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Revised Version

A GROEBNER BASIS FOR THE 2×2 DETERMINANTAL IDEAL MOD t^2

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Abstract. In an earlier paper ([6]) we had begun a study of the components and dimensions of the spaces of $(k-1)$-th order jets of the classical determinantal varieties: these are the varieties $\mathcal{Z}_{r,k}^{m,n}$ obtained by considering generic $m \times n$ $(m \leq n)$ matrices over rings of the form $F[t]/(t^k)$, and for some fixed r, setting the coefficients of powers of t of all $r \times r$ minors to zero. In this paper, we consider the case where $r = k = 2$, and provide a Groebner basis for the ideal $\mathcal{I}_{2,2}^{m,n}$ which defines the tangent bundle to the classical 2×2 determinantal variety. We use the results of these Groebner basis calculations to describe the components of the varieties $\mathcal{Z}_{r,4}^{m,n}$ where r is arbitrary. (The components of $\mathcal{Z}_{r,2}^{m,n}$ and $\mathcal{Z}_{r,3}^{m,n}$ were already described in [6].)

1. Introduction

Let F be an algebraically closed field and $\mathbf{A}_F^k$ the affine space of dimension k over F. By a *variety* in $\mathbf{A}_F^k$ we will mean the zero set of a collection of polynomials over F in k variables; in particular, our varieties are not assumed irreducible. In the paper [6], we had begun a study of the components and dimensions of the following varieties that are very closely related to the classical determinantal varieties: Consider the truncated polynomial ring $F[t]/(t^k)$ $(k = 1, 2, 3, \dots)$, and let $X(t) = (x_{i,j}(t))_{i,j}$ be the generic $m \times n$ $(m \leq n)$ matrix over this ring; thus, the (i,j) entry of X is of the form $x_{i,j}(t) = x_{i,j}^{(0)} + x_{i,j}^{(1)} t + \cdots + x_{i,j}^{(k-1)} t^{k-1}$, where for various i, j and l the $x_{i,j}^{(l)}$ are variables. Let $\mathcal{I}_{r,k}^{m,n}$ be the ideal of $R = F[x_{i,j}^{(l)} \mid 1 \leq i \leq m,\ 1 \leq j \leq n,\ 0 \leq l < k]$ generated by the coefficients of powers of t in each $r \times r$ minor of the generic matrix $X(t)$, and define $\mathcal{Z}_{r,k}^{m,n} \subseteq \mathbf{A}_F^{nmk}$ to be the zero set of $\mathcal{I}_{r,k}^{m,n}$.

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(When $k = 1$, of course, we simply recover the classical determinantal varieties.)

These varieties are just the spaces of $(k-1)$-th order jets of the classical determinantal varieties. Our interest in them was sparked by the fact that a special case of these varieties had arisen in some previous work on commuting matrices ([9]).

We had shown in [6] that $\mathcal{Z}_{r,k}^{m,n}$ is irreducible when $r = m$, and reducible (with at least $1 + \lfloor k/2 \rfloor$ components) when $r < m$. When $r = 2 < m$, we had explicitly determined the components, and shown that there are exactly $1 + \lfloor k/2 \rfloor$ of them.

In this paper, we provide a Groebner basis of the ideal $\mathcal{I} := \mathcal{I}_{2,2}^{m,n}$ that defines the 2×2 determinantal variety mod t^2, for $2 \le m \le n$. Since these varieties may be interpreted as the tangent bundle over the classical 2×2 determinantal variety (see [6, §1] or [1, AG §16.2] for instance) they are of independent geometric interest. Note that in the classical case ($k = 1$) the defining minors already form a Groebner basis (see [10], also [3] and [8]). In our case ($k = 2$), however, Groebner bases have to contain other polynomials in addition to the defining ones (see Remark 2.3 ahead).

Essential portions of our Groebner basis calculations were done with the computer algebra system Singular ([5]). To simplify our computations, we actually compute a Groebner basis of a related ideal $\mathcal{I}_0$ and deduce a Groebner basis for $\mathcal{I}$ from this. The two Groebner bases together show that the ideal $\mathcal{I}_0$ is precisely the ideal of one of the two components of $\mathcal{Z}_{2,2}^{m,n}$. We then use this result, along with general facts about $\mathcal{Z}_{r,k}^{m,n}$ from [6], to provide a complete description of the components of $\mathcal{Z}_{r,k}^{m,n}$ in the case $k = 4$ and r arbitrary. (The components for $k = 2$ and $k = 3$ were already described in [6].)

2. Groebner basis Computations

We will first switch to an easier notation: we will write $x_{i,j}$ and $y_{i,j}$, $(1 \le i \le m,\ 1 \le j \le n)$, instead of $x_{i,j}^{(0)}$ and $x_{i,j}^{(1)}$, respectively, and we will write $R = F[x_{i,j},\ y_{i,j} \mid 1 \le i \le m, 1 \le j \le n]$. Expanding each 2×2 minor of the generic matrix $(x_{i,j} + t y_{i,j})_{i,j}$, we see that our ideal $\mathcal{I}$ is defined by the family of polynomials $\delta_{[i,j][k,l]}$ and $\epsilon_{[i,j][k,l]}$, where

$$\delta_{[i,j][k,l]} = det \begin{pmatrix} x_{i,k} & x_{i,l} \\ x_{j,k} & x_{j,l} \end{pmatrix} = x_{i,k}x_{j,l} - x_{i,l}x_{j,k}, \tag{1}$$

and $\epsilon_{[i,j][k,l]}$ is its "polarization," i.e.,

$$\epsilon_{[i,k][j,l]} = x_{i,k}y_{j,l} + y_{i,k}x_{j,l} - x_{i,l}y_{j,k} - y_{i,l}x_{j,k}. \tag{2}$$

We will consider the graded reverse lexicographic order (*grevlex*) on the monomials on R given by the following scheme: $y_{1,1} > y_{1,2} > \cdots > y_{1,n} > y_{2,1} > \cdots > y_{2,n} > \cdots > y_{m,n} > x_{1,1} > x_{1,2} > \cdots > x_{1,n} > x_{2,1} > \cdots > x_{2,n} > \cdots > x_{m,n}$. For a polynomial $f \in R$, we will write $lm(f)$ for its leading monomial. We introduce the following auxiliary polynomials:

$$\rho_{[i,j,k][p,q,r]} = det \begin{pmatrix} y_{i,p} & y_{i,q} & y_{i,r} \\ y_{j,p} & y_{j,q} & y_{j,r} \\ x_{k,p} & x_{k,q} & x_{k,r} \end{pmatrix}, \tag{3}$$

$$\lambda_{[i,j,k][p,q,r]} = det \begin{pmatrix} y_{i,p} & y_{i,q} & x_{i,r} \\ y_{j,p} & y_{j,q} & x_{j,r} \\ y_{k,p} & y_{k,q} & x_{k,r} \end{pmatrix}, \tag{4}$$

$$\psi_{[i,j,k][p,q,r]} = det \begin{pmatrix} y_{i,p} & y_{i,q} & y_{i,r} \\ y_{j,p} & y_{j,q} & y_{j,r} \\ y_{k,p} & y_{k,q} & y_{k,r} \end{pmatrix}. \tag{5}$$

Note that at this stage the relative order of the indices i, j, k and p, q, r can be arbitrary; for instance, $\rho_{[1,3,2][1,2,3]}$ will stand for the determinant of a matrix whose middle row consists of $y_{3,*}$ and third row consists of $x_{2,*}$.

We will assume that the characteristic of F is not 2 in what follows:

Theorem 2.1. *A Groebner basis for $\mathcal{I} = \mathcal{I}_{2,2}^{m,n}$ $(2 \leq m \leq n)$ with respect to the grevlex ordering described above consists of the five families of polynomials $\Delta = \{\delta_{[i,j][k,l]} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\}$, $\mathcal{E} = \{\epsilon_{[i,j][k,l]} \mid 1 \leq i < j \leq m,\ 1 \leq k < l \leq n\}$, $\mathcal{R} = \{\rho_{[i,j,k][p,q,r]} \mid 1 \leq i < j \leq k \leq m,\ 1 \leq p < q < r \leq n\}$, $\Lambda = \{\lambda_{[i,j,k][p,q,r]} \mid 1 \leq i < j < k \leq m,\ 1 \leq p < q \leq r \leq n\}$, and $\Xi = \{x_{s,t}\psi_{[i,j,k][p,q,r]} \mid 1 \leq s \leq m,\ 1 \leq t \leq n,\ 1 \leq i < j < k \leq m,\ 1 \leq p < q < r \leq n\}$.*

Remark 2.2. When $m = 2$, it is to be understood above that a Groebner basis for $\mathcal{I}_{2,2}^{2,n}$ consists of all $\delta_{[1,2][k,l]}$ and all $\epsilon_{[1,2][k,l]}$ with $1 \leq k < l \leq n$, and all $\rho_{[1,2,2][p,q,r]}$ with $1 \leq p < q < r \leq n$. (In particular, when $m = n = 2$, this is a special case of [6, Theorem 3.3] that the defining polynomials $\delta_{[1,2][1,2]}$ and $\epsilon_{[1,2][1,2]}$ form a Groebner basis for $\mathcal{I}_{2,2}^{2,2}$. This is an exceptional situation, see Remark 2.3 ahead.)

Proof. We will write $\mathcal{G}$ for the family of polynomials in the statement of the theorem. We first show that the polynomials $\rho_{[i,j,k][p,q,r]}$, $\lambda_{[i,j,k][p,q,r]}$, and $x_{s,t}\psi_{[i,j,k][p,q,r]}$, with indices as in the theorem, indeed are in $\mathcal{I}$. Actually, we will show more: all polynomials $\rho_{[i,j,k][p,q,r]}$, $\lambda_{[i,j,k][p,q,r]}$, and $x_{s,t}\psi_{[i,j,k][p,q,r]}$, *with no restriction on the indices,* are in $\mathcal{I}$. We have:

$$\rho_{[i,j,k][p,q,r]} - \rho_{[i,k,j][p,q,r]} + \rho_{[i,k,j][p,q,r]} - \rho_{[j,k,i][p,q,r]} + \rho_{[j,k,i][p,q,r]} + \rho_{[i,j,k][p,q,r]} = 2\rho_{[i,j,k][p,q,r]}.$$

However, it is easy to see that each of the three pairs in the left hand side is in $\mathcal{I}$: for instance,

$$\rho_{[i,j,k][p,q,r]} - \rho_{[i,k,j][p,q,r]} = y_{i,p}\epsilon_{[j,k][q,r]} - y_{i,q}\epsilon_{[j,k][p,r]} + y_{i,r}\epsilon_{[j,k][p,q]}.$$

Similarly, the second pair can be rewritten as a sum of products of $y_{k,*}$ and $\epsilon_{[i,j][*,*]}$, while the third pair, which equals, $\rho_{[j,k,i][p,q,r]} - \rho_{[j,i,k][p,q,r]}$, can be rewritten as a sum of products of $y_{j,*}$ and $\epsilon_{[i,k][*,*]}$. Since the characteristic of F is not 2 by assumption, we find that indeed $\rho_{[i,j,k][p,q,r]} \in \mathcal{I}$.

A similar computation with the sum $\lambda_{[i,j,k][p,q,r]} - \lambda_{[i,j,k][p,r,q]} + \lambda_{[i,j,k][p,r,q]} - \lambda_{[i,j,k][q,r,p]} + \lambda_{[i,j,k][q,r,p]} + \lambda_{[i,j,k][p,q,r]}$ shows that all possible polynomials $\lambda_{[i,j,k][p,q,r]}$ are in $\mathcal{I}$.

As for the polynomials $x_{s,t}\psi_{[i,j,k][p,q,r]}$, we expand the determinant of the matrix

$$\begin{pmatrix} x_{s,t} & y_{s,p} & y_{s,q} & y_{s,r} \\ x_{i,t} & y_{i,p} & y_{i,q} & y_{i,r} \\ x_{j,t} & y_{j,p} & y_{j,q} & y_{j,r} \\ x_{k,t} & y_{k,p} & y_{k,q} & y_{k,r} \end{pmatrix}$$

in two ways, once along the top row, and once along the last column. Equating the two, we find

$$x_{s,t}\psi_{[i,j,k][p,q,r]} - y_{s,p}\lambda_{[i,j,k][q,r,t]} + y_{s,q}\lambda_{[i,j,k][p,r,t]} - y_{s,r}\lambda_{[i,j,k][p,q,t]} = -y_{s,r}\lambda_{[i,j,k][p,q,t]} + y_{i,r}\lambda_{[s,j,k][p,q,t]} - y_{j,r}\lambda_{[s,i,k][p,q,t]} + y_{k,r}\lambda_{[s,i,j][p,q,t]}.$$

Since we have already shown that all possible $\lambda_{[i,j,k][p,q,r]}$ are in $\mathcal{I}$, it follows that $x_{s,t}\psi_{[i,j,k][p,q,r]} \in \mathcal{I}$.

Remark 2.3. It is already clear that, unlike in the classical case, the generating polynomials in Δ and $\mathcal{E}$ do not suffice as a Groebner basis for $\mathcal{I}$ (except when $m = n = 2$). Namely, the leading monomial of any $\delta_{[i,j][k,l]}$ $(i < j,\ k < l)$ is $x_{i,l}x_{j,k}$, while the leading monomial of $\epsilon_{[i,j][k,l]}$ $(i < j,\ k < l)$ is $x_{i,k}y_{j,l}$. On the other hand, the leading monomial of $\rho_{[i,j,k][p,q,r]}$ $(i < j \leq k,\ p < q < r)$ is $y_{i,r}y_{j,q}x_{k,p}$, and such a leading term is not divisible by the leading monomial of any $\delta_{[i,j][k,l]}$ or $\epsilon_{[i,j][k,l]}$.

The proof that the given families form a Groebner basis of $\mathcal{I}$ follows from the following theorem. This result helps us reducing the complexity of the Groebner basis computations (see Remark 2.7).

Theorem 2.4. *Assume $m \geq 3$, and let $\mathcal{I}_0$ denote the ideal of $R = F[x_{i,j}, y_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ defined by the polynomials of the families Δ, $\mathcal{E}$, and $\Psi = \{\psi_{[i,j,k|p,q,r]} \mid 1 \leq i < j < k \leq m,\ 1 \leq p < q < r \leq n\}$. Then a Groebner basis of $\mathcal{I}_0$ with respect to the grevlex ordering above consists of the polynomials in the families Δ, $\mathcal{E}$, $\mathcal{R}$, Λ, and Ψ.*

We will prove this theorem ahead, but we will now show how the proof of Theorem 2.1 follows from Theorem 2.4. In order to show that the polynomials in $\mathcal{G}$ form a Groebner basis for $\mathcal{I}$ we need to show that all possible pairs of S-polynomials $S(\alpha,\ \beta)$, where α and β range over $\mathcal{G}$, can be written as $\sum f_\gamma \gamma$, f_γ a polynomial in R, and γ ranging over $\mathcal{G}$, with $lm(f_\gamma \gamma) \leq lm(S(\alpha,\ \beta))$ (see e.g. [4, Chapter 2, §9, Theorem 3]). We will adopt the notation of [4] and write $S(\alpha,\ \beta) \rightarrow_{\mathcal{G}} 0$ when this happens. We assume first that $m \geq 3$ and we write $\mathcal{H}$ for the family of polynomials of Theorem 2.4 above, i.e. $\mathcal{H} = \Delta \cup \mathcal{E} \cup \mathcal{R} \cup \Lambda \cup \Psi$. Since the polynomials of $\mathcal{H}$ form a Groebner basis for $\mathcal{I}_0$, and since $\mathcal{I} \subset \mathcal{I}_0$, we must have $S(\alpha, \beta) \rightarrow_{\mathcal{H}} 0$ for all $\alpha, \beta \in \mathcal{G}$ by the generalized division algorithm (see [4, Chapter 2, §3, Theorem 3] for instance). Write $\mathcal{U}$ for the family of polynomials in Δ, $\mathcal{E}$, $\mathcal{R}$ and Λ. Then, for $\alpha, \beta \in \mathcal{G}$ we write

$$S(\alpha, \beta) = \sum_{\gamma \in \mathcal{U}} f_\gamma \gamma + \sum_{\psi \in \Psi} f_\psi \psi$$

with $lm(f_\gamma \gamma),\ lm(f_\psi \psi) \leq lm(S(\alpha,\ \beta))$.

We break up each f_ψ as $g_\psi + h_\psi$, where h_ψ contains all the monomials of f_ψ that only involve the $y_{i,j}$ or are constant, so g_ψ contains all the monomials of f_ψ divisible by at least one of the $x_{i,j}$. Since every monomial of every polynomial in $\mathcal{G}$ is divisible by some $x_{i,j}$, every monomial of $S(\alpha, \beta)$ will also be divisible by some $x_{i,j}$. By choice, every monomial in every $\gamma \in \mathcal{U}$ and every monomial in every g_ψ is divisible by some $x_{i,j}$ while no monomial of any of the polynomials $h_\psi \psi$ is divisible by any $x_{i,j}$. Hence, setting all $x_{i,j} = 0$, we find $\sum_{\psi \in \Psi} h_\psi \psi = 0$. We thus have the rewrite

$$S(\alpha, \beta) = \sum_{\gamma \in \mathcal{U}} f_\gamma \gamma + \sum_{\psi \in \Psi} g_\psi \psi.$$

Moreover, $lm(g_\psi) \le lm(f_\psi)$ as the monomials of g_ψ come from f_ψ, so $lm(g_\psi \psi) \le lm(f_\psi \psi) \le lm(S(\alpha,\ \beta))$. Further breaking up each g_ψ as $\sum m$, where m runs through the monomials of g_ψ, and writing $m\psi$ as $m' x_{i,j}\psi$ for some $x_{i,j}$ that necessarily divides m, we get a rewrite of $S(\alpha, \beta)$ in terms of the polynomials in $\mathcal{G}$. Moreover, for any such monomial m appearing in a g_ψ, $lm(m) \le lm(g_\psi)$, so $lm(m' x_{i,j}\psi) = lm(m\psi) \le lm(g_\psi \psi) \le lm(S(\alpha,\ \beta))$. Thus, $S(\alpha,\ \beta) \to_{\mathcal{G}} 0$ as desired.

In the case where $m = 2$, we may embed the ambient ring $R = F[x_{i,j}, y_{i,j} \mid 1 \le i \le 2, 1 \le j \le n]$ in the ring $R' = F[x_{i,j}, y_{i,j} \mid 1 \le i \le 3, 1 \le j \le n]$ and work there. Given α and β from the families in Remark 2.2, we have a rewrite $S(\alpha, \beta) = \sum_{\gamma \in \mathcal{H}} f_\gamma \gamma$, where the family $\mathcal{H}$ is from the larger ring R', and $lm(f_\gamma \gamma) \le lm(S(\alpha, \beta))$. Since the polynomial $S(\alpha, \beta)$ does not involve any of the variables $x_{3,j}$ or $y_{3,j}$, the sum of all the monomials on the right side of the equation $S(\alpha, \beta) = \sum_{\gamma \in \mathcal{H}} f_\gamma \gamma$ that are divisible by some $x_{3,j}$ or $y_{3,j}$ must be zero. Throwing these out, we find that we are left precisely with a sum of polynomials of the form $g_\gamma \gamma$, where γ comes from one of the families in Remark 2.2, and where g_γ is obtained from the corresponding f_γ by throwing away any monomial divisible by some $x_{3,j}$ or $y_{3,j}$. It follows that $lm(g_\gamma) \le lm(f_\gamma)$ so $lm(g_\gamma \gamma) \le lm(f_\gamma \gamma) \le lm(S(\alpha, \beta))$. We thus have the desired rewrite showing that indeed the polynomials from the families in Remark 2.2 form a Groebner basis for $\mathcal{I}^{2,n}_{2,2}$.

The rest of the proof of Theorem 2.1 therefore consists of proving Theorem 2.4.

Proof of Theorem 2.4:

We need to show that all possible pairs of S-polynomials $S(\alpha,\ \beta)$, where α and β range over these families in $\mathcal{H}$, can be written as $\sum f_\gamma \gamma$, f_γ a polynomial in R, and γ ranging over $\mathcal{H}$, with $lm(f_\gamma \gamma) \le lm(S(\alpha,\ \beta)$. We will dispose of some computations right away by invoking the theorem proved by Sturmfels (and others, see [3, 8, 10]) cited earlier about the classical determinantal ideals $\mathcal{I}^{m,n}_{r,1}$: the defining determinantal polynomials for the classical determinantal varieties form a Groebner basis for $\mathcal{I}^{m,n}_{r,1}$ under the grevlex order, with the mn variables ordered in any manner such that the leading monomial of any $r \times r$ determinant is the product of the entries on the diagonal from the top right to the bottom left corner. We call this product the *antidiagonal term.* (Note that although Sturmfels' proof is for one particular (lexicographic) order, the proof works for any order that ensures that the leading monomial is always the antidiagonal term. The key to Sturmfels' proof, and the only place where the specific order comes into the picture, is in Lemma 6 of [10], and this in turn depends only on the fact that the leading term of any $r \times r$ determinant is the antidiagonal term.)

Since the polynomials in Δ are just the defining polynomials of the classical 2×2 determinantal variety in the variables $x_{i,j}$, and since our order on the $x_{i,j}$ ensures that the leading term of any 2×2 determinant is always the antidiagonal term, we find $S = S(\delta_{[i,j][k,l]},\ \delta_{[i',j'][k',l']}) \rightarrow_{\Delta} 0$. Trivially therefore, $S \rightarrow_{\mathcal{H}} 0$ as well. Similarly, we find by Sturmfels' result that $S = S(\psi_{[i,j,k][p,q,r]},\ \psi_{[i',j',k'][p',q',r']}) \rightarrow_{\Psi} 0$. Trivially therefore, $S \rightarrow_{\mathcal{H}} 0$.

We next dispose of S-polynomials of pairs of polynomials of the form $S(\mathcal{R}, \mathcal{R})$, $S(\mathcal{R}, \Psi)$, $S(\Lambda, \Lambda)$, and $S(\Lambda, \Psi)$ as well. (Here $S(\mathcal{A}, \mathcal{B})$ stands for the set of all S-polynomials $S(\alpha, \beta)$ for $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.) Given a pair $\rho_{[i,j,k][p,q,r]}$ and $\rho_{[i',j',k'][p',q',r']}$ whose S-polynomial we need, we arrange our variables in a $2m \times n$ matrix with the upper $m \times n$ block consisting of the $y_{i,j}$ and the lower $m \times n$ block consisting of the $x_{i,j}$, and then choose the submatrix M consisting of the y variables from the rows i, j, i', j', and columns p, q, r, p', q', and r' and the x variables from the row k, k' and columns p, q, r, p', q', and r'. (The understanding here is that when some of the rows and columns defined by the two ρ polynomials are equal we use the variables from these rows and columns just once.) Working within this submatrix M, we find once again that the variable order on the $y_{i,j}$ and $x_{i,j}$ that we have chosen guarantees that the leading term of any 3×3 determinant of M is always the antidiagonal term, so Sturmfels' result shows that $S = S(\rho_{[i,j,k][p,q,r]}, \rho_{[i',j',k'][p',q',r']})$ can be rewritten as $\sum_{\gamma} f_{\gamma}\gamma$, with γ ranging over all 3×3 minors of M, and $lm(f_{\gamma}\gamma) \leq lm(S)$. Such minors are of three kinds: (1) All three rows consist of y variables: these are just the polynomials ψ that are already in $\mathcal{H}$. (2) The first two rows consist of y variables and the last row consists of x variables: these are just the polynomials ρ that are already in $\mathcal{H}$. (3) The first row consists of y variables and the last two rows consist of x variables: such a γ can be expanded (along the top row) as a sum $y_s\delta_s$ where y_s ranges over the y variables in the first row, and of course, $\delta_s \in \mathcal{H}$. Since the monomials of $y_s\delta_s$ all appear in γ, i.e., no cancelation occurs, it follows that $lm(y_s\delta_s) \leq lm(\gamma)$. So $lm(f_\gamma y_s \delta_s) \leq lm(f_\gamma \gamma) \leq lm(S)$. This implies that $S \rightarrow_{\mathcal{H}} 0$.

To compute the S-polynomial of a pair $\rho_{[i,j,k][p,q,r]}$ and $\psi_{[i',j',k'][p',q',r']}$, we choose the submatrix M consisting of the y variables from the rows i, j, i', j', k' and columns p, q, r, p', q', and r' and the x variables from the row k and columns p, q, r, p', q', and r'. Working within this submatrix M, Sturmfels' result shows that $S = S(\rho_{[i,j,k][p,q,r]}, \psi_{[i',j',k'][p',q',r']})$ can be rewritten as $\sum_\gamma f_\gamma \gamma$, with γ ranging over all 3×3 determinants of M, and $lm(f_\gamma \gamma) \leq lm(S)$. Such determinants are of two kinds: (1) The first kind has all three rows consisting of the y variables: these are just the polynomials ψ that are already part of $\mathcal{H}$. (2) The second kind has the top two rows consisting of the y variables and the last row consisting of the x variables: these are just the polynomials ρ that already part of $\mathcal{H}$. It follows therefore that $S \rightarrow_{\mathcal{H}} 0$.

An analogous reduction works for $S(\Lambda, \Lambda)$ and $S(\Lambda, \Psi)$. We first arrange our variables in a $m \times 2n$ matrix with the left $m \times n$ block consisting of the $y_{i,j}$ and the right $m \times n$ block consisting of the $x_{i,j}$, and then choose an appropriate submatrix M as above. The order on the $y_{i,j}$ and $x_{i,j}$ that we have chosen once again guarantees that the leading term of any 3×3 determinant of M is always the antidiagonal term. Sturmfels' result now applies to M and ensures analogously that all $S(\Lambda, \Lambda) \rightarrow_{\mathcal{H}} 0$ and $S(\Lambda, \Psi) \rightarrow_{\mathcal{H}} 0$.

Finally, all computations of $S(\Delta, \Psi)$ can be eliminated since the leading terms of any $\alpha \in \Delta$ and $\beta \in \Psi$ are obviously relatively prime, so $S(\alpha, \beta) \rightarrow_{\mathcal{H}} 0$ anyway ([4, Chapter 2, §9, Proposition 4]).

We now proceed to show that the remaining eight cases of S-polynomials reduce to zero as well: we need to check the S-polynomials of elements of the following pairs of families: $S(\Delta, \mathcal{E})$, $S(\Delta, \mathcal{R})$, $S(\Delta, \Lambda)$, $S(\mathcal{E}, \mathcal{E})$, $S(\mathcal{E}, \mathcal{R})$, $S(\mathcal{E}, \Lambda)$, $S(\mathcal{E}, \Psi)$, and $S(\mathcal{R}, \Lambda)$. For each pair of families in this list, there is a large number of cases to check, depending on the relative position of the variables in the two polynomials whose S-polynomial is being computed.

We first observe the following:

Lemma 2.5. *Suppose one has a rewrite* $S(\alpha', \beta') = \sum_{\gamma'} f'_{\gamma'} \gamma'$, *with* $\gamma' \in \mathcal{H}$ *and with* $lm(f'_{\gamma'}\gamma') \leq lm(S(\alpha', \beta'))$. *Suppose that* α' *and* β' *are defined by various products of* $x_{i,j}$ *and* $y_{i,j}$ *arising from a fixed grid* $T' = \{(i'_\xi, j'_\eta) \mid 1 \leq \xi, \eta \leq p\}$ *with* $i'_1 < i'_2 < \cdots < i'_p$ *and*

$j'_1 < j'_2 < \cdots < j'_p$. Suppose, too, that all the variables that appear in all the polynomials $f'_{\gamma'}$ and γ' in the rewrite above also arise from the same grid T'. Let $T = \{(i_\xi, j_\eta) \mid 1 \leq \xi, \eta \leq p\}$ be another grid with $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_p$. Write α, β, γ, f_γ for the corresponding polynomials obtained by substituting x_{i_ξ, j_η} for $x_{i'_\xi, j'_\eta}$ and y_{i_ξ, j_η} for $y_{i'_\xi, j'_\eta}$. Then $S(\alpha, \beta) = \sum_\gamma f_\gamma \gamma$, and $lm(f_\gamma \gamma) \leq lm(S(\alpha, \beta))$. Moreover, $\gamma \in \mathcal{H}$.

Proof. The map from the subrings $W' = F[x_{i',j'}, y_{i',j'} \mid (i', j') \in T']$ to $W = F[x_{i,j}, y_{i,j} \mid (i, j) \in T]$ that sends $x_{i'_\xi, j'_\eta}$ to x_{i_ξ, j_η} and $y_{i'_\xi, j'_\eta}$ to y_{i_ξ, j_η} is a ring homomorphism that preserves the monomial order. Moreover, $\mathcal{H}$, by definition, is closed under such maps. □

We will use this lemma as follows: While computing all possible S-polynomials $S(\alpha, \beta)$ where α and β, for instance, come from the families Δ and $\mathcal{R}$ respectively, it is sufficient to consider just the submatrix $(x_{i,j} + ty_{i,j})_{i,j}$, $1 \leq i, j \leq 4$, and to show that for all possible pairs $\alpha \in \Delta$ and $\beta \in \mathcal{R}$ coming from this 4×4 submatrix, $S(\alpha, \beta) \to_{\mathcal{H}'} 0$, where $\mathcal{H}'$ is the subset of $\mathcal{H}$ consisting of all polynomials arising from this submatrix. This is sufficient because if the variables in $\alpha \in \Delta$ and $\beta \in \mathcal{R}$, for general α and β, are disjoint from each other, then their leading terms will be relatively prime, and we will have $S(\alpha, \beta) {\to}_{\mathcal{H}} 0$ anyway ([4, Chapter 2, §9, Proposition 4]). Otherwise, the rows that define α and β must have at least one member in common, and similarly for the columns that define α and β, hence, the variables in α and β arise from at most $2 + 3 - 1 = 4$ rows and columns. It follows then that α and β are the images of some α' and β' defined on the upper left 4×4 grid under a map of the sort described in the lemma, and the lemma then applies. (As an example, to rewrite $S(\delta_{[10,13][3,5]}, \rho_{[5,7,10][5,7,10]})$ it is sufficient to rewrite $S(\delta_{[3,4][1,2]}, \rho_{[1,2,3][2,3,4]})$ in terms of polynomials in Δ, $\mathcal{E}$, $\mathcal{R}$, Λ, and Ψ that only involve variables from the upper left 4×4 grid.) Similar considerations apply to other pairs of families, and we find that while computing $S(\Delta, \mathcal{E})$ and $S(\mathcal{E}, \mathcal{E})$ it is sufficient to work with a generic 3×3 matrix $(x_{i,j} + ty_{i,j})_{i,j}$, while computing $S(\Delta, \mathcal{R})$, $S(\Delta, \Lambda)$, $S(\mathcal{E}, \mathcal{R})$, $S(\mathcal{E}, \Lambda)$, and $S(\mathcal{E}, \Psi)$, it is sufficient to work with a 4×4 matrix, and while computing $S(\mathcal{R}, \Lambda)$, it is sufficient to work with a 5×5 matrix.

We have done these computations using the computational algebra package Singular ([5]) and have checked that $S(\alpha, \beta) {\to}_{\mathcal{H}} 0$ for all pairs.

$$
\begin{aligned}
S(\delta_{[1,2][1,2]}, \epsilon_{[1,2][2,3]}) &= (y_{2,2})\delta_{[1,2][1,3]} + (-y_{2,1})\delta_{[1,2][2,3]} \\
&\quad + (-x_{2,3})\epsilon_{[1,2][1,2]} + (x_{2,2})\epsilon_{[1,2][1,3]} \\
S(\delta_{[1,2][1,2]}, \epsilon_{[1,3][2,3]}) &= (y_{3,2})\delta_{[1,2][1,3]} + (-y_{3,1})\delta_{[1,2][2,3]} + (-y_{1,3})\delta_{[2,3][1,2]} \\
+(y_{1,2})\delta_{[2,3][1,3]} &+ (-y_{1,1})\delta_{[2,3][2,3]} + (-x_{2,3})\epsilon_{[1,3][1,2]} + (x_{2,2})\epsilon_{[1,3][1,3]} \\
S(\delta_{[1,2][1,2]}, \epsilon_{[2,3][1,2]}) &= (y_{2,2})\delta_{[1,3][1,2]} + (-y_{1,2})\delta_{[2,3][1,2]} \\
&\quad + (-x_{3,2})\epsilon_{[1,2][1,2]} + (x_{2,2})\epsilon_{[1,3][1,2]} \\
S(\delta_{[1,2][1,2]}, \epsilon_{[2,3][1,3]}) &= (-y_{3,1})\delta_{[1,2][2,3]} + (y_{2,3})\delta_{[1,3][1,2]} + (y_{2,1})\delta_{[1,3][2,3]} \\
+(-y_{1,3})\delta_{[2,3][1,2]} &+ (-y_{1,1})\delta_{[2,3][2,3]} + (-x_{3,2})\epsilon_{[1,2][1,3]} + (x_{2,2})\epsilon_{[1,3][1,3]} \\
S(\delta_{[1,2][1,3]}, \epsilon_{[2,3][1,2]}) &= (y_{3,1})\delta_{[1,2][2,3]} + (y_{2,2})\delta_{[1,3][1,3]} + (-y_{2,1})\delta_{[1,3][2,3]} \\
+(-y_{1,2})\delta_{[2,3][1,3]} &+ (y_{1,1})\delta_{[2,3][2,3]} + (-x_{3,3})\epsilon_{[1,2][1,2]} + (x_{2,3})\epsilon_{[1,3][1,2]} \\
S(\delta_{[1,3][1,2]}, \epsilon_{[1,2][2,3]}) &= (y_{2,2})\delta_{[1,3][1,3]} + (-y_{2,1})\delta_{[1,3][2,3]} + (y_{1,3})\delta_{[2,3][1,2]} \\
+(-y_{1,2})\delta_{[2,3][1,3]} &+ (y_{1,1})\delta_{[2,3][2,3]} + (-x_{3,3})\epsilon_{[1,2][1,2]} + (x_{3,2})\epsilon_{[1,2][1,3]}
\end{aligned}
$$

TABLE 1. $S(\Delta, \mathcal{E})$

(See Remark 2.7 about the complexity of computations involved.) The process essentially consists of dividing $S(\alpha, \beta)$ by the polynomials in $\mathcal{H}$, using the generalized division algorithm, as described, for instance, in [4, Chapter 2, §3].

The cases $S(\Delta, \mathcal{E})$ and $S(\mathcal{E}, \mathcal{E})$ are particularly simple to enumerate, since in each case, we only have to work with a 3×3 grid. The six relations shown in Table 1 (in which the equality as well as the order relation between the leading monomials on the right side and the left side can easily be verified by hand) take care of all $S(\alpha, \beta)$ with $\alpha \in \Delta$ and $\beta \in \mathcal{E}$. (For instance, $S(\delta_{[1,3][1,2]}, \epsilon_{[1,3][2,3]})$ can be obtained from $S(\delta_{[1,2][1,2]}, \epsilon_{[1,2][2,3]})$ with an application of Lemma 2.5.)

Similarly, the nine relations shown in Table 2 take care of all $S(\alpha, \beta)$ with $\alpha, \beta \in \mathcal{E}$. (Note that the polynomials in $\mathcal{R}$ and Λ do not appear in the $S(\Delta, \Delta)$ and $S(\Delta, \mathcal{E})$ calculations, but do appear in $S(\mathcal{E}, \mathcal{E})$ calculations.)

The number of separate cases to consider in the remaining cases $S(\Delta, \mathcal{R})$, $S(\Delta, \Lambda)$, $S(\mathcal{E}, \mathcal{R})$, $S(\mathcal{E}, \Lambda)$, $S(\mathcal{E}, \Psi)$, and $S(\mathcal{R}, \Lambda)$, even though we are working in a matrix of maximum size 5×5, is simply too large to be able to report every computation in this paper. (Again, we

$$\begin{array}{rcl}
S(\epsilon_{[1,2][1,2]}, \epsilon_{[1,2][1,3]}) & = & (-y_{2,1})\epsilon_{[1,2][2,3]} + (-1)\rho_{[1,2,2][1,2,3]} \\
S(\epsilon_{[1,2][1,2]}, \epsilon_{[1,3][1,2]}) & = & (-y_{1,2})\epsilon_{[2,3][1,2]} + (-1)\lambda_{[1,2,3][1,2,2]} \\
S(\epsilon_{[1,2][1,2]}, \epsilon_{[1,3][1,3]}) & = & (-y_{2,1})\epsilon_{[1,3][2,3]} + (-y_{1,2})\epsilon_{[2,3][1,3]} \\
+(y_{1,1})\epsilon_{[2,3][2,3]} & + & (-1)\rho_{[1,2,3][1,2,3]} + (-1)\lambda_{[1,2,3][1,2,3]} \\
S(\epsilon_{[1,2][1,3]}, \epsilon_{[1,2][2,3]}) & = & (y_{1,3})\delta_{[1,2][1,2]} + (-y_{1,2})\delta_{[1,2][1,3]} \\
 & & + (y_{1,1})\delta_{[1,2][2,3]} + (x_{1,3})\epsilon_{[1,2][1,2]} \\
S(\epsilon_{[1,2][1,3]}, \epsilon_{[1,3][1,2]}) & = & (y_{3,1})\epsilon_{[1,2][2,3]} + (-y_{1,3})\epsilon_{[2,3][1,2]} \\
 & & + (1)\rho_{[1,2,3][1,2,3]} + (-1)\lambda_{[1,2,3][1,2,3]} \\
S(\epsilon_{[1,2][1,3]}, \epsilon_{[1,3][1,3]}) & = & (-y_{1,3})\epsilon_{[2,3][1,3]} + (-1)\lambda_{[1,2,3][1,3,3]} \\
S(\epsilon_{[1,3][1,2]}, \epsilon_{[2,3][1,2]}) & = & (y_{3,1})\delta_{[1,2][1,2]} + (-y_{2,1})\delta_{[1,3][1,2]} \\
 & & + (y_{1,1})\delta_{[2,3][1,2]} + (x_{3,1})\epsilon_{[1,2][1,2]} \\
S(\epsilon_{[1,3][1,3]}, \epsilon_{[2,3][2,3]}) & = & (y_{3,1})\delta_{[1,2][2,3]} + (-y_{2,1})\delta_{[1,3][2,3]} \\
+(y_{1,3})\delta_{[2,3][1,2]} & + & (-y_{1,2})\delta_{[2,3][1,3]} + (2y_{1,1})\delta_{[2,3][2,3]} \\
+(-x_{3,3})\epsilon_{[1,2][1,2]} & + & (x_{3,2})\epsilon_{[1,2][1,3]} + (x_{2,3})\epsilon_{[1,3][1,2]} \\
S(\epsilon_{[1,3][2,3]}, \epsilon_{[2,3][1,3]}) & = & (y_{3,2})\delta_{[1,2][1,3]} + (-y_{2,3})\delta_{[1,3][1,2]} \\
+(-y_{2,1})\delta_{[1,3][2,3]} & + & (y_{1,2})\delta_{[2,3][1,3]} + (x_{3,2})\epsilon_{[1,2][1,3]} + (-x_{2,3})\epsilon_{[1,3][1,2]}
\end{array}$$

TABLE 2. $S(\mathcal{E}, \mathcal{E})$

refer to Remark 2.7 for the complexity of computations involved.) The complete computations are available from the authors ([7]): for each pair of polynomials α and β, we present the polynomial $S(\alpha, \beta)$, its leading monomial listed explicitly, each divisor $\gamma \in \mathcal{H}$ and each quotient f_γ in the rewrite $S(\alpha, \beta) = \sum_\gamma f_\gamma \gamma$ (with $\gamma \in \mathcal{H}$) listed explicitly, and the leading monomial of each product $f_\gamma \gamma$ listed explicitly. If desired, the reader can easily check from these listings the equality $S(\alpha, \beta) = \sum_\gamma f_\gamma \gamma$ (as well as the relationship $\text{LM}(f_\gamma \gamma) \leq \text{LM} S(\alpha, \beta)$) for any pair α, β. □

We immediately have the following:

Corollary 2.6. *Both $\mathcal{I}$ and $\mathcal{I}_0$ are radical.*

Proof. This is clear, since the leading terms of a Groebner basis for the two ideals are square free (see the proof of [6, Theorem 3.4] for instance). □

In the next section we will recall some facts about $\mathcal{Z}_{2,2}^{m,n}$ that will immediately show that the ideal $\mathcal{I}_0$ is one of the two primary components of $\mathcal{I}$. This will be used in describing the components of $\mathcal{Z}_{r,k}^{m,n}$ in the case $k = 4$.

Remark 2.7. If one were to check directly, i.e., without referring to Theorem 2.4, that the elements of $\mathcal{G}$ form a Groebner basis for $\mathcal{I}$, one could eliminate computation of $S(\Xi, \Xi)$ by using Sturmfels' result for the classical case, but one would have to compute polynomials in $S(\mathcal{R}, \Xi)$ and $S(\Lambda, \Xi)$. Each of these involve computing with a 6×6 matrix. The number of different S-polynomials to consider is of the order of 10^6. By contrast, the most complicated case for $\mathcal{I}_0$ is $S(\mathcal{R}, \Lambda)$, which involves a 5×5 matrix. In this case we only have to consider on the order of 10^4 S-polynomials: a hundred-fold reduction in complexity. Still, the massive undertaking of checking the reduction of all S-polynomials could not be done without help of a computer.

3. The components for $k = 4$

We recall that in [6], we determined the components for $\mathcal{Z}_{r,k}^{m,n}$ for $k = 2$ and $k = 3$, for all values of $r < m$. (Recall from [6] that when $r = m$ the variety $\mathcal{Z}_{m,k}^{m,n}$ is irreducible for all values of k. Also, in the case $k = 3$ and $r < m$, we described the components for all but finitely many values of (m, n).) We will use our Groebner basis for $\mathcal{I}$ and $\mathcal{I}_0$ from the previous section to determine the components of $\mathcal{Z}_{r,k}^{m,n}$ in the case $k = 4$, for all values of $r < m$.

We need to first recall some theorems from [6]. We also need to revert to the notation of writing our generic matrix as $(x_{i,j}(t))_{i,j}$, where $x_{i,j}(t) = x_{i,j}^{(0)} + x_{i,j}^{(1)}t + \cdots + x_{i,j}^{(k-1)}t^{k-1}$. We recall from the introduction that $R = F[x_{i,j}^{(l)} \mid 1 \le i \le m,\ 1 \le j \le n,\ 0 \le l < k]$.

We have the following basic decomposition:

Theorem 3.1. ([6, Theorem 2.8]) *The variety $\mathcal{Z}_{r,k}^{m,n}$ (for $r \ge 2$) is the union of two subvarieties Z_0 and Z_1. The variety Z_0 is the closure of any of the open sets $U_{i,j}$ ($1 \le i \le m$, $1 \le j \le n$), where $x_{i,j}^{(0)}$ is nonzero. Moreover, it is the closure of the open set U where all $x_{i,j}^{(0)}$ are nonzero. Alternatively, the variety Z_0 is the union of the components of $\mathcal{Z}_{r,k}^{m,n}$ that correspond to minimal primes of $\mathcal{I}$ that do not contain some (hence any) $x_{i,j}^{(0)}$. Such components always exist, and are in one-to-one correspondence with the components of the variety*

$\mathcal{Z}_{r-1,k}^{m-1,n-1}$. *The correspondence preserves the codimension (in* $\mathbf{A}^{mnk}$ *and* $\mathbf{A}^{(m-1)(n-1)k}$ *respectively) of the components, and in fact,* Z_0 *is birational to* $\mathcal{Z}_{r-1,k}^{m-1,n-1} \times \mathbf{A}^{(m+n-1)k}$. *The variety* Z_1 *is the subvariety of* $\mathcal{Z}_{r,k}^{m,n}$ *where all* $x_{i,j}^{(0)}$ *are zero, and is isomorphic to* $\mathcal{Z}_{r,k-r}^{m,n} \times \mathbf{A}^{mn(r-1)}$ *when* $k > r$*, and isomorphic to* $\mathbf{A}^{mn(k-1)}$ *when* $k \le r$. Z_1 *will be wholly contained in* Z_0 *precisely when there are no minimal primes of* $\mathcal{I}$ *that contain some (hence all)* $x_{i,j}^{(0)}$.

We also have the following, which explicitly describes the correspondence between Z_0 and $\mathcal{Z}_{r-1,k}^{m-1,n-1}$ and is the basis for the proof of the theorem above:

Theorem 3.2. *(*[6, Theorem 2.3]*) Assume* $r \ge 2$. *Let* $z_{i,j}^{(l)}$, $1 \le i \le m-1$, $1 \le j \le n-1$, $0 \le l < k$ *be a new set of variables, and write* T *for the ring* $F[z_{i,j}^{(l)} \mid 1 \le i \le m-1,\ 1 \le j \le n-1, 0 \le l < k]$, *and* T' *for the ring* $F[z_{i,j}^{(l)}, x_{1,n}^{(l)}, \dots, x_{m,n}^{(l)}, x_{m,1}^{(l)}, \dots, x_{m,n-1}^{(l)} \mid 1 \le i \le m-1,\ 1 \le j \le n-1, 0 \le l < k]$. *Also, write* Z *for the* $m-1 \times n-1$ *matrix* $(z_{i,j}(t))_{i,j}$ *over* $T[t]/(t^k)$, *where* $z_{i,j}(t) = \sum_{l=0}^{k-1} z_{i,j}^{(l)} t^l$. *We have an isomorphism*

$$R[(x_{m,n}^{(0)})^{-1}] \cong T'[(x_{m,n}^{(0)})^{-1}],$$

given by

$$\begin{aligned} f\colon\quad x_{i,n}^{(l)} &\to x_{i,n}^{(l)}\ 1 \le i \le m \\ x_{m,j}^{(l)} &\to x_{m,j}^{(l)}\ 1 \le j \le n-1 \\ x_{i,j}^{(l)} &\to z_{i,j}^{(l)} + q_{i,j}^{(l)}(x_{m,j}^{(p)}, x_{i,n}^{(r)}, x_{m,n}^{(s)}, (x_{m,n}^{(0)})^{-1}),\ 0 \le p, r < l,\ 1 \le s < l, \\ &\quad \textit{for } 1 \le i \le m-1,\ 1 \le j \le n-1,\ 0 \le l < k. \end{aligned}$$

(Here, the $q_{i,j}^{(l)}$ *are polynomials in the indicated variables.)*

Under this isomorphism, the localization of $\mathcal{I}$ *at* $x_{m,n}^{(0)}$ *corresponds to the localization of the ideal* $\mathcal{I}_{r-1,k}^{m-1,n-1} T'$ *at* $x_{m,n}^{(0)}$, *where* $\mathcal{I}_{r-1,k}^{m-1,n-1}$ *is the ideal of* T *determined by the coefficients of powers of* t *of the various* $(r-1)\times(r-1)$ *minors of the matrix* Z. *Moreover, this induces a one-to-one correspondence between the prime ideals* P *of* R *that are minimal over* $\mathcal{I}$ *and do not contain* $x_{m,n}^{(0)}$ *and the prime ideals* Q *of* T *that are minimal over* $\mathcal{I}_{r-1,k}^{m-1,n-1}$. *If* P *corresponds to* Q *then the codimension of* P *in* R *equals the codimension of* Q *in* T.

We recall that the motivation behind this theorem is row-reduction on the matrix $(x_{i,j}(t))_{i,j}$ under the assumption that $x_{m,n}^{(0)}$ (and hence

$x_{m,n}(t)$) is invertible, and that the point of the isomorphism is the assignment

$$z_{i,j}(t) = x_{i,j}(t) - x_{m,j}(t)x_{i,n}(t)x_{m,n}^{-1}(t) \tag{6}$$

for $1 \leq i \leq m-1,\ 1 \leq j \leq n-1$.

We now determine the primary decomposition of $\mathcal{I} = \mathcal{I}_{2,2}^{m,n}$. We recall from [6, Theorem 7.1] that $\mathcal{Z}_{2,2}^{m,n}$ has precisely two components, namely the subvarieties Z_0 and Z_1 of Theorem 3.1 above. Write J_0 and J_1 for the ideals of Z_0 and Z_1 respectively. Since $\mathcal{I}$ is radical (Corollary 2.6 above), $\mathcal{I} = J_0 \cap J_1$. Since the component Z_1 is the subvariety defined by $\mathcal{I}$ and all the $x_{i,j}^{(0)}$, and since $\mathcal{I}$ is already contained in the (prime) ideal generated by the $x_{i,j}^{(0)}$, we find that J_1 is precisely the ideal generated by the $x_{i,j}^{(0)}$.

The component Z_0 is the closure of the open set where $x_{m,n}^{(0)}$ is nonzero, so J_0 is the radical of the ideal $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$. By standard Groebner basis facts (see [11, Lemma 12.1]–note that the hypothesis on the basis being reduced is not necessary there), a Groebner basis of $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$ with respect to the chosen monomial order is given by factoring out all powers of $x_{m,n}^{(0)}$ from every polynomial in a Groebner basis of $\mathcal{I}$. (Note that the fact that $x_{m,n}$ is the variables with the least weight in the order of the variables is important here.) The polynomials in our Groebner basis of $\mathcal{I}$ that are divisible by $x_{m,n}^{(0)}$ (note the change of notation between §2 and §3) are the various $x_{m,n}^{(0)}\psi_{[i,j,k][p,q,r]}$, so factoring $x_{m,n}^{(0)}$ from them yields the polynomials in Ψ. Once all polynomials of the family Ψ are in the Groebner basis of $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$, one no longer needs the other polynomials $x_{s,t}^{(0)}\psi_{[i,j,k][p,q,r]}$, $(s,t) \neq (m,n)$, in the Groebner basis, since $lm(x_{s,t}^{(0)}\psi_{[i,j,k][p,q,r]}) = x_{s,t}^{(0)} lm(\psi_{[i,j,k][p,q,r]})$. It follows that a Groebner basis of $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$ is given by the polynomials in Δ, $\mathcal{E}$, $\mathcal{R}$, Λ, and Ψ. Since these polynomials are all in $\mathcal{I}_0$ and generate $\mathcal{I}_0$, we find that $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$ equals the ideal $\mathcal{I}_0$ of the previous section. Moreover, $\mathcal{I}_0$ is radical (Corollary 2.6), so J_0 is precisely $\mathcal{I}_0$. (This shows as well that $\mathcal{I}_0$ is prime.)

We split this off for future reference as:

Proposition 3.3. *The ideal $\mathcal{I} = \mathcal{I}_{2,2}^{m,n}$ is the intersection of the two prime ideals $\mathcal{I}_0$ and the ideal generated by all $x_{i,j}^{(0)}$. Moreover, $\mathcal{I}_0 = (\mathcal{I} : (x_{m,n}^{(0)})^\infty)$.*

We now proceed to determine the components of $\mathcal{Z}_{r,k}^{m,n}$ when $k = 4$. We recall from [6, Proposition 6.3] that when $k < r$, the subvariety

Z_1 is wholly contained in Z_0, and the components of $\mathcal{Z}_{r,k}^{m,n}$ and their codimensions in $\mathbf{A}^{mnk}$ are hence determined by those of $\mathcal{Z}_{r-1,k}^{m-1,n-1}$ and their codimensions in $\mathbf{A}^{(m-1)(n-1)k}$. Thus, it is sufficient to determine the components of $\mathcal{Z}_{4,4}^{m,n}$, $\mathcal{Z}_{3,4}^{m,n}$, and $\mathcal{Z}_{2,4}^{m,n}$. But of these, the last has already been determined in [6, Theorem 5.1], which we state below as:

Theorem 3.4. *(*[6, Theorem 5.1]*) The variety $\mathcal{Z}_{2,4}^{m,n}$ has three components X_0, X_1, and X_2. The component X_0 is the closure of the subset where some $x_{i,j}^{(0)}$ is nonzero, and has codimension $4(m-1)(n-1)$ in $\mathbf{A}^{4mn}$. The component X_1 is the closure, in the subvariety where all $x_{i,j}^{(0)}$ are zero, of the open set where some $x_{i,j}^{(1)}$ is nonzero; this component has codimension $2(m-1)(n-1)+mn$. The third component X_2 is the subvariety where all $x_{i,j}^{(0)}$ and $x_{i,j}^{(1)}$ are zero; this has codimension $2mn$.*

Finally, we recall the notation $\mathbf{u}_i(t)$ for the i-th row of the matrix $(x_{i,j}(t))_{i,j}$: this is an element of $(R[t]/t^k)^n$. We write $\mathbf{u}_i(t) = \sum_{l=0}^{k-1} \mathbf{u}_i^{(l)} t^l$, so the various $\mathbf{u}_i^{(l)}$ are row vectors from R^n, and we refer to $\mathbf{u}_i^{(l)}$ as being "of degree l."

We begin with $\mathcal{Z}_{3,4}^{m,n}$:

Theorem 3.5. *In the variety $\mathcal{Z}_{3,4}^{m,n}$ (for all values of m, n with $3 < m \leq n$), the subvariety Z_1 of Theorem 3.1 above is wholly contained in Z_0. Therefore the components of $\mathcal{Z}_{3,4}^{m,n}$ are in one-to-one correspondence with the components of $\mathcal{Z}_{2,4}^{m-1,n-1}$ as in Theorem 3.1, and by Theorem 3.4 these have codimensions $4(m-2)(n-2)$, $2(m-2)(n-2)+(m-1)(n-1)$, and $2(m-1)(n-1)$ in $\mathbf{A}^{4mn}$.*

Proof. Let us write X_0', X_1', and X_2' for the three components of Z_0 corresponding to the three components X_0, X_1, and X_2 of $\mathcal{Z}_{2,4}^{m-1,n-1}$. We will prove that the subvariety Z_1 is contained in the component X_2', by showing that the ideal of X_2' is contained in the ideal of Z_1. We will need the primary decomposition $\mathcal{I}_{2,2}^{m,n}$ that we just established to prove the containment of ideals.

The ideal of Z_1 is easy to determine: the equations that define Z_1 are those of $I_{3,4}^{m,n}$ along with all $x_{i,j}^{(0)} = 0$. The equations of $I_{3,4}^{m,n}$ are

$$\mathbf{u}_{i_1}(t) \wedge \mathbf{u}_{i_2}(t) \wedge \mathbf{u}_{i_3}(t) = \mathbf{0}$$

for all $1 \leq i_1 < i_2 < i_3 \leq m$, which expands to the following four equations:

$$\begin{aligned}
\mathbf{u}_{i_1}^{(0)} \wedge \mathbf{u}_{i_2}^{(0)} \wedge \mathbf{u}_{i_3}^{(0)} &= 0,\\
\sum_{\circlearrowright} \mathbf{u}_{i_1}^{(1)} \wedge \mathbf{u}_{i_2}^{(0)} \wedge \mathbf{u}_{i_3}^{(0)} &= 0,\\
\sum_{\circlearrowright} \mathbf{u}_{i_1}^{(2)} \wedge \mathbf{u}_{i_2}^{(0)} \wedge \mathbf{u}_{i_3}^{(0)} + \sum_{\circlearrowright} \mathbf{u}_{i_1}^{(1)} \wedge \mathbf{u}_{i_2}^{(1)} \wedge \mathbf{u}_{i_3}^{(0)} &= 0,\\
\sum_{\circlearrowright} \mathbf{u}_{i_1}^{(3)} \wedge \mathbf{u}_{i_2}^{(0)} \wedge \mathbf{u}_{i_3}^{(0)} + \sum_{\circlearrowright} \mathbf{u}_{i_1}^{(2)} \wedge \mathbf{u}_{i_2}^{(1)} \wedge \mathbf{u}_{i_3}^{(0)} + \mathbf{u}_{i_1}^{(1)} \wedge \mathbf{u}_{i_2}^{(1)} \wedge \mathbf{u}_{i_3}^{(1)} &= 0
\end{aligned}$$

for all $1 \leq i_1 < i_2 < i_3 \leq m$. (Here, the notation $\sum_{\circlearrowright} \mathbf{u}_{i_1}^{(d_1)} \wedge \mathbf{u}_{i_2}^{(d_2)} \wedge \mathbf{u}_{i_3}^{(d_3)}$ stands for sums of all possible terms $\mathbf{u}_{i_1}^{(d_1)} \wedge \mathbf{u}_{i_2}^{(d_2)} \wedge \mathbf{u}_{i_3}^{(d_3)}$, with i_1, i_2, i_3 fixed and the d_i permuted.) It is clear that on setting all $\mathbf{u}_{i_1}^{(0)}$ to zero, we are just left with the classical determinantal equations $\mathbf{u}_{i_1}^{(1)} \wedge \mathbf{u}_{i_2}^{(1)} \wedge \mathbf{u}_{i_3}^{(1)} = 0$ on the variables $x_{i,j}^{(1)}$, and the ideal defined by these is well-known to be prime (and in particular, radical). Hence, the ideal of Z_1 is given by the $x_{i,j}^{(0)}$ and the various $\mathbf{u}_{i_1}^{(1)} \wedge \mathbf{u}_{i_2}^{(1)} \wedge \mathbf{u}_{i_3}^{(1)}$ for $1 \leq i_1 < i_2 < i_3 \leq m$.

As for the ideal of X_2', let us first compute the ideal of X_2 in the ring $T = F[z_{i,j}^{(l)} \mid 1 \leq i \leq m-1,\ 1 \leq j \leq n-1, 0 \leq l < 4]$ – see Theorem 3.4 and the remarks following that theorem for the setup. The variety X_2 is defined by $I_{2,4}^{m-1,n-1}$ along with all $z_{i,j}^{(0)} = 0$ and all $z_{i,j}^{(1)} = 0$. The ideal $I_{2,4}^{m-1,n-1}$ is generated by the following equations (here the various $\mathbf{w}_i$ are the rows of the $(m-1) \times (n-1)$ generic matrix $(z_{i,j}(t)\,)_{i,j}$):

$$\begin{aligned}
\mathbf{w}_{i_1}^{(0)} \wedge \mathbf{w}_{i_2}^{(0)} &= 0,\\
\sum_{\circlearrowright} \mathbf{w}_{i_1}^{(1)} \wedge \mathbf{w}_{i_2}^{(0)} &= 0,\\
\sum_{\circlearrowright} \mathbf{w}_{i_1}^{(2)} \wedge \mathbf{w}_{i_2}^{(0)} + \sum_{\circlearrowright} \mathbf{w}_{i_1}^{(1)} \wedge \mathbf{w}_{i_2}^{(1)} &= 0,\\
\sum_{\circlearrowright} \mathbf{w}_{i_1}^{(3)} \wedge \mathbf{w}_{i_2}^{(0)} + \sum_{\circlearrowright} \mathbf{w}_{i_1}^{(2)} \wedge \mathbf{w}_{i_2}^{(1)} &= 0.
\end{aligned}$$

Clearly, setting all $z_{i,j}^{(0)} = 0$ and all $z_{i,j}^{(1)} = 0$ makes these equations vanish, so the X_2 is defined by all $z_{i,j}^{(0)}$ and all $z_{i,j}^{(1)}$. This is a radical ideal, so this is the ideal of X_2, and it is of course prime.

Write J for this ideal. We now compute the image of the ideal JT' in the ring $R[(x_{m,n}^{(0)})^{-1}]$ under the isomorphism of Theorem 3.4. Recall

Equation (6) that $z_{i,j}(t) = x_{i,j}(t) - x_{m,j}(t)x_{i,n}(t)x_{m,n}^{-1}(t)$ for $1 \le i \le m-1,\ 1 \le j \le n-1$, which we rewrite as

$$x_{m,n}(t)z_{i,j}(t) = x_{i,j}(t)x_{m,n}(t) - x_{m,j}(t)x_{i,n}(t).$$

Comparing coefficients of the constant term and the t term on both sides, we find

$$x^{(0)}_{m,n}z^{(0)}_{i,j} = x^{(0)}_{i,j}x^{(0)}_{m,n} - x^{(0)}_{m,j}x^{(0)}_{i,n} \tag{7}$$

and

$$x^{(1)}_{m,n}z^{(0)}_{i,j} + x^{(0)}_{m,n}z^{(1)}_{i,j} = x^{(1)}_{i,j}x^{(0)}_{m,n} + x^{(0)}_{i,j}x^{(1)}_{m,n} - x^{(1)}_{m,j}x^{(0)}_{i,n} - x^{(0)}_{m,j}x^{(1)}_{i,n}. \tag{8}$$

Recall that $x^{(0)}_{m,n}$ is invertible in T'. Thus, Equation (7) shows that $x^{(0)}_{m,n}z^{(0)}_{i,j} = \delta_{[i,m][j,n]}$, so the ideal generated by just the $z^{(0)}_{i,j}$ goes over to the ideal generated by the various $\delta_{[i,m][j,n]}$, $1 \le i \le m-1$, $1 \le j \le n-1$. Equation (8) then shows that $x^{(0)}_{m,n}z^{(1)}_{i,j} = \epsilon_{[i,m][j,n]} - x^{(1)}_{m,n}\delta_{[i,m][j,n]}/x^{(0)}_{m,n}$, so the ideal generated by both the $z^{(0)}_{i,j}$ and the $z^{(1)}_{i,j}$ goes over to the ideal generated by the various $\delta_{[i,m][j,n]}$ and the various $\epsilon_{[i,m][j,n]}$, $1 \le i \le m-1$, $1 \le j \le n-1$.

Write $\widetilde{K}$ for this ideal. We claim that $\widetilde{K}$ contains *all* $\delta_{[i,p][j,q]}$ and $\epsilon_{[i,p][j,q]}$, for all $1 \le i,p \le m$ and $1 \le j,q \le n$. We first consider the case $p = m$ and $q < n$ for given $i < p$ and $j < q$. Then consider the following matrix

$$\begin{pmatrix} x_{i,j}(t) & x_{i,q}(t) & x_{i,n}(t) \\ x_{m,j}(t) & x_{m,q}(t) & x_{m,n}(t) \\ x_{m,j}(t) & x_{m,q}(t) & x_{m,n}(t) \end{pmatrix}.$$

The determinant of this matrix is zero, and expanding along the bottom row and considering the constant term of the determinant, we find $x^{(0)}_{m,n}\delta_{[i,m][j,q]}$ is in $\widetilde{K}$, and since $x^{(0)}_{m,n}$ is invertible, $\delta_{[i,m][j,q]}$ is in $\widetilde{K}$. Similarly, considering the t term and using the fact that $\delta_{[i,m][j,q]}$ is already in $\widetilde{K}$, we find $\epsilon_{[i,m][j,q]}$ is in $\widetilde{K}$ as well. A similar proof with the n-th column duplicated shows that $\delta_{[i,p][j,n]}$ and $\epsilon_{[i,p][j,n]}$ are in $\widetilde{K}$ for $i < p < m$ and $j < n$. Finally, given $1 \le i,p < m$ and $1 \le j,q < n$, consider the matrix

$$\begin{pmatrix} x_{i,j}(t) & x_{i,q}(t) & x_{i,n}(t) \\ x_{p,j}(t) & x_{p,q}(t) & x_{p,n}(t) \\ x_{m,j}(t) & x_{m,q}(t) & x_{m,n}(t) \end{pmatrix}.$$

Expanding the determinant in two ways, once along the bottom row and once along the top row, then comparing the constant terms and the coefficients of t on both sides and invoking what we just proved in the special cases where $p = m$ or $q = n$, we find that $\delta_{[i,p][j,q]}$ and $\epsilon_{[i,p][j,q]}$ are in $\widetilde{K}$ as claimed.

Note that $\widetilde{K}$ is radical since it is the image of JT'. Note too that its generators are those of $\mathcal{I}_{2,2}^{m,n}$. It follows from Theorems 3.1 and 3.4 that the ideal of X_2 is the pullback of $\mathcal{I} = \mathcal{I}_{2,2}^{m,n}$ under localization at $x_{m,n}^{(0)}$, i.e., the ideal $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$. We have seen in the computation of the primary decomposition of $\mathcal{I}$ above (see Proposition 3.3) that $(\mathcal{I} : (x_{m,n}^{(0)})^\infty)$ is precisely the ideal $\mathcal{I}_0$ of the previous section. The generators of $\mathcal{I}_0$ are the polynomials in Δ, $\mathcal{E}$ and Ψ, all of which are clearly in the ideal of Z_1, proving that the subvariety Z_1 is contained in the component X_2'. The last statement of the theorem is now just an application of Theorems 3.1 and 3.4.

□

Theorem 3.6. *The variety $\mathcal{Z}_{4,4}^{m,n}$, for all values of m, n with $4 < m \leq n$, except possibly the pairs $(5,5)$, $(5,6)$, $(5,7)$, $(5,8)$, $(5,9)$, $(5,10)$, $(5,11)$, $(5,12)$, $(6,6)$, $(6,7)$, and $(6,8)$, has four components. One is the subvariety Z_1, which has codimension mn in $\mathbf{A}^{4mn}$, while the other three are those that arise from Z_0 (as in Theorem 3.5), and have codimensions $4(m-3)(n-3)$, $2(m-3)(n-3) + (m-2)(n-2)$, and $2(m-2)(n-2)$ in $\mathbf{A}^{4mn}$.*

Proof. This is an easy consequence of Theorem 3.5 and dimension-counting. By Theorem 3.1, the subvariety Z_1 is simply $\mathbf{A}^{3mn}$, which of course has codimension mn in $\mathbf{A}^{4mn}$. The components of the subvariety Z_0 on the other hand are in one-to-one correspondence with those of $\mathcal{Z}_{3,4}^{m-1,n-1}$, so by Theorem 3.5 above, Z_1 has three components, with codimensions $4(m-3)(n-3)$, $2(m-3)(n-3) + (m-2)(n-2)$, and $2(m-2)(n-2)$ in $\mathbf{A}^{4mn}$. (Note that the correspondence between the components of Z_0 and those of $\mathcal{Z}_{3,4}^{m-1,n-1}$ preserves codimensions in $\mathbf{A}^{4mn}$ and $\mathbf{A}^{4(m-1)(n-1)}$ respectively.) It is easy to see that for all $m, n \geq 5$ except for the pairs indicated, mn is smaller than any of these three codimensions coming from Z_0, so Z_1 is a separate component. □

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