COMMON FIXED POINTS AND COMMON EIGENVECTORS FOR SETS OF MATRICES

J. BERNIK, R. DRNOVŠEK, T. KOŠIR, T. LAFFEY, G. MACDONALD, R. MESHULAM, M. OMLADIČ, AND H. RADJAVI

Abstract. The following questions are studied: Under what conditions does the existence of a (nonzero) fixed point for every member of a semigroup of matrices imply a common fixed point for the entire semigroup? What is the smallest number k such that the existence of a common fixed point for every k members of a semigroup implies the same for the semigroup? If every member has a fixed space of dimension at least k, what is the best that can be said about the common fixed space? We also consider analogues of these questions with general eigenspaces replacing fixed spaces.

1. Introduction

Let $\mathcal{S}$ be a set of matrices. For most of our study, $\mathcal{S}$ will be a multiplicative semigroup or group. If every member of $\mathcal{S}$ has a (nonzero) fixed point, under what conditions does there exist a common fixed point for all members of $\mathcal{S}$? The general question was considered in [2] and partial answers were given. Among other things, we extend the results of that paper. We show, for example, that a semigroup of nonnegative monomial matrices has a common fixed point if every member has a fixed point. By "monomial" is meant that each column and each row of the matrix has at most one nonzero entry, and "nonnegative" is entry-wise. By the fixed space of a matrix we mean, as usual, the set of of all its fixed vectors together with zero. The common fixed space for a set of matrices has the obvious definition. We propose to record results, some affirmative and some negative, concerning the following following type of questions:

(1) Is there a fixed k such that if every k members of $\mathcal{S}$ have a common fixed point, then so does the whole set $\mathcal{S}$? We are of course interested in small k, compared to the matrix size. We also consider the natural extension of this question to general eigenvectors, i.e., with eigenvalues not necessarily corresponding to 1. We show that for a semigroup in $M_n(F)$, if every n members have a common eigenvector (in other words, have a common one-dimensional invariant subspace), then so does the entire semigroup.

Date: April 16, 2004.

2000 *Mathematics Subject Classification.* Primary. 15A30, 20M20. Secondary. 15A18.

Key words and phrases. common eigenvectors, fixed points, groups and semigroups of matrices, reducibility.

(2) If the fixed space of each member of a semigroup $\mathcal{S}$ is at least k-dimensional, what is the best possible bound for the dimension of the common fixes space of $\mathcal{S}$? One of our results is that for the case of bounded groups, $k = (n + r - 1)/2$ yields an r-dimensional common fixed space.

(3) Is there a "small" k such that the hypothesis of the preceding question implies reducibility for a semigroup or a group? We give examples of irreducible groups in $M_n(F)$ with the property that the dimension of the fixed space of any member is at least $n/9$.

2. Common Fixed Points

Throughout the paper, we assume that F is a field. We denote by $\mathcal{S}$ a set of matrices in $M_n(F)$ such that each element $S \in \mathcal{S}$ has 1 in the spectrum $\sigma(S)$, i.e., each element of $\mathcal{S}$ has a fixed point. Here a vector v is a fixed point of S if $v \neq 0$ and $Sv = v$. The set of all fixed points of a matrix S together with the zero vector is denoted by $\mathcal{F}_S$. Note that $\mathcal{F}_S$ is the vector space $\mathcal{F}_S = \ker(S - I)$. We write $f_S = \dim \mathcal{F}_S$.

We start with the following Helly type theorem (see e.g. [1]). Its proof is an easy consequence of the observation that the intersection of a family of subspaces of F^n is nontrivial iff the intersection of any n subspaces is nontrivial.

Theorem 2.1. *Suppose that every n members of a set $\mathcal{S} \subset M_n(F)$ have a common fixed point. Then there is a common fixed point for all elements of $\mathcal{S}$.*

The following example shows that the assumption that every $(n-1)$ members of a set $\mathcal{S} \subset M_n(F)$ have a common fixed point does not lead to the conclusion that the whole set $\mathcal{S}$ has a common fixed point. Therefore, the bound n in Theorem 2.1 is best possible, even if we assume additionally that $\mathcal{S}$ is a semigroup.

Example 2.2. Suppose that V is a vector space over F of dimension n. Denote by V^* the dual of V, i.e., the vector space of all linear functionals on V. We fix a nonzero vector $u \in V$. Let

$$\mathcal{S} = \{I + u \otimes y;\ y \in V^*\}.$$

Here $(u \otimes y)v = y(v)u$ for $v \in V$. Note that $\mathcal{S}$ is a semigroup since

$$(I + u \otimes y)(I + u \otimes z) = I + u \otimes (y + (1 + y(u))z).$$

The set of fixed points of $I + u \otimes y$ is equal to the kernel of y, so that any $n - 1$ elements of $\mathcal{S}$ has a common fixed point, while $\mathcal{S}$ does not. □

The special case when $k = 1$ of the following result has essentially been proved in [2, Thm. 2.12].

Theorem 2.3. *Suppose that $\mathcal{G} \subset M_n(\mathbb{C})$ is a bounded group such that $f_G \geq \frac{n+k-1}{2}$ for all $G \in \mathcal{G}$, where $1 \leq k \leq n$. Then $\mathcal{G}$ is simultaneously similar to a group of unitary matrices having a k-dimensional space of common fixed points.*

Proof. By a well-known theorem (see e.g. [9, Thm 3.1.5]), $\mathcal{G}$ is simultaneously similar to a group of unitary matrices. So, we may assume that $\mathcal{G}$ is a group of unitary matrices acting on the vector space $V = \mathbb{C}^n$. By [2, Thm. 2.12], $\mathcal{G}$ has a common fixed point $v_1 \in V$. Let U be the orthogonal complement of v_1, and let P be the orthogonal projection on U. Then

$$\mathcal{G}_1 = \{PG|_U : G \in \mathcal{G}\}$$

is a group of unitary operators on U such that

$$f_{G_1} \geq \frac{n+k-1}{2} - 1 = \frac{(n-1)+(k-1)-1}{2}$$

for all $G_1 \in \mathcal{G}_1$. If $k > 1$, then apply [2, Thm. 2.12] for $\mathcal{G}_1$ to get a common fixed point $v_2 \in U$ which is a common fixed point for $\mathcal{G}$ as well. Continuing in this way we obtain after k steps common fixed points $v_1, v_2, \ldots, v_k$ that generate the desired k-dimensional space of common fixed points. □

The following example shows that in the absence of additional hypotheses in Theorem 2.3, the number k in the conclusion cannot be improved.

Example 2.4. Choose $r \geq 2$ and let $n = 2^r - 1$. We write $U = \mathbb{Z}_2^r$. Then the cardinality of $\widehat{U} = U \backslash \{0\}$ is equal to n. We enumerate the rows and columns of matrices in $M_n(F)$ by the elements of $\widehat{U}$. Here F is a field of characteristic not equal to 2. Now let $\mathcal{S}$ be the set of all diagonal matrices $D_x \in M_n(F)$, $x \in U$, such that for $y \in \widehat{U}$ the (y, y) diagonal element of D_x is equal to $(-1)^{x^T y}$. It is straightforward to check that $D_{x_1} D_{x_2} = D_{x_1+x_2}$ and that $\mathcal{S}$ is an abelian group isomorphic to the additive group $(U, +)$. For every $y \in \widehat{U}$ there are elements $x_1, x_2 \in U$ such that $x_1^T y = 0$ and $x_2^T y = 1$. Then it follows that every element of $\mathcal{S}$ has a fixed point and that there is no common fixed point for all elements of $\mathcal{S}$.

Let us compute the dimensions $f_x = f_{D_x}$ of the vector spaces $\mathcal{F}_x = \mathcal{F}_{D_x}$. If $x = 0$ then $D_0 = I$ and $f_0 = n$. Assume next that $x \neq 0$. Then $\varphi_x : U \to \mathbb{Z}_2$ defined by $\varphi_x(y) = x^T y$ is a nonzero linear functional. Its kernel is a subspace of dimension $r - 1$ and the cardinality of $\ker \varphi_x \backslash \{0\}$ is $2^{r-1} - 1$. Therefore $f_x = \frac{n-1}{2}$.

Now choose $k \geq 1$ and write $n_k = n + k$. Let $\mathcal{S}_k$ be a matrix subgroup of $Gl_{n_k}(F)$ that consists of all the matrices

$$D_{x,k} = \begin{bmatrix} D_x & 0 \\ 0 & I_k \end{bmatrix}, \ x \in U,$$

where I_k is the $k \times k$ identity matrix. Then

$$f_{D_{x,k}} \geq \frac{n-1}{2} + k = \frac{n_k + k - 1}{2}$$

and $\mathcal{S}_k$ has exactly k-dimensional space of common fixed points. □

3. Common fixed points for semigroups of nonnegative matrices

In this section we improve Theorem 3.2 (and its corollaries) from [2].

Lemma 3.1. *Let $\mathcal{S}$ be a semigroup of nonnegative diagonal matrices. If each member of $\mathcal{S}$ has a fixed point, then $\mathcal{S}$ has a common fixed point.*

Proof. Denote by $k \geq 1$ the minimum of the set $\{f_S : S \in \mathcal{S}\}$ and choose $A \in \mathcal{S}$ such that $f_A = k$. Then we may assume that $A = I \oplus A_2$, where I denotes the identity of order k and A_2 is a diagonal matrix whose diagonal entries are not equal to 1. If $B = B_1 \oplus B_2$ is an arbitrary member of $\mathcal{S}$, then we conclude from $f_{A^m B} \geq k$ ($m \in \mathbb{N}$) that $B_1 = I$, and so $\mathcal{S}$ has k-dimensional fixed space. □

Lemma 3.2. *Let $\mathcal{G}$ be a group of nonnegative matrices. If the identity matrix is the only diagonal matrix in $\mathcal{G}$ then $\mathcal{G}$ is diagonally similar to a permutation group. In particular, $\mathcal{G}$ has a common fixed point.*

Proof. By [9, Lem. 5.1.11], $\mathcal{G}$ is monomial, i.e., each row and column in every member has precisely one nonzero entry. We choose a nondiagonal element G in $\mathcal{G}$. Since G is monomial it follows that G is (up to a permutational similarity) a direct sum of matrices of the form

$$G_j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_{jl_j} \\ a_{j1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{j2} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{j,l_j-1} & 0 \end{bmatrix},$$

Denote by d_j the product $\prod_{i=1}^{l_j} a_{ji}$ which is positive. Suppose that m is a positive integer such that G^m is diagonal, i.e., $G^m = I$. Then each l_j divides m. Since $G_j^{kl_j} = (d_j)^k I$ for all positive integers k it follows that $d_j = 1$. Thus each G_j is diagonally similar to a permutation matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

In particular $1 \in \sigma(G_j)$ for all j and therefore $1 \in \sigma(G)$. Since $\mathcal{G}$ is monomial, the space V, upon which its acts, decomposes into direct sum of standard subspaces $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$, where each V_j is invariant under $\mathcal{G}$ and the restriction $\mathcal{G}_j$ of $\mathcal{G}$ to V_j is indecomposable. Note that nonnegativity of the entries implies that each $\mathcal{G}_j$ has a trivial diagonal subgroup. The same argument as above shows that $1 \in \sigma(A)$ for all $A \in \mathcal{G}_j$. Then it follows by [2, Thm. 3.2] that each $\mathcal{G}_j$, and therefore $\mathcal{G}$, is diagonally similar to a permutation group, and as a consequence has a common fixed point. □

Theorem 3.3. *Let $\mathcal{S}$ be a semigroup of nonnegative monomial matrices such that $1 \in \sigma(S)$ for all $S \in \mathcal{S}$. Then $\mathcal{S}$ has a common fixed point.*

Proof. By Lemma 3.1, the diagonal subsemigroup $\mathcal{D}$ of $\mathcal{S}$ has a nontrivial common fixed space F. We assume F maximal possible. We claim that $\mathcal{S}$ leaves F invariant. Suppose not. Then there is an element $S \in \mathcal{S}$ and a standard subspace F_2 such that F_2 is invariant for S, $F_2 \cap F \neq 0$, $F_2 \cap F^\perp \neq 0$, $S(F_2 \cap F) \not\subset F$ and

$$S|_{F_2} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_l \\ a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{l-1} & 0 \end{bmatrix}$$

with respect to some standard basis $e_1, \ldots, e_l$ of F_2 where $a_j \geq 0$ for $j = 1, 2, \ldots, l$. Note that we can assume that $e_1 \in F$ and $e_l \in F^\perp$. Write $d(S) = \prod_{i=1}^{l} a_i$ and note it is nonnegative. Now, if $d(S) \neq 1$, then some power of S, say S^m, is diagonal, but $S^m e_1 = (d(S))^{m/l} e_1 \neq e_1$, a contradiction. If $d(S) = 1$, let $D \in \mathcal{D}$ be such that $De_l = be_l$ with $b \neq 1$, which exists since $e_l \in F^\perp$. Note that $SDS^{l-1}e_1 = be_1$. Since some power of SDS^{l-1} is diagonal and b is nonnegative, we get that $e_1 \notin F$, a contradiction. Therefore F is invariant for all $S \in \mathcal{S}$.

Since $\mathcal{S}$ is monomial, each member of $\mathcal{S}$ has the form $S = S_1 \oplus S_2$ with respect to the decomposition $F \oplus F^\perp$. We consider the semigroup $\mathcal{S}_1 = \{S_1 : S = S_1 \oplus S_2 \in \mathcal{S}\}$ of (monomial) non-negative matrices. We claim that the subset of diagonal matrices $\mathcal{D}_1$ of $\mathcal{S}_1$ contains only the identity matrix. If D_1 is in $\mathcal{D}_1$, then $D_1 \oplus S_2 \in \mathcal{S}$ for some S_2. Clearly, there is a positive integer m such that $(D_1 \oplus S_2)^m \in \mathcal{D}$, and so $D_1^m = I$. But by positivity it has to be that $D_1 = I$ proving the claim. Now it follows that $\mathcal{S}_1$ is actually a group: for every element $S_1 \in \mathcal{S}_1$ we have $S_1^m \in \mathcal{D} = \{I\}$ for some positive integer m and so S_1 is invertible and $S_1^{-1} = S_1^{m-1} \in \mathcal{S}_1$.

Finally Lemma 3.2 implies that $\mathcal{S}_1$, and thus also $\mathcal{S}$, has a common fixed point. □

Corollary 3.4. *Let $\mathcal{G}$ be a group of nonnegative matrices such that $1 \in \sigma(G)$ for all $G \in \mathcal{G}$. Then $\mathcal{G}$ has a common fixed point.*

Proof. By [9, Lem. 5.1.11] $\mathcal{G}$ is monomial. □

4. Irreducible groups of matrices with eigenvalue one

Theorem 2.3 tells us that if $f_G \geq \frac{n}{2}$ for all elements G of a bounded matrix group $\mathcal{G} \subset M_n(\mathbb{C})$ then there is a common fixed point for all elements of $\mathcal{G}$. Here we study a related question for the reducibility of $\mathcal{G}$: What is the best possible bound k such that $f_G \geq k$ implies reducibility of $\mathcal{G}$? In Remark 2.13 of [2] it was shown that $k = \sqrt{n+1}$ is not sufficient. The following example shows that even $k = \frac{n}{9}$ is not sufficient.

Example 4.1. Let F be a field of characteristic not equal to 2 and such that the primitive cubic roots of 1 are in F. Let $\mathcal{G}$ be the subgroup of $Gl_3(F)$ generated by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the set

$$\left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix};\ \alpha, \beta, \gamma \in \{1, -1\},\ \alpha\beta\gamma = 1 \right\}.$$

Then the possible sets (counting multiplicities) of eigenvalues of elements of $\mathcal{G}$ are: $S_1 = \{1, 1, 1\}$, $S_2 = \{1, -1, -1\}$ and $S_3 = \{1, \omega, \omega^2\}$, where ω is a primitive cubic root of 1. Hence every element of $\mathcal{G}$ has a fixed point. Notice also that all the eigenvalues are semisimple. Since the linear span of $\mathcal{G}$ is $M_3(F)$ the group $\mathcal{G}$ is irreducible. Now let $\mathcal{G}_k \subset M_n(F)$, $n = 3^k$, be a tensor (or Kronecker) product of k copies of $\mathcal{G}$. Since $\mathcal{G}_1 = \mathcal{G}$ is irreducible also $\mathcal{G}_k$ is irreducible for all $k \geq 2$. Since every eigenvalue of an element of $\mathcal{G}$ is semisimple also every eigenvalue of an element of $\mathcal{G}_k$ is semisimple. Thus f_G is equal to the multiplicity of 1 in the spectrum of $G \in \mathcal{G}_k$. Let us compute this multiplicity. If $G = A_1 \otimes A_2 \otimes \cdots \otimes A_k \in \mathcal{G}_k$ then the spectrum of G is the set of all the products $\alpha_1\alpha_2 \cdots \alpha_k$ where α_j is an eigenvalue of A_j. Assume that a is the number of A_j that have the spectrum equal to S_1, b the number of A_j with the spectrum equal to S_2 and c the number of A_j with the spectrum equal to S_3. Then

$$f_G = 3^a \cdot \frac{3^b + (-1)^b}{2} \cdot 3^{c-1}.$$

To explain the second factor in this product, denote by x_b the multiplicity of 1 in the spectrum of the tensor product of A_j with the spectrum equal to S_2. Then we easily obtain the following recursive relation

$$x_b = x_{b-1} + 2(3^{b-1} - x_{b-1}) = 2 \cdot 3^{b-1} - x_{b-1}.$$

Since $x_1 = 1$, its solution is $x_b = (3^b + (-1)^b)/2$ as asserted. Now we have the desired lower bound

$$f_G \geq 3^{a+b+c-2} = \frac{n}{9}.$$

□

In [2] the authors gave an example of an infinite irreducible subgroup $\mathcal{G}$ of $GL_8(\mathbb{C})$ such that $1 \in \sigma(G)$ for every $G \in \mathcal{G}$. The following is an example of an irreducible finite subgroup of $GL_8(\mathbb{C})$ with this spectral property.

Example 4.2. Let G be the Frobenius group of order 72. It is well known (see [8]) that G is a semidirect product of its Frobenius kernel H, isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, and its Frobenius complement Q, isomorphic to the quaternion group of order 8. In particular, Q acts on H by conjugation and this action is regular and transitive on the set $H - \{e\}$.

Let F be a field that contains the primitive third root ω of unity. Then the group X of characters $\chi : H \to F^*$ is non-trivial and is in fact isomorphic to H. The action of Q on X is also regular and transitive on the set of nontrivial characters of H. Let $\chi \in X$ be a non-trivial character of H, acting on a one dimensional H-module V, and consider the induced representation of G. This is a representation of degree 8 and we can take the set gv, $g \in Q$, $0 \neq v \in V$, for the basis of the corresponding G-module (see [4]). Given $h \in H$ we have

$$h \cdot (gv) = g(g^{-1}hg) \cdot v = \chi(g^{-1}hg)gv,$$

so H is a group of diagonal matrices in this induced representation. It follows from the discussion above that the corresponding characters $h \mapsto \chi(g^{-1}hg)$, $g \in Q$, are exactly all the non-trivial characters of H. Consequently, for every $h \in H$, $h \neq e$, exactly two of them assume value one and the F-algebra, spanned by the image of H under this representation, is the whole algebra of diagonal matrices. Since Q acts transitively on the basis vectors, the induced representation is (absolute) irreducible. Finally, the complement Q is mapped to permutation matrices under this representation, so one is in the spectrum of the corresponding matrices. The same holds for the whole group G since every $g \in G$, $g \notin H$, is conjugate to an element of Q.

Explicitly, the Frobenius kernel H is generated by the matrices

$$h_1 = \mathrm{diag}(\omega, \omega^2, \omega, \omega^2, \omega, \omega^2, 1, 1) \quad \text{and} \quad h_2 = \mathrm{diag}(1, 1, \omega, \omega, \omega^2, \omega^2, \omega, \omega^2),$$

and the Frobenius complement Q by the permutation matrices g_1 and g_2, corresponding to the permutations (1234)(5678) and (1836)(2745) respectively. □

5. Common eigenvectors for sets

Theorem 5.1. *Let $\mathcal{S} \subset M_n(F)$ be a set with the property that every $(n+1)$ members of $\mathcal{S}$ have a common eigenvector. Then there is a common eigenvector for all elements of $\mathcal{S}$.*

Proof. Pick $A_1 \in \mathcal{S}$ and let L_1, L_2, ... be the eigenspaces corresponding to distinct eigenvalues of A_1, so that their dimensions add up to t_1, where $1 \leq t_1 \leq n$. If for fixed i every member of $\mathcal{S}$ has L_i as an eigenspace, we are done. Otherwise, there is $A_2 \in \mathcal{S}$ such that the pair $\{A_1, A_2\}$ has common eigenspaces $L_1^{(2)}$, $L_2^{(2)}$, ..., the dimensions of them add up to t_2. Since each $L_j^{(2)}$ is contained in some L_i and at least one properly, we have $t_2 \leq t_1 - 1$. If for fixed j every member of $\mathcal{S}$ has $L_j^{(2)}$ as an eigenspace, we are done. Otherwise, find $A_3 \in \mathcal{S}$ such that the common eigenspaces $L_1^{(3)}$, $L_2^{(3)}$, ... for $\{A_1, A_2, A_3\}$ have dimensions adding up to $t_3 \leq t_2 - 1$. However, this process must stop after at most n steps, when we get a common eigenvector for $\mathcal{S}$. □

The following example proves that $(n+1)$ in Theorem 5.1 cannot be decreased.

Example 5.2. Let e_1, ..., e_n be the standard basis vectors of F^n (with $n \geq 2$). Define the set $\mathcal{S} = \{S_1, S_2, \ldots, S_n, T\}$ of $n \times n$ matrices by $S_j = e_1 \otimes (e_1 + e_2 + \ldots + e_j)$ for $j = 1, 2, \ldots, n$ and $T = (e_1 - e_2) \otimes e_1$. Then every n members of $\mathcal{S}$ have a common eigenvector, but $\mathcal{S}$ does not.

Theorem 5.3. *Let $\mathcal{S} \subset M_n(F)$ be a set of matrices with a singleton spectrum. If every n members of $\mathcal{S}$ have a common eigenvector, then there is a common eigenvector for all elements of $\mathcal{S}$.*

Proof. If every member of $\mathcal{S}$ is a multiply of the identity, we are done. Otherwise, pick $A_1 \in \mathcal{S}$ that is not a multiple of the identity, and let L_1 be the eigenspace of A_1 of the dimension t_1. Clearly, we have $t_1 \leq n-1$, and we can proceed as in the proof of Theorem 5.1, but in this case the process must stop after at most $(n-1)$ steps. □

The following example shows that n in the last theorem cannot be replaced by a smaller number.

Example 5.4. Let e_1, ..., e_n be the standard basis vectors of F^n (with $n \geq 2$). Define the set $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of nilpotent matrices by $S_1 = e_n \otimes e_1$ and $S_j = e_1 \otimes e_j$ for $j = 2, 3, \ldots, n$. Then every $(n-1)$ members of $\mathcal{S}$ have a common kernel, but $\mathcal{S}$ does not.

6. Common eigenvectors for semigroups

The purpose of this section is to prove that a semigroup $\mathcal{S} \subset M_n(F)$, $n \geq 2$, with the property that any n of its members share an eigenvector, has a common eigenvector. Before we proceed with the proof, let us remark that, in general, this is the best possible result for semigroups. Consider any irreducible semigroup $\mathcal{S} \subset M_n(F)$ of elements of rank at most one. Then clearly any $n-1$ elements of this semigroup share an eigenvector since they share a common kernel. It is conceivable though, that under some additional restrictions placed on the ranks of the elements of the semigroup $\mathcal{S} \subset M_n(F)$, the minimal number k such that any k-tuple of elements in $\mathcal{S}$ sharing an eigenvector implies that the whole semigroup has a common eigenvector should be much smaller than $n-1$. In particular, it seems reasonable to conjecture that for a semigroup $\mathcal{S} \subset GL_n(F)$ already any two of its members sharing an eigenvector would imply the whole semigroup $\mathcal{S}$ to have a common eigenvector. (This conjecture has been recently disproved by J. Okninski. A proof will appear elsewhere.)

In what follows, we regard matrices as operators acting on the space of column vectors. If $\mathcal{E} \subset M_n(F)$ is a nonempty set of matrices, we call a common invariant subspace M *triangulated* (for $\mathcal{E}$), if the restriction of $\mathcal{E}$ to M is (simultaneously) triangularizable. It is easy to see that any common invariant subspace of a triangulated space is again triangulated. Furthermore, given an invariant space N for $\mathcal{E}$ and an invariant subspace $L < N$, then N is triangulated for $\mathcal{E}$ iff both L and N/L are triangulated for the restriction of $\mathcal{E}$ to L and the induced action of $\mathcal{E}$ on N/L respectively.

We begin with the following simple observation.

Lemma 6.1. *Let $\mathcal{E} \subset M_n(F)$ be a nonempty set of matrices. Then there exists a unique maximal triangulated subspace (possibly trivial) M for $\mathcal{E}$.*

Proof. Let $\{M_\alpha\}$ be the set of all the triangulated subspaces for $\mathcal{E}$ (including the trivial one). We claim that the linear span of all its elements is the desired maximal triangulated subspace M for $\mathcal{E}$. To this end it suffices to show that the span of any two triangulated invariant subspaces M_1 and M_2 for $\mathcal{E}$ is again triangulated. Now, $M_1 \cap M_2$ is triangulated for $\mathcal{E}$. Since the induced action of $\mathcal{E}$ on $(M_1+M_2)/M_1$ is equivalent to the induced action on $M_2/(M_1\cap M_2)$ which is triangulated, the assertion follows. $\square$

Before we can prove the main result, we need the following result which is a slight generalization of [9, Cor. 4.2.14].

Lemma 6.2. *If every pair in a semigroup $\mathcal{S}\subset M_n(F)$ is triangularizable, then so is $\mathcal{S}$ itself.*

Proof. If every pair $A,B\in\mathcal{S}$ is triangularizable, then $AB-BA$ is nilpotent and the result follows by [6, Thm. B]. $\square$

Theorem 6.3. *Let $\mathcal{S}\subset M_n(F)$, $n\geq 2$, be a semigroup with the property that any n of its members share an eigenvector. Then there exists a common eigenvector for the entire semigroup $\mathcal{S}$.*

Proof. Observe that Lemma 6.2 implies that $\mathcal{S}$ is triangularizable if $n=2$.

We now use induction on n. So suppose $n\geq 3$ and that $\mathcal{S}$ is not triangularizable since otherwise we are done. By Lemma 6.2 we can assume there exist two elements $S_1,S_2\in\mathcal{S}$ that are not simultaneously triangularizable. Let M_2 be the unique maximal triangulated subspace for the pair $\{S_1,S_2\}$. Observe that by the hypothesis we have $1\leq\dim M_2\leq n-2$.

Assume now that a set $\{S_1,\ldots,S_k\}$, $2\leq k\leq n-1$, of elements in $\mathcal{S}$ has been found such that the associated maximal triangulated subspace M_k for this set satisfies $1\leq\dim M_k\leq n-k$. If for each $S\in\mathcal{S}\backslash\{S_1,\ldots,S_k\}$ the subspace M_k is also the maximal triangulated subspace for $\{S_1,\ldots,S_k,S\}$, then M_k is clearly invariant for $\mathcal{S}$ so we may consider the restriction $\mathcal{S}|_{M_k}$ of $\mathcal{S}$ to M_k. Observe that in this case $\mathcal{S}|_{M_k}$ has the property that every $\dim M_k$ (in fact, $n-k$) members have a common eigenvector, since they share an eigenvector with the chosen elements $S_1,\ldots,S_k$ which must be in M_k. By induction, the theorem follows.

If not, there is an element $S=:S_{k+1}\in\mathcal{S}$ such that the associated maximal triangulated subspace M_{k+1} of $\{S_1,\ldots,S_k,S_{k+1}\}$ satisfies $1\leq\dim M_{k+1}\leq\dim M_k-1$ and we can proceed. So there must exist a $k_0\leq n-1$ for which the first alternative occurs which proves the theorem. $\square$

Acknowledgements. This project started during the 3rd Linear Algebra Workshop in June 2002 at Bled, Slovenia. J. Bernik, R. Drnovšek, T. Košir and M. Omladič were supported in part by the Ministry of Education, Science, and Sport of Slovenia.

References

[1] R. V. Benson, *Euclidean geometry and convexity*, McGraw-Hill, New York-Toronto (1966).

[2] J. Bernik, R. Drnovšek, T. Košir, M. Omladič, H. Radjavi, Irreducible Semigroups of Matrices with Eigenvalue One, *Semigroup Forum* 67 (2003), 271–287.

[3] J. Bernik, J. Okninski, On Semigroups of Matrices with Eigenvalue One in Small Dimensions, submitted.

[4] C. W. Curtis and I. Reiner, *Representation theory of finite grops and associative algebras*, Wiley, New York (1962).

[5] R. Goodman and N.R. Wallach, *Representations and invariants of the classical groups,* Cambridge University Press (1998).

[6] R.M. Guralnick, Triangularization of Sets of Matrices, *Linear and Multilinear Algebra* 9 (1980), 133-140.

[7] I. Kaplansky, The Engel-Kolchin theorem revisited. In : *Contributions to Algebra*, edited by H. Bass, P. J. Cassidy, J. Kovacic, Academic Press, New York (1977), 233–237.

[8] D. S. Passman, *Permutation groups*, Benjamin, New York (1968).

[9] H. Radjavi and P. Rosenthal, *Simultaneous triangularization*, Springer-Verlag, Berlin, Heidelberg, New York (2000).

J. Bernik, R. Drnovšek, T. Košir, and M. Omladič : Department of Mathematics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: janez.bernik@fmf.uni-lj.si, roman.drnovsek@fmf.uni-lj.si, tomaz.kosir@fmf.uni-lj.si, matjaz.omladic@fmf.uni-lj.si

T. Laffey : Department of Mathematical Science, National University of Ireland, Dublin, Belfield, Dublin 4, Ireland

E-mail address: thomas.laffey@ucd.ie

G. MacDonald : Department of Mathematics and Computer Science, University of Prince Edward Island, Charlottetown, Prince Edward Island, Canada, C1A 4P3

E-mail address: gmacdonald@upei.ca

R. Meshulam : Department of Mathematics, Technion–Israel Institute of Technology, Haifa 32000, Israel

E-mail address: meshulam@techunix.technion.ac.il

H. Radjavi : Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5

E-mail address: radjavi@mathstat.dal.ca