Common Jordan Chains of Matrices

Tomaž Košir
E. K. University of Ljubljana
Department of Mathematics
Jadranska 19, 61000 Ljubljana, Yugoslavia

Submitted by Peter Lancaster

ABSTRACT

A necessary and sufficient condition for any set of matrices to have an eigenvector in common is given; and the connection between common divisors of matrix polynomials and common Jordan chains of their first companion matrices is studied.

1. INTRODUCTION

A computable criterion for two matrices to have an eigenvector in common was given by Shemesh in [1]. Our Theorem 1 is a natural generalization of this result for any finite set of matrices. We also give a necessary and sufficient condition for any set of matrices to have an eigenvector in common.

In Section 3 we study common Jordan chains of matrices. The main result of the paper is given in Section 4, where we characterize common right divisors of matrix polynomials by common Jordan chains of their first companion matrices.

2. COMMON EIGENVECTORS

Let us first fix the notation. For $n \geqslant 1$ the symbol $\mathbb{N}_n$ will denote the set $\{1,2,\dots,n\}$ and $\mathbb{N}_0 = \{0\}$. The symbol $\Lambda(n_1, n_2, \dots, n_i)$ will be used for the Cartesian product $\mathbb{N}_{n_1-1} \times \cdots \times \mathbb{N}_{n_i-1}$ and the symbol $\Phi_{i,j}$ for the set of all injective transformations $\varphi:\mathbb{N}_i \to \mathbb{N}_j$ such that $\varphi(1) < \varphi(2) < \cdots < \varphi(i)$. We will use the symbol Π_i for the set of all permutations π of order i such that $\pi(1) \neq 1$.

Let $C_1, C_2, \dots, C_i$ be complex $n \times n$ matrices, and let p_j be the degree of the minimal polynomial of the matrix C_j for $j = 1, 2, \dots, i$. Define

$$k(C_1, C_2, \dots, C_i) = \bigcap_K \bigcap_{\pi \in \Pi_i} \ker\left(C_1^{k_1} C_2^{k_2} \cdots C_i^{k_i} - C_{\pi(1)}^{k_{\pi(1)}} C_{\pi(2)}^{k_{\pi(2)}} \cdots C_{\pi(i)}^{k_{\pi(i)}}\right),$$

where the first intersection runs over all elements $K = (k_1, k_2, \dots, k_i)$ of the set $\Lambda(p_1, p_2, \dots, p_i)$.

THEOREM 1. *The complex $n \times n$ matrices $C_1, C_2, \dots, C_r$ have a common eigenvector if and only if*

$$\mathcal{N} = \bigcap_{i=2}^{r} \bigcap_{\varphi \in \Phi_{i,r}} k\left(C_{\varphi(1)}, C_{\varphi(2)}, \dots, C_{\varphi(i)}\right) \neq \{0\}. \tag{1}$$

Proof. For every nonzero vector $x \in \mathcal{N}$ we define the subspace $\mathcal{N}_x = \{p(C_1, C_2, \dots, C_r)x;\ p \in \mathcal{P}_r\}$, where $\mathcal{P}_r$ denotes the set of all complex polynomials in r noncommutative variables. As in the proof of [1, Theorem 3.1.], it can be shown that the matrices $C_1, C_2, \dots, C_r$ leave the subspace $\mathcal{N}_x$ invariant and commute on it. Consequently, they have a common eigenvector in $\mathcal{N}_x$. ∎

REMARK. The criterion for any set of matrices to have an eigenvector in common can also be given. Let $\{C_\alpha\}_{\alpha \in A}$ be any set of matrices and $B \subset A$ a finite subset. We define for the finite family of matrices $\{C_\beta\}_{\beta \in B}$ the subspace $\mathcal{N}_B$ as in (1). Then the matrices $\{C_\alpha\}_{\alpha \in A}$ have a common eigenvector if and only if the subspace

$$\mathcal{N} = \bigcap_B \mathcal{N}_B$$

is not trivial. The intersection runs over all finite subsets $B \subset A$. If the space $\mathcal{N}$ is not trivial, then there exists a finite subset $B_0 \subset A$ (not necessarily unique) such that $\mathcal{N} = \mathcal{N}_{B_0}$ and for every pair of sets B_1 and B_2, where $B_1 \subset B_0 \subset B_2 \subset A$, it holds that $\mathcal{N}_{B_0} = \mathcal{N}_{B_2} = \mathcal{N}$ and $\mathcal{N}_{B_1} \neq \mathcal{N}$.

3. COMMON JORDAN CHAINS

DEFINITION. A finite sequence of nonzero vectors $x_1, x_2, \ldots, x_k$ is said to be *a common Jordan chain of matrices* $C_1, C_2, \ldots, C_r$ *at the same eigenvalue* λ_0 if the following relations hold:

$$C_j x_1 = \lambda_0 x_1,$$

$$C_j x_i = \lambda_0 x_i + x_{i-1}$$

for $j = 1,2,\ldots,r$ and for $i = 2,3,\ldots,k$.

If a sequence $x_1, x_2, \ldots, x_k$ is a Jordan chain for matrices $C_1, C_2, \ldots, C_r$ (not necessarily at the same eigenvalue), then it lies in the subspace $\mathcal{N}$ defined by (1). We could check this using the definition of the subspace $\mathcal{N}$. But we will limit our interest to common Jordan chains at the same eigenvalue, since they are connected in a certain way with common right divisors of matrix polynomials (see Theorem 3 below).

THEOREM 2. *If the subspace* $\mathcal{M} \subset \mathbb{C}^n$ *is the subspace spanned by all common Jordan chains at the same eigenvalues of matrices* $C_1, C_2, \ldots, C_r$, *then*

$$\mathcal{M} = \mathcal{N} \cap \mathcal{N}' \tag{2}$$

where $\mathcal{N}' = \bigcap_{i=1}^r \bigcap_{j=i}^r \ker(C_i - C_j)$ *and* $\mathcal{N}$ *is given by* (1).

Proof. For every vector $x \in \mathcal{M}$, for every permutation $\pi \in \Pi_r$, and for $k_1, k_2, \ldots, k_r \in \mathbb{N} \cup \{0\}$ it holds that

$$C_1^{k_1} C_2^{k_2} \cdots C_r^{k_r} x = C_1^S x = C_{\pi(1)}^{k_{\pi(1)}} C_{\pi(2)}^{k_{\pi(2)}} \cdots C_{\pi(r)}^{k_{\pi(r)}} x,$$

where $S = \sum_{i=1}^r k_i$. Indeed, this is true for every member of a common Jordan chain at the same eigenvalue, and therefore for every linear combination of these vectors. Thus, we have proved the relation $\mathcal{M} \subset \mathcal{N} \cap \mathcal{N}'$. The matrices $C_1, C_2, \ldots, C_r$ leave the subspace $\mathcal{N} \cap \mathcal{N}'$ invariant, and they commute on it. Since $C_1 x = C_i x$ for every $x \in \mathcal{N} \cap \mathcal{N}'$ and for $i = 2,3,\ldots,r$, every Jordan chain of the matrix C_1 in the subspace $\mathcal{N} \cap \mathcal{N}'$ is a common Jordan chain at the same eigenvalue of the matrices $C_1, C_2, \ldots, C_r$, and the theorem follows. ∎

4 COMMON DIVISORS OF MATRIX POLYNOMIALS

Let $M(\lambda) = I\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a monic matrix polynomial, where I, $A_{m-1}, A_{m-2}, \ldots, A_0$ are complex $n \times n$ matrices.

Let us introduce some definitions analogous to those in [2].

DEFINITION. A finite sequence of vectors $x_1, x_2, \ldots, x_k$ (with x_1 nonzero) is called a *Jordan chain for* $M(\lambda)$ at the eigenvalue λ_0 if the following equations are satisfied:

$$\sum_{j=0}^{p} \frac{1}{j!} M^{(j)}(\lambda_0) x_{p+1-j} = 0$$

for $p = 0, 1, \ldots, k-1$.

The $mn \times mn$ matrix

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{m-1} \end{bmatrix}$$

is called the *first companion matrix* of the polynomial $M(\lambda)$. The pair of matrices (X, J), where $X \in \mathbb{C}^{n \times mn}$ and $J \in \mathbb{C}^{mn \times mn}$ is a Jordan matrix, is called the *canonical pair* of the polynomial $M(\lambda)$ if the following relation holds

$$C = \hat{X} J \hat{X}^{-1},$$

where

$$\hat{X} = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix}.$$

For more information see [2, p. 8].

A subspace $\mathscr{S} \subset \mathbb{C}^{mn}$ is called a *supporting subspace* of the matrix polynomial $M(\lambda)$ if

(i) the subspace $\mathscr{S}$ is invariant for the first companion matrix C

(ii) the matrix $P_k\tilde{X}_1$ is invertible, where the matrix $\tilde{X}_1 \in \mathbb{C}^{mn \times kn}$ is a submatrix of the matrix $\hat{X}$ such that the columns of $\tilde{X}_1$ span the subspace $\mathscr{S}$, and the matrix $P_k = [I_k \ 0] \in \mathbb{C}^{kn \times mn}$ is the projection on the first kn components.

REMARK. Let $\mathscr{S}$ be a supporting subspace of the matrix polynomial $M(\lambda)$. We have chosen the matrix $\tilde{X}_1$ so that it is a submatrix of the matrix $\hat{X}$. Therefore $\tilde{X}_1$ is of the form

$$\tilde{X}_1 = \begin{bmatrix} X_1 \\ X_1 J_1 \\ \vdots \\ X_1 J_1^{m-1} \end{bmatrix},$$

where $X_1 \in \mathbb{C}^{n \times kn}$ and $J_1 \in \mathbb{C}^{kn \times kn}$ is a Jordan matrix. Using [2, Corollary 2, p. 29] and [2, Theorem 2] it can be shown that the pair of matrices (X_1, J_1) determines uniquely the right divisor of $M(\lambda)$.

Now we are interested in common divisors of the matrix polynomials $M_1(\lambda), M_2(\lambda), \dots, M_r(\lambda)$. We can assume that these polynomials are of the same degree. (If they are not, we can multiply each of them by a corresponding polynomial $I\lambda^k$.)

REMARK. Let the matrix polynomial $M_i(\lambda)$ have the first companion matrix C_i for $i = 1, 2, \dots, r$. If a sequence $x_1, x_2, \dots, x_k$ is a Jordan chain for every matrix C_i, then $x_1, x_2, \dots, x_k$ is a common Jordan chain at the same eigenvalue of matrices $C_1, C_2, \dots, C_r$. We get this result from [1, Theorem 4.1.] using induction.

THEOREM 3. *Let the matrix polynomials $M_1(\lambda), M_2(\lambda), \dots, M_r(\lambda)$ be of the same degree, denote by $C_1, C_2, \dots, C_r$ their first companion matrices, and define the subspace $\mathscr{M}$ by (2). Then there exists a one-to-one correspondence between supporting subspaces lying in $\mathscr{M}$ and common right divisors of the polynomials $M_1(\lambda), M_2(\lambda), \dots, M_r(\lambda)$.*

Proof. Let $\mathscr{S}$ be a supporting subspace lyng in $\mathscr{M}$. Then $\mathscr{S}$ is a supporting subspace for all polynomials $M_1(\lambda), M_2(\lambda), \dots, M_r(\lambda)$, since for every vector $x \in \mathcal{M}$ it holds that $C_i x = C_j x$ for $i, j = 1,2,\dots,r$. By [3, Corollary 3.13] the subspace $\mathcal{S}$ determines uniquely a common right divisor. It is obvious that this common right divisor determines the same supporting subspace $\mathcal{S}$. ∎

REMARK. Let $M^H(\lambda)$ denote the transposed and conjugated polynomial. Then a left divisor $L(\lambda)$ of the matrix polynomial $M(\lambda)$ can be viewed as a right divisor $L^H(\lambda)$ of the polynomial $M^H(\lambda)$. Therefore, we can describe common left divisors of matrix polynomials $M_1(\lambda), M_2(\lambda),\dots, M_r(\lambda)$ by common right divisors of polynomials $M_1^H(\lambda), M_2^H(\lambda),\dots, M_r^H(\lambda)$.

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