GEOMETRIC ASPECTS OF MULTIPARAMETER SPECTRAL THEORY

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Abstract. The paper contains a geometric description of the dimension of the total root subspace of a regular multiparameter system in terms of the intersection multiplicities of its determinantal hypersurfaces. The new definition of regularity used is proved to restrict to the familiar definition in the linear case. A decomposability problem is also considered. It is shown that the joint root subspace of a multiparameter system may be decomposable even when the root subspace of each member is indecomposable.

1. Introduction

In this paper, we are mainly interested in a geometric description of the dimensions of the root subspaces (which are also called generalized or algebraic eigenspaces) of regular multiparameter systems of arbitrary degree. Binding and Browne studied in [3] a special class of linear two-parameter systems over the real numbers, and showed that the dimension of the root subspace at a given eigenvalue is equal to the sum of the orders of contact of the eigencurves at that eigenvalue. In this special case our results reduce to those of [3]. In [3] analytical methods were used, while we are using results from algebraic geometry [17] and commutative algebra [1,16].

The study of eigenspaces and root subspaces is one of the central topics of multiparameter spectral theory. Its importance stems primarily from classical analysis, where the understanding of root subspaces yields various completeness and expansion results. The literature on the subject is extensive and for further details we refer to the books [2,9,14,19,21]. Beside this 'classical' motivation our topic may also be of algebraic and geometric interest. For example, the somewhat related problem of linearization of polynomials in several variables was studied in [5,6,20].

An n-parameter system $\mathbf{f}$ is a system of n endomorphisms $f_i : A \otimes V_i \to A \otimes V_i$ of free $\mathcal{A}$-modules, where we assume that A is a commutative regular Noetherian algebra of Krull dimension $\dim A = n$ over a field F and V_i are finite dimensional vector spaces over F. Typically, $A = F[\mathbf{x}]$ is the polynomial algebra in n variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and f_i are polynomials in $\mathbf{x}$ whose coefficients are linear maps on V_i. We say that $\mathbf{f}$ is *regular* if the determinants $\{\det f_i | i = 1, 2, \dots, n\}$ form a regular sequence in A. Another equivalent definition is that $\mathbf{f}$ is *regular* if the Koszul complex $K_A(\det \mathbf{f})$ is acyclic [17]. We show that in the particular case when $A = F[\mathbf{x}]$ and all the f_i are linear polynomials this definition coincides with the usual definition of regularity that is used, for instance, by Atkinson in [2]. Our definition of regularity not only extends the familiar one to the nonlinear case, but is also natural in the context of the commutative algebra used in our proofs. Another advantage is that it can be localized and so we can speak of local regularity at each eigenvalue. Since our results are local in nature, it suffices to assume local regularity of the systems. One of the crucial results is that regularity of the system $\mathbf{f}$ at a point $\mathfrak{m}$ in its spectrum $\sigma(\mathbf{f})$ implies acyclicity

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of its Koszul complex $K_{\mathcal{A}}(\mathbf{f})$, where $\mathcal{A}$ is the localization of A at $\mathfrak{m}$. In [10] we proved that the root subspace of a linear multiparameter system at a point $\mathfrak{m} \in \sigma(\mathbf{f})$ is given by the cotensor product

$$\mathcal{N} = \mathcal{M}_1^0 \otimes^{\mathcal{A}^0} \mathcal{M}_2^0 \otimes^{\mathcal{A}^0} \dots \otimes^{\mathcal{A}^0} \mathcal{M}_n^0,$$

where $\mathcal{M}_i$ is the cokernel of f_i localized at $\mathfrak{m}$ and where ^{0} is the contravariant functor which takes a module into its dual comodule [12]. The left adjoint of ^{0} is * and the composite ^{0*} is the completion functor for the cofinite topology. Hence, for every $\mathcal{A}^0$-comodule $\mathcal{N}$ the dual $\mathcal{M} = \mathcal{N}^*$ is a module over a complete local ring so that we can apply Serre's multiplicity theory [17] to compute the length $l^{\mathcal{A}}$ of $\mathcal{M}$. Our main result is that

$$l^{\mathcal{A}}(\mathcal{M}) = l^{\mathcal{A}}(\mathcal{A}/\langle \det f_1, \dots, \det f_n \rangle),$$

which in turn is the intersection multiplicity of the system of determinantal hypersurfaces at the point $\mathfrak{m}$. In the case $A = F[\mathbf{x}]$ and $\mathfrak{m} = \langle x_1 - \lambda_1, x_2 - \lambda_2, \dots, x_n - \lambda_n \rangle$ the residue class field of $\mathcal{A}$ is the fixed underlying field F, therefore the length coincides with the dimension of $\mathcal{M}$ as a vector space over F. For a general maximal ideal $\mathfrak{m}$ of $\mathcal{A}$ we have $\dim_F \mathcal{M} = l^{\mathcal{A}}(\mathcal{M})[\mathcal{A}/\mathfrak{m} : F]$. In the paper we mostly consider the case when $\mathcal{A}/\mathfrak{m} \cong F$.

For the terminology from commutative algebra used in this paper we refer to the books of Matsumura [16] and Serre [17], while for the terminology concerning the coalgebra dual of the polynomial algebra we refer to our papers [10,11]. All algebras and coalgebras are over a fixed base field F. If A is a commutative F-algebra and M is an A-module then $\mathrm{Max}(A)$ and $\mathrm{Spec}(A)$ are the maximal ideal spectrum and the prime ideal spectrum of A, while $\mathrm{Var}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) | M_{\mathfrak{p}} \neq 0\} = \{\mathfrak{p} \in \mathrm{Spec}(A) | \operatorname{ann} M \subseteq \mathfrak{p}\}$ is the variety of M.

2. Spectral Properties of an Endomorphism

For any regular commutative Noetherian algebra A over the field F and any finite dimensional vector space V there is a canonical isomorphism of A-algebras

$$A \otimes \mathrm{End}_F(V) \cong \mathrm{End}_A(A \otimes V).$$

For every element $f \in A \otimes \mathrm{End}_F(V)$ consider the exact sequence

$$A \otimes V \xrightarrow{f} A \otimes V \to M \to 0$$

and its localization at a maximal ideal $\mathfrak{m}$ of A

$$\mathcal{A} \otimes V \xrightarrow{f} \mathcal{A} \otimes V \to \mathcal{M} \to 0,$$

where the Noetherian local algebra $\mathcal{A} = A_{\mathfrak{m}}$ has maximal ideal $\mathfrak{m}\mathcal{A}$ and the residue class field $k = \mathcal{A}/\mathfrak{m}\mathcal{A}$ is a finite field extension of F. The coalgebra dual $\mathcal{A}^0$ is a colocal coalgebra and $\widehat{\mathcal{A}} = \mathcal{A}^{0*}$ is the completion of $\mathcal{A}$ in its $\mathfrak{m}\mathcal{A}$-adic topology. The comodule dual sequence

$$0 \to \mathcal{M}^0 \to \mathcal{A}^0 \otimes V \xrightarrow{f^0} \mathcal{A}^0 \otimes V$$

is exact, since the functor ${}^0 : \mathrm{Mod}_{\mathcal{A}}^{op} \to \mathrm{Comod}_{\mathcal{A}^0}$ is a right adjoint [12]. Our main interest lies with $\mathcal{M}^0$. But $\mathcal{M}^{0*0} = \mathcal{M}^0$ and the completion functor ${}^{0*} = \widehat{\mathcal{A}} \otimes_{\mathcal{A}} - : \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{\widehat{\mathcal{A}}}$ is exact, so that we may always replace $\mathcal{A}$ by its completion $\mathcal{A}^{0*}$. Since A is regular the localization $\mathcal{A} = A_{\mathfrak{m}}$ is also regular [17, p. IV-41, Prop. 23], and $\mathcal{A}$ is a unique factorization domain [17, IV-39].

The *spectrum* of the A-endomorphism $f : A \otimes V \to A \otimes V$ is defined by

$$\sigma(f) = \operatorname{Max}(A) \cap \operatorname{Var}(M).$$

In this paper we study spectral properties of the root subspaces of one and several endomorphisms and we will therefore mainly consider the localized exact sequences above.

For an element $p \in A$ we denote by $\langle p \rangle$ the ideal generated by p in the localized algebra $\mathcal{A} = A_{\mathfrak{m}}$. If V is finite dimensional and $\det f$ is a regular element of $\mathcal{A}$ then $f(\operatorname{adj} f) = (\det f) \otimes 1$. The commutative diagram with exact rows of $\mathcal{A}$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A} \otimes V & \xrightarrow{(\det f)\otimes 1} & \mathcal{A} \otimes V & \longrightarrow & \mathcal{A}/\langle \det f \rangle \otimes V & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \operatorname{adj} f} & & \big\| & & \big\downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \mathcal{A} \otimes V & \xrightarrow{f} & \mathcal{A} \otimes V & \longrightarrow & \mathcal{M} & \longrightarrow & 0
\end{array}$$

induces a commutative diagram with exact rows of $\mathcal{A}^0$-comodules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}^0 & \longrightarrow & \mathcal{A}^0 \otimes V & \xrightarrow{f^0} & \mathcal{A}^0 \otimes V & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \pi^0} & & \big\| & & \big\downarrow{\scriptstyle (\operatorname{adj} f)^0} & & \\
0 & \longrightarrow & (\mathcal{A}/\langle \det f \rangle)^0 \otimes V & \longrightarrow & \mathcal{A}^0 \otimes V & \xrightarrow{(\det f)^0 \otimes 1} & \mathcal{A}^0 \otimes V & \longrightarrow & 0
\end{array}$$

and therefore $(\mathcal{A}/\langle \det f \rangle)^0 \cong \ker((\det f)^0)$. Observe that the $\mathcal{A}$-module map $\pi : \mathcal{A}/\langle \det f \rangle \otimes V \to \mathcal{M}$ is surjective.

2.1. Theorem. *Suppose that $\mathcal{A}$ is regular, hence a unique factorization domain, and that $\det f = p_1^{l_1} p_2^{l_2} \dots p_r^{l_r}$ is the primary decomposition in $\mathcal{A}$ of the determinant of $f \in \mathcal{A} \otimes \operatorname{End}_F(V)$. If $\mathfrak{p}_i = \langle p_i \rangle$ for $i = 1, 2, \dots, r$ then $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$ is the set of minimal primes associated with $\mathcal{M}$, $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\} \subset \operatorname{Ass} \mathcal{M} \subset \operatorname{Var} \mathcal{M}$, and $l_i = l_{\mathfrak{p}_i}(\mathcal{M})$ is the length of the $\mathcal{A}_{\mathfrak{p}_i}$-module $\mathcal{M}_{\mathfrak{p}_i}$. Moreover, if $\dim \mathcal{A} = 2$, then $\operatorname{Ass} \mathcal{M} = \operatorname{Var} \mathcal{M} \setminus \{\mathfrak{m}\}$.*

Proof. For every prime ideal $\mathfrak{p}$ of $\mathcal{A}$ the sequence of $\mathcal{A}_{\mathfrak{p}}$-modules

$$\mathcal{A}_{\mathfrak{p}} \otimes V \xrightarrow{f_{\mathfrak{p}}} \mathcal{A}_{\mathfrak{p}} \otimes V \to \mathcal{M}_{\mathfrak{p}} \to 0$$

is exact and $f(\operatorname{adj} f) = (\det f) \otimes 1$ has localization $f_{\mathfrak{p}}(\operatorname{adj} f_{\mathfrak{p}}) = (\det f_{\mathfrak{p}}) \otimes 1$. Observe that $\det f_{\mathfrak{p}}$ is invertible in $\mathcal{A}_{\mathfrak{p}}$ if and only if $p_i \notin \mathfrak{p}$ for $i = 1, 2, \dots, r$, i.e. $\det f_{\mathfrak{p}}$ is not invertible in $\mathcal{A}_{\mathfrak{p}}$ if and only if $p_i \in \mathfrak{p}$ for some i. Hence, $\mathcal{M}_{\mathfrak{p}} \neq 0$ if and only if $p_i \in \mathfrak{p}$ for some i. Moreover, $\mathcal{M}_{\mathfrak{p}_i} \neq 0$ for all i, so that $\mathfrak{p}_i \in \operatorname{Var} \mathcal{M}$ for every i. This implies that $\mathfrak{p} \in \operatorname{Var} \mathcal{M}$ if and only if $p_i \in \mathfrak{p}$ and hence $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r\} \subset \operatorname{Ass} \mathcal{M}$. If $\mathfrak{p} \in \operatorname{Var} \mathcal{M}$ and $\dim \mathcal{A}/\mathfrak{p} = n - 1$ then $\mathfrak{p} = \mathfrak{p}_i$ for some i. If $\dim \mathcal{A} = 2$ then $\operatorname{Ass} \mathcal{M} = \operatorname{Var} \mathcal{M} \setminus \{\mathfrak{m}\}$. This is because $\mathcal{M}_{\mathfrak{p}} \neq 0$ if and only $p_i \in \mathfrak{p}$ for some i, therefore $\mathfrak{p}_i \subset \mathfrak{p} \subset \mathfrak{m}$, which implies that $\mathfrak{p}_i = \mathfrak{p}$ or $\mathfrak{p} = \mathfrak{m}$.

Now observe that $\dim \mathcal{A}_{\mathfrak{p}_i} = 1$, since every nonzero principal prime ideal of a unique factorization domain has height one. Thus, $\mathcal{A}_{\mathfrak{p}_i}$ is a discrete valuation ring, i.e. a local Noetherian domain whose maximal ideal is principal, while

$$\dim \mathcal{M} = \dim \mathcal{A}/\operatorname{ann} \mathcal{M} = \dim \mathcal{A}/\mathfrak{p}_i = \operatorname{coht} \mathfrak{p}_i = \dim \mathcal{A} - 1.$$

In the exact sequence of $\mathcal{A}_{\mathfrak{p}_i}$-modules

$$\mathcal{A}_{\mathfrak{p}_i} \otimes V \xrightarrow{f_{\mathfrak{p}_i}} \mathcal{A}_{\mathfrak{p}_i} \otimes V \to \mathcal{M}_{\mathfrak{p}_i} \to 0$$

the module $\mathcal{M}_{\mathfrak{p}_i}$ is not zero and $\det f_{\mathfrak{p}_i} = c_i p_i^{l_i}$ for some invertible element c_i in $\mathcal{A}_{\mathfrak{p}_i}$. Since $\mathcal{A}_{\mathfrak{p}_i}$ is a discrete valuation ring, in particular a principal ideal domain, the map $f_{\mathfrak{p}_i}$ can be diagonalized, i.e. $\mathcal{M}_{\mathfrak{p}_i} \cong \oplus_j \mathcal{A}_{\mathfrak{p}_i}/p_i^{d_{ij}}\mathcal{A}$ for some integers $d_{i0} \geq d_{i1} \geq \ldots \geq d_{is}$, and $l_{\mathfrak{p}_i}(\mathcal{M}) = l(\mathcal{M}_{\mathfrak{p}_i}) = \sum_{j=1}^{s} d_{ij} = l_i$. $\square$

Let us mention at this point that for every $\mathcal{A}$-module map $f : \mathcal{A} \otimes V \to \mathcal{A} \otimes V$ the equality $f(\operatorname{adj} f) = (\det f) \otimes 1$ implies that $\langle \det f \rangle \subseteq \operatorname{ann} \mathcal{M} \subset \cap_i \mathfrak{p}_i = \langle p_1 p_2 \cdots p_r \rangle$, where $\det f = p_1^{l_1} p_2^{l_2} \ldots p_r^{l_r}$ is the primary decomposition in $\mathcal{A}$ and $\mathfrak{p}_i = \langle p_i \rangle$.

2.2. Theorem. *If* $\mathfrak{m} \in \sigma(f)$ *is* simple, *i.e. if* $\mathcal{M}/\mathfrak{m}\mathcal{M} \cong \mathcal{A}/\mathfrak{m}\mathcal{A}$, *then* $\mathcal{M} \cong \mathcal{A}/\langle \det f \rangle$.

Proof. The maximal ideal of $\mathcal{A} = A_{\mathfrak{m}}$ is $\mathfrak{m}\mathcal{A}$ and $\mathcal{A}/\mathfrak{m}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} \cong \mathcal{M}/\mathfrak{m}\mathcal{M}$. Every homomorphism $\theta : \mathcal{A}/\mathfrak{m}\mathcal{A} \to \mathcal{M}/\mathfrak{m}\mathcal{M}$ can be lifted to an $\mathcal{A}$-module map $\chi : \mathcal{A} \to \mathcal{M}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\chi} & \mathcal{M} \\ \eta \downarrow & & \downarrow \eta \\ \mathcal{A}/\mathfrak{m}\mathcal{A} & \xrightarrow{\theta} & \mathcal{M}/\mathfrak{m}\mathcal{M} \end{array}$$

commutes, and so $1 \otimes_{\mathcal{A}} \chi = \theta$ (via the natural isomorphism $\mathcal{A}/\mathfrak{m}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A}/\mathfrak{m}\mathcal{A}$). Tensoring the exact sequence of $\mathcal{A}$-modules

$$\mathcal{A} \xrightarrow{\chi} \mathcal{M} \to \operatorname{cok} \chi \to 0$$

by $\mathcal{A}/\mathfrak{m}\mathcal{A}$ over $\mathcal{A}$ gives a commutative diagram

$$\begin{array}{ccccccc} \mathcal{A}/\mathfrak{m}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A} & \xrightarrow{1 \otimes_{\mathcal{A}} \chi} & \mathcal{A}/\mathfrak{m}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{M} & \longrightarrow & \mathcal{A}/\mathfrak{m}\mathcal{A} \otimes_{\mathcal{A}} \operatorname{cok} \chi & \longrightarrow & 0 \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\ \mathcal{A}/\mathfrak{m}\mathcal{A} & \xrightarrow{\theta} & \mathcal{M}/\mathfrak{m}\mathcal{M} & \longrightarrow & \operatorname{cok} \chi/\mathfrak{m} \operatorname{cok} \chi & \longrightarrow & 0 \end{array}$$

with exact rows. If $\theta : \mathcal{A}/\mathfrak{m}\mathcal{A} \to \mathcal{M}/\mathfrak{m}\mathcal{M}$ is an isomorphism, then $\operatorname{cok} \chi/\mathfrak{m} \operatorname{cok} \chi = 0$. By the Nakayama Lemma [1] we must conclude that $\operatorname{cok} \chi = 0$, so that $\chi : \mathcal{A} \to \mathcal{M}$ is surjective and $\ker \chi = \operatorname{ann} \mathcal{M}$. If $\det f = p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r}$ is the primary decomposition then it follows from Theorem 2.1 that $\mathfrak{p}_i = \langle p_i \rangle \in \operatorname{Ass} \mathcal{M}$. Then $\mathcal{A}_{\mathfrak{p}_i}$ is a discrete valuation ring for each i and hence $(\operatorname{ann} \mathcal{M})_{\mathfrak{p}_i} = \operatorname{ann}(\mathcal{M}_{\mathfrak{p}_i}) = p_i^{s_i} \mathcal{A}_{\mathfrak{p}_i}$ for some $s_i \leq l_i$. On the other hand, because $\mathcal{M}_{\mathfrak{p}_i}$ can be generated as an $\mathcal{A}_{\mathfrak{p}_i}$-module by a single element, it follows that the map $f_{\mathfrak{p}_i}$ in the exact sequence

$$\mathcal{A}_{\mathfrak{p}_i} \otimes V \xrightarrow{f_{\mathfrak{p}_i}} \mathcal{A}_{\mathfrak{p}_i} \otimes V \to \mathcal{M}_{\mathfrak{p}_i} \to 0$$

has a diagonal matrix representation with diagonal $(p_i^{s_i}, 1, \ldots, 1)$. But then $s_i = l_i$, since $\det(f_{p_i}) = c_i p_i^{l_i}$ for some invertible element c_i in $\mathcal{A}_{\mathfrak{p}_i}$, and so $\operatorname{ann} \mathcal{M}_{\mathfrak{p}_i} = p_i^{l_i} \mathcal{A}_{\mathfrak{p}_i}$. Moreover, if $a \in \operatorname{ann} \mathcal{M}$, i.e. $a\mathcal{M} = 0$, then $a\mathcal{M}_{\mathfrak{p}_i} = 0$, hence $a \in \operatorname{ann} \mathcal{M}_{\mathfrak{p}_i} = p_i^{l_i} \mathcal{A}_{\mathfrak{p}_i}$ for every i, which implies that each $p_i^{l_i}$ divides a in $\mathcal{A}$, so that $a \in \langle \det f \rangle$. We conclude that $\ker \chi = \operatorname{ann} \mathcal{M} = \langle \det f \rangle$ and $\mathcal{A}/\langle \det f \rangle \cong \mathcal{M}$. $\square$

The above theorem remains valid when $\mathcal{A}$ and $\mathcal{M}$ are replaced by $\mathcal{A}/\mathfrak{m}^i\mathcal{A}$ and $\mathcal{M}/\mathfrak{m}^i\mathcal{M}$ for $i \geq 1$. It follows that the isomorphism $\mathcal{A}/\langle \det f \rangle \cong \mathcal{M}$ is an isomorphism of filtered $\mathcal{A}$-modules.

3. Finite System of Endomorphisms

Consider a system $\mathbf{f} = (f_1, f_2, \dots, f_n)$ of endomorphisms $f_i \in \mathrm{End}_A(A \otimes V_i)$, where the V_i are finite dimensional vector spaces over the field F. Note that $n = \dim A$. For each f_i consider the exact sequence

$$A \otimes V_i \xrightarrow{f_i} A \otimes V_i \to M_i \to 0$$

and its localization

$$\mathcal{A} \otimes V_i \xrightarrow{f_i} \mathcal{A} \otimes V_i \to \mathcal{M}_i \to 0$$

at a maximal ideal $\mathfrak{m}$ of A. Then the comodule dual sequence

$$0 \to \mathcal{M}_i^0 \to \mathcal{A}^0 \otimes V_i \xrightarrow{f_i^0} \mathcal{A}^0 \otimes V_i$$

is also exact. The joint spectrum of $\mathbf{f}$ is the intersection

$$\sigma(\mathbf{f}) = \cap_{i=1}^n \sigma(f_i) = \mathrm{Max}(A) \cap_{i=1}^n \mathrm{Var}(M_i)$$

and consists of all maximal ideals $\mathfrak{m}$ of A satisfying $(M_i)_\mathfrak{m} \neq 0$ for $i = 1, 2, \dots, n$. If $A = F[\mathbf{x}]$ and $\mathfrak{m} = \langle x_1 - \lambda_1, x_2 - \lambda_2, \dots, x_n - \lambda_n \rangle$ then $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is an eigenvalue of $\mathbf{f}$ in the sense usual in multiparameter spectral theory [2,3,10,11,14]. In this case we will frequently change the notation and write $\boldsymbol{\lambda}$ for the ideal $\mathfrak{m}$.

The Koszul complex $K_A(\mathbf{f})$ associated with $\mathbf{f}$ is defined recursively by

$$K_A(f_i) : A \otimes V_i \xrightarrow{f_i} A \otimes V_i$$

and

$$K_A(f_1, \dots, f_i, f_{i+1}) = K_A(f_1, \dots, f_i) \otimes_A K_A(f_{i+1}).$$

Its homology is denoted by $H_*(\mathbf{f})$. In a similar way we construct the Koszul complex $K_A(\det\ \mathbf{f})$ of the n-tuple $\det\ \mathbf{f} = (\det f_1, \det f_2, \dots, \det f_n)$ of elements of A recursively by

$$K_A(\det f_i) : A \xrightarrow{\det f_i} A$$

and

$$K_A(\det f_1, \dots, \det f_i, \det f_{i+1}) = K_A(\det f_1, \det f_2, \dots, \det f_i) \otimes_A K_A(\det f_{i+1})$$

and its homology is $H_*(\det\ \mathbf{f})$. Since localization and completion are exact and preserve tensor products, we see that $K_A(\mathbf{f})_\mathfrak{m} \cong K_\mathcal{A}(\mathbf{f})$ and $K_A(\det\ \mathbf{f})_\mathfrak{m} \cong K_\mathcal{A}(\det\ \mathbf{f})$ at the maximal ideal $\mathfrak{m}$ of A. Moreover, $H_*(K_\mathcal{A}(\mathbf{f})) \cong H_*(K_A(\mathbf{f}))_\mathfrak{m}$ and in particular

$$H_0(K_\mathcal{A}(\mathbf{f})) \cong H_0(K_A(\mathbf{f})_\mathfrak{m}) \cong \mathcal{M}_1 \otimes_\mathcal{A} \mathcal{M}_2 \otimes_\mathcal{A} \ldots \otimes_\mathcal{A} \mathcal{M}_n.$$

A chain complex K is called *acyclic* if $H_p(K) = 0$ for all $p \neq 0$. The system $\mathbf{f}$ is called *regular* if the Koszul complex $K_A(\det\ \mathbf{f})$ is acyclic. Locally, we say that $\mathbf{f}$ is *regular* at a maximal ideal $\mathfrak{m} \in \sigma(\mathbf{f})$ if $K_\mathcal{A}(\det\ \mathbf{f})$ is acyclic. By [17, IV-5-12] it is equivalent to say that $\mathbf{f}$ is *regular* (*locally regular*, respectively) if $(\det f_1, \det f_2, \dots, \det f_n)$ is a regular sequence in A (in $\mathcal{A}$, respectively). An element $\mathfrak{m} \in \sigma(\mathbf{f})$ is called *simple* if $\mathcal{M}_i / \mathfrak{m}\mathcal{M}_i \cong \mathcal{A}/\mathfrak{m}\mathcal{A}$ for $i = 1, 2, \dots, n$. The following two results are immediate consequences of Theorem 2.2.

3.1. Corollary. *A point $\mathfrak{m} \in \sigma(\mathbf{f})$ is simple if and only if*

$$\mathcal{M} \cong \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{M}_n \cong \mathcal{A}/\langle \det f_1, \det f_2, \ldots, \det f_n \rangle$$

as filtered $\mathcal{A}$-modules. $\square$

3.2. Corollary. *If $\mathfrak{m} \in \sigma(\mathbf{f})$ is simple, then $\mathcal{M}$ is (up to isomorphisms of filtered $\mathcal{A}$-modules) uniquely determined by the determinants $\det f_1, \det f_2, \ldots, \det f_n$ in $\mathcal{A}$. In particular, the length of the $\mathcal{A}$-module $\mathcal{M}/\mathfrak{m}^k\mathcal{M}$ depends on the determinants only.* $\square$

The Koszul complex $K_{\mathcal{A}^0}(\mathbf{f}^0)$ associated with the sequence of $\mathcal{A}^0$-comodule maps $\mathbf{f}^0$ is the cochain complex defined recursively by

$$K_{\mathcal{A}^0}(f_i^0) : \mathcal{A}^0 \otimes V_i \to \mathcal{A}^0 \otimes V_i$$

and

$$K_{\mathcal{A}^0}(f_1^0, \ldots, f_i^0, f_{i+1}^0) = K_{\mathcal{A}^0}(f_1^0, \ldots, f_i^0) \otimes^{\mathcal{A}^0} K_{\mathcal{A}^0}(f_{i+1}^0).$$

Since the functor ${}^0 : \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Comod}_{\mathcal{A}^0}$ is exact [12] and preserves tensors, it follows that $K_{\mathcal{A}^0}(\mathbf{f}^0) \cong K_{\mathcal{A}}(\mathbf{f})^0$ and $H^*(K_{\mathcal{A}^0}(\mathbf{f}^0)) \cong H_*(K_{\mathcal{A}}(\mathbf{f}))^0$. In particular,

$$H^0(K_{\mathcal{A}^0}(\mathbf{f}^0)) \cong H_0(K_{\mathcal{A}}(\mathbf{f}))^0 \cong \mathcal{M}_1^0 \otimes^{\mathcal{A}^0} \mathcal{M}_2^0 \otimes^{\mathcal{A}^0} \ldots \otimes^{\mathcal{A}^0} \mathcal{M}_n^0$$

is the 'total root space' of the system $\mathbf{f}$ at $\mathfrak{m}$.

Our main tool relating the geometric aspects of the determinants $\det \mathbf{f}$ to the module theoretic aspects of the $\mathcal{M}_i$ is Serre's multiplicity theory [17]. In addition, the following theorem is a crucial step toward our main result.

3.3. Theorem. *If the system $\mathbf{f}$ is regular at a point $\mathfrak{m} \in \sigma(\mathbf{f})$ then :*

(1) *the Koszul complex $K_{\mathcal{A}}(f_1, f_2, \ldots, f_i)$ is acyclic and $\mathcal{A}$-free for each $i = 1, 2, \ldots, n$,*
(2) *$\mathrm{Tor}_j^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_{i-1}, \mathcal{M}_i) = 0$ for $i = 2, 3, \ldots, n$ and $j \neq 0$.*

Proof. By definition, $\mathbf{f}$ is regular at the point $\mathfrak{m}$ if $K_{\mathcal{A}}(\det \mathbf{f})$ is acyclic. By [17, IV-5-12] it is equivalent to say that $\det \mathbf{f} = (\det f_1, \det f_2, \ldots, \det f_n)$ is a regular sequence in $\mathcal{A}$. Since $\det f_i \neq 0$ and $f_i(\mathrm{adj}\, f_i) = (\det f_i) \otimes 1 = (\mathrm{adj}\, f_i) f_i$ we have the commutative diagram with exact rows and columns of $\mathcal{A}$-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} \otimes V_i & \xrightarrow{f_i} & \mathcal{A} \otimes V_i & \longrightarrow & \mathcal{M}_i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{adj}\, f_i} & & \downarrow{\scriptstyle \mu_i} & & \\
0 & \longrightarrow & \mathcal{A} \otimes V_i & \xrightarrow{(\det f_i)\otimes 1} & \mathcal{A} \otimes V_i & \longrightarrow & \mathcal{A}/\langle \det f_i \rangle \otimes V_i & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{N}_i & = & \mathcal{N}_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

for each $i = 1, 2, \dots, n$. This means in particular that

$$0 \to \mathcal{A} \otimes V_i \xrightarrow{f_i} \mathcal{A} \otimes V_i \to \mathcal{M}_i \to 0$$

is a free $\mathcal{A}$-resolution of $\mathcal{M}_i$, so that

$$\operatorname{dh}_{\mathcal{A}} \mathcal{M}_i = 1.$$

But $\mathcal{A}$ is a regular Noetherian local algebra, so we also have [17, IV-35-43]

$$\operatorname{gldh} \mathcal{A} = \dim \mathcal{A} = n \quad \text{and} \quad \operatorname{dh}_{\mathcal{A}} \mathcal{M}_i + \operatorname{codh}_{\mathcal{A}} \mathcal{M}_i = n.$$

We conclude that $\dim \mathcal{M}_i = n - 1 = \operatorname{codh}_{\mathcal{A}} \mathcal{M}_i$ and similarly $\dim \mathcal{N}_i = n - 1 = \operatorname{codh}_{\mathcal{A}} \mathcal{N}_i$, so that each $\mathcal{M}_i$ and each N_i is a Cohen-Macaulay $\mathcal{A}$-module [17, IV-18]. Moreover, we also have the commutative diagram with exact rows and columns of $\mathcal{A}$-modules

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{A} \otimes V_i & \xrightarrow{\operatorname{adj} f_i} & \mathcal{A} \otimes V_i & \longrightarrow & \mathcal{N}_i & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \nu_i} & & \\
0 & \longrightarrow & \mathcal{A} \otimes V_i & \xrightarrow{(\det f_i)\otimes 1} & \mathcal{A} \otimes V_i & \longrightarrow & \mathcal{A}/\langle \det f_i \rangle \otimes V_i & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{M}_i & = & \mathcal{M}_i & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

for each $i = 1, 2, \dots, n$.

We want to show by induction on i that in fact the Koszul complex $K_{\mathcal{A}}(f_1, f_2, \dots, f_i)$ for $i = 1, 2, \dots, n$, is a free resolution of $\mathcal{M}^{[i]} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_i$, i.e. that $K_{\mathcal{A}}(f_1, \dots, f_i)$ is acyclic. So suppose that $\mathbf{X}_i = K_{\mathcal{A}}(f_1, \dots, f_i)$ is acyclic and let $\mathbf{Y}_i = K_{\mathcal{A}}(\det f_1, \det f_2, \dots, \det f_i)$, which is acyclic by hypothesis. Then by induction the map of complexes

$$\mathbf{F}_i : \mathbf{X}_i \to \mathbf{Y}_i,$$

is defined as the tensor product of the injective maps $F_j : K_{\mathcal{A}}(f_j) \to K_{\mathcal{A}}(\det f_j)$ given by the commutative diagrams

$$\begin{array}{ccc}
\mathcal{A} \otimes V_j & \xrightarrow{f_j} & \mathcal{A} \otimes V_j \\
\| & & \downarrow{\scriptstyle \operatorname{adj} f_j} \\
\mathcal{A} \otimes V_j & \xrightarrow{(\det f_j)\otimes 1} & \mathcal{A} \otimes V_j
\end{array},$$

so that

$$\mathbf{F}_{i+1} = \mathbf{F}_i \otimes_{\mathcal{A}} F_{i+1} : \mathbf{X}_{i+1} = \mathbf{X}_i \otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1}) \to \mathbf{Y}_i \otimes_{\mathcal{A}} K_{\mathcal{A}}(\det f_{i+1}) = \mathbf{Y}_{i+1}.$$

This map of complexes induces a map of exact sequences in homology

$$\begin{array}{ccccc} H_0(H_p(\mathbf{X}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1})) & \longrightarrow & H_p(\mathbf{X}_{i+1}) & \longrightarrow & H_1(H_{p-1}(\mathbf{X}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1})) \\ \downarrow & & \downarrow & & \downarrow \\ H_0(H_p(\mathbf{Y}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(\det f_{i+1})) & \longrightarrow & H_p(\mathbf{Y}_{i+1}) & \longrightarrow & H_1(H_{p-1}(\mathbf{Y}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(\det f_{i+1})) \end{array}$$

for $p \geq 0$ [17, p. IV-2, Prop. 1], where the left hand and the right hand horizontal arrows are injective and surjective, respectively. When $\mathbf{X}_i$ is acyclic, and since $\mathbf{Y}_i$ is acyclic by hypothesis, the diagram is trivial for $p > 1$, so that

$$H_p(\mathbf{X}_{i+1}) = 0$$

for $p > 1$. The case $p = 0$ reduces to the commutative square

$$\begin{array}{ccc} H_0(H_0(\mathbf{X}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1})) & \xrightarrow{\cong} & H_0(\mathbf{X}_{i+1}) \\ \downarrow & & \downarrow \\ H_0(H_0(\mathbf{Y}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(\det f_{i+1})) & \xrightarrow{\cong} & H_0(\mathbf{Y}_{i+1}) \end{array}$$

and it gives a canonical map

$$H_0(\mathbf{F}_{i+1}) = \mu^{[i+1]} : \mathcal{M}^{[i+1]} = H_0(\mathbf{X}_{i+1}) \to H_0(\mathbf{Y}_{i+1}) = \mathcal{A}^{[i+1]} \otimes V^{(i+1)},$$

where $\mathcal{M}^{[i+1]} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{M}_{i+1}$, $\mathcal{A}^{[i+1]} = \mathcal{A}/\langle \det f_1, \ldots, \det f_{i+1}\rangle \cong \mathcal{A}/\langle \det f_1\rangle \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{A}/\langle \det f_{i+1}\rangle$, $\mu^{[i+1]} = \mu_1 \otimes_{\mathcal{A}} \mu_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mu_{i+1}$, and $V^{(i+1)} = V_1 \otimes \ldots \otimes V_{i+1}$. When $p = 1$ we get the commutative square

$$\begin{array}{ccc} H_1(\mathbf{X}_{i+1}) & \xrightarrow{\cong} & H_1(H_0(\mathbf{X}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1})) \\ \downarrow & & \downarrow \\ H_1(\mathbf{Y}_{i+1}) & \xrightarrow{\cong} & H_1(H_0(\mathbf{Y}_i)\otimes_{\mathcal{A}} K_{\mathcal{A}}(\det f_{i+1})) \end{array}.$$

The left hand vertical arrow of the first of the above two diagrams is the 'cokernel' and the right hand vertical arrow of the second diagram is the 'kernel' of

$$\begin{array}{ccc} H_0(\mathbf{X}_i)\otimes_{\mathcal{A}} (\mathcal{A}\otimes V_{i+1}) & \xrightarrow{1\otimes_{\mathcal{A}} f_{i+1}} & H_0(\mathbf{X}_i)\otimes_{\mathcal{A}} (\mathcal{A}\otimes V_{i+1}) \\ {\scriptstyle H_0(\mathbf{F}_i)\otimes_{\mathcal{A}} 1}\downarrow & & \downarrow{\scriptstyle H_0(\mathbf{F}_i)\otimes_{\mathcal{A}} \operatorname{adj} f_{i+1}} \\ H_0(\mathbf{Y}_i)\otimes_{\mathcal{A}} (\mathcal{A}\otimes V_{i+1}) & \xrightarrow{1\otimes_{\mathcal{A}} \det f_{i+1}} & H_0(\mathbf{Y}_i)\otimes_{\mathcal{A}} (\mathcal{A}\otimes V_{i+1}) \end{array}$$

in which the bottom map

$$\det f_{i+1} : \mathcal{A}^{[i]} \otimes V^{(i+1)} \to \mathcal{A}^{[i]} \otimes V^{(i+1)}$$

is injective by hypothesis. If $H_0(\mathbf{F}_i)$ were also injective then we could conclude that the top map is injective and hence also that

$$H_1(\mathbf{X}_{i+1}) \xrightarrow{\cong} H_1(H_0(\mathbf{X}) \otimes_{\mathcal{A}} K_{\mathcal{A}}(f_{i+1})) = 0.$$

To make the induction procedure work one must now be able to show that the 'kernel' vanishes, i.e. that $H_0(\mathbf{F}_{i+1}) : H_0(\mathbf{X}_{i+1}) \to H_0(\mathbf{Y}_{i+1})$, is injective.

To begin we will show that $H_0(\mathbf{F}_i) \otimes_{\mathcal{A}} \operatorname{adj} f_{i+1}$ is injective. For this it suffices to prove that

$$1 \otimes_{\mathcal{A}} \operatorname{adj} f_{i+1} : H_0(\mathbf{Y}_i) \otimes_{\mathcal{A}} (\mathcal{A} \otimes V_{i+1}) \to H_0(\mathbf{Y}_i) \otimes_{\mathcal{A}} (\mathcal{A} \otimes V_{i+1})$$

is injective. The map $\det f_{i+1} : \mathcal{A}^{[i]} \to \mathcal{A}^{[i]}$ is injective by hypothesis. From the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^{[i]} \otimes V_{i+1} & \xrightarrow{f_{i+1}} & \mathcal{A}^{[i]} \otimes V_{i+1} \\ \| & & \downarrow{\scriptstyle \operatorname{adj} f_{i+1}} \\ \mathcal{A}^{[i]} \otimes V_{i+1} & \xrightarrow{\det f_{i+1}} & \mathcal{A}^{[i]} \otimes V_{i+1} \end{array}$$

we find that $f_{i+1} : \mathcal{A}^{[i]} \otimes V_{i+1} \to \mathcal{A}^{[i]} \otimes V_{i+1}$ is injective. But the diagram

$$\begin{array}{ccc} \mathcal{A}^{[i]} \otimes V_{i+1} & \xrightarrow{\operatorname{adj} f_{i+1}} & \mathcal{A}^{[i]} \otimes V_{i+1} \\ \| & & {\scriptstyle f_{i+1}}\downarrow \\ \mathcal{A}^{[i]} \otimes V_{i+1} & \xrightarrow{\det f_{i+1}} & \mathcal{A}^{[i]} \otimes V_{i+1} \end{array}$$

also commutes so that

$$\operatorname{adj} f_{i+1} : \mathcal{A}^{[i]} \otimes V_{i+1} \to \mathcal{A}^{[i]} \otimes V_{i+1}$$

is also injective.

Let us first show that $\mu^{[i+1]} = H_0(\mathbf{F}_{i+1}) : H_0(\mathbf{X}_{i+1}) \to H_0(\mathbf{Y}_{i+1})$ is injective for $i = 0, 1$. For $i = 0$ this is clear from the first diagram in this proof. The case $i = 1$ is a bit more complicated. Consider the diagram

$$\begin{array}{ccccc} \mathcal{M}_1 \otimes V_2 & \xrightarrow{f_2} & \mathcal{M}_1 \otimes V_2 & \longrightarrow & \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \\ {\scriptstyle \mu_1 \otimes_{\mathcal{A}} 1}\downarrow & & {\scriptstyle \mu_1 \otimes_{\mathcal{A}} 1}\downarrow & & {\scriptstyle \mu_1 \otimes_{\mathcal{A}} 1}\downarrow \\ \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \xrightarrow{f_2} & \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \longrightarrow & (\mathcal{A}/\langle \det f_1 \rangle \otimes V_1) \otimes_{\mathcal{A}} \mathcal{M}_2, \\ \| & & {\scriptstyle \operatorname{adj} f_2}\downarrow & & {\scriptstyle 1 \otimes_{\mathcal{A}} \mu_2}\downarrow \\ \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \xrightarrow{\det f_2} & \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \longrightarrow & \mathcal{A}^{[2]} \otimes V^{(2)} \end{array}$$

where all the vertical maps, except possibly the right hand top vertical map, are injective, the left hand horizontal maps are injective and the right hand horizontal maps are surjective. The composite of the right hand vertical maps is $H_0(\mathbf{F}_2)$. Moreover, we have the exact sequence

$$\operatorname{Tor}_1^{\mathcal{A}}(\mathcal{N}_1, \mathcal{M}_2) \to \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \xrightarrow{\mu_1 \otimes_{\mathcal{A}} 1} (\mathcal{A}/\langle \det f_1 \rangle \otimes V_1) \otimes_{\mathcal{A}} \mathcal{M}_2 \to \mathcal{N}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \to 0,$$

and so it suffices to show that $\operatorname{Tor}_1^{\mathcal{A}}(\mathcal{N}_1, \mathcal{M}_2) = 0$. But this follows from the commutative diagram

$$\begin{array}{ccccc} \mathcal{N}_1 \otimes V_2 & \xrightarrow{f_2} & \mathcal{N}_1 \otimes V_2 & \longrightarrow & \mathcal{N}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \\ \downarrow{\scriptstyle \mu_1 \otimes_{\mathcal{A}} 1} & & \downarrow & & \downarrow \\ \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \xrightarrow{\det f_2} & \mathcal{A}/\langle \det f_1 \rangle \otimes V^{(2)} & \longrightarrow & \mathcal{A}^{[2]} \otimes V^{(2)} \end{array}$$

in which the left hand vertical map, the left hand bottom horizontal map and hence also the left hand top map are injective. Now let $i \geq 1$.

Induction hypothesis: For each j such that $0 \leq j \leq i$ and every subset $J \subset [n]$ with $[j] \cap J = \emptyset$ and $|J| = i - j$ the map

$$\mu^{[j]} : \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \to \mathcal{A}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V^{(j)}$$

is injective. Here $[j] = \{1, 2, \ldots, j\}$ and $|J|$ is the cardinality of J. Furthermore, $\mathcal{A}^J$ is the tensor product over $\mathcal{A}$ of the quotients $\mathcal{A}/\langle \det f_k \rangle$ with $k \in J$, i.e. $\mathcal{A}^J = \mathcal{A}/\mathcal{J}$, where $\mathcal{J}$ is the ideal of $\mathcal{A}$ generated by $\{\det f_k | k \in J\}$.

If $k \notin [j] \cup J$ then by the induction hypothesis and the regularity condition the vertical maps and the bottom horizontal map in the commutative square

$$\begin{array}{ccc} \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J & \xrightarrow{\det f_k} & \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \\ {\scriptstyle \mu^{[j]}} \big\downarrow & & \big\downarrow {\scriptstyle \mu^{[j]}} \\ \mathcal{A}^{[j] \cup J} \otimes V^{(j)} & \xrightarrow{\det f_k} & \mathcal{A}^{[j] \cup J} \otimes V^{(j)} \end{array}$$

are injective and hence so is the top horizontal map. Furthermore, since $(\operatorname{adj} f_k) f_k = \det f_k = f_k(\operatorname{adj} f_k)$, it follows that in the commutative diagram with exact rows

$$\begin{array}{ccccc} \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V_k & \xrightarrow{f_k} & \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V_k & \longrightarrow & \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{M}_k \otimes_{\mathcal{A}} \mathcal{A}^J \\ \| & & {\scriptstyle \operatorname{adj} f_k} \big\downarrow & & \big\downarrow {\scriptstyle \mu_k} \\ \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V_k & \xrightarrow{\det f_k} & \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V_k & \longrightarrow & \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}_k \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V_k, \end{array}$$

in which the right hand horizontal maps are surjective and the left hand bottom map is injective, the maps f_k and $\operatorname{adj} f_k$ are injective and hence so is μ_k. Moreover, for every l such that $1 \leq l \leq j$ and $J_l = (l+1, \ldots, j, k) \cup J$ it follows by the induction hypothesis and the regularity condition that the vertical maps and the bottom horizontal map in the commutative square

$$\begin{array}{ccc} \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} & \xrightarrow{\det f_l} & \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \\ {\scriptstyle \mu^{[l-1]}} \big\downarrow & & \big\downarrow {\scriptstyle \mu^{[l-1]}} \\ \mathcal{A}^{[l-1] \cup J_l} \otimes V^{(l-1)} & \xrightarrow{\det f_l} & \mathcal{A}^{[l-1] \cup J_l} \otimes V^{(l-1)} \end{array}$$

are injective and hence so is the top horizontal map. In the commutative diagram with exact rows

$$\begin{array}{ccccc} \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \otimes V_l & \xrightarrow{f_l} & \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \otimes V_l & \longrightarrow & \mathcal{M}^{[l]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \\ \| & & {\scriptstyle \operatorname{adj} f_l} \big\downarrow & & \big\downarrow {\scriptstyle \mu_l} \\ \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \otimes V_l & \xrightarrow{\det f_l} & \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \otimes V_l & \longrightarrow & \mathcal{M}^{[l-1]} \otimes_{\mathcal{A}} \mathcal{A}_l \otimes_{\mathcal{A}} \mathcal{A}^{J_l} \otimes V_l \end{array}$$

the map $\det f_l$ is injective and the right hand horizontal maps are surjective. Since $(\operatorname{adj} f_l) f_l = \det f_l = f_l(\operatorname{adj} f_l)$, it follows from the above commutative diagram that the maps f_l and $\operatorname{adj} f_l$ are also injective and hence so is μ_l.

Now, choosing $k = i + 1$, we find by composition that

$$\mu^{[j]} : \mathcal{M}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \to \mathcal{A}^{[j]} \otimes_{\mathcal{A}} \mathcal{A}^J \otimes V^{(j)}$$

is injective for $j = 0, 1, 2, \ldots, i + 1$ and any $J \subset [n]$ with $[j] \cap J = \emptyset$ and $|J| = i + 1 - j$. $\square$

4. Euler Characteristic and Intersection Multiplicities

As in the preceeding sections

$$\dim \mathcal{M} = \dim \mathcal{A}/\operatorname{ann}_{\mathcal{A}} \mathcal{M} = \sup\{\dim(\mathcal{A}/\mathfrak{p})|\mathfrak{p} \in \operatorname{Ass}(\mathcal{M})\}$$

is the Krull dimension of an $\mathcal{A}$-module $\mathcal{M}$. Moreover, since $\mathcal{A}$ is a Noetherian (complete regular) local algebra over a field F, the 'intersection theorem'

$$\dim \mathcal{N} \leq \operatorname{dh}_{\mathcal{A}} \mathcal{M} + \dim(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$$

of Peskine-Szpiro [4, Cor. 9.4.6] holds for finitely generated $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$.

4.1. Lemma. *If* $\mathbf{f}$ *is regular then the* $\mathcal{A}$*-module* $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n$ *has finite length and for* $i = 1, 2, \ldots, n$ *:*

1. $\dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_i) = n - i$*,*
2. $\dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_i) + \dim \mathcal{M}_{i+1} = \dim \mathcal{A} + \dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_{i+1})$*, where* $\mathcal{M}_{n+1} = \mathcal{A}$*,*
3. $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_i$ *is a Cohen-Macaulay module.*

Proof. As in the proof of Theorem 3.3, we denote by $\mathcal{M}^{[i]}$ the product $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_i$, $i = 1, 2, \ldots, n$. The module $\mathcal{M}^{[n]}$ is also denoted by $\mathcal{M}$. If $\mathbf{f}$ is regular then $\det \mathbf{f} = (\det f_1, \det f_2, \ldots, \det f_n)$ is a regular sequence in $\mathcal{A}$, so that $\mathcal{A}/\langle \det \mathbf{f} \rangle$ has finite length. The surjection $\mathcal{A}/\langle \det \mathbf{f} \rangle \otimes V \to \mathcal{M}$ implies that $\mathcal{M}$ has finite length, since $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is finite dimensional over F. By Theorem 3.3 we have the inequalities

$$\operatorname{dh}_{\mathcal{A}} \mathcal{M}^{[i]} \leq i \quad \text{and} \quad \operatorname{dh}_{\mathcal{A}}(\mathcal{M}_i \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n) \leq n - i + 1$$

for $i = 1, 2, \ldots, n$. Moreover, $\dim \mathcal{M} = 0$ since $\mathcal{M}$ has finite length, and $\dim \mathcal{M}_i = n - 1$ by Theorem 2.1, Applying the theorem of Peskine and Szpiro [4, Thm. 9.4.5] to $\mathcal{M} = \mathcal{M}^{[i]} \otimes_{\mathcal{A}} (\mathcal{M}_{i+1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n)$ we get that $\dim \mathcal{M}^{[i]} \leq \operatorname{dh}_{\mathcal{A}}(\mathcal{M}_{i+1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n) \leq n - i$. Now $\mathcal{M}^{[i]} \cong \mathcal{M}^{[i]} \otimes_{\mathcal{A}} \mathcal{A}$, hence [4, Cor. 9.4.6] gives

$$n = \dim \mathcal{A} \leq \operatorname{dh}_{\mathcal{A}} \mathcal{M}^{[i]} + \dim(\mathcal{M}^{[i]} \otimes_{\mathcal{A}} \mathcal{A}) \leq i + n - i = n$$

for $i = 1, 2, \ldots, n$. But then we must conclude that $\operatorname{dh}_{\mathcal{A}} \mathcal{M}^{[i]} = i$ and $\dim \mathcal{M}^{[i]} = n - i$ for each i, and therefore

$$\dim \mathcal{M}^{[i]} + \dim \mathcal{M}_{i+1} = 2n - i - 1 = \dim \mathcal{A} + \dim \mathcal{M}^{[i+1]}$$

for $i = 1, 2, \ldots, n-1$. The equality $\operatorname{dh}_{\mathcal{A}} \mathcal{M}^{[i]} + \operatorname{codh}_{\mathcal{A}} \mathcal{M}^{[i]} = n$ [17, IV-35, Prop. 21] implies that $\operatorname{codh}_{\mathcal{A}} \mathcal{M}^{[i]} = n - i = \dim \mathcal{M}^{[i]}$, so that each $\mathcal{M}^{[i]}$ is a Cohen-Macaulay module. □

4.2. Corollary. *If* $\mathbf{q} = \{q_1, q_2, \ldots, q_n\}$ *is a regular sequence in* $\mathcal{A}$ *and* p_i *is a prime divisor of* q_i *for each* i *then the sequence* $\mathbf{p} = \{p_1, p_2, \ldots, p_n\}$ *is also regular.*

Proof. Since $\mathcal{A}/\langle q_1, \ldots, q_n \rangle$ has finite length also $\mathcal{A}/\langle p_1, \ldots, p_n \rangle$ has finite length and $\dim \mathcal{A}/\langle p_i \rangle = n - 1$. Observe that Theorem 3.3 is still valid if we replace $\mathcal{M}_i$ by $\mathcal{A}/\langle p_i \rangle$, f_i by p_i, $\operatorname{adj} f_i$ by q_i/p_i, and $\det f_i$ by q_i. Now taking $\mathcal{M}_i = \mathcal{A}/\langle p_i \rangle$ in Lemma 4.1 we see that $\mathbf{p}$ is a regular sequence. □

Next we recall some of the definitions and the properties of the theory of cycles [17]. If $\mathfrak{q}$ is a prime ideal of $\mathcal{A}$ then $\operatorname{ht}\mathfrak{q} = \dim \mathcal{A}_{\mathfrak{q}}$ and $\operatorname{coht}\mathfrak{q} = \dim \mathcal{A}/\mathfrak{q}$. Moreover, $\operatorname{ht}\mathfrak{q} + \operatorname{coht}\mathfrak{q} = \dim\mathcal{A} = n$, since $\mathcal{A}$ is a finitely generated F-algebra and an integral domain. The free abelian group

$$Z(\mathcal{A}) = \left\{ \sum z(\mathfrak{q})\mathfrak{q} | z(\mathfrak{q}) \in \mathbb{Z} \right\}$$

generated by $\operatorname{Spec}(\mathcal{A})$ is the group of cycles of $\mathcal{A}$. If

$$Z_a(\mathcal{A}) = \left\{ \sum_{\operatorname{coht}\mathfrak{q}=a} z(\mathfrak{q})\mathfrak{q} | z(\mathfrak{q}) \in \mathbb{Z} \right\}$$

is the subgroup of cycles of coheight a, then $Z(\mathcal{A}) = \oplus Z_a(\mathcal{A})$. A cycle $z = \sum z(\mathfrak{q})\mathfrak{q}$ is called positive if and only if $z(\mathfrak{q}) \geq 0$ for all $\mathfrak{q}$. Grading by height or codimension gives $Z^{\alpha}(\mathcal{A}) = Z_{n-\alpha}(\mathcal{A}) = \left\{ \sum_{\operatorname{ht}\mathfrak{q}=\alpha} z(\mathfrak{q})\mathfrak{q} | z(\mathfrak{q}) \in \mathbb{Z} \right\}$.

If $\dim\mathcal{A} = n$ and $a + b = n + c$ then the cycle product

$$\cdot : Z_a(\mathcal{A}) \otimes Z_b(\mathcal{A}) \to Z_c(\mathcal{A})$$

is defined by linearity

$$z_a \cdot z_b = \sum_{\operatorname{coht}\mathfrak{r}=c} z_a(\mathfrak{p}) z_b(\mathfrak{q}) \chi^{\mathcal{A}_{\mathfrak{r}}}(\mathcal{A}_{\mathfrak{r}}/\mathfrak{p}, \mathcal{A}_{\mathfrak{r}}/\mathfrak{q})\mathfrak{r},$$

i.e. $\mathfrak{p}\cdot\mathfrak{q} = \sum_{\operatorname{coht}\mathfrak{r}=c} \chi^{\mathcal{A}_{\mathfrak{r}}}(\mathcal{A}_{\mathfrak{r}}/\mathfrak{p}, \mathcal{A}_{\mathfrak{r}}/\mathfrak{q})\mathfrak{r}$. Grading by height or codimension gives

$$\cdot : Z^{\alpha}(\mathcal{A}) \otimes Z^{\beta}(\mathcal{A}) \to Z^{\alpha+\beta}(\mathcal{A}),$$

where

$$z^{\alpha} \cdot z^{\beta} = \sum_{\operatorname{ht}\mathfrak{r}=\alpha+\beta} z^{\alpha}(\mathfrak{p}) z^{\beta}(\mathfrak{q}) \chi^{\mathcal{A}_{\mathfrak{r}}}(\mathcal{A}_{\mathfrak{r}}/\mathfrak{p}, \mathcal{A}_{\mathfrak{r}}/\mathfrak{q})\mathfrak{r},$$

i.e. $\mathfrak{p}\cdot\mathfrak{q} = \sum_{\operatorname{ht}\mathfrak{r}=\alpha+\beta} \chi^{\mathcal{A}_{\mathfrak{r}}}(\mathcal{A}_{\mathfrak{r}}/\mathfrak{p}, \mathcal{A}_{\mathfrak{r}}/\mathfrak{q})\mathfrak{r}$.

For any finitely generated $\mathcal{A}$-module $\mathcal{M}$ and every integer a, consider the cycle [17, V-1-2]

$$z_a(\mathcal{M}) = \sum_{\operatorname{coht}\mathfrak{q}=a} l_{\mathfrak{q}}(\mathcal{M})\mathfrak{q},$$

where $l_{\mathfrak{q}}(\mathcal{M}) = l^{\mathcal{A}_{\mathfrak{q}}}(\mathcal{M}_{\mathfrak{q}})$, $\operatorname{coht} q = \dim \mathcal{A}/\mathfrak{q}$, and the sum is over all prime ideals $\mathfrak{q}$ of coheight a. If $\dim\mathcal{M} \leq a$, $\dim\mathcal{N} \leq b$ and $\dim(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \leq c$ with $a + b = n + c$ then the cycles $z_a(\mathcal{M})$ and $z_b(\mathcal{N})$ are defined, intersect properly, and Serre's theorem [17,V-22] says that the intersection cycle is given by

$$z_a(\mathcal{M}) \cdot z_b(\mathcal{N}) = z_c(\operatorname{Tor}^{\mathcal{A}}(\mathcal{M},\mathcal{N})) = \sum_i (-1)^i z_c(\operatorname{Tor}_i^{\mathcal{A}}(\mathcal{M},\mathcal{N})),$$

where the coefficient of a prime ideal $\mathfrak{p} \in V(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N})$ with $\operatorname{coht}\mathfrak{p} = c$ is the Euler characteristic

$$\chi_{\mathfrak{p}}(\mathcal{M},\mathcal{N}) = \sum_i (-1)^i l(\operatorname{Tor}_i^{\mathcal{A}}(\mathcal{M},\mathcal{N})_{\mathfrak{p}}),$$

which is additive in $\mathcal{M}$ and $\mathcal{N}$.

Some of the properties of the cycle function z_a are :

(1) $z_a(\mathcal{M}) \geq 0$.
(2) $z_a(\mathcal{M}) = 0$ if $\dim \mathcal{M} \leq a-1$: If $\dim \mathcal{A}/\operatorname{ann}\mathcal{M} = \dim \mathcal{M} \leq a-1$ and $\operatorname{coht} q = a$ then $\operatorname{ann}\mathcal{M}$ is not in q and hence $\mathcal{M}_q = 0$.
(3) z_a is additive: If $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{L} \to 0$ is exact then so is $0 \to \mathcal{N}_q \to \mathcal{M}_q \to \mathcal{L}_q \to 0$. Hence, $l(\mathcal{M}_q) = l(\mathcal{N}_q) + l(\mathcal{L}_q)$ so that $z_a(\mathcal{M}) = z_a(\mathcal{N}) + z_a(\mathcal{L})$.
(4) Universal property: Every additive function on $K_a(\mathcal{A})$ taking positive values in an ordered abelian group factors through z_a.
(5) If $\mathcal{A}$ is a domain and $\dim \mathcal{A} = n$, then $Z_n(\mathcal{A}) \cong \mathbb{Z}$ and $z_n : K_n(\mathcal{A}) \to \mathbb{Z}$ is the rank function.

4.3. Theorem. *Let* $\mathbf{f}$ *be an* n*-parameter system,* $\mathfrak{m} \in \sigma(\mathbf{f})$ *and let* I *be the ideal* $\langle \det f_1, \det f_2, \dots, \det f_n \rangle$ *in* $\mathcal{A} = A_{\mathfrak{m}}$*. If* $\mathbf{f}$ *is regular at* $\mathfrak{m}$ *then* $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_n$ *has finite length and*

$$l^{\mathcal{A}}(\mathcal{M}) = \sum_{p_i | \det f_i} l_{p_1}(\mathcal{M}_1) \dots l_{p_n}(\mathcal{M}_n) l^{\mathcal{A}}(\mathcal{A}/\langle p_1 \rangle, \mathcal{A}/\langle p_2 \rangle, \dots, \mathcal{A}/\langle p_n \rangle) = l^{\mathcal{A}}(\mathcal{A}/I),$$

where the length $l_{p_i}(\mathcal{M}_i)$ *of the localization of* $\mathcal{M}_i$ *at* p_i *is equal to the multiplicity of* p_i *in* $\det f_i$ *and* $l^{\mathcal{A}}(\mathcal{A}/\langle p_1 \rangle, \mathcal{A}/\langle p_2 \rangle, \dots, \mathcal{A}/\langle p_n \rangle) = \chi^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2, \dots, p_n \rangle) = i(p_1, p_2, \dots, p_n)$ *is the intersection multiplicity of the hypersurfaces defined by the irreducible polynomials* p_i *at the point* $\mathfrak{m}$.

Proof. Lemma 4.1 says that

$$n - s = \dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s) = \sup \{\operatorname{coht} \mathfrak{p} | \mathfrak{p} \in \operatorname{Ass}(\mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s)\}.$$

In particular, $\mathcal{M}$ has finite length and the cycles $z_{n-1}(\mathcal{M}_i)$ intersect properly. In fact, since $\operatorname{Tor}^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s, \mathcal{M}_{s+1}) = \mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s$ by Theorem 3.3(2) and

$$\dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_s) + \dim \mathcal{M}_{s+1} = \dim \mathcal{A} + \dim(\mathcal{M}_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_{s+1})$$

by Lemma 4.1, the cycles $z_{n-s}(\mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s)$ and $z_{n-1}(\mathcal{M}_{s+1})$ intersect properly for each $s = 1, 2, \dots, n-1$ [17, p. V-22, Prop. 1]. Now one proves by induction on s that

$$z_{n-1}(\mathcal{M}_1) \cdot z_{n-1}(\mathcal{M}_2) \cdot \dots \cdot z_{n-1}(\mathcal{M}_s) = z_{n-s}(\mathcal{M}_1 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_s).$$

In particular, for $s = n$ we get

$$z_{n-1}(\mathcal{M}_1) \cdot z_{n-1}(\mathcal{M}_2) \cdot \dots \cdot z_{n-1}(\mathcal{M}_n) = z_0(\mathcal{M}) = l^{\mathcal{A}}(\mathcal{M})\mathfrak{m}.$$

By Theorem 2.1 the prime ideals of coheigth $n-1$ in $\operatorname{Var}(\mathcal{M}_i)$ are exactly the principal primes generated by the prime divisors of $\det f_i$, and so

$$z_{n-1}(\mathcal{M}_i) = \sum_{\operatorname{coht} \mathfrak{q} = n-1} l_{\mathfrak{q}}(\mathcal{M}_i)\mathfrak{q} = \sum_{p | \det f_i} l_{\langle p \rangle}(\mathcal{M}_i)\langle p \rangle,$$

where p runs all over prime divisors of $\det f_i$. The associativity of the cycle product shows that the coefficient of $\mathfrak{m}$ on the left hand side of the equation is exactly

$$\sum_{p_i | \det f_i} l_{p_1}(\mathcal{M}_1) \dots l_{p_n}(\mathcal{M}_n) \chi(\mathcal{A}/\langle p_1 \rangle, \dots, \mathcal{A}/\langle p_n \rangle).$$

The same procedure applied to the system $\det \mathbf{f}$ shows that this is also equal to $l(\mathcal{A}/\langle \det \mathbf{f}\rangle)$. Furthermore, it follows by Corollary 4.2 that we may replace the Euler characteristic $\chi^{\mathcal{A}}(\mathcal{A}/\langle p_1\rangle,\dots,\mathcal{A}/\langle p_n\rangle)$ by the length $l^{\mathcal{A}}(\mathcal{A}/\langle p_1,\dots,p_n\rangle)$. $\square$

If the Koszul complex $K_A(\mathbf{f})$ of the n-parameter system $\mathbf{f}$ on the affine space $\mathbb{A}^n$ is acyclic, then so is $K_{\mathcal{A}}(\mathbf{f})$ at every point $\boldsymbol{\lambda}$, since localization and completion are exact. In this case the assertions of Theorem 4.3 hold for every point $\boldsymbol{\lambda}$ in $\mathbb{A}^n$, i.e. at the ideal $\mathfrak{m}$ generated by the polynomials $x_i - \lambda_i$, $i = 1, 2, \dots, n$.

Note that if the residue class field $\mathcal{A}/\mathfrak{m}\mathcal{A}$ of $\mathcal{A}$ is isomorphic to the base field F then the length of the $\mathcal{A}$-module $\mathcal{M}$ coincides with the (vector space) dimension of $\mathcal{M}$ over F, i.e. $l^{\mathcal{A}}(\mathcal{M}) = \dim_F(\mathcal{M})$. Since $\mathcal{M}$ is finite dimensional, there is a positive integer, and hence a least positive integer r, such that $\mathfrak{m}^r\mathcal{M} = 0$, i.e. $\mathfrak{m}^r \subset \operatorname{ann}_{\mathcal{A}} \mathcal{M}$. On the other hand, since the $\det f_i$ form a regular sequence, there is an integer, hence a least integer s, such that $\mathfrak{m}^s \subset I$, i.e. $\operatorname{ann}_{\mathcal{M}} \mathfrak{m}^s = \mathcal{M}$ (see [18, p. 186, Lemma 1]). The string of inclusions

$$I \subseteq \sum_{j=1}^{n} \operatorname{ann}_{\mathcal{A}} \mathcal{M}_j \subseteq \operatorname{ann}_{\mathcal{A}} \mathcal{M} \subset \mathfrak{m}$$

implies that $r \le s$, with equality if $I = \operatorname{ann}_{\mathcal{A}} \mathcal{M}$.

Observe that Theorem 4.3 generalizes a result of Binding and Browne [3], who proved for a particular class of linear two-parameter systems with $F = \mathbb{R}$ that the dimension of the root subspace (in our notation of $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$) at the simple point $\boldsymbol{\lambda}$ equals the sum of the orders of contact of the eigencurves (counting multiplicities) passing through $\boldsymbol{\lambda}$. For the two-parameter case and also for a simple point in the spectrum we have a more direct proof of Theorem 4.3 avoiding the use of Theorem 3.3.

4.4. Theorem. *If the 2-parameter system* $\mathbf{f}$ *is regular at the point* $\mathfrak{m} \in \sigma(\mathbf{f})$ *then* $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$ *has finite length and*

$$l^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2) = \sum_{p_i \mid \det f_i} l_{p_1}(\mathcal{M}_1) l_{p_2}(\mathcal{M}_2) l^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2\rangle) = l^{\mathcal{A}}(\mathcal{A}/\langle \det f_1, \det f_2\rangle),$$

where

$$l^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2\rangle) = i(p_1, p_2)$$

is the intersection multiplicity of the algebraic curves defined by the irreducible polynomials p_1 *and* p_2 *at the point* $\mathfrak{m}$.

Proof. By 2.1 there are canonical surjective $\mathcal{A}$-module maps $\pi_i : \mathcal{A}/\langle \det f_i\rangle \otimes V_i \to \mathcal{M}_i$ for each $i = 1, 2$ and hence a surjection of $\mathcal{A}$-modules $\mathcal{A}/I \otimes V \to \mathcal{M}$, where I is the ideal $\langle \det f_1, \det f_2\rangle \subset \mathcal{A}$ and $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$. The quotient $\mathcal{A}/I$ has finite length, i.e. $\operatorname{Ass} \mathcal{A}/I = \{\mathfrak{m}\}$, since $K_{\mathcal{A}}(\det \mathbf{f})$ is acyclic. Now $\mathcal{M}$ has finite length, since $\mathcal{A}/I$ has it and since V is finite-dimensional. It now suffices to observe that $\dim \mathcal{A} = 2$ and $\dim \mathcal{M}_i = 1$ for $i = 1, 2$. Since $\cup_{\mathfrak{p} \in \operatorname{Ass} \mathcal{M}_i} \mathfrak{p}$ is the set of zero-divisors for $\mathcal{M}_i$ in $\mathcal{A}$ and $\mathfrak{m} \notin \operatorname{Ass} \mathcal{M}_i$ there is a non-zerodivisor $a_i \in \mathfrak{m}$ for $\mathcal{M}_i$, hence $\dim \mathcal{M}_i/a_i\mathcal{M}_i = 0$. This means that $\operatorname{codh} \mathcal{M}_i = \dim \mathcal{M}_i$, i.e. that $\mathcal{M}_i$ is a Cohen-Macaulay $\mathcal{A}$-module, and we conclude by [17, V-19] that $\operatorname{Tor}_k^{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_2) = 0$ for $k > 0$. This implies that $\chi_{\mathfrak{m}}(\operatorname{Tor}^{\mathcal{A}}(\mathcal{M}_1, \mathcal{M}_2)) = l^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2)$. For the rest see [8, 18 p.186], or prove it directly as follows. The pair $(\det f_1, \det f_2)$ forms a regular sequence in $\mathcal{A}$, and so $\det f_1$ and $\det f_2$ must be relatively prime. Moreover, every pair of irreducible divisors $\{p_1, p_2\}$ of $\{\det f_1, \det f_2\}$ forms a regular sequence in the Noetherian regular local (hence Cohen-Macaulay) algebra $\mathcal{A}$ [8 p. 40, 18]. It follows by [17, IV-5] that the Koszul complex $K_{\mathcal{A}}(p_1, p_2) \cong K_{\mathcal{A}}(p_1) \otimes_{\mathcal{A}} K_{\mathcal{A}}(p_2)$ is a free resolution of the $\mathcal{A}$-module $\mathcal{A}/\langle p_1, p_2\rangle$. We conclude that

$$\mathrm{Tor}^{\mathcal{A}}(\mathcal{A}/\langle p_1\rangle, \mathcal{A}/\langle p_2\rangle) = \mathcal{A}/\langle p_1\rangle \otimes_{\mathcal{A}} \mathcal{A}/\langle p_2\rangle \cong \mathcal{A}/\langle p_1, p_2\rangle$$

by [17, V-26], so that $\chi^{\mathcal{A}}(\mathcal{A}/\langle p_1\rangle, \mathcal{A}/\langle p_2\rangle) = l^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2\rangle)$. □

4.5. Theorem. *If $\mathfrak{m}$ is a simple regular point for the multiparameter system $\mathbf{f}$ then*

$$\mathcal{M} \cong \mathcal{A}/\langle \det f_1, \dots, \det f_n\rangle$$

and

$$l^{\mathcal{A}}(\mathcal{M}) = \sum_{p_i \,|\, \det f_i} l_{p_1}(\mathcal{M}_1) \dots i_{p_n}(M_n) l^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2, \dots, p_n\rangle) = l^{\mathcal{A}}(\mathcal{A}/I),$$

where $I = \langle \det f_1, \det f_2, \dots, \det f_n\rangle$ and $l^{\mathcal{A}}(\mathcal{A}/\langle p_1, p_2, \dots, p_n\rangle = i(p_1, p_2, \dots, p_n)$ is the intersection multiplicity of the hypersurfaces defined by the irreducible polynomials p_i at the point $\mathfrak{m}$.

Proof. It suffices to show that the Koszul complex $K_{\mathcal{A}}(\mathbf{f})$ is acyclic. The Koszul complex $K_{\mathcal{A}}(\det \mathbf{f})$ is acyclic by assumption. It follows in particular that $\det f_i \neq 0$ and hence that $f_i : \mathcal{A} \otimes V_i \to \mathcal{A} \otimes V_i$ is injective for $i = 1, 2, \dots, n$ so that $K_{\mathcal{A}}(f_i)$ is a free resolution of the $\mathcal{A}$-module $\mathcal{M}_i$. By Theorem 2.2 we have that $\mathcal{M}_i \cong \mathcal{A}/\langle \det f_i\rangle$. Now we prove by induction that $K_{\mathcal{A}}(\mathbf{f})$ is acyclic. Assume that $Y_i = K_{\mathcal{A}}(f_1) \otimes_{\mathcal{A}} K_{\mathcal{A}}(f_2) \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} K_{\mathcal{A}}(f_i)$ is a free resolution of $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_i \cong \mathcal{A}/\langle \det f_1, \det f_2, \dots, \det f_i\rangle$. The acyclicity of $K_{\mathcal{A}}(\det \mathbf{f})$ implies that

$$\mathrm{Tor}_j^{\mathcal{A}}(\mathcal{A}/\langle \det f_1, \det f_2, \dots, \det f_i\rangle, \mathcal{A}/\langle \det f_{i+1}\rangle) = 0$$

for $j \neq 0$ and therefore that

$$H_j^{\mathcal{A}}(Y_{i+1}) = \mathrm{Tor}_j^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{M}_i, \mathcal{M}_{i+1}) = 0$$

for $j \neq 0$. We conclude that Y_{i+1} is acyclic and the proof is complete when $i = n - 1$. □

5. The Linear Multiparameter Case

A *linear n-parameter system* $\mathbf{f}$ is a system of linear polynomials

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} A_{ij} x_j - A_{i0}, \ (i = 1, 2, \dots, n)$$

in n variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$, where we assume that the coefficients A_{ij} are linear maps acting on a finite-dimensional vector space V_i over a field F. From now on we assume that F is infinite. The linear map A_{ij} induces a linear map $A_{ij}^{\dagger}$ on the vector space $V = V_1 \otimes V_2 \otimes \dots \otimes V_n$ by acting on the i-th tensor factor. The determinant Δ_0 of the matrix

$$\begin{pmatrix} A_{11}^{\dagger} & A_{12}^{\dagger} & \cdots & A_{1n}^{\dagger} \\ A_{21}^{\dagger} & A_{22}^{\dagger} & \cdots & A_{2n}^{\dagger} \\ \vdots & \vdots & & \vdots \\ A_{n1}^{\dagger} & A_{n2}^{\dagger} & \cdots & A_{nn}^{\dagger} \end{pmatrix}$$

is a linear transformation on V. It is well defined because any two entries from distinct rows in the above matrix commute. In a similar way linear transformations Δ_i $(i = 1, 2, \ldots, n)$ on V are defined by replacing the i-th column in the matrix by $\left[A_{k0}^{\dagger}\right]_{k=1}^{n}$.

Recall that for each f_i we have the short exact sequence

$$F[\mathbf{x}] \otimes V_i \xrightarrow{f_i} F[\mathbf{x}] \otimes V_i \longrightarrow M_i \longrightarrow 0$$

and its localization at a maximal ideal m in the variety $\operatorname{Var} M_i$

$$\mathcal{A} \otimes V_i \xrightarrow{f_i} \mathcal{A} \otimes V_i \longrightarrow \mathcal{M}_i \longrightarrow 0.$$

5.1. Theorem. *For a linear multiparameter system* $\mathbf{f}$ *over an infinite field* F *the following are equivalent :*

1. $\mathbf{f}$ *is regular, i.e.* $K_A(\det \mathbf{f})$ *is acyclic,*
2. *there exists* $\alpha_i \in F$ $(i = 0, 1, \ldots, n)$ *such that* $\sum_{i=0}^{n} \alpha_i \Delta_i$ *is invertible,*
3. *the spectrum* $\sigma(\mathbf{f})$ *is finite,*
4. *the (Krull) dimension* $\dim F[\mathbf{x}] / \langle \det f_1, \det f_2, \ldots, \det f_n \rangle = 0$,
5. *the sequence* $(\det f_1, \det f_2, \ldots, \det f_n)$ *is a regular sequence in* $F[\mathbf{x}]$.

Each of the above statements implies :

(6) *the Koszul complex* $K_{\mathcal{A}}(\mathbf{f})$ *is acyclic for all* $m \in \sigma(\mathbf{f})$ *and* $H^0_{\mathcal{A}}(\mathbf{f})$ *has finite length.*

Proof. The implication (2) $\Rightarrow$ (3) was proved by Atkinson in [2, Thm. 6.8.1] and the inverse implication (3) $\Rightarrow$ (2) follows from [2, Thm. 8.7.2]. (Note that in the proof of [2, Thms. 8.2.1 and 8.7.2] the fact that F is an infinite field is used. This is the case for instance, if $\operatorname{char} F = 0$.) To show that (3) $\Leftrightarrow$ (4) note that the spectrum $\sigma(\mathbf{f})$ coincides with the set of all maximal ideals containing $\det \mathbf{f}$. This set is finite if and only if $\dim F[\mathbf{x}] / \langle \det f_1, \det f_2, \ldots, \det f_n \rangle = 0$. The equivalencies (1) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (5) are proved by Serre in [17, p. III-11, Prop. 6] and [17, p. IV-5, Prop. 3], respectively.

Finally it follows by Theorem 3.3 that (1) $\Rightarrow$ (6). □

Each of the equivalent statements (1), (3), (4) or (5) could be used as a definition of regularity of a linear multiparameter system. Theorem 5.1 implies that in the linear case our definition of regularity is equivalent with the standard one, i.e. to the statement (2) in Theorem 5.1 (see e.g. [2]).

The following is a restatement of Theorem 4.3 for the linear case.

5.2. Theorem. *If* $\mathbf{f}$ *is a regular linear* n*-parameter system on the affine space* $\mathbb{A}^n$ *then the* $\mathcal{A}$*-module* $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{M}_n$ *has finite length and*

$$l^{\mathcal{A}}(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \mathcal{M}_n) = \sum_{p_i \mid \det P_i} l_{p_1}(\mathcal{M}_1) \ldots l_{p_n}(\mathcal{M}_n) l^{\mathcal{A}}(\mathcal{A} / \langle p_1, p_2, \ldots, p_n \rangle),$$

where

$$l^{\mathcal{A}}(\mathcal{A} / \langle p_1, p_2, \ldots, p_n \rangle) = i(p_1, p_2, \ldots, p_n)$$

is the intersection multiplicity of the hypersurfaces defined by the irreducible polynomials p_i *at the point* $m \in \sigma(\mathbf{f})$.

5.3. Example. Let us consider the linear 2-parameter system $\mathbf{f}$, where the matrices $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are given by

$$\begin{pmatrix} x_1+x_2+1 & 0 & 0 & 0 \\ 2 & 2x_1+x_2+1 & 1-x_2 & 2 \\ 0 & x_1+x_2 & x_1+x_2 & x_2-1 \\ 0 & 0 & 0 & x_1+2x_2 \end{pmatrix} \text{ and } \begin{pmatrix} x_2-1 & x_1-1 & 0 \\ x_1 & x_1 & 0 \\ 0 & 1 & x_2 \end{pmatrix}.$$

This system is simple at the point $m = \langle x_1, x_2 \rangle$, i.e. the joint eigenspace at $\boldsymbol{\lambda} = (0,0)$ is 1-dimensional, so by Corollary 4.5 we have

$$\mathcal{M} = \mathcal{M}_1 \otimes_A \mathcal{M}_2 \cong \mathcal{A}/\langle \det f_1, \det f_2 \rangle,$$

where $\det f_1 = 2(x_1+x_2+1)(x_1+2x_2)(x_1+x_2)^2$ and $\det f_2 = x_1(x_2-x_1)x_2$. A direct calculation shows that for $i = 1, 2, \dots$ the lengths of the filtered modules $\mathcal{M}/m^i\mathcal{M}$ are $1, 3, 6, 8, 9, 9, \dots$ respectively, and thus $l^{\mathcal{A}}(\mathcal{M}) = \dim_F \mathcal{M} = 9$. Let $p_{11} = x_1 + 2x_2$, $p_{12} = x_1 + x_2$, $p_{21} = x_1$, $p_{22} = x_2 - x_1$ and $p_{23} = x_2$. By Theorem 5.2 we see next that

$$\dim_F \mathcal{M} = \sum_{j=1}^{3} l^{\mathcal{A}}(\mathcal{A}/\langle p_{11}, p_{2j}\rangle) + 2\sum_{j=1}^{3} l^{\mathcal{A}}(\mathcal{A}/\langle p_{12}, p_{2j}\rangle = 1+1+1+2+2+2 = 9.$$

5.4. Example. Some of the irreducible curves may be singular. Here is an example of such a linear 2-parameter system $\mathbf{f} = (f_1(\mathbf{x}), f_2(\mathbf{x}))$ in $\mathbb{A}^5$:

$$f_1(x_1,x_2) = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 \\ -1 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 1 & x_1 \end{pmatrix}, \quad f_2(x_1,x_2) = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 \\ 1 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix},$$

with the determinants $\det f_1 = (x_1^3 - x_2^2)(x_1^2 - x_2)$ and $\det f_2 = x_1(x_1^4 - x_2^3)$. The irreducible components are then $p_{11} = x_1^3 - x_2^2$, $p_{12} = x_1^2 - x_2$, $p_{21} = x_1$ and $p_{22} = x_1^4 - x_2^3$. The point $(0,0)$ is then singular point for the curves of p_{11} and p_{22}. Next it follows that

$$\begin{aligned} i(p_{11},p_{21}) &= l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^3 - x_2^2, x_1\right\rangle\right) = l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1, x_2^2\right\rangle\right) = 2, \\ i(p_{11},p_{22}) &= l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^3 - x_2^2, x_1^4 - x_2^3\right\rangle\right) \\ &= 2l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^3 - x_2^2, x_2\right\rangle\right) + l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^3 - x_2^2, x_1 - x_2\right\rangle\right) = 6 + 2 = 8, \end{aligned}$$

since $x_1^4 - x_2^3 = x_1\left(x_1^3 - x_2^2\right) + x_2^2(x_1 - x_2)$, and

$$\begin{aligned} i(p_{12},p_{21}) &= l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^2 - x_2, x_1\right\rangle\right) = l^{\mathcal{A}}(\mathcal{A}/\langle x_1, x_2\rangle) = 1, \\ i(p_{12},p_{22}) &= l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^2 - x_2, x_1^4 - x_2^3\right\rangle\right) \\ &= 2l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^2 - x_2, x_2\right\rangle\right) + l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^2 - x_2, x_1 - x_2\right\rangle\right) \\ +l^{\mathcal{A}}\left(\mathcal{A}/\left\langle x_1^2 - x_2, x_1 + x_2\right\rangle\right) &= 2 + 1 + 1 = 4, \end{aligned}$$

since $x_1^4 - x_2^3 = x_1^2\left(x_1^2 - x_2\right) + x_2(x_1 - x_2)(x_1 + x_2)$. Because $l_{p_{ij}}(\mathcal{M}_i) = 1$ for $i = 1, 2$ and $j = 1, 2$, we conclude that $\dim_F(\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2) = 15$.

6. Blow-up at a Singular Point

The results in Theorem 4.3 and Theorem 5.2 provide a nice geometric interpretation of the dimension of the total root space at a point $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, i.e. at the ideal $m = \langle x_1 - \lambda_1, x_2 - \lambda_2, \ldots, x_n - \lambda_n \rangle$, in the spectrum of a multiparameter system as long as all varieties associated with the irreducible factors of the det f_i are non-singular at that point. In the singular case the results still hold, but are not so intuitive anymore. As a remedy we propose to blow up the affine space $\mathbb{A}^n$ at the point $\boldsymbol{\lambda}$ (see [13,18]).

Let us assume for simplicity that $\boldsymbol{\lambda} = (0, 0, \ldots, 0) \in \mathbb{A}^n$; the blow up at another point is obtained by an affine change of coordinates. Consider the subvariety

$$\mathbf{B} = \{(\mathbf{x}, \mathbf{y}) \,|x_i y_j = x_j y_i, 1 \le i, j \le n\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$

where $\mathbb{P}^{n-1}$ is the $n-1$-dimensional projective space over F, and the projection map

$$\pi : \mathbf{B} \to \mathbb{A}^n \times \mathbb{P}^{n-1} \xrightarrow{\text{proj}} \mathbb{A}^n$$

defined by $\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}$. The 'diagonal' map

$$\rho : \mathbb{A}^n / \{\boldsymbol{\lambda}\} \to \mathbf{B} / \pi^{-1}(\boldsymbol{\lambda}),$$

given by $\rho(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$, is easily seen to be inverse to the corresponding restriction of π, and moreover

$$\pi^{-1}(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}\} \times \mathbb{P}^{n-1} \cong \mathbb{P}^{n-1}.$$

For any line $L_{\mathbf{a}}$ passing through $\boldsymbol{\lambda}$, described by the parametric equation $\mathbf{x} = \mathbf{a}t$, one has $\rho(\mathbf{x}) = (\mathbf{x}, \mathbf{a})$ for $t \neq 0$, and $\pi^{-1}(L_{\mathbf{a}}) \cap (\{\boldsymbol{\lambda}\} \times \mathbb{P}^{n-1}) = \{(\mathbf{0}, \mathbf{a})\}$. We may extend ρ to

$$\rho_{\mathbf{a}} : \mathbb{A}^n \to \mathbf{B}$$

by defining $\rho(\boldsymbol{\lambda}) = (\mathbf{0}, \mathbf{a})$. Choosing various lines, i.e. as $\mathbf{a}$ varies through $\mathbb{P}^{n-1}$, we get all possible points of $\boldsymbol{\lambda} \times \mathbb{P}^{n-1}$, thus blowing up $\boldsymbol{\lambda}$ to $\{\boldsymbol{\lambda}\} \times \mathbb{P}^{n-1} = \pi^{-1}(\boldsymbol{\lambda})$. This gives a bijective correspondence between lines through $\boldsymbol{\lambda}$ in $\mathbb{A}^n$ and points of the 'exceptional divisor' $E = \pi^{-1}(\boldsymbol{\lambda})$. Observe that $\mathbf{B} = (\mathbf{B} \setminus \pi^{-1}(\boldsymbol{\lambda})) \cup \pi^{-1}(\boldsymbol{\lambda})$ and that $\mathbf{B} \setminus \pi^{-1}(\boldsymbol{\lambda}) \cong \mathbb{A}^n \setminus \{\boldsymbol{\lambda}\}$ is irreducible. Hence the closure $\overline{\mathbf{B} \setminus \{x_1 = 0\}}$ is irreducible as well. But $\pi^{-1}(L_{\mathbf{a}})$, and thus also $\pi^{-1}(L_{\mathbf{a}}) \cap \pi^{-1}(\boldsymbol{\lambda})$, are in $\overline{\mathbf{B} \setminus \pi^{-1}(\boldsymbol{\lambda})}$ for every $\mathbf{a} \in \mathbb{P}^{n-1}$, so that $\pi^{-1}(\boldsymbol{\lambda}) \subset \overline{\mathbf{B} \setminus \pi^{-1}(\boldsymbol{\lambda})}$. Therefore, $\overline{\mathbf{B} \setminus \pi^{-1}(\boldsymbol{\lambda})} = \mathbf{B}$ and $\mathbf{B}$ is irreducible.

If $Y \subset \mathbb{A}^n$ is a closed subvariety passing through $\boldsymbol{\lambda}$ then the 'blow-up' of a curve Y at $\boldsymbol{\lambda}$ is the closure $Y_{\mathbf{B}} = \overline{\pi^{-1}(Y \setminus \{\boldsymbol{\lambda}\})}$ in $\mathbf{B}$. It is called the 'strict transform' of Y under the blow-up $\pi : \mathbf{B} \to \mathbb{A}^n$:

$$\begin{array}{ccc} Y_{\mathbf{B}} & \longrightarrow & \mathbf{B} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi.} \\ Y & \longrightarrow & \mathbb{A}^n \end{array}$$

The total inverse image in $\mathbf{B}$ of the variety Y of an irreducible polynomial $p(\mathbf{x}) \in F[\mathbf{x}]$ passing through $\mathbf{0}$ in $\mathbb{A}^n$ is

$$\pi^{-1}(Y) = \{(\mathbf{x}, \mathbf{y}) \,|p(\mathbf{x}) = 0; x_i y_j = x_j y_i, 1 \le i, j \le n\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

The projective space $\mathbb{P}^{n-1}$ is covered by the n open sets $U_i = \{\mathbf{y}|y_i \neq 0\}$. If $y_1 \neq 0$, then we may set $y_1 = 1$ and use the remaining coordinates as affine parameters, so that

$$\pi^{-1}(Y) \cap (\mathbb{A}^n \times U_1) = \{(\mathbf{x}, \mathbf{z})\,|p(\mathbf{x}) = 0, \mathbf{x} = x_1(1, \mathbf{z})\} \subset \mathbb{A}^{2n-1}.$$

The substitution $\mathbf{x} = x_1(1, \mathbf{z})$, then gives

$$p(\mathbf{x}) = x_1^{e_p} q(x_1, \mathbf{z}),$$

where q is irreducible and not divisible by x_1 and e_p is the largest power of the maximal ideal m of $\mathcal{A}$ containing $p(\mathbf{x})$. The two irreducible components give $E = \{(\mathbf{x}, \mathbf{z})\,|\mathbf{x} = \mathbf{0}\}$ and $B_Y = \{(\mathbf{x}, \mathbf{z})\,|q(x_1, \mathbf{z}) = 0, \mathbf{x} = x_1(1, \mathbf{z})\}$, so that $B_Y \cap E = \{\mathbf{z}|q(0, \mathbf{z}) = 0\}$ is a variety of dimension $n - 2$.

6.1. Proposition. *Let $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_n(\mathbf{x})$ be a regular sequence in $F[\mathbf{x}]$ and let $Y_1, Y_2, \ldots, Y_n$ be the corresponding hypersurfaces in $\mathbb{A}^n$. Then $E \cap_{i=1}^n Y_{\mathbf{B}}$ is a finite set and*

$$i(\boldsymbol{\lambda};\, Y_1 Y_2 \ldots Y_n;\, \mathbb{A}^n) = \Pi_{i=1}^n e_{\boldsymbol{\lambda}}(Y_i) + \sum_{\boldsymbol{\mu} \in E} i(\boldsymbol{\mu};\, \mathbf{B}_{Y_1}, \mathbf{B}_{Y_2}, \ldots, \mathbf{B}_{Y_n};\, \mathbf{B}),$$

where $e_{\boldsymbol{\lambda}}(Y_i) = e_{p_i}$ is the largest power of the maximal ideal m of $\mathcal{A}$ containing $p_i(\mathbf{x})$.

Proof. If $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_n(\mathbf{x})$ form a regular sequence of polynomials in $F[\mathbf{x}]$, then the intersection multiplicity

$$i(\boldsymbol{\lambda};\, Y_1 Y_2 \ldots Y_n;\, \mathbb{A}^n) = l^{\mathcal{A}}\left(\mathcal{A}/\left\langle p_1, p_2, \ldots, p_n\right\rangle\right)$$

of the corresponding hypersurfaces $Y_1, Y_2, \ldots, Y_n$ at the point $\boldsymbol{\lambda}$ in $\mathbb{A}^n$ is finite, and hence $E \cap_{i=1}^n \mathbf{B}_{Y_i}$ is a finite set. Moreover, if $\pi : \mathbf{B} \to \mathbb{A}^n$ is the blow-up of $\mathbb{A}^n$ at the point $\boldsymbol{\lambda}$ with the exceptional divisor E, then the inverse image divisor of Y_i on B is $\pi^*(Y_i) = e_{\boldsymbol{\lambda}}(Y_i)E + \mathbf{B}_{Y_i}$ [8, p. 82]. The assertion now follows from [8, p. 124]. See also [18, IV.3]. $\square$

6.2. Example. As an illustration let us consider again the same 2-parameter system $\mathbf{f}$ in $\mathbb{A}^5$ as in Example 5.4 :

$$f_1(x_1, x_2) = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 \\ -1 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 1 & x_1 \end{pmatrix}, \quad f_2(x_1, x_2) = \begin{pmatrix} x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & 0 \\ 1 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix},$$

with the determinants $\det f_1 = (x_1^3 - x_2^2)(x_1^2 - x_2)$ and $\det f_2 = x_1(x_1^4 - x_2^3)$. Then, with $x_1 z = x_2$, one obtains $p_{11} = x_1^2(x_1 - z^2)$, $p_{12} = x_1(x_1 - z)$, $p_{21} = x_1$ and $p_{22} = x_1^3(x_1 - z^3)$, and we see that

$$\begin{aligned} i\,(Y_{11}, Y_{21}) &= e(Y_{11})e(Y_{21}) + i(Z_{11}, Z_{21}) = 2 + 0 = 2, \\ i\,(Y_{11}, Y_{22}) &= e(Y_{11})e(Y_{22}) + i(Z_{11}, Z_{22}) = 6 + 2 = 8, \\ i\,(Y_{12}, Y_{21}) &= e(Y_{12})e(Y_{21}) + i(Z_{12}, Z_{21}) = 1 + 0 = 1, \\ i\,(Y_{12}, Y_{22}) &= e(Y_{12})e(Y_{22}) + i(Z_{12}, Z_{22}) = 3 + 1 = 4. \end{aligned}$$

Since all $l_{p_{ij}}(M_i) = 1$ we conclude that $\dim_F M_1 \otimes_A M_2 = 15$, which coincides with the result obtained in Example 5.4.

7. (In)decomposability of the Associated Modules

7.1. Proposition. *If the module* $\mathcal{M} = \mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathcal{M}_n$ *is indecomposable then all the modules* $\mathcal{M}_i$ *are indecomposable.*

Proof. It is clear that if $\mathcal{M}_i$ is decomposable then $\mathcal{M}$ is decomposable since $\oplus$ and $\otimes_{\mathcal{A}}$ are distributive. $\square$

7.2. Proposition. *If* $m \in \sigma(\mathbf{f})$ *is simple then* $\mathcal{M}$*, and also each* $\mathcal{M}_i$*, is indecomposable.*

Proof. The decomposition of $\mathcal{M} = \mathcal{K} \oplus \mathcal{L}$ induces a decomposition of $\mathcal{M}^{(0)} = \mathcal{K}^{(0)} \oplus \mathcal{L}^{(0)}$, where $\mathcal{M}^{(0)} = \mathcal{M}/m\mathcal{M}$. Since $\mathcal{M}^{(0)}$ is one-dimensional one of $\mathcal{K}^{(0)}$ and $\mathcal{L}^{(0)}$ is 0. By the Nakayama Lemma then one of the modules $\mathcal{K}$ or $\mathcal{L}$ is 0. The proof for $\mathcal{M}_i$ is the same. $\square$

In general however, the converse of Proposition 7.1 does not hold :

7.3. Example. Consider the two-parameter system

$$f_1(\mathbf{x}) = \begin{pmatrix} x_1 & 0 & x_1 & x_2 \\ 0 & x_1 & 0 & 0 \\ 1 & 0 & x_1 & 0 \\ 0 & 1 & 0 & x_1 \end{pmatrix} \text{ and } f_2(\mathbf{x}) = \begin{pmatrix} x_2 & 0 & x_2 & x_1 \\ 0 & x_2 & 0 & 0 \\ 1 & 0 & x_2 & 0 \\ 0 & 1 & 0 & x_2 \end{pmatrix}.$$

Then $\det f_1(\mathbf{x}) = x_1^3(x_1 - 1)$ and $\det f_2(\mathbf{x}) = x_2^3(x_2 - 1)$. Because Δ_0 is invertible it follows by Theorem 5.1 that the system $\mathbf{f} = (f_1, f_2)$ is regular. All the irreducible factors of the determinantes $\det f_1$ and $\det f_2$ are linear. Conting the multiplicities of these irreducible factors it follows by Theorem 5.2 that the root subspace at $(0,0)$ has dimension 9.

One checks directly that

$$\mathcal{M}_1^{0(0)} = \left\{ e_{00} \otimes \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in F \right\}$$

and

$$\mathcal{M}_1^{0(1)} = \left\{ e_{00} \otimes \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} + e_{01} \otimes \begin{pmatrix} 0 \\ 0 \\ \varepsilon \\ \alpha \end{pmatrix} - e_{10} \otimes \begin{pmatrix} 0 \\ 0 \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta, \gamma, \delta, \varepsilon \in F \right\}.$$

Let $\mathcal{U}$ be the subcomodule of $\mathcal{M}_1^0$ 'generated' by the element

$$u = e_{00} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + e_{01} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - e_{10} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathcal{U} = \left\{ e_{00} \otimes \begin{pmatrix} \alpha \\ 0 \\ \beta \\ \gamma \end{pmatrix} + e_{01} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix} - e_{10} \otimes \begin{pmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{pmatrix} : \alpha, \beta, \gamma \in F \right\}.$$

Since $\mathcal{U}$ is the smallest subcomodule of $\mathcal{M}_1^0$ containing u it is indecomposable. Note also that $\mathcal{M}_1^{0(0)} \subset \mathcal{U}$ and therefore $\mathcal{U}^{(0)} = \mathcal{M}_1^{0(0)}$. Suppose that $\mathcal{M}_1^0$ is decomposable, i.e. $\mathcal{M}_1^0 = \mathcal{K} \oplus \mathcal{L}$. Then $\mathcal{U} = (\mathcal{U} \cap \mathcal{K}) \oplus (\mathcal{U} \cap \mathcal{L})$. From the relation $\mathcal{U}^{(0)} = \mathcal{M}_1^{0(0)}$. it follows that $\mathcal{U}^{(0)} = \mathcal{K}^{(0)} \oplus \mathcal{L}^{(0)}$ and hence either $\mathcal{K}^{(0)} = 0$ or $\mathcal{L}^{(0)} = 0$. But then it follows by the dual Nakayama Lemma [12] that either $\mathcal{K} = 0$ or $\mathcal{L} = 0$. Hence $\mathcal{M}_1^0$ is indecomposable. The proof that $\mathcal{M}_2^0$ is indecomposable is the same only the indices $i = 1$ and $i = 2$ are interchanged.

Let R be the root subspace of the associated system Γ at $\boldsymbol{\lambda} = (0,0)$. Obviously R is invariant for both Γ_1 and Γ_2. We use the main results of [15] to construct a basis $\mathcal{B}$ for R such that the pair (Γ_1, Γ_2) restricted to R is in the canonical form

$$\Gamma_1|_R = \begin{pmatrix} 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

and

$$\Gamma_2|_R = \begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Now it is easy to check that $R = R_1 \oplus R_2$, where R_1 is spanned by the first 8 vector of the basis $\mathcal{B}$ and R_2 is spanned by the last vector in $\mathcal{B}$, and that both R_1 and R_2 are invariant for Γ_1 and Γ_2. Then R is decomposable as a module over the algebra generated by Γ_1 and Γ_2. Since $R = \varepsilon^\dagger \left(\mathcal{M}_1^{(0)} \otimes^{\mathcal{B}_0} \mathcal{M}_2^{(0)} \right)$ by the main theorem of [10] it follows that $\mathcal{M}_1^{(0)} \otimes^{\mathcal{B}_0} \mathcal{M}_2^{(0)}$, and thus also $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$, is decomposable.

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