0)$. This enables us to absorb the initial block of zeros into the blocks of types (L,p) and (M,p). So for instance, pair of matrices

$$\left(\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\right)$$

has set of invariants $\mathcal{I}=\{(L,0),(M,0),(M,0),(J(\infty),1),(J(0),2)\}$.

Suppose that a pair of commuting $N\times N$ matrices (A_1,A_2) is such that $A_1^2=A_2^2=A_1A_2=0$. Under similarity they can be reduced simultaneously to the strict block upper-triangular form

$$A_i=\begin{bmatrix}0&0&B_i\\0&0&0\\0&0&0\end{bmatrix},\ i=1,2, \tag{20}$$

where we may assume that B_1 and B_2 are such that the matrix $\begin{bmatrix}B_1 & B_2\end{bmatrix}$ has linearly independent rows, and that the matrix $\begin{bmatrix}B_1\\ B_2\end{bmatrix}$ has linearly independent columns. Suppose that $\left(\widehat{B}_1, \widehat{B}_2\right)$ is the Kronecker canonical form of the pair (B_1, B_2) and that invertible matrices P and Q are such that $\widehat{B}_i = PB_iQ$ for $i = 1, 2$. Then

$$UA_iU^{-1} = \begin{bmatrix} 0 & \widehat{B}_i \\ 0 & 0 \end{bmatrix}, \tag{21}$$

where $U = \begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^{-1} \end{bmatrix}$, is a canonical form for (A_1, A_2) under simultaneous similarity of matrices.

Next we introduce the notion of a Kronecker basis for a pair of matrices (A, B). With every invariant $\iota \in \mathcal{I}$ we associate a *Kronecker chain*$\mathcal{C}_\iota$ of linearly independent vectors as follows :
If $\iota = (L, p)$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p+1}\right\}$ and

$$\begin{array}{rcll} Bu_1 & = & 0, & \\ Bu_{i+1} & = & Au_i, & i \in \underline{p}, \\ 0 & = & Au_{p+1}\,. & \end{array}$$

If $\iota = (M, 0)$ then $\mathcal{C}_\iota = \emptyset$ and if $\iota = (M, p)$, $p \geq 1$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$Bu_i \quad = \quad Au_{i+1}, \quad i \in \underline{p-1}.$$

If $\iota = (J(\alpha), p)$, $\alpha \in \mathbb{C}$, then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$\begin{array}{rcll} (B - \alpha A)\, u_1 & = & 0, & \\ (B - \alpha A)\, u_{i+1} & = & Au_i, & i \in \underline{p-1}. \end{array} \tag{22}$$

And finally, if $\iota = (J(\infty), p)$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$\begin{array}{rcll} Au_1 & = & 0, & \\ Au_{i+1} & = & Bu_i, & i \in \underline{p-1}. \end{array} \tag{23}$$

The union of all Kronecker chains $\mathcal{C} = \cup_{\iota \in \mathcal{I}} \mathcal{C}_\iota$ of a pair of matrices (A, B) is called a *Kronecker basis* of (A, B).

To unify the treatment of pairs of invariants of types (L, p) and $(J(\alpha), q)$ with pairs of invariants $(J(\alpha), p)$ and $(J(\alpha), q)$ we define the *α-shift*

$$\mathcal{C}_\iota(\alpha) = \left\{u_i(\alpha),\ i \in \underline{p+1}\right\}$$

of a Kronecker chain$\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p+1}\right\}$ for $\iota = (L, p)$. The chain $\mathcal{C}_\iota(\alpha)$ is determined by the relations

$$\sum_{i=0}^{p} \lambda^i u_{i+1}(\alpha) = \sum_{i=0}^{p} (\lambda - \alpha)^i u_{i+1},$$

if $\alpha \in \mathbb{C}$, and

$$\sum_{i=0}^{p} \lambda^i u_{i+1}(\infty) = \sum_{i=0}^{p} \lambda^{p-i} u_{i+1}.$$

Here λ is an indeterminate. Then we check directly that vectors $u_i(\alpha)$, $i \in \underline{p+1}$ satisfy relations (22) if $\alpha \in \mathbb{C}$ and relations (23) if $\alpha = \infty$. Note also that the chains $\mathcal{C}_\iota$ and $\mathcal{C}_\iota(\alpha)$ span the same subspace.

4 The Matrix Equation $AXD^T - BXC^T = 0$

Now we consider the matrix equation (5). Suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are the sets of invariants of the pairs (A, B) and (C, D), respectively, and $\mathcal{C}_1$ and $\mathcal{C}_2$ their corresponding Kronecker bases. Now we define a subset $\mathcal{J}$ in the set of pairs of invariants $\mathcal{I}_1 \times \mathcal{I}_2$. A pair (ι_1, ι_2) is in the set $\mathcal{J}$ if one of the following holds :

(i) $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$,

(iia) $\iota_1 = (L, p_1)$, $\iota_2 = (M, p_2)$ and $p_1 < p_2$,

(iib) $\iota_1 = (M, p_1)$, $\iota_2 = (L, p_2)$ and $p_1 > p_2$,

(iiia) $\iota_1 = (L, p_1)$ and $\iota_2 = (J(\alpha), p_2)$,

(iiib) $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (L, p_2)$ and

(iiic) $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\alpha), p_2)$.

With a pair of invariants $(\iota_1, \iota_2) \in \mathcal{J}$ we associate a set $\mathcal{U}_{(\iota_1,\iota_2)}$ of matrices as follows :

(i) if $\iota_1 = (L, p_1)$, $\iota_2 = (L, p_2)$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1+1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2+1}\right\}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i_1+i_2=l+1} u_{1i_1} u_{2i_2}^T,\ l \in \underline{p_1+p_2+1}\right\};$$

(iia) if $\iota_1 = (L, p_1)$, $\iota_2 = (M, p_2)$, where $p_1 < p_2$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1+1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2}\right\}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i=0}^{p_1} u_{1,i+1} u_{2,l+i}^T,\ l \in \underline{p_2-p_1}\right\};$$

(iib) if $\iota_1 = (M, p_1)$, $\iota_2 = (L, p_2)$, where $p_1 > p_2$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2+1}\right\}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i=0}^{p_2} u_{1,l+i} u_{2,i+1}^T,\ l \in \underline{p_1-p_2}\right\};$$

(iiia) if $\iota_1 = (L, p_1)$, $\iota_2 = (J(\alpha), p_2)$, where $\alpha \in \mathbb{C} \cup \{\infty\}$, $\mathcal{C}_{1\iota_1}(\alpha) = \{u_{1i}(\alpha),\ i \in \underline{p_1 + 1}\}$ is the α-shift of the Kronecker chain $\mathcal{C}_{1\iota_1}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2}\right\}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i_1+i_2=l+1} u_{1i_1}(\alpha)\, u_{2i_2}^T,\ l \in \underline{p_2}\right\};$$

(iiib) if $\iota_1 = (J(\alpha), p_1)$, where $\alpha \in \mathbb{C} \cup \{\infty\}$, $\iota_2 = (L, p_2)$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1}\right\}$ and $\mathcal{C}_{2\iota_2}(\alpha) = \left\{u_{2i}(\alpha),\ i \in \underline{p_2 + 1}\right\}$ is the α-shift of the Kronecker chain $\mathcal{C}_{2\iota_2}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i_1+i_2=l+1} u_{1i_1} u_{2i_2}(\alpha)^T,\ l \in \underline{p_1}\right\};$$

(iiic) if $\iota_1 = (J(\alpha), p_1)$, $\iota_2 = (J(\alpha), p_2)$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2}\right\}$ then

$$\mathcal{U}_{(\iota_1,\iota_2)} = \left\{U_l;\ U_l = \sum_{i_1+i_2=l+1} u_{1i_1} u_{2i_2}^T,\ l \in \underline{\min\{p_1, p_2\}}\right\}.$$

Using the above setting we have the following theorem :

Theorem 4 *A basis $\mathcal{U}$ for the space of solutions of the matrix equation $AXD^T - BXC^T = 0$, consists of the union of all the sets $\mathcal{U}_{(\iota_1,\iota_2)}$ for pairs of invariants (ι_1, ι_2) in the set $\mathcal{J}$.*

The proof uses standard methods and so we only outline the main steps. First we reduce the matrix equation (5) into blocks, one for each pair of invariants $(\iota_1, \iota_2) \in \mathcal{I}_2 \times \mathcal{I}_2$ (cf. [30, 32]). Then we show directly that for invariants $(\iota_1, \iota_2) \in \mathcal{J}$ the matrices in $\mathcal{U}_{(\iota_1,\iota_2)}$ are linearly independent solutions of (5). Next we consider the linear map $X \mapsto AXD^T - BXC^T$ and show that the dimension of its kernel is equal to the number of elements in $\mathcal{U}$, and so $\mathcal{U}$ is a basis as required. (For a special case $X \mapsto AX - XC^T$ this is studied in [35, §5].)

Corollary 5 *The matrix equation $AXD^T - BXC^T = 0$ has only the trivial solution $X = 0$ if and only if either*

(i) *there are no invariants of type (L, p) in the sets $\mathcal{I}_1$ and $\mathcal{I}_2$ and there is no pair of invariants $(J(\alpha), p_1) \in \mathcal{I}_1$ and $(J(\beta), p_2) \in \mathcal{I}_2$ with $\alpha = \beta$, or*

(ii) *one of the sets of invariants $\mathcal{I}_i$, where i is either 1 or 2, consists only of invariants of the type (M, p_1), while there might be invariants of the type (L, p_2) in the other set of invariants but any of them is such that $p_2 \geq p$, where $p = \max\{p_1,\ (M, p_1) \in \mathcal{I}_i\}$.*

5 The System of Matrix Equations $AX+BY=0$ and $XC^T+YD^T=0$

Suppose that $\mathcal{U}=\cup_{(\iota_1,\iota_2)\in\mathcal{J}}\mathcal{U}_{(\iota_1,\iota_2)}$ is a basis for the space of solutions of the matrix equation (5) as described in Theorem 4. We write $\mathcal{J}'$ for the set of all the pairs $(\iota_1,\iota_2)\in\mathcal{J}$ that are different from the cases $(\iota_1,\iota_2)=((L,p),(M,p+1))$ and $(\iota_1,\iota_2)=((M,p+1),(L,p))$, i.e., if the cases **(iia)** and **(iib)** of §4 are replaced by

(iia') $\iota_1=(L,p_1)\,,\ \iota_2=(M,p_2)$ and $p_1<p_2+1$,

(iib') $\iota_1=(M,p_1)\,,\ \iota_2=(L,p_2)$ and $p_1+1>p_2$.

We associate with every pair of invariants $(\iota_1,\iota_2)\in\mathcal{J}'$ a set of pairs of matrices $\mathcal{U}^2_{(\iota_1,\iota_2)}$ as follows. Here the matrices U_l are defined in (i)-$(iiic)$ for different cases of pairs (ι_1,ι_2) :

(i) If $\iota_1=(L,p_1)$ and $\iota_2=(L,p_2)$ then

$$\mathcal{U}^2_{(\iota_1,\iota_2)}=\left\{\begin{bmatrix} U_{l-1}\\ -U_l\end{bmatrix},\ l\in\underline{p_1+p_2+2}\right\}$$

where $U_0=U_{p_1+p_2+2}=0$,

(ii) If $\iota_1=(L,p_1)$, $\iota_2=(M,p_2)$ and $p_1+2\le p_2$ or $\iota_1=(M,p_1)$, $\iota_2=(L,p_2)$ and $p_1\ge p_2+2$ then

$$\mathcal{U}^2_{(\iota_1,\iota_2)}=\left\{\begin{bmatrix} -U_{l+1}\\ U_l\end{bmatrix},\ l\in\underline{|p_1-p_2|-1}\right\},$$

(iii) In cases (iiia), (iiib) or (iiic), if $\alpha\in\mathbb{C}$, then

$$\mathcal{U}^2_{(\iota_1,\iota_2)}=\left\{\begin{bmatrix} \alpha U_l+U_{l-1}\\ -U_l\end{bmatrix},\ l\in\underline{p}\right\},$$

and if $\alpha=\infty$ then

$$\mathcal{U}^2_{(\iota_1,\iota_2)}=\left\{\begin{bmatrix} -U_l\\ U_{l-1}\end{bmatrix},\ l\in\underline{p}\right\}.$$

Here

$$p=\begin{cases} p_2, & \text{if } \iota_1=(L,p_1) \text{ and } \iota_2=(J(\alpha),p_2)\,,\\ p_1 & \text{if } \iota_1=(J(\alpha),p_1) \text{ and } \iota_2=(L,p_2)\,,\\ \min\{p_1,p_2\} & \text{if } \iota_1=(J(\alpha),p_1) \text{ and } \iota_2=(J(\alpha),p_2)\,,\end{cases}$$

and we assume that $U_0=0$.

In the above setting we have the following result :

Theorem 6 *The space of solutions of the pair of matrix equations $AX+BY=0$ and $XC^T+YD^T=0$ has a basis $\mathcal{U}^2=\cup_{(\iota_1,\iota_2)\in\mathcal{J}'}\mathcal{U}^2_{(\iota_1,\iota_2)}$, where the sets $\mathcal{U}^2_{(\iota_1,\iota_2)}$ are given above.*

Let us give a short sketch of a proof. If $AX+BY=0$ and $XC^T+YD^T=0$ then both X and Y are solutions of the matrix equation $AXD^T-BXC^T=0$. By Theorem 4 it follows that $X=\sum_{U\in\mathcal{U}}\gamma_U U$ and $Y=\sum_{U\in\mathcal{U}}\delta_U U$. The relations between the scalars γ_U and δ_U follow from the equalities $AX+BY=0$ and $XC^T+YD^T=0$ using the reduction into blocks as for the proof of Theorem 4.

6 A Canonical Basis for the Second Root Subspace of Two-parameter Systems

Now we return to study two-parameter systems (7). We use the notation introduced in §2. Theorem 2 tells that we can associate to a basis for the kernel of $\mathcal{D}_0^\lambda$ a basis for the second root subspace $\mathcal{N}$. The elements of $\ker\mathcal{D}_0^\lambda$ satisfy the system of matrix equations (17) and (18) for $\mathbf{B}_1$ and $\mathbf{B}_2$ that is of type (4). Now we apply the canonical construction from the previous section to construct a canonical basis for $\mathcal{N}$. In this basis the matrices for $A_i=(\Gamma_i-\lambda_i I)\,|_{\mathcal{N}}$, $i=1,2$ are in the canonical form (21). To get the precise form (21) we suppose that $\mathcal{I}_i$ is the set of invariants of the pair $\left(V_{i2}^\lambda,V_{i1}^\lambda\right)$ (note the order) and $\mathcal{C}_i=\cup_{\iota\in\mathcal{I}_i}\mathcal{C}_{i\iota_i}$ is the corresponding Kronecker basis. Further we assume that set of invariants $\mathcal{J}'$ and corresponding basis $\mathcal{U}^2$ are defined as in the previous section. (Note that $\mathcal{U}^2$ is a basis for $\ker\mathcal{D}_0^\lambda$.) Next we define a mapping η on the set of pairs of invariants $\mathcal{J}'$ by

$$\eta(\iota_1,\iota_2)=\begin{cases}(L,p_1+p_2+1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (i),\\ (M,p_2-p_1-1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iia'),\\ (M,p_1-p_2-1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iib'),\\ (J(-\alpha),p_2), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiia),\\ (J(-\alpha),p_1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiib),\\ (J(-\alpha),\min\{p_1,p_2\}), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiic).\end{cases}\tag{24}$$

Here we assume that $-\infty=\infty$. The set of invariants $\mathcal{I}=\{\eta(\iota_1,\iota_2),\ (\iota_1,\iota_2)\in\mathcal{J}'\}$ is called the *set of invariants for* $\boldsymbol{\lambda}$. We write $\mathcal{U}^2_\iota=\mathcal{U}^2_{(\iota_1,\iota_2)}$ if $\iota=\eta(\iota_1,\iota_2)$. It turns out that the set of invariants $\mathcal{I}$ is equal to the set of invariants of a pair of matrices $\widehat{\mathbf{C}}=\left(\widehat{C}_1,\widehat{C}_2\right)$, that correspond to the matrices $A_i=(\Gamma_i-\lambda_i I)\,|_{\mathcal{N}}$, $i=1,2$, when in the canonical form (21).

In the rest of this section we describe the construction of a canonical basis for $\mathcal{N}$ using the basis $\mathcal{U}^2$ for $\ker\mathcal{D}_0^\lambda$. We discuss each of three different types of invariants $\iota\in\mathcal{I}$ separately. We discuss the case $\iota=(L,p)$ in detail and we only outline the construction for the other two cases since it is similar. We construct a vector in $\mathcal{N}/\mathcal{M}$ for every element in $\mathcal{U}^2_\iota$. In order for A_i, $i=1,2$ to be in the precise canonical form (21) this construction differs from the one given in (14). For the same reason we also change bases for $\ker W_i(\boldsymbol{\lambda})$, and hence for $\mathcal{M}$: if $\mathcal{C}_{i\iota_i}=\left\{u_{il};\ l\in\underline{p}\right\}$ is a Kronecker chain for $\left(V_{i2}^\lambda,V_{i1}^\lambda\right)$ then we define

$\mathcal{C}'_{i\iota_i}$ to be the set of vectors

$$v_{i0}^l = X_{i0} u_{il}, \; l \in \underline{p}. \tag{25}$$

Because $\mathcal{C}_i = \cup_{\iota_i \in \mathcal{I}_i} \mathcal{C}_{i\iota_i}$ is a basis for $\mathbb{C}^{q_i}$ it follows that $\mathcal{C}'_i = \cup_{\iota_i \in \mathcal{I}_i} \mathcal{C}'_{i\iota_i}$ is a basis for $\ker W_i(\boldsymbol{\lambda})$.

6.1 A Basis Corresponding to an Invariant $\iota = (L, p)$

Theorem 7 *Suppose $\iota = (L,p) = \eta(\iota_1, \iota_2)$ where $\iota_1 = (L, p_1) \in \mathcal{I}_1$ and $\iota_2 = (L, p_2) \in \mathcal{I}_2$ and $p = p_1 + p_2 + 1$. Then there exist vectors $v_i^{l_i} \in \mathbb{C}^{n_i}$, $i = 1, 2$ and $l_i \in \underline{p_i + 1}$ such that*

$$V_{i1}^1 v_{i0}^1 + W_i(\boldsymbol{\lambda}) v_{i1}^1 = 0, \tag{26}$$

$$V_{i1} v_{i0}^{l_i+1} - V_{i2} v_{i0}^{l_i} + W_i(\boldsymbol{\lambda}) v_{i1}^{l_i+1} = 0 \qquad \text{for } l_i \in \underline{p_i} \tag{27}$$

and

$$-V_{i2} v_{i0}^{p_i} + W_i(\boldsymbol{\lambda}) v_{i1}^{p_i+1} = 0. \tag{28}$$

Then the vectors

$$z_0^l = (-1)^l \sum_{i_1+i_2=l+1} v_{10}^{i_1} \otimes v_{20}^{i_2}, \qquad l \in \underline{p} \tag{29}$$

and

$$z_1^l = (-1)^{l+1} \sum_{i_1+i_2=l+1} \left(v_{11}^{i_1} \otimes v_{20}^{i_2} + v_{10}^{i_1} \otimes v_{21}^{i_2} \right), \qquad l \in \underline{p+1} \tag{30}$$

are linearly independent. It also follows that

$$(\Gamma_1 - \lambda_1 I) z_1^l = z_0^l \tag{31}$$

and

$$(\Gamma_2 - \lambda_2 I) z_1^l = z_0^{l-1} \tag{32}$$

for $k \in \underline{p+1}$, where $z_0^0 = z_0^{p+1} = 0$.

Proof. Suppose that $\mathcal{C}_{i\iota_i} = \left\{ u_{il};\; l \in \underline{p_i + 1} \right\}$, $i = 1, 2$. Because $\iota_i = (L, p_i)$ it follows from the definition of a Kronecker chain that

$$\begin{array}{rcl} V_{i1}^{\lambda} u_{i1} & = & 0, \\ V_{i1}^{\lambda} u_{i,k+1} & = & V_{i2}^{\lambda} u_{ik}, \qquad k \in \underline{p_i}, \\ 0 & = & V_{i2}^{\lambda} u_{i,p_i+1}. \end{array}$$

Recall that $V_{ij}^{\lambda} = Y_{i0}^* V_{ij} X_{i0}$, where columns of Y_{i0} form a basis for $\ker W_i(\boldsymbol{\lambda})^*$. Then it follows that vectors $V_{i1} v_{i0}^1$, $V_{i1} v_{i0}^{l+1} - V_{i2} v_{i0}^l$, $l \in \underline{p_i}$, and $-V_{i2} v_{i0}^{p_i+1}$ belong to $(\ker W_i(\boldsymbol{\lambda})^*)^{\perp}$. Because $(\ker W_i(\boldsymbol{\lambda})^*)^{\perp} = \operatorname{im} W_i(\boldsymbol{\lambda})$ there exist vectors $v_{i1}^l \in \mathbb{C}^{n_i}$ such that (26), (27) and (28) hold. Then we construct vectors (29) and (30). We write vector $x_1 \otimes y_2 - x_2 \otimes y_1$ also in the determinantal form as $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^{\otimes}$. Then it follows that

$$(\Delta_1 - \lambda_1 \Delta_0) z_1^l = (-1)^l \sum_{k=1}^{l} \begin{vmatrix} W_1(\boldsymbol{\lambda}) v_{11}^k & V_{12} v_{10}^k \\ W_2(\boldsymbol{\lambda}) v_{21}^{l+1-k} & V_{22} v_{20}^{l+1-k} \end{vmatrix}^{\otimes} =$$

$$= (-1)^l \sum_{k=1}^{l} \begin{vmatrix} V_{11}v_{10}^k & V_{12}v_{10}^k \\ V_{21}v_{20}^{l+1-k} & V_{22}v_{20}^{l+1-k} \end{vmatrix}^{\otimes} + (-1)^{l-1} \sum_{k=1}^{l-1} \begin{vmatrix} V_{12}v_{10}^k & V_{12}v_{10}^k \\ V_{22}v_{20}^{l-k} & V_{22}v_{20}^{l-k} \end{vmatrix}^{\otimes} = \Delta_0 z_0^l$$

and

$$(\Delta_2 - \lambda_2\Delta_0)\, z_1^l = (-1)^l \sum_{k=1}^{l} \begin{vmatrix} V_{11}v_{10}^k & W_1(\lambda)\, v_{11}^k \\ V_{21}v_{20}^{l+1-k} & W_2(\lambda)\, v_{21}^{l+1-k} \end{vmatrix}^{\otimes} =$$

$$= (-1)^l \sum_{k=1}^{l} \begin{vmatrix} V_{11}v_{10}^k & V_{11}v_{10}^k \\ V_{21}v_{20}^{l+1-k} & V_{21}v_{20}^{l+1-k} \end{vmatrix}^{\otimes} + (-1)^{l-1} \sum_{k=1}^{l-1} \begin{vmatrix} V_{11}v_{10}^k & V_{12}v_{10}^k \\ V_{21}v_{20}^{l-k} & V_{22}v_{20}^{l-k} \end{vmatrix}^{\otimes} = \Delta_0 z_0^{l-1}$$

for $l \in \underline{p+1}$. Hence (31) and (32) hold. Here we assume $z_0^0 = z_0^{p+1} = 0$. The vectors z_0^l, $l \in \underline{p}$ are linearly independent because the vectors v_{i0}^l, $l \in \underline{p_i}$ are linearly independent. Then it follows from (31) and (32) that the vectors $\left\{z_0^l;\ l \in \underline{p}\right\} \cup \left\{z_1^l;\ l \in \underline{p+1}\right\}$ are linearly independent. □

If we restrict the transformations $\Gamma_1 - \lambda_1 I$ and $\Gamma_2 - \lambda_2 I$ to the joint invariant subspace $\mathcal{S}$ spanned by the vectors $\left\{z_0^l;\ l \in \underline{p}\right\} \cup \left\{z_1^l;\ l \in \underline{p+1}\right\}$ then we have

$$(\Gamma_1 - \lambda_1 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & F_p \\ 0 & 0 \end{bmatrix} \text{ and } (\Gamma_2 - \lambda_2 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & G_p \\ 0 & 0 \end{bmatrix}. \tag{33}$$

Note that the invariant of the pair of matrices (F_p, G_p) is (L, p). We also remark that the relations (26)-(28) relate the elements in $\mathcal{U}_\iota^2$ to the vectors z_1^l in the same way as the relations (12) and (13) relate $\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$ to the vector (14). Here vectors (25) are used. The particular form of vectors (29) and (30) is chosen in order that (31) and (32) hold, i.e., that we have (33).

6.2 A Basis Corresponding to an Invariant $\iota = (M, p)$

Suppose that $\iota = (M,p) = \eta(\iota_1, \iota_2)$, where $\iota_1 = (L, p_1) \in \mathcal{I}_1$, $\iota_2 = (M, p_2) \in \mathcal{I}_2$ and $p = p_2 - p_1 - 1\,(\geq 1)$, and that vectors v_{i0}^l are defined by (25). The basis for the case $\iota_1 = (M, p_1) \in \mathcal{I}_1$, $\iota_2 = (L, p_2) \in \mathcal{I}_2$ and $p = p_1 - p_2 - 1$ is obtained symmetrically, interchanging $i = 1$ and $i = 2$.

Theorem 8 *If $\iota = (M,p) \in \mathcal{I}$ is as above then there exist vectors $v_{11}^l \in \mathbb{C}^{n_1}$, $l \in \underline{p_1+2}$ and $v_{21}^l \in \mathbb{C}^{n_2}$, $l \in \underline{p_2-1}$ such that*

$$V_{11}v_{10}^1 + W_1(\boldsymbol{\lambda})\, v_{11}^1 = 0,$$

$$V_{11}v_{10}^{l+1} - V_{12}v_{10}^l + W_1(\boldsymbol{\lambda})\, v_{11}^{l+1} = 0, \quad l \in \underline{p_1},$$

$$-V_{12}v_{10}^{p_1} + W_1(\boldsymbol{\lambda})\, v_{11}^{p_1+2} = 0,$$

and

$$V_{21}v_{20}^l - V_{22}v_{20}^{l+1} + W_2(\boldsymbol{\lambda})\, v_{21}^l = 0, \quad l \in \underline{p_2-1}.$$

Then the vectors

$$z_0^l = (-1)^l \sum_{k=0}^{p_1} v_{10}^{k+1} \otimes v_{20}^{l+k}, \quad l \in \underline{p+1},$$

and

$$z_1^l = (-1)^l \left(\sum_{k=0}^{p_1} v_{10}^{k+1} \otimes v_{21}^{l+k} + \sum_{k=0}^{p_1+1} v_{10}^{k+1} \otimes v_{21}^{l+k} \right), \quad l \in \underline{p}$$

are linearly independent. Furthermore, we have

$$(\Gamma_1 - \lambda_1 I) z_1^l = z_0^l$$

and

$$(\Gamma_2 - \lambda_2 I) z_1^l = z_0^{l+1}$$

for $l \in \underline{p}$.

The proof is very similar to the proof of Theorem 7 and we will omit it.

If we restrict the transformations $\Gamma_1 - \lambda_1 I$ and $\Gamma_2 - \lambda_2 I$ to the joint invariant subspace $\mathcal{S}$ spanned by the vectors $\left\{z_0^l;\ l \in \underline{p+1}\right\} \cup \left\{z_1^l;\ l \in \underline{p}\right\}$, described in the above theorem, then we have

$$(\Gamma_1 - \lambda_1 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & F_p^T \\ 0 & 0 \end{bmatrix} \text{ and } (\Gamma_2 - \lambda_2 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & G_p^T \\ 0 & 0 \end{bmatrix}.$$

Note that the invariant of the pair of matrices $\left(F_p^T, G_p^T\right)$ is (M, p).

6.3 A Basis Corresponding to an Invariant $\iota = (J(\alpha), p)$

Suppose $\iota = (J(\alpha), p) = \eta(\iota_1, \iota_2)$, where $\iota_1 = (J(-\alpha), p_1) \in \mathcal{I}_1$, $\iota_2 = (J(-\alpha), p_2) \in \mathcal{I}_2$, $p = \min\{p_1, p_2\}$ and $\alpha \in \mathbb{C}$. The basis for the case $\iota_1 = (L, p_1)$ and $\iota_2 = (J(-\alpha), p_2)$ is obtained by the same arguments as in the case $\iota_1 = (J(-\alpha), p_1)$ and $\iota_2 = (J(-\alpha), p_2)$ using the α-shift $\mathcal{C}_{1\iota_1}(-\alpha) = \left\{u_{1i}(-\alpha),\ i \in \underline{p_1+1}\right\}$ and writing $u_{1i}(-\alpha) = 0$ for $i \geq p_1 + 1$ if $p_2 > p_1 + 1$. Then the case $\iota_1 = (J(-\alpha), p_1)$ and $\iota_2 = (L, p_2)$ is analogous, we need only to interchange $i = 1$ and $i = 2$. The cases $\alpha = \infty$ are the same as the cases $\alpha = 0$ after interchanging V_{i1} and V_{i2} for $i = 1, 2$.

Theorem 9 *If $\iota_1 = (J(-\alpha), p_1)$ and $\iota_2 = (J(-\alpha), p_2)$ then there exist vectors $v_{i1}^l \in \mathbb{C}^{n_i}$, $l \in \underline{p}$ such that*

$$U_i(\alpha) v_{i0}^1 + W_i(\boldsymbol{\lambda}) v_{i1}^1 = 0$$

and

$$U_i(\alpha) v_{i0}^{l+1} - V_{i2} v_{i0}^l + W_i(\boldsymbol{\lambda}) v_{i1}^{l+1} = 0,\ l \in \underline{p-1}.$$

Here $U_i(\alpha) = V_{i1} + \alpha V_{i2}$. The vectors

$$z_0^l = (-1)^l \sum_{k=1}^{l} v_{10}^k \otimes v_{20}^{l+1-k}, \quad l \in \underline{p}$$

and

$$z_1^l = (-1)^l \sum_{k=1}^{l} \left(v_{11}^k \otimes v_{20}^{l+1-k} + v_{10}^k \otimes v_{21}^{l+1-k} \right), \quad l \in \underline{p}$$

are linearly independent. Furthermore

$$(\Gamma_1 - \lambda_1 I)\, z_1^l = z_0^l$$

and

$$(\Gamma_2 - \lambda_2 I)\, z_1^l = \alpha z_0^l + z_0^{l-1}$$

for $l \in \underline{p}$*. Here we assume that* $z_0^0 = 0$.

To prove the theorem we apply the same arguments as in the proof of Theorem 7.

If we restrict the transformations $\Gamma_1 - \lambda_1 I$ and $\Gamma_2 - \lambda_2 I$ to the joint invariant subspace $\mathcal{S}$ spanned by the vectors $\left\{ z_0^l;\ l \in \underline{p} \right\} \cup \left\{ z_1^l;\ l \in \underline{p} \right\}$ given in the above theorem. Then we have

$$(\Gamma_1 - \lambda_1 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix} \text{ and } (\Gamma_2 - \lambda_2 I)\mid_{\mathcal{S}} = \begin{bmatrix} 0 & J_p(\alpha) \\ 0 & 0 \end{bmatrix},$$

and the corresponding invariant is $(J(\alpha), p)$.

6.4 A Canonical Basis for the Second Root Subspace

For every element in the set of invariants $\mathcal{I}$ for $\boldsymbol{\lambda}$ we construct vectors z_0^l and z_1^l as explained in Theorems 7–9. They are linearly independent (cf. Theorem 2). We denote the set of these vectors by $\mathcal{B}_2'$ and by $\mathcal{N}'$ the subspace they span. Then $\mathcal{N}'$ is invariant for both Γ_i and we have that $\mathcal{N}' \subset \mathcal{N}$. If $\mathcal{N}' \neq \mathcal{N}$ we complete the set $\mathcal{B}_2'$ by a set of vectors, say $\mathcal{B}''$, to the basis $\mathcal{B}_2$ for $\mathcal{N}$. We write $\mathcal{N}'' = \mathcal{L}(\mathcal{B}'')$. Because the vectors z_1^l are as many as $\dim\ker \mathcal{D}_0^\lambda$ and are linearly independent it follows from Theorem 2 that we can assume that $\mathcal{N}'' \subset \mathcal{M}$. Then we write the pair of restricted transformation $A_i = (\Gamma_i - \lambda_i I)\mid_{\mathcal{N}}$ in the form (9) using the basis $\mathcal{B}_2$. It follows from Theorems 7–9 that matrices B_i of (9) have the form

$$B_i = \begin{bmatrix} \widehat{B}_i \\ 0 \end{bmatrix}, \tag{34}$$

where

$$\widehat{B}_i = \begin{bmatrix} B_{i1} & & 0 & & & \cdots & & & 0 \\ & \ddots & & & & & & & \vdots \\ 0 & & B_{ir_1} & 0 & & & \cdots & & 0 \\ & & 0 & B_{i,r_1+1} & & & & & \\ & & & & \ddots & & & & \vdots \\ \vdots & & \vdots & & & B_{i,r_1+r_2} & & & \\ & & & & & & B_{i,r_1+r_2+1} & & 0 \\ & & & & & & & \ddots & \\ 0 & \cdots & 0 & & \cdots & & 0 & & B_{i,r_1+r_2+r_3} \end{bmatrix}. \tag{35}$$

The first r_1 blocks B_{ij}, $i=1,2$ in the form (35) correspond to the invariants (L,p_j) in the set $\mathcal{I}$ (hence $B_{1j}=F_{p_j}$ and $B_{2j}=G_{p_j}$). The next r_2 blocks in (35) correspond to the invariants (M,p_j) in the set $\mathcal{I}$ and the last r_3 blocks B_{ij} correspond to the invariants $(J(\alpha),p_j)$. The rows of 0 at the bottom in (34) are as many as there are vectors in the set $\mathcal{N}''$. Note that the matrices A_i of (9), where B_i are in the form given by (34) and (35), are in the canonical form (21). Note also that the set of invariants of the pair of matrices $\left(\widehat{B}_1,\widehat{B}_2\right)$ equals $\mathcal{I}$.

We illustrate the preceding discussion with two examples.

Example 10 Consider again the two-parameter system of Example 3 (cf. also [6, Ex. 7.1]). The sets of invariants for the pairs $\left(V_{12}^{\lambda_0},V_{11}^{\lambda_0}\right)$ and $\left(V_{22}^{\lambda_0},V_{21}^{\lambda_0}\right)$ (see (19)) that correspond to the eigenvalue $\boldsymbol{\lambda}_0=(0,0)$ are $\{(L,0),(M,0)\}$ and $\{(L,1),(M,0)\}$, respectively. The set of invariants for $\boldsymbol{\lambda}_0$ is $\{(L,2)\}$. Kronecker chains associated with the invariants $(L,0)$ and $(L,1)$ are $[1]$ and $\left\{\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\right\}$, respectively. Then we have that

$$v_{10}^1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ v_{20}^1=\begin{bmatrix}0\\1\\0\end{bmatrix}\text{ and }v_{20}^2=\begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix}.$$

We find vectors

$$v_{11}^1=\begin{bmatrix}0\\1\\-1\end{bmatrix},\ v_{11}^2=\begin{bmatrix}0\\-1\\1\end{bmatrix}\text{ and }v_{21}^1=\begin{bmatrix}0\\0\\0\end{bmatrix},\ v_{21}^2=\begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix},\ v_{21}^3=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

such that

$$V_{11}v_{10}^1-V_{10}v_{11}^1=0,\ -V_{12}v_{10}^1-V_{10}v_{11}^2=0$$

and

$$V_{21}v_{20}^1-V_{20}v_{21}^1=0,\ V_{21}v_{20}^2-V_{22}v_{20}^1-V_{20}v_{21}^2=0\text{ and }-V_{22}v_{20}^2-V_{20}v_{21}^3=0$$

hold. Then it follows from Theorems 2 and 7 that the vectors

$$z_0^1=\begin{bmatrix}0\\-1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix},\ z_0^2=\begin{bmatrix}\frac{1}{2}\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix};\ z_1^1=\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\-1\\0\end{bmatrix},\ z_1^2=\begin{bmatrix}0\\0\\-\frac{1}{2}\\-\frac{1}{2}\\1\\0\\\frac{1}{2}\\-1\\0\end{bmatrix}\text{ and }z_1^3=\begin{bmatrix}0\\0\\0\\-\frac{1}{2}\\0\\0\\\frac{1}{2}\\0\\0\end{bmatrix}$$

form a basis for $\mathcal{N}$. Note that the above method to construct vectors of a basis for $\mathcal{N}$ differs from the method given in Theorems 1 and 2, and which is used in Example 3, hence also the bases constructed in the two examples are different. □

Example 11 Suppose that we are given matrices

$$V_{11}^{\lambda}=\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix},\ V_{12}^{\lambda}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

and

$$V_{21}^{\lambda}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix}\text{ and }V_{22}^{\lambda}=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

Then we form a two-parameter system

$$W_1(\boldsymbol{\lambda})=\begin{bmatrix}I_6&V_{11}^{\lambda}\\0&I_6\end{bmatrix}\lambda_1+\begin{bmatrix}0&V_{12}^{\lambda}\\0&0\end{bmatrix}\lambda_2-\begin{bmatrix}0&0\\I_6&0\end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda})=\begin{bmatrix}0&V_{21}^{\lambda}\\0&0\end{bmatrix}\lambda_1+\begin{bmatrix}I_5&V_{22}^{\lambda}\\0&I_5\end{bmatrix}\lambda_2-\begin{bmatrix}0&0\\I_5&0\end{bmatrix}.$$

From the structure of the above two-parameter system it follows that it is nonsingular and that $\boldsymbol{\lambda}=(0,0)$ is an eigenvalue. We also find that the matrices V_{ij}^{λ}, $i,j=1,2$, are precisely as defined by (8) if we choose $X_{10}=\begin{bmatrix}0\\I_6\end{bmatrix}$, $Y_{10}=\begin{bmatrix}I_6\\0\end{bmatrix}$, $X_{20}=\begin{bmatrix}0\\I_5\end{bmatrix}$ and $Y_{20}=\begin{bmatrix}I_5\\0\end{bmatrix}$. The set of invariants for $\left(V_{12}^{\lambda},V_{11}^{\lambda}\right)$ is $\{(L,2),(M,3)\}$ and the set of invariants for $\left(V_{22}^{\lambda},V_{21}^{\lambda}\right)$ is $\{(L,1),(M,1),(J(2),2)\}$. The set of pairs of invariants $\mathcal{J}'$ has three elements $\{(L,2),(L,1)\}$, $\{(L,2),(J(2),2)\}$ and $\{(M,3),(L,1)\}$. Applying the mapping η defined by (24) we find that the set of invariants for $\boldsymbol{\lambda}_0$ is $\{(L,4),(M,1),(J(2),2)\}$. □

We conclude with a remark. Note that an arbitrary set of invariants that contains no invariants $(L,0)$ and $(M,0)$ is a set of invariants for some pair of matrices (B_1,B_2) associated with a pair of commuting matrices A_1,A_2 as given in (20). However, not every set of invariants (even if it does not contain any invariants $(L,0)$ and $(M,0)$) is a set of invariants, say $\mathcal{I}$, for an eigenvalue of a two-parameter system. For instance, the number of invariants of the type (M,p) in $\mathcal{I}$ can not be larger then twice the number of invariants of the type (L,p) in $\mathcal{I}$. (To prove this claim observe that matrices $V_{i2}^{\lambda},V_{i1}^{\lambda}$ are square matrices and therefore their set of invariants $\mathcal{I}_i$ contains equal numbers, say l_i, of invariants of types (L,p) and (M,p). Then $\mathcal{I}=\eta(\mathcal{J}')$ contains exactly l_1l_2 invariants of type (L,p) and at most $2l_1l_2$ invariants of type (M,p) – see (24), and sections 4 and 5.) To characterize the sets $\mathcal{I}$ that are possible sets of invariants for eigenvalues of two-parameter systems is, at this time, an open problem.

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