ROOT VECTORS FOR GEOMETRICALLY SIMPLE MULTIPARAMETER EIGENVALUES

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Abstract. A class of multiparameter eigenvalue problems involving (generally) non self-adjoint and unbounded operators is studied. Bases for lower order root subspaces, at geometrically simple eigenvalues of Fredholm type of arbitrary finite index, are computed in terms of the underlying multiparameter system.

1. Introduction

We consider an *n-parameter system* ($n \geq 2$) of the form

$$W_i(\boldsymbol{\lambda}) = \sum_{j=0}^{n} A_{ij}\lambda_j,\ i \in \underline{n}, \tag{1.1}$$

where A_{ij} $(j \in \underline{n})$ are bounded linear operators acting on a Hilbert space H_i $(i \in \underline{n})$ over the complex numbers, A_{i0} are closed densely defined operators with domain $\mathcal{D}(A_{i0}) \subseteq H_i$, $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ are parameters, and $\underline{n} = \{1, 2, \ldots, n\}$. Eigenvalue problems

$$W_i(\boldsymbol{\lambda}) x_i = 0,\ x_i \neq 0, \boldsymbol{\lambda} \in \mathbb{C}\backslash\{0\}, \tag{1.2}$$

arise in various applications: classically they arise in the theory of boundary value problems for partial differential equations after separation of variables. For background on multiparameter eigenvalue problems we refer to [2, 6, 7, 20, 22]. We assume that H_i are separable Hilbert spaces over $\mathbb{C}$ to allow for complex eigenvalues which can occur even when all the A_{ij} are self-adjoint. However, for $\boldsymbol{\lambda} \in \mathbb{R}^{n+1}$ all the calculations can be performed over $\mathbb{R}$.

In applications to boundary value problems for partial differential equations the A_{ij}, $i, j \in \underline{n}$, are multiplication operators and A_{i0}, $i \in \underline{n}$, are differential operators. Then solutions of the boundary value problems are given in terms of Fourier type series over a complete system of eigenfunctions and associated functions. In the abstract setup, the completeness is naturally studied in Hilbert space tensor product $H = \bigotimes_{i=1}^{n} H_i$ by means of certain determinantal operators Δ_j. For $j = 0, 1, \ldots, n$, the operator Δ_j is (up to the sign) the tensor determinant of the array $[A_{ik}]_{i=1,k=0}^{n\ \ \ n}$ with the j-th column omitted. In order to proceed we introduce certain regularity and solvability assumptions. If $\dim H_i < \infty$ for all i then the existence of a linear combination of Δ_j, which is an invertible operator, suffices [2]. In infinite dimensions and when unbounded operators are involved (as in the applications to boundary value problems) several sets of assumptions mostly for self-adjoint cases are used in the literature (see e.g. [2, 6, 20, 22]). Here we use a setup introduced in [4] which encompasses most of the others and does not require self-adjointness. After these assumptions are made the completeness problem reduces to

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the description – in terms of the original operators A_{ij} – of bases for a joint root subspace of a certain commuting n-tuple of operators Γ_j, $j = 0, 1, \ldots, n-1$. (If Δ_n is one-to-one then $\Gamma_j = \Delta_n^{-1}\Delta_j$.) Specifically, for $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{C}^n$ the root subspace $\mathcal{R}_l$, $l = 0, 1, 2, \ldots$, is the subspace of vectors annihilated by all the products of $(l+1)$-tuples of operators from the set $\{\Gamma_j - \lambda_j I\}_{j=0}^{n-1}$. The joint eigenspace $\mathcal{R}_0$ is described in terms of elements of the spaces H_i and maps A_{ij} in [2]. The root subspace $\mathcal{R}_l$ is described and its basis is constructed for $l = 1$ in [4], and for general l in [9] (see also [10]). However, for general eigenvalues $\boldsymbol{\lambda}$ and general l, the construction of bases via the cotensor product [9, Thms. 5.1 and 5.2] is technically involved. So, it is natural to consider eigenvalue problems (1.2) with additional properties which yield simpler description of a basis for $\mathcal{R}_l$. For instance, in [3] Binding proved a completeness result for real eigenvalues of elliptic multiparameter systems. His results yield a method for construction of bases for the corresponding root subspaces. In [15] constructed bases for root subspaces of nonderogatory eigenvalues in a finite-dimensional setting. (See [19] for a discussion of numerical implementation of results in [15].) In applications to boundary value problems all the eigenvalues of (1.2) are *geometrically simple*, i.e. $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})) = 1$ for all i. Here $\mathcal{N}(A)$ is the nullspace of operator A. In the paper we consider such eigenvalues and construct bases for the corresponding root subspaces $\mathcal{R}_2$ and $\mathcal{R}_3$. We remark that our results can be derived from the construction in [9], however, we do not follow that route. Instead we use techniques of linear algebra as in [13], where our results were first proved in finite-dimensional setup. We also remark that our method can be extended for general l but proofs similar to those in [13] require long technical calculations and we do not reproduce them here. Note that our results are new even in finite-dimensions. In infinite-dimensions we assume that $W_i(\boldsymbol{\lambda})$ are Fredholm operators of arbitrary (finite) index [21] (Fredholmness with index 0 is automatic in finite-dimensions). Hence we allow dimensions $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})^*)$ to be arbitrary positive integers. With slight modifications our approach would apply even when some (or all) of $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})^*)$ were infinite. The two-parameter version of our results for general l is given in [5]. Further connections of our results with boundary value problems will appear elsewhere.

One might observe looking at the main results on root subspaces in this paper and in [4, 5, 13] that there is a close relation between the root subspaces and multiplicative structure of the algebra generated by the commuting maps Γ_j. This is indeed so. Suppose that $\mathcal{R} = \cup_{l=0}^{\infty}\mathcal{R}_l$ is the root subspace at an eigenvalue of (1.1), that $\mathcal{R}$ is finite-dimensional, and that $\mathcal{A}$ is the commutative subalgebra generated by the restrictions $\Gamma_j|_{\mathcal{R}}$, $j = 0, 1, \ldots, n-1$, in the algebra of all linear maps on $\mathcal{R}$. Then $\mathcal{R}$ is a module over $\mathcal{A}$. Moreover, if the corresponding eigenvalue is geometrically simple then $\mathcal{A}$ and $\mathcal{R}$ are isomorphic as $\mathcal{A}$-modules, i.e., $\mathcal{R}$ is a free $\mathcal{A}$-module of rank 1. We refer to [11, 18] for proofs of these and further results on multiparameter eigenvalue problems from a point of view of commutative algebra and algebraic geometry.

We conclude the introduction with a short setup of the paper. In next section we introduce our regularity and solvability assumptions. In §3 we recall several results needed later. Our main results are in sections 4 and 5, where bases for root subspaces $\mathcal{R}_2$ and $\mathcal{R}_3$, respectively, of a geometrically simple eigenvalue are described. In the last section we discuss two examples.

2. Regularity and Solvability Assumptions

In the rest of the paper we use the basic setup introduced in the first paragraph of §1. The operators A_{ij}, $j \in \underline{n}$, induce operators $A_{ij}^{\dagger}$ on the Hilbert space tensor product $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ by means of

$$A_{ij}^{\dagger}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes A_{ij}x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$$

on decomposable tensors, extended by linearity and continuity to the whole of H. Similarly A_{i0} induces an operator $A_{i0}^{\dagger}$ with domain $\mathcal{D}\left(A_{i0}^{\dagger}\right) \subset H$ (see [20, §2.3]). We denote by $\mathcal{D}$ the intersection $\bigcap_{i=1}^{n} \mathcal{D}\left(A_{i0}^{\dagger}\right)$, which is a dense subspace of H since $\mathcal{D}\left(A_{10}\right) \otimes \mathcal{D}\left(A_{20}\right) \otimes \cdots \otimes \mathcal{D}\left(A_{n0}\right) \subseteq \mathcal{D}$. The operator Δ_0 on H is defined by

$$\Delta_0 = \det \left[A_{ij}^{\dagger}\right]_{i,j=1}^{n} \tag{2.1}$$

and operators Δ_i $(i \in \underline{n})$ on $\mathcal{D}$ by replacing the i-th column in (2.1) by $\left[-A_{i0}^{\dagger}\right]_{i=1}^{n}$.

In what follows we make two regularity assumptions and a solvability assumption.

Assumption I We assume that the operator Δ_n has a bounded inverse.

Remark 2.1. *Actually a weaker assumption that the operator $\Delta_n + \alpha\Delta_i$ has a bounded inverse for some $\alpha \in \mathbb{C}$ and some index i would suffice. Then Assumption I follows by a shift in parameters. In the finite-dimensional case Assumption I follows by a shift in parameters if there is a linear combination of Δ_i's which is a nonsingular operator. This can be also formulated in terms of polynomials* $\det W_i(\boldsymbol{\lambda})$ *in $n+1$ variables $\lambda_0, \lambda_1, \ldots, \lambda_n$ (cf.* [2, Ch. 8] *and* [11]*).*

Now we normalize our multiparameter system by assuming that $\lambda_n = 1$, and from now we write $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ and

$$W_i(\boldsymbol{\lambda}) = \sum_{j=0}^{n-1} A_{ij}\lambda_j + A_{in} \tag{2.2}$$

for $i \in \underline{n}$.

Next we define the notions of eigenvalues and spectra. An n-tuple $\boldsymbol{\lambda} \in \mathbb{C}^n$ is called an *eigenvalue of a multiparameter system* (2.2) if all $W_i(\boldsymbol{\lambda})$ are singular. The set of all eigenvalues is called the *spectrum* of (2.2), and it is denoted by $\sigma(\mathbf{W})$.

Assumption II A given eigenvalue $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ of (2.2) is *geometrically simple*, i.e., operators $W_i(\boldsymbol{\lambda})$ are Fredholm [21] and $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})) = 1$ for all i.

Remark 2.2. *Assumption II is satisfied for example, in several cases arising from boundary value problems, e.g. of Sturm-Liouville type (see* [4]*). Note that in infinite-dimensional settings it might happen that* $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})) = 1$ *but* $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})^*) \neq 1$*. However, due to Fredholmness assumption it follows that* $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})^*) < \infty$*. In finite-dimensions the Fredholmness assumption is automatic, moreover an eigenvalue is geometrically simple if and only if* $\dim \mathcal{N}(W_i(\boldsymbol{\lambda})) = 1$ *for all i (cf.* [15]*).*

Let $\mathcal{A}_{ij}^{\dagger}$ denote the restriction of $A_{ij}^{\dagger}$ to $\mathcal{D}$. The array $\mathcal{A} = \left[\mathcal{A}_{ij}^{\dagger}\right]_{i=1,\ j=0}^{n \quad n}$ then defines a linear map $\mathcal{A} : \mathcal{D}^{n+1} \longrightarrow H^n$. Here H^n is the direct sum of n copies of H. Omitting the j-th column we get a transformation $\mathcal{A}_j$ acting on the (algebraic) direct sum $\mathcal{D}^n$ for $j = 0, 1, \ldots, n$. Note that $\Delta_j = (-1)^j \det \mathcal{A}_j$ for $j \in \underline{n}$. Next we define the transformations $\mathcal{B}_j : \mathcal{D}^n \to H^n$ adjugate to $\mathcal{A}_j$, so $(\mathcal{B}_j)_{lk}$ is the (k,l)-th cofactor of $\mathcal{A}_j$. We denote by $\mathcal{C}_j$ the j-th column of $\mathcal{A}$. Now we state the solvability assumption.

Assumption III The equation $\mathcal{A}_n\mathbf{y} = \mathcal{C}_n x$ has a solution $\mathbf{y} \in \mathcal{D}^n$ for all $x \in \mathcal{D}$.

The linear transformations $\Gamma_j = \Delta_n^{-1}\Delta_j : \mathcal{D} \to \mathcal{D}(\subseteq H)$, $(j = 0, 1, \ldots, n-1)$ are called the *associated transformations* of a multiparameter system (2.2). Assumption III implies [4, Thm.

3.2] that the linear transformations Γ_i commute on $\mathcal{D}$, i.e., $\Gamma_i\Gamma_j x = \Gamma_j\Gamma_i x$ for all $x \in \mathcal{D}$ and i, $j = 0, 1, \ldots, n-1$, and that

$$\sum_{j=0}^{n-1} A_{ij}^{\dagger}\Gamma_j x + A_{in}^{\dagger} x = 0 \tag{2.3}$$

for all $x \in \mathcal{D}$ and $i \in \underline{n}$. Note that (2.3) can be viewed as a generalization of Cramer's rule for a system of linear equations.

An n-tuple $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{C}^n$ is called an *eigenvalue of the system of commuting linear transformations* $\{\Gamma_j\}_{j=0}^{n-1}$ if

$$\mathcal{R}_0 = \bigcap_{j=0}^{n-1} \mathcal{N}(\Gamma_j - \lambda_j I) \neq \{0\}.$$

The set of all the eigenvalues is called the *spectrum* of the system $\{\Gamma_j\}_{j=0}^{n-1}$.

Assumptions I-III imply [4, §4] that the spectrum of (1.1) and that of its associated system of linear transformations $\{\Gamma_j\}_{j=0}^{n-1}$ coincide and that for a given eigenvalue $\boldsymbol{\lambda}$ we have

$$\mathcal{R}_0 = \mathcal{N}(W_1(\boldsymbol{\lambda})) \otimes \mathcal{N}(W_2(\boldsymbol{\lambda})) \otimes \cdots \otimes \mathcal{N}(W_n(\boldsymbol{\lambda})). \tag{2.4}$$

The subspace $\mathcal{R}_0$ is called the *eigenspace* corresponding to $\boldsymbol{\lambda}$.

For $l \geq 1$ we define *root subspaces*

$$\mathcal{R}_l = \bigcap_{m \in \mathbf{M}_{l+1}} \mathcal{N}[m(\Gamma_0 - \lambda_0 I, \Gamma_1 - \lambda_1 I, \ldots, \Gamma_{n-1} - \lambda_{n-1} I)], \tag{2.5}$$

where $\mathbf{M}_{l+1}$ is the set of all monomials in n variables of degree $l+1$. Using linearity it is easy to see that $\mathcal{R}_l$ is equal to

$$\bigcap_{m \in \mathbf{R}_{l+1}} \mathcal{N}[m(\Gamma_0 - \lambda_0 I, \Gamma_1 - \lambda_1 I, \ldots, \Gamma_{n-1} - \lambda_{n-1} I)],$$

where $\mathbf{R}_{l+1}$ is the set of all homogeneous polynomials in n variables of degree $l+1$.

The least integer l, if it exists, such that $\mathcal{R}_l = \mathcal{R}_{l+1}$ is called the *ascent of* $\boldsymbol{\lambda}$. We write K for the ascent if it exists, and assume $K = \infty$ otherwise. The subspace

$$\mathcal{R} = \bigcup_{l=0}^{K} \mathcal{R}_l$$

is called the *root subspace* of $\mathbf{W}$ at $\boldsymbol{\lambda}$. For $l \geq 1$ we denote by d_l the difference $\dim \mathcal{R}_l - \dim \mathcal{R}_{l-1}$. Observe that if $\boldsymbol{\lambda}$ is geometrically simple then it follows from (2.4) that $d_0 := \dim \mathcal{R}_0 = 1$. Our Assumption III guarantees that the root vectors described in this paper in terms of operators A_{ij} span the root subspaces $\mathcal{R}_l$ for $l = 1, 2, 3$. Actually, a weaker assumption that the root subspace $\mathcal{R}_l$ is contained in the subspace

$$\mathcal{K} = \{x \in \mathcal{D} : \text{there exist } \mathbf{y} \in \mathcal{D}^n \text{ such that } \mathcal{A}_n \mathbf{y} = \mathcal{C}_n x\}$$

is sufficient. Without such an assumption our method would yield only a basis for the subspace $\mathcal{R}_l \cap \mathcal{K}$ of $\mathcal{R}_l$.

The main ideas in the proof of the following result are essentially the same as those in the proof of [4, Lem. 5.2]. Here, we include the proof for the sake of completeness.

Lemma 2.3. *Assume that $\mathcal{R}_0$ is finite-dimensional. Then for $l \geq 1$ the subspaces $\mathcal{R}_l$ are also finite-dimensional.*

Proof. We use the induction on l. Suppose that $\mathcal{R}_k$ are finite-dimensional for $k < l$ and that $l \geq 1$. For a monomial $m \in \mathbf{M}_l$ we write $m(\mathbf{\Gamma}, \boldsymbol{\lambda}) = m(\Gamma_0 - \lambda_0 I, \Gamma_1 - \lambda_1 I, \ldots, \Gamma_{n-1} - \lambda_{n-1} I)$. The subspace $\mathcal{R}_l$ is invariant for all $m(\mathbf{\Gamma}, \boldsymbol{\lambda})$. Since $(\Gamma_i - \lambda_i I)\, m(\mathbf{\Gamma}, \boldsymbol{\lambda}) u = 0$ for $i = 0, 1, \ldots, n-1$, and all $u \in \mathcal{R}_l$, it follows that the range $\mathcal{I}\left(m(\mathbf{\Gamma}, \boldsymbol{\lambda})\,|_{\mathcal{R}_l}\right)$ is a subspace of $\mathcal{R}_0$. So, it is finite-dimensional. Then each kernel $\mathcal{N}\left(m(\mathbf{\Gamma}, \boldsymbol{\lambda})\,|_{\mathcal{R}_l}\right)$ has finite codimension in $\mathcal{R}_l$, i.e., the orthogonal complement $\mathcal{Q}_m$ of $\mathcal{N}\left((m(\mathbf{\Gamma}, \boldsymbol{\lambda}))\,|_{\mathcal{R}_l}\right)$ in $\mathcal{R}_l$ is finite-dimensional. Hence the linear span $\mathcal{Q}$ of the $\mathcal{Q}_m$, $m \in \mathbf{M}_l$ is finite dimensional. By definition we have $\mathcal{R}_{l-1} = \bigcap_{m \in \mathbf{M}_{l-1}} \mathcal{N}\left(m(\mathbf{\Gamma}, \boldsymbol{\lambda})\,|_{\mathcal{R}_l}\right)$, and therefore $\mathcal{R}_l = \mathcal{R}_{l-1} \oplus \mathcal{Q}$. Hence, $\mathcal{R}_l$ also has finite dimension. □

3. Preliminaries

3.1. The Root Subspace $\mathcal{R}_1$. In this section we introduce some further notation and recall some of the results from [4, 14, 17] that are needed later.

From now we fix an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ (which is geometrically simple). Suppose that $x_{i0} \in \mathcal{N}(W_i(\boldsymbol{\lambda}))$ $(i = 1, 2, \ldots, n)$ are nonzero vectors. By (2.4) it follows that the vector $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ spans $\mathcal{R}_0$. Further, we choose vectors $y_{i0}^{k_i} \in \mathcal{N}(W_i(\boldsymbol{\lambda})^*)$ $(k_i = 1, 2, \ldots, n_i^*,\ i = 1, 2, \ldots, n)$, such that $\{y_{i0}^1, y_{i0}^2, \ldots, y_{i0}^{n_i^*}\}$ is a basis for $\mathcal{N}(W_i(\boldsymbol{\lambda})^*)$. By Assumption II it follows that the dimensions

$$n_i^* = \dim \mathcal{N}(W_i(\boldsymbol{\lambda})^*),\ i \in \underline{n},$$

are finite. For $i \in \underline{n}$ and $j = 0, 1, \ldots, n-1$ we write

$$A_{ij}^0 = \begin{bmatrix} \langle A_{ij} x_{i0}, y_{i0}^1 \rangle & \langle A_{ij} x_{i0}, y_{i0}^2 \rangle & \cdots & \left\langle A_{ij} x_{i0}, y_{i0}^{n_i^*} \right\rangle \end{bmatrix}^T$$

and

$$B_0 = \begin{bmatrix} A_{10}^0 & A_{11}^0 & \cdots & A_{1,n-1}^0 \\ A_{20}^0 & A_{21}^0 & \cdots & A_{2,n-1}^0 \\ \vdots & \vdots & & \vdots \\ A_{n0}^0 & A_{n1}^0 & \cdots & A_{n,n-1}^0 \end{bmatrix}.$$

Here we use $\langle x, y \rangle$ to denote the scalar product of vectors x and y. It is clear from the context which Hilbert space H_i or H is meant. Note that B_0 is a $n^* \times n$ matrix, where $n^* = \left(\sum_{i=1}^n n_i^*\right)$.

The following result is a special case of [4, Thm. 6.3]. Recall that for a positive integer k we denote by $\underline{k}$ the set of integers $\{1, 2, \ldots, k\}$.

Theorem 3.1. *Assume that $\boldsymbol{\lambda}$ is a geometrically simple eigenvalue of multiparameter system $\mathbf{W}$ and that $\left\{z_0; z_1^1, z_1^2, \ldots, z_1^{d_1}\right\}$ is a basis for the subspace $\mathcal{R}_1$ such that*

$$(\Gamma_i - \lambda_i I)\, z_1^k = a_i^k z_0$$

for $i = 0, 1, \ldots, n-1$ and $k \in \underline{d_1}$. Then $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{d_1}\right\}$, where $\mathbf{a}^k = \left[a_0^k, a_1^k, \ldots, a_{n-1}^k\right]^T$, is a basis for the kernel of B_0.

Conversely, suppose that $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^d\right\}$, where $\mathbf{a}^k = \left[a_0^k, a_1^k, \ldots, a_{n-1}^k\right]^T$, is a basis for $\mathcal{N}(B_0)$. Then there exist vectors $x_{i1}^k \in \mathcal{D}(A_{i0})$ such that

$$U_i\left(\mathbf{a}^k\right) x_{i0} + W_i(\boldsymbol{\lambda})\, x_{i1}^k = 0 \tag{3.1}$$

for $i \in \underline{n}$ and $k \in \underline{d}$, where

$$U_i\left(\mathbf{a}^k\right) = \sum_{j=0}^{n-1} a_j^k A_{ij}.$$

Furthermore, the vectors

$$z_1^k = \sum_{i=1}^{n} x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1}^k \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0} \tag{3.2}$$

are such that

$$\left(\Gamma_i - \lambda_i I\right) z_1^k = a_i^k z_0$$

for all i and k and $\left\{z_0;\ z_1^1, z_1^2, \ldots, z_1^d\right\}$ is a basis for the second root subspace $\mathcal{R}_1$; hence $d_1 = d$.

3.2. Commutative Arrays. Next we recall notation and some results from [14, 17]. Let us first explain how the general setup introduced in this subsection is used later in the paper.

By Lemma 2.3 each root subspace $\mathcal{R}_l$, $l = 1, 2, \ldots$, is finite-dimensional. It is an invariant subspace for $\Gamma_i - \lambda_i I$, $i = 0, 1, \ldots, n-1$. The restricted operators $A_i = \left(\Gamma_i - \lambda_i I\right)|_{\mathcal{R}_l}$ form an n-tuple of commutative nilpotent operators on a finite-dimensional Hilbert space $\mathcal{R}_l$. We choose a basis for $\mathcal{R}_l$ and we identify A_i, $i = 0, 1, \ldots, n-1$, with an n-tuple of commutative nilpotent $N \times N$ matrices, where $N = \dim \mathcal{R}_l$.

In the rest of the subsection we choose the notation so that it corresponds as much as possible to the notation already introduced for the special commutative n-tuple A_i, $i = 0, 1, \ldots, n-1$, as above.

In general, we consider a set $\mathbf{A} = \{A_i,\ i = 0, 1, \ldots, n-1\}$ of commutative nilpotent $N \times N$ matrices. The set $\mathbf{A}$ is viewed also as a cubic array of dimensions $N \times N \times n$ [14, 17]. For $l \geq 0$ we write

$$\mathcal{N}\mathbf{A}^l = \bigcap_{m \in \mathbf{M}_{l+1}} \mathcal{N}\left[m\left(A_0, A_1, \ldots, A_{n-1}\right)\right].$$

Suppose that $M = \min_l \left\{\mathcal{N}\mathbf{A}_l = \mathbb{C}^N\right\}$. Then

$$\{0\} \subset \mathcal{N}\mathbf{A}^0 \subset \mathcal{N}\mathbf{A}^1 \subset \cdots \subset \mathcal{N}\mathbf{A}^M = \mathbb{C}^N \tag{3.3}$$

is a filtration of the vector space $\mathbb{C}^N$. Further we write

$$D_l = \dim \mathcal{N}\mathbf{A}^l \quad \text{and} \quad d_l = D_l - D_{l-1} \tag{3.4}$$

for $l = 0, 1, \ldots, M$. Here $D_{-1} = 0$. Then there exists a basis

$$\mathcal{B} = \left\{z_0^1, z_0^2, \ldots, z_0^{d_0};\ \ z_1^1, z_1^2, \ldots, z_1^{d_1};\ \ \ldots\ \ ;\ \ z_M^1, z_M^2, \ldots, z_M^{d_M}\right\}$$

for $\mathbb{C}^N$ such that for $l = 0, 1, \ldots, M$, the set

$$\mathcal{B}_l = \left\{z_0^1, z_0^2, \ldots, z_0^{d_0};\ \ z_1^1, z_1^2, \ldots, z_1^{d_1};\ \ \ldots\ \ ;\ \ z_l^1, z_l^2, \ldots, z_l^{d_l}\right\}$$

is a basis for $\mathcal{N}\mathbf{A}^l$. We call a basis $\mathcal{B}$ with the latter property a *filtered basis*. A set of commutative nilpotent matrices $\mathbf{A}$ is then simultaneously reduced to a special upper-triangular form and viewed as a cubic array

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{A}^{02} & \cdots & \mathbf{A}^{0,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} & \cdots & \mathbf{A}^{1,M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \tag{3.5}$$

where

$$\mathbf{A}^{kl} = \begin{bmatrix} \mathbf{a}_{11}^{kl} & \mathbf{a}_{12}^{kl} & \cdots & \mathbf{a}_{1,d_l}^{kl} \\ \mathbf{a}_{21}^{kl} & \mathbf{a}_{22}^{kl} & \cdots & \mathbf{a}_{2,d_l}^{kl} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{d_k,1}^{kl} & \mathbf{a}_{d_k,2}^{kl} & \cdots & \mathbf{a}_{d_k,d_l}^{kl} \end{bmatrix}$$

is a cubic array of dimensions $d_k \times d_l \times n$ and $\mathbf{a}_{ij}^{kl} \in \mathbb{C}^n$. The *row* and *column cross-sections* of $\mathbf{A}^{kl}$ are

$$R_i^{kl} = [\, \mathbf{a}_{i1}^{kl} \quad \mathbf{a}_{i2}^{kl} \quad \cdots \quad \mathbf{a}_{i,d_l}^{kl} \,], \; i \in \underline{d_k}, \tag{3.6}$$

and

$$C_j^{kl} = [\, \mathbf{a}_{1j}^{kl} \quad \mathbf{a}_{2j}^{kl} \quad \cdots \quad \mathbf{a}_{d_k,j}^{kl} \,], \; j \in \underline{d_l}. \tag{3.7}$$

These are matrices of dimensions $n \times d_l$ and $n \times d_k$, respectively. We wish to remark that the matrices C_j^{kl} of (3.7) are denoted by $\left(C_j^{kl}\right)^T$ in [14, 17]. We changed the notation for notational simplicity since in our paper only the matrices in (3.7) appear.

We call a matrix A *symmetric* if $A = A^T$. In [14, Cor. 1] we observed that $\mathbf{A}$ is commutative if and only if certain products of row and column cross-sections are symmetric.

A commutative array $\mathbf{A}$ is called *geometrically simple* if $d_0 = 1$, i.e., if the joint kernel of matrices in $\mathbf{A}$ has the dimension equal to 1.

Now we state a theorem used latter in the text that follows from [14, Thm. 2] and results of [17, §4].

Theorem 3.2. *Assume that* $\mathbf{A}$ *is a geometrically simple commutative array in form* (3.5) *with* $M = 3$. *Then there exists a set*

$$\left\{ T_{2f} = \left[t_{fij}^2 \right]_{i,j=1}^{d_1}; \; f \in \underline{d_2} \right\}$$

of linearly independent symmetric matrices such that $C_f^{12} = R_1^{01} T_{2f}$ *for* $f \in \underline{d_2}$.

Furthermore, there exists a set

$$\left\{ T_{3f} = \begin{bmatrix} T_{3f}^1 & T_{3f}^2 \\ \left(T_{3f}^2\right)^T & 0 \end{bmatrix}; \; f \in \underline{d_3} \right\} \tag{3.8}$$

of symmetric $(d_1 + d_2) \times (d_1 + d_2)$ *matrices such that :*

(i) *matrices* $T_{3f}^2 = \left[t_{fgh}^{32} \right]_{g=1,h=1}^{d_1 \quad d_2}; f \in \underline{d_3}$, *are linearly independent,*

(ii) $C_f^{23} = R_1^{01} T_{3f}^2$ *and* $C_f^{13} = R_1^{01} T_{3f}^1 + R_1^{02} \left(T_{3f}^2\right)^T$ *for* $f \in \underline{d_3}$,

(iii) for $h_1, h_2, h_3 \in \underline{d_1}$ and $f \in \underline{d_3}$ there is

$$\sum_{g=1}^{d_2} t^2_{gh_1h_2}\, t^{32}_{fh_3g} = \sum_{g=1}^{d_2} t^2_{gh_1h_3}\, t^{32}_{fh_2g}. \tag{3.9}$$

Remark 3.3. *The symmetry of matrices T_{mf} for $m = 2, 3$ and $f \in \underline{d_m}$ is equivalent to the commutativity of $\mathbf{A}$ with $M = 3$. This follows from the fact that when the array $\mathbf{A}$ is geometrically simple, i.e. $d_0 = 1$, the nonzero entries of matrices T_{mf} are structure constants for multiplication in the commutative algebra generated by the matrices A_i and the identity matrix* [17, Thm. 3]. *In addition, the structure constants satisfy relations* (3.9) *and also higher order symmetries* [17, Cor. 4]. *These arise since the products of three or more matrices of $\mathbf{A}$ do not depend on the order of multiplication.*

4. A Basis for Root Subspace $\mathcal{R}_2$

We suppose that $\boldsymbol{\lambda}$ is a geometrically simple eigenvalue and that $\mathbf{a}_1^{01}, \mathbf{a}_2^{01}, \ldots, \mathbf{a}_{d_1}^{01}$ form a basis for $\mathcal{N}(B_0)$. Further, we assume that the columns of the matrix

$$\mathbf{b}_0 = \begin{bmatrix} \mathbf{b}_1^{01} & \mathbf{b}_2^{01} & \cdots & \mathbf{b}_{d_1^*}^{01} \end{bmatrix} \in \mathbb{C}^{n^* \times d_1^*}$$

form a basis for $\mathcal{N}(B_0^*)$. Observe that $d_1^* = n^* - n + d_1$. We restrict our attention to the root subspace $\mathcal{R} = \mathcal{R}_2$, which is finite-dimensional by Lemma 2.3. We bring the restricted linear transformations $(\Gamma_i - \lambda_i I)|_{\mathcal{R}}$, that are commuting and nilpotent, with respect to a filtered basis to the form (3.5) with $M = 2$. It follows from Theorem 3.2 that for every column cross-section C_f^{12}, $f \in \underline{d_2}$ there exists a unique symmetric matrix T_f such that $R_1^{01} T_f = C_f^{12}$. We choose vectors z_1^k, $k \in \underline{d_1}$, as in Theorem 3.1, such that $\mathcal{B}_1 = \left\{ z_0, z_1^1, z_1^2, \ldots, z_1^{d_1} \right\}$ is a basis for $\mathcal{R}_1$ and $(\Gamma_i - \lambda_i I)\, z_1^k = a_{ki}^{01} z_0$ for $i = 0, 1, \ldots, n-1$.

Now we define matrices A_{ij}^{1k} for $k \in \underline{d_1}$, $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, n$ by

$$A_{ij}^{1k} = \begin{bmatrix} \left\langle A_{ij} x_{i1}^k, y_{i0}^1 \right\rangle & \left\langle A_{ij} x_{i1}^k, y_{i0}^2 \right\rangle & \cdots & \left\langle A_{ij} x_{i1}^k, y_{i0}^{n_i^*} \right\rangle \end{bmatrix}^T,$$

and matrices B_{1k}, $k \in \underline{d_1}$ by

$$B_{1k} = \begin{bmatrix} A_{10}^{1k} & A_{11}^{1k} & \cdots & A_{1,n-1}^{1k} \\ A_{20}^{1k} & A_{21}^{1k} & \cdots & A_{2,n-1}^{1k} \\ \vdots & \vdots & & \vdots \\ A_{n0}^{1k} & A_{n1}^{1k} & \cdots & A_{n,n-1}^{1k} \end{bmatrix}.$$

Further we construct matrix $S \in \mathbb{C}^{n^* \times (d_1+1)d_1/2}$ as follows : for $p \in \underline{\frac{(d_1+1)d_1}{2}}$ we can uniquely choose numbers k and l so that $k \geq l \geq 1$ and $p = \frac{(k-1)k}{2} + l$. Then the p-th column of S is equal to $B_{1,k}\mathbf{a}_l^{01} + B_{1,l}\mathbf{a}_k^{01}$ if $k \neq l$ and to $B_{1,k}\mathbf{a}_k^{01}$ otherwise. We also write

$$\mathcal{S}_2 = \mathbf{b}_0^* S.$$

Further we identify the subspace Θ_2 of symmetric $d_1 \times d_1$ matrices with the vector space $\mathbb{C}^{(d_1+1)d_1/2}$ via the isomorphism $\psi : \Theta_2 \longrightarrow \mathbb{C}^{(d_1+1)d_1/2}$ defined by

$$\psi(T) = \begin{bmatrix} t_{11} & t_{12} & t_{22} & t_{13} & t_{23} & t_{33} & \ldots & t_{1d_1} & t_{2d_1} & \cdots & t_{d_1 d_1} \end{bmatrix}^T \tag{4.1}$$

for

$$T=\left[\begin{array}{cccc} t_{11} & t_{12} & \cdots & t_{1d_1} \\ t_{12} & t_{22} & \cdots & t_{2d_1} \\ \vdots & \vdots & & \vdots \\ t_{1d_1} & t_{2d_1} & \cdots & t_{d_1d_1} \end{array}\right]\in\Theta_2.$$

We denote by $\mathcal{D}_i'$ the intersection $\mathcal{D}\left(A_{i0}\right)\cap\left(\mathcal{N}\left(W_i(\boldsymbol{\lambda}_i)\right)\right)^{\perp}$, where $\left(\mathcal{N}\left(W_i(\boldsymbol{\lambda}_i)\right)\right)^{\perp}$ is the orthogonal complement of $\mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)\right)$ in H_i. The following is a technical observation, which enables us to shorten proofs but has no significance otherwise. In actual calculations of basis vectors we can chose vectors $x_{ik}^g\in\mathcal{D}\left(A_{i0}\right)$ (i.e. not necessarily $x_{ik}^g\in\mathcal{D}_i'$) that satisfy all other conditions. Note also that by a similar argument as in Lemma 4.1 we can assume that $x_{i1}^k\in\mathcal{D}_i'$ in Theorem 3.1.

Lemma 4.1. *Suppose that a vector $x_i\in\left(\mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)^*\right)\right)^{\perp}$. Then there exists a vector $v_i\in\mathcal{D}_i'$ such that $x_i=W_i\left(\boldsymbol{\lambda}\right)v_i$.*

Proof. Suppose that $x_i\in\left(\mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)^*\right)\right)^{\perp}$. Because $\left(\mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)^*\right)\right)^{\perp}=\mathcal{R}\left(W_i\left(\boldsymbol{\lambda}\right)\right)$ it follows that there exists a vector $u_i\in\mathcal{D}\left(A_{i0}\right)$ such that $x_i=W_i\left(\boldsymbol{\lambda}\right)u_i$. By the definition of the direct sum of vector spaces we can find vectors $v_i\in\mathcal{D}_i'$ and $w_i\in\mathcal{N}\left(W_i\left(\boldsymbol{\lambda}\right)\right)$ such that $v_i+w_i=u_i$. Then it follows that $x_i=W_i\left(\boldsymbol{\lambda}\right)v_i$. □

Proposition 4.2. *Suppose that $\mathbf{t}\in\mathcal{N}\left(\mathcal{S}_2\right)\setminus\{0\}$ and $T=\left[t_{ij}\right]_{i,j=1}^{d_1}=\psi^{-1}\left(\mathbf{t}\right)\in\Theta_2$. Then there exists an n-tuple $\mathbf{a}^{02}\in\mathbb{C}^n$ such that*

$$\sum_{k,l=1}^{d_1}t_{kl}B_{1k}\mathbf{a}_l^{01}+B_0\mathbf{a}^{02}=0, \tag{4.2}$$

and there exist vectors $x_{i2}\in\mathcal{D}_i'$, for $i=1,2,\ldots,n$, such that

$$U_i\left(\mathbf{a}^{02}\right)x_{i0}+\sum_{k=1}^{d_1}U_i\left(\mathbf{a}_k^{12}\right)x_{i1}^k+W_i\left(\boldsymbol{\lambda}\right)x_{i2}=0, \tag{4.3}$$

where $\mathbf{a}_k^{12}=\sum_{l=1}^{d_1}t_{kl}\mathbf{a}_l^{01}$. Then the vector

$$z_2=\sum_{s=1}^{n}x_{10}\otimes\cdots\otimes x_{s2}\otimes\cdots\otimes x_{n0}+\sum_{k,l=1}^{d_1}t_{kl}\sum_{s=1}^{n-1}\sum_{t=s+1}^{n}x_{10}\otimes\cdots\otimes x_{s1}^k\otimes\cdots\otimes x_{t1}^l\otimes\cdots\otimes x_{n0} \tag{4.4}$$

is in $\mathcal{R}_2\backslash\mathcal{R}_1$, and

$$\left(\Gamma_i-\lambda_iI\right)z_2=\sum_{k=1}^{d_1}a_{ki}^{12}z_1^k+a_i^{02}z_0 \tag{4.5}$$

for $i=0,1,\ldots,n-1$.

Conversely, if $z_2\in\mathcal{R}_2\backslash\mathcal{R}_1$ and (4.5) holds then $\psi\left(T\right)\in\mathcal{N}\left(\mathcal{S}_2\right)$, where T is the unique symmetric matrix such that $\left[\begin{array}{cccc}\mathbf{a}_1^{12} & \mathbf{a}_2^{12} & \cdots & \mathbf{a}_{d_1}^{12}\end{array}\right]=R_1^{01}T$. Furthermore, there exist vectors $x_{i2}\in\mathcal{D}(A_{i0}),\ i=1,2,\ldots,n$, such that (4.3) and (4.4) hold.

Proof. Because $\mathbf{t} \in \mathcal{N}(\mathcal{S}_2)$ and $T = \psi^{-1}(\mathbf{t})$ it follows that

$$\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} \mathbf{b}_0^* B_{1l} \mathbf{a}_k^{01} = 0.$$

Hence $\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} B_{1l} \mathbf{a}_k^{01} \in (\mathcal{N}(B_0^*))^{\perp}$ and therefore there exists $\mathbf{a}^{02} \in \mathbb{C}^n$ such that the equality (4.2) holds. By definition of the matrices B_0 and B_{1k} it follows that

$$\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} \sum_{j=0}^{n-1} \left\langle A_{ij} a_{kj}^{01} x_{i1}^{l}, y_{i0}^{h_i} \right\rangle + \sum_{j=0}^{n-1} \left\langle A_{ij} a_j^{02} x_{i0}, y_{i0}^{h_i} \right\rangle = 0$$

for $h_i \in \underline{n_i^*}$ and $i \in \underline{n}$. Thus $U_i(\mathbf{a}^{02}) x_{i0} + \sum_{k=1}^{d_1} U_i(\mathbf{a}_k^{12}) x_{i1}^k \in (\mathcal{N}(W_i(\boldsymbol{\lambda})^*))^{\perp}$ and by Lemma 4.1 there exist vectors $x_{i2} \in \mathcal{D}_i'$ such that (4.3) hold for $\mathbf{a}_k^{12} = \sum_{l=1}^{d_1} t_{kl} \mathbf{a}_l^{01}$. Next we form the vector z_2 as in (4.4). For vectors $x_{ij} \in H_i$, $i, j \in \underline{n}$ we write

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}^{\otimes} = \sum_{\sigma \in \Pi_n} \operatorname{sgn} \sigma \, x_{1\sigma(1)} \otimes x_{2\sigma(2)} \otimes \cdots \otimes x_{n\sigma(n)}, \tag{4.6}$$

where Π_n is the set of all permutations of order n and $\operatorname{sgn}\sigma$ is the signature of permutation σ. Observe that for the "vector determinant" (4.6) usual determinantal properties with respect to column operations hold. These properties hold for the "operator determinants" of the form (2.1) as well. In particular, we have that

$$(-1)^n (\Delta_0 - \lambda_0 \Delta_n) = \begin{vmatrix} -W_1(\boldsymbol{\lambda})^{\dagger} & A_{11}^{\dagger} & \cdots & A_{1,n-1}^{\dagger} \\ -W_2(\boldsymbol{\lambda})^{\dagger} & A_{21}^{\dagger} & \cdots & A_{2,n-1}^{\dagger} \\ \vdots & \vdots & & \vdots \\ -W_n(\boldsymbol{\lambda})^{\dagger} & A_{n1}^{\dagger} & \cdots & A_{n,n-1}^{\dagger} \end{vmatrix}.$$

To reduce the technical complexity of the proof we set $i = 0$ in the following calculation. For $i \geq 1$ the calculation is similar to the one given. Using all the prepared notations and relations, we have

$$(-1)^n (\Delta_0 - \lambda_0 \Delta_n) z_2 = \sum_{s=1}^{n} \begin{vmatrix} 0 & A_{11} x_{10} & \cdots & A_{1,n-1} x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{s-1,1} x_{s-1,0} & \cdots & A_{s-1,n-1} x_{s-1,0} \\ -W_s(\boldsymbol{\lambda}) x_{s2} & A_{s1} x_{s2} & \cdots & A_{s,n-1} x_{s2} \\ 0 & A_{s+1,1} x_{s+1,0} & \cdots & A_{s+1,n-1} x_{s+1,0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} x_{n0} & \cdots & A_{n,n-1} x_{n0} \end{vmatrix}^{\otimes} +$$

$$+ \sum_{k,l=1}^{d_1} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} \begin{vmatrix} 0 & A_{11} x_{10} & \cdots & A_{1,n-1} x_{10} \\ \vdots & \vdots & & \vdots \\ -W_s(\boldsymbol{\lambda}) x_{s1}^k & A_{s1} x_{s1}^k & \cdots & A_{s,n-1} x_{s1}^k \\ \vdots & \vdots & & \vdots \\ -W_t(\boldsymbol{\lambda}) x_{t1}^l & A_{t1} x_{t1}^l & \cdots & A_{t,n-1} x_{tn}^l \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1} x_{n0} & \cdots & A_{n,n-1} x_{n0} \end{vmatrix}^{\otimes} =$$

$$= a_0^{02} \begin{vmatrix} A_{10}x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ A_{20}x_{20} & A_{21}x_{20} & \cdots & A_{2,n-1}x_{20} \\ \vdots & \vdots & & \vdots \\ A_{n0}x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+ \sum_{k,l=1}^{d_1} t_{kl} \left(\sum_{s=1}^{n} \begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ U_s\left(\mathbf{a}_l^{01}\right) x_{s1}^k & A_{s1}x_{s1}^k & \cdots & A_{s,n-1}x_{s1}^k \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} + \right.$$

$$+ \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} \left(\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{n0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{s1}x_{s1}^k & \cdots & A_{s,n-1}x_{s1}^k \\ \vdots & \vdots & & \vdots \\ U_t\left(\mathbf{a}_l^{01}\right) x_{t0} & A_{t1}x_{t0} & \cdots & A_{t,n-1}x_{t0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} + \right.$$

$$\left.\left. + \begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{n0} \\ \vdots & \vdots & & \vdots \\ U_s\left(\mathbf{a}_k^{01}\right) x_{s0} & A_{s1}x_{s0} & \cdots & A_{s,n-1}x_{s0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{t1}x_{t1}^l & \cdots & A_{t,n-1}x_{t1}^l \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} \right) \right) =$$

(since $t_{kl} = t_{lk}$)

$$= a_0^{02} (-1)^n \Delta_n z_0 + \sum_{k,l=1}^{d_1} t_{kl} \sum_{s=1}^{n} \begin{vmatrix} U_1\left(\mathbf{a}_l^{01}\right) x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ U_s\left(\mathbf{a}_l^{01}\right) x_{s1}^k & A_{s1}x_{s1}^k & \cdots & A_{s,n-1}x_{s1}^k \\ \vdots & \vdots & & \vdots \\ U_n\left(\mathbf{a}_l^{01}\right) x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} =$$

$$= a_0^{02} (-1)^n \Delta_n z_0 + \sum_{k=1}^{d_1} a_{k0}^{12} (-1)^n \Delta_n z_1^k.$$

Conversely, suppose that $z_2 \in \mathcal{R}_2/\mathcal{R}_1$ and that (4.5) holds. Then there exist a symmetric matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1d_1} \\ t_{12} & t_{22} & \cdots & t_{2d_1} \\ \vdots & \vdots & & \vdots \\ t_{1d_1} & t_{2d_1} & \cdots & t_{d_1d_1} \end{bmatrix} \in \Theta_2$$

such that for $i = 0, 1, \ldots n-1$, we have

$$\mathbf{a}_k^{12} = \sum_{l=1}^{d_2} t_{kl} \mathbf{a}_l^{01} \tag{4.7}$$

by Theorem 3.2. Next it follows from (2.3) that

$$\sum_{j=0}^{n-1} A_{ij}^{\dagger} \left(\Gamma_j - \lambda_j I\right) z_2 + W_i \left(\boldsymbol{\lambda}\right)^{\dagger} z_2 = 0$$

and from (4.5) that

$$\sum_{j=0}^{n-1} A_{ij}^{\dagger} \left(\sum_{k=1}^{d_1} a_{kj}^{12} z_1^k + a_j^{02} z_0 \right) + W_i \left(\boldsymbol{\lambda}\right)^{\dagger} z_2 = 0. \tag{4.8}$$

For $i \in \underline{n}$ we choose vectors $v_i \in H_i$ so that $\langle x_{i0}, v_i \rangle = 1$ and $\left\langle x_{i1}^k, v_i \right\rangle = 0$ for $k \in \underline{d_1}$. This is possible because $\operatorname{Span}\{x_{i0}\} \cap \mathcal{D}_i' = \{0\}$. After multiplying (4.8) by $v_1 \otimes \cdots \otimes v_{i-1} \otimes y_{i0}^{h_i} \otimes v_{i+1} \otimes \cdots \otimes v_n$ on the right we get

$$\sum_{j=0}^{n-1} \left\langle A_{ij} \sum_{k=1}^{d_1} a_{kj}^{12} x_{i1}^k, y_{i0}^{h_i} \right\rangle + \sum_{j=0}^{n-1} \left\langle A_{ij} a_j^{02} x_{i0}, y_{i0}^{h_i} \right\rangle = 0 \tag{4.9}$$

for $h_i \in \underline{n_i^*}$ and all i. Hence, by Lemma 4.1, there exist vectors $x_{i2} \in \mathcal{D}_i'$ such that (4.3) hold. Now we form the vector

$$z_2^1 = \sum_{s=1}^{n} x_{10} \otimes \cdots \otimes x_{s2} \otimes \cdots \otimes x_{n0} + \sum_{k,l=1}^{d_1} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{t1}^l \otimes \cdots \otimes x_{n0}.$$

The same calculation as in the first part of the proof shows that

$$\left(\Gamma_i - \lambda_i I\right) z_2^1 = \sum_{k=1}^{d_1} a_{ki}^{12} z_1^k + a_i^{02} z_0$$

for $i = 0, 1, \ldots, n-1$. It follows that $z_2^1 - z_2 \in \mathcal{R}_0$ and so there exists a number $\delta \in \mathbb{C}$ such that $z_2 = z_2^1 + \delta z_0$. We replace one of the vectors x_{i2} in z_2^1 by $x_{i2} + \delta x_{i0}$, say we use the vector $x_{12} + \delta x_{10}$ in place of x_{12}, to obtain the required form of z_2, i.e.,

$$z_2 = \sum_{s=1}^{n} x_{10} \otimes \cdots \otimes x_{s2} \otimes \cdots \otimes x_{n0} + \sum_{k,l=1}^{d_2} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{t1}^l \otimes \cdots \otimes x_{n0}.$$

Note that here we might lose the condition that $x_{12} \in \mathcal{D}_i'$.

It remains to be shown that $\psi(T) \in \mathcal{N}(\mathcal{S}_2)$. The equalities (4.9) can be written in matrix form as

$$\sum_{k=1}^{d_1} B_{1k} \mathbf{a}_k^{12} + B_0 \mathbf{a}^{02} = 0.$$

Multiplication on the left-hand side by the matrix $\mathbf{b}_0^*$ yields

$$\sum_{k=1}^{d_1} \mathbf{b}_0^* B_{1k} \mathbf{a}_k^{12} = 0$$

and then also

$$\sum_{k=1}^{d_1}\sum_{l=1}^{d_1}\mathbf{b}_0^* B_{1k}\mathbf{a}_l^{01} t_{kl} = 0. \tag{4.10}$$

Finally, we note that the relation (4.10) is equivalent to $\psi(T) \in \mathcal{N}(\mathcal{S}_2)$. □

Theorem 4.3. *Suppose that $\mathcal{T} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d\}$ is a basis for $\mathcal{N}(\mathcal{S}_2)$ and that*

$$T_{2k} = \left[t_{rs}^k\right]_{r,s=1}^{d_1} = \psi^{-1}(\mathbf{t}_k).$$

Then there exists n-tuples $\mathbf{a}_k^{02} \in \mathbb{C}^n$ such that

$$\sum_{r,s=1}^{d_1} t_{rs}^k B_{1r}\mathbf{a}_s^{01} + B_0\mathbf{a}_k^{02} = 0, \tag{4.11}$$

and there exist vectors $x_{i2}^k \in \mathcal{D}_i'$, for $i \in \underline{n}$ such that

$$U_i\left(\mathbf{a}_k^{02}\right)x_{i0} + \sum_{r=1}^{d_1} U_i\left(\mathbf{a}_{rk}^{12}\right)x_{i1}^r + W_i(\boldsymbol{\lambda})\,x_{i2}^k = 0, \tag{4.12}$$

where $\mathbf{a}_{rk}^{12} = \sum_{s=1}^{d_1} t_{rs}^k\mathbf{a}_s^{01}$. Then vectors

$$z_2^k = \sum_{s=1}^{n} x_{10}\otimes\cdots\otimes x_{s2}^k\otimes\cdots\otimes x_{n0} + \sum_{r,s=1}^{d_1} t_{rs}^k \sum_{t=1}^{n-1}\sum_{u=t+1}^{n} x_{10}\otimes\cdots\otimes x_{t1}^r\otimes\cdots\otimes x_{u1}^s\otimes\cdots\otimes x_{n0}, \tag{4.13}$$

$k \in \underline{d}$, and

$$\mathcal{B}_2 = \left\{z_0; z_1^1, z_1^2, \ldots, z_1^{d_1}; z_2^1, z_2^2, \ldots, z_2^d\right\} \tag{4.14}$$

is a basis for $\mathcal{R}_2$.

Conversely, if $z_2^1, z_2^2, \ldots, z_2^{d_2}$ are such that $\left\{z_0; z_1^1, z_1^2, \ldots, z_1^{d_1}; z_2^1, z_2^2, \ldots, z_2^{d_2}\right\}$ is a basis for $\mathcal{R}_2$ and $T_{21}, T_{22}, \ldots, T_{2d_2}$ are symmetric matrices such that $C_k^{12} = R_1^{01}T_{2k},\ k \in \underline{d_2}$, then

$$\{\psi(T_{21}), \psi(T_{22}), \ldots, \psi(T_{2d_2})\}$$

is a basis for $\mathcal{N}(\mathcal{S}_2)$ and $d = d_2$.

Proof. The theorem follows using the correspondence between $\mathbf{t}$ and z_2 as described in Proposition 4.2, and the fact that z_2^k are linearly independent if and only if T_k are linearly independent. □

From the proof of Proposition 4.2 it follows that vectors z_2^k in basis $\mathcal{B}_2$ can be chosen so that they are of the form (4.13) with $x_{i2}^k \in \mathcal{D}_i'$. We assume hereafter that this is the case.

5. The Root Subspace $\mathcal{R}_3$

We choose vectors $z_3^f \in \mathcal{D},\ f \in \underline{d_3}$ so that $\mathcal{B}_3 = \mathcal{B}_2 \cup \left\{z_3^f,\ f \in \underline{d_3}\right\}$ is a filtered basis for the root subspace $\mathcal{R}_3$, which is finite-dimensional by Lemma 2.3. We put the restrictions $(\Gamma_i - \lambda_i I)\,|_{\mathcal{R}_3}$ with respect to the basis $\mathcal{B}_3$ to the form (3.5). By Theorem 3.2 there exist symmetric matrices $T_{3f},\ f \in \underline{d_3}$, in the form (3.8) such that (3.9) holds and for all $i = 0, 1, \ldots, n-1$, we have

$$(\Gamma_i - \lambda_i I)\,z_3^f = \sum_{k=1}^{2}\sum_{g=1}^{d_k} a_{gfi}^{k3}\,z_k^g + a_{fi}^{03}\,z_0, \tag{5.1}$$

and

$$\mathbf{a}_{gf}^{13} = \sum_{h=1}^{d_1} t_{fhg}^{31}\, \mathbf{a}_h^{01} + \sum_{h=1}^{d_2} t_{fgh}^{32}\, \mathbf{a}_h^{02} \text{ and } \mathbf{a}_{gf}^{23} = \sum_{h=1}^{d_1} t_{fhg}^{32}\, \mathbf{a}_h^{01}. \tag{5.2}$$

Note that (5.1) is the expansion of the vector $(\Gamma_i - \lambda_i I)\, z_3^f$ with respect to the basis $\mathcal{B}_2$ given in the array (3.5).

We use the notation of Theorem 3.2. We assume that vectors z_2^k (in form (4.4)) and matrices $T_{2k},\ k \in \underline{d_2}$, are given as in Theorem 4.3 and matrices $T_{3f},\ f \in \underline{d_3}$ as in Theorem 3.2. For $f \in \underline{d_3}$ the entries of the $d_1 \times d_2$ matrix T_{3f}^2 satisfy the d_1^3 conditions (3.9), i.e.

$$\sum_{g=1}^{d_2} t_{gh_1h_2}^2\, t_{fh_3g}^{32} - \sum_{g=1}^{d_2} t_{gh_1h_3}^2\, t_{fh_2g}^{32} = 0 \tag{5.3}$$

for $h_1, h_2, h_3 \in \underline{d_1}$. We write the matrix $T_{3f}^2 = \left[t_{fgh}^{32}\right]_{g=1,h=1}^{d_1 \quad d_2}$ also as a column

$$\mathbf{t}_f^{32} = \left[\begin{array}{ccccccccc} t_{f11}^{32} & \cdots & t_{fd_11}^{32} & t_{f12}^{32} & \cdots & t_{fd_12}^{32} & \cdots & t_{f1d_2}^{32} & \cdots & t_{fd_1d_2}^{32} \end{array}\right]^T \tag{5.4}$$

and the symmetric matrix T_{3f}^1 as a column

$$\mathbf{t}_f^{31} = \left[\begin{array}{ccccccccc} t_{f11}^{31} & t_{f12}^{31} & t_{22}^{31} & t_{f13}^{31} & \cdots & t_{f33}^{31} & \cdots & t_{f1d_1}^{31} & \cdots & t_{fd_1d_1}^{31} \end{array}\right]^T. \tag{5.5}$$

Thus we split the entries of a matrix T_{3f} into two column vectors $\mathbf{t}_f^{31}$ and $\mathbf{t}_f^{32}$. We denote by Θ_3 the set of all symmetric matrices in the form (3.8). The mapping $\psi_3 : \Theta_3 \to \mathbb{C}^{d_1d_2} \oplus \mathbb{C}^{d_1(d_1+1)/2}$ given by $\psi_3\left(T_{3f}\right) = \left(\mathbf{t}_f^{31}, \mathbf{t}_f^{32}\right)$ is a generalization of the linear transformation ψ defined by (4.1). It is bijective and therefore it has an inverse. The inverse maps two vectors $\mathbf{t}_f^{31}$ and $\mathbf{t}_f^{32}$ to a matrix $T_{3f} \in \Theta_3$. We use this inverse mapping in Lemma 5.2. We also write

$$\psi_{3j}(T_{3f}) = \mathbf{t}_f^{3j} \text{ for } j = 1, 2. \tag{5.6}$$

Note that $\psi_{3j},\ j = 1, 2$, are surjective linear transformations, $\psi_{31} : \Theta_3 \to \mathbb{C}^{d_1(d_1+1)/2}$ and $\psi_{32} : \Theta_3 \to \mathbb{C}^{d_1d_2}$.

Now we view relations (5.3) as a system of equations for the entries of T_{3f} and we write them in matrix form as

$$S^{21}\mathbf{t}_f^{31} + S^{22}\mathbf{t}_f^{32} = 0. \tag{5.7}$$

The entries of the matrices S^{21} and S^{22} are determined by (5.3): Matrices S^{21} and S^{22} have d_1^3 rows; thus $S^{21} \in \mathbb{C}^{d_1^3 \times d_1d_2}$ and $S^{22} \in \mathbb{C}^{d_1^3 \times d_1(d_1+1)/2}$. The rows in S^{21} and S^{22} are ordered lexicographically by all triples $(h_1, h_2, h_3) \in \underline{d_1} \times \underline{d_1} \times \underline{d_1}$.

Further we *want* the entries of the matrix T_{3f} and of the n-tuple

$$\mathbf{a}_f^{03} = \left[\begin{array}{cccc} a_{f1}^{03} & a_{f2}^{03} & \cdots & a_{fn}^{03} \end{array}\right]^T$$

to satisfy the n^* scalar relations

$$\sum_{k=1}^{2}\sum_{g=1}^{d_k}\sum_{h=1}^{d_1} t_{fhg}^{3k} \left\langle U_i\left(\mathbf{a}_h^{01}\right) x_{ik}^g, y_{i0}^{l_i}\right\rangle + \sum_{g=1}^{d_1}\sum_{h=1}^{d_2} t_{fgh}^{32} \left\langle U_i\left(\mathbf{a}_h^{02}\right) x_{i1}^g, y_{i0}^{l_i}\right\rangle +$$

$$+\sum_{j=0}^{n-1} a_j^{03} \left\langle A_{ij}x_{i0}, y_{i0}^{l_i}\right\rangle = 0 \tag{5.8}$$

for $l_i \in \underline{n_i^*}$ and $i \in \underline{n}$. These can be written equivalently in matrix form

$$S^{11}\mathbf{t}_f^{31} + S^{12}\mathbf{t}_f^{32} + B_0\mathbf{a}_f^{03} = 0. \tag{5.9}$$

Again the entries of the matrices $S^{11} \in \mathbb{C}^{n^* \times d_1 d_2}$ and $S^{12} \in \mathbb{C}^{n^* \times d_1(d_1+1)/2}$ are determined by the equations (5.8). The ordering of rows in S^{11} and S^{12} is chosen so that it is compatible with the ordering of rows in B_0.

We multiply the equation (5.9) by the matrix $\mathbf{b}_0^*$ on the left-hand side and we obtain

$$\mathbf{b}_0^* S^{11}\mathbf{t}_f^{31} + \mathbf{b}_0^* S^{12}\mathbf{t}_f^{32} = 0. \tag{5.10}$$

We choose a matrix $\mathbf{b}_3$ so that its columns form a basis for the kernel of

$$\begin{bmatrix} \mathbf{b}_0^* S^{11} \\ S^{21} \end{bmatrix}^*.$$

Then we define a matrix

$$\mathcal{S}_3 = \mathbf{b}_3^* \begin{bmatrix} \mathbf{b}_0^* S^{12} \\ S^{22} \end{bmatrix}.$$

Next we prove three auxiliary results.

Lemma 5.1. *In the above setting it follows that* $\mathbf{t}_f^{32} \in \mathcal{N}(\mathcal{S}_3)$ *for* $f \in \underline{d_3}$ *and* $\dim \mathcal{N}(\mathcal{S}_3) \geq d_3$.

Proof. By Theorem 3.2 it follows that the entries of the matrices T_{3f} satisfy the conditions (5.3). We put the entries of these matrices into two columns $\mathbf{t}_f^{31}$ and $\mathbf{t}_f^{32}$ as in (5.4) and (5.5) via the isomorphism ψ_3. Then we have $S^{21}\mathbf{t}_f^{31} + S^{22}\mathbf{t}_f^{32} = 0$. Relation (2.3) implies

$$\sum_{j=0}^{n-1} A_{ij}^\dagger \left(\Gamma_j - \lambda_j I\right) z_3^f + W_i\left(\boldsymbol{\lambda}\right)^\dagger z_3^f = 0$$

for $i \in \underline{n}$. From relations (5.1) it follows that

$$\sum_{k=1}^{2}\sum_{g=1}^{d_k} U_i\left(\mathbf{a}_{gf}^{k3}\right)^\dagger z_k^g + U_i\left(\mathbf{a}_f^{03}\right)^\dagger z_0 + W_i\left(\boldsymbol{\lambda}\right)^\dagger z_3^f = 0. \tag{5.11}$$

Because we assumed $x_{ik}^g \in \mathcal{D}_i'$ and $\mathcal{D}_i' \cap \operatorname{Span}\{x_{i0}\} = \{0\}$ it follows that there exist vectors $v_i \in H_i$ such that $\langle x_{i0}, v_i\rangle = 1$ and $\langle x_{ik}^g, v_i\rangle = 0$ for $k = 1, 2$ and $g \in \underline{d_k}$. We multiply the equality (5.11) by a vector $v_1 \otimes \cdots \otimes v_{i-1} \otimes y_{i0}^{h_i} \otimes v_{i+1} \otimes \cdots \otimes v_n$ on the right-hand side. Then it follows, using the structure of vectors z_k^g, $k = 0, 1, 2$, that

$$\sum_{k=1}^{2}\sum_{g=1}^{d_k} \left\langle U_i\left(\mathbf{a}_{gf}^{k3}\right) x_{ik}^g + U_i\left(\mathbf{a}_f^{03}\right) x_{i0}, y_{i0}^{h_i}\right\rangle = 0 \tag{5.12}$$

for $i \in \underline{n}$ and $f \in \underline{d_3}$. Now we apply relations (5.2) to obtain (5.8). The vectors $\mathbf{t}_f^{31}$, $\mathbf{t}_f^{32}$ and the n-tuple $\mathbf{a}_f^{03}$ are such that equation (5.9) holds for all f. Since

$$\mathbf{b}_0^* B_0 = 0 \text{ and } \mathbf{b}_3^* \begin{bmatrix} \mathbf{b}_0^* S^{11} \\ S^{21} \end{bmatrix}$$

it follows that the vectors $\mathbf{t}_f^{32}$, $f \in \underline{d_3}$ are elements of the kernel of $\mathcal{S}_3$. They are linearly independent by Theorem 3.2 and so we have $d_3 \leq \dim \mathcal{N}(\mathcal{S}_3)$. □

Lemma 5.2. *Suppose that* $\mathbf{t}_1^{32}$ *is an element of the kernel* $\mathcal{N}(\mathcal{S}_3)$*. Then there exist a vector* $\mathbf{t}_1^{31}$ *and an* n*-tuple* $\mathbf{a}_1^{03}$ *such that (5.7) and (5.9) with* $f=1$ *hold. Furthermore there exist vectors* $x_{i3}^1 \in \mathcal{D}_i'$*,* $i \in \underline{n}$ *such that*

$$\sum_{k=1}^{2}\sum_{g=1}^{d_k} U_i\left(\mathbf{a}_{g1}^{k3}\right) x_{ik}^g + U_i\left(\mathbf{a}_1^{03}\right) x_{i0} + W_i(\boldsymbol{\lambda})\, x_{i3}^1 = 0, \tag{5.13}$$

where $\mathbf{a}_{g1}^{k3}$ *are given by (5.2) for* $f=1$ *and* $T_{31} = \psi_3^{-1}\left(\mathbf{t}_1^{31}, \mathbf{t}_1^{32}\right)$*.*

Proof. From the structure of the matrix $\mathcal{S}_3$ it follows that for an element $\mathbf{t}_1^{32} \in \mathcal{N}(\mathcal{S}_3)$ there exist a vector $\mathbf{t}_1^{31}$ and a n-tuple $\mathbf{a}_1^{03}$ such that relations (5.7) and (5.9) hold: Namely, relation

$$0 = \mathcal{S}_3 \mathbf{t}_1^{32} = \mathbf{b}_3^* \begin{bmatrix} \mathbf{b}_0^* S^{12} \\ S^{22} \end{bmatrix} \mathbf{t}_1^{32}$$

implies that $\begin{bmatrix} \mathbf{b}_0^* S^{12} \\ S^{22} \end{bmatrix} \mathbf{t}_1^{32}$ is orthogonal to the kernel of $\begin{bmatrix} \mathbf{b}_0^* S^{11} \\ S^{21} \end{bmatrix}^*$. We denote by $U^\perp$ the orthogonal complement of a subspace U. Since $(\mathcal{N}(A^*))^\perp = \mathcal{R}(A)$ for a linear transformation A between two finite-dimensional Hilbert spaces, it follows that there is an element $\mathbf{t}_f^{31} \in \mathbb{C}^{d_1 d_2}$ such that (5.7) and (5.10) for $f=1$ hold. Then we have

$$\mathbf{b}_0^* \left(S^{11}\mathbf{t}_1^{31} + S^{12}\mathbf{t}_1^{32}\right) = 0.$$

Thus the vector $\left(S^{11}\mathbf{t}_1^{31} + S^{12}\mathbf{t}_1^{32}\right)$ is orthogonal to the kernel of B_0^* and so, there is an element $\mathbf{a}_1^{03} \in \mathbb{C}^n$ such that (5.9) holds.

We associate with the pair of vectors $\mathbf{t}_1^{32}$ and $\mathbf{t}_1^{31}$, using the inverse of the isomorphism ψ_3^{-1}, a symmetric matrix T_{31}. The relations (5.9) can be written equivalently in the form (5.12) for $f=1$. Then it follows for every i that

$$\sum_{k=1}^{2}\sum_{g=1}^{d_k} U_i\left(\mathbf{a}_{g1}^{k3}\right) x_{ik}^g + U_i\left(\mathbf{a}_1^{03}\right) x_{i0} \in \left(\mathcal{N}\left(W_i(\boldsymbol{\lambda})^*\right)\right)^\perp$$

and hence it follows from Lemma 4.1 that there exists a vector $x_{i3}^1 \in \mathcal{D}_i'$ such that (5.13) holds. □

The vector z_3^1 is defined by

$$\begin{aligned} z_3^1 = {} & \sum_{j=1}^{n} x_{10} \otimes \cdots \otimes x_{j3}^1 \otimes \cdots \otimes x_{n0} + \sum_{\substack{j,k=1\\ j<k}}^{n} \sum_{h_1,h_2=1}^{d_1} t_{1h_1h_2}^{31} x_{10} \otimes \cdots \otimes x_{j1}^{h_1} \otimes \cdots \otimes x_{k1}^{h_2} \otimes \cdots \otimes x_{n0} \\ & + \sum_{\substack{j,k=1\\ j\neq k}}^{n} \sum_{h_1=1}^{d_1} \sum_{h_2=1}^{d_2} t_{1h_1h_2}^{32} x_{10} \otimes \cdots \otimes x_{j1}^{h_1} \otimes \cdots \otimes x_{k2}^{h_2} \otimes \cdots \otimes x_{n0} + \\ & + \sum_{\mathbf{j}\in \underline{n}_3} \sum_{\mathbf{h}\in \underline{d_1}^3} \left(\sum_{g=1}^{d_2} t_{gh_1h_2}^{2} t_{1h_3g}^{32} \right) x_{10} \otimes \cdots \otimes x_{j_11}^{h_1} \otimes \cdots \otimes x_{j_21}^{h_2} \otimes \cdots \otimes x_{j_31}^{h_3} \otimes \cdots \otimes x_{n0}, \end{aligned}$$

where the latter summation does not occur for $n=2$. Here $\underline{n}_3$ is the set of all triples $\mathbf{j} = (j_1, j_2, j_3)$ of three pairwise distinct indices $j_1, j_2, j_3 \in \underline{n}$ and $\underline{d_1}^3$ is the set of all triples $\mathbf{h} = (h_1, h_2, h_3)$ of indices in $\underline{d_1}$.

Lemma 5.3. *In the above setting we have*

$$(\Gamma_i - \lambda_i I)\, z_3^1 = \sum_{k=1}^{2} \sum_{g=1}^{d_k} a_{g1i}^{k3}\, z_k^g + a_{1i}^{03} z_0 \tag{5.14}$$

for $i = 0, 1, \ldots, n-1$.

The proof is a direct calculation, similar to the proof of Proposition 4.2. Because it is rather technical we include it in Appendix.

Suppose that $\mathcal{T} = \left\{ \mathbf{t}_f^{32},\ f \in \underline{d} \right\}$ is a basis for the kernel of $\mathcal{S}_3$ where $d = \dim \mathcal{N}(\mathcal{S}_3)$. Let vectors x_{i3}^f, $i \in \underline{n}$, matrix T_{3f}, n-tuple $\mathbf{a}_f^{03}$ and vector z_3^f be associated with $\mathbf{t}_f^{32}$ as described in Lemmas 5.1–5.3 for $f = 1$. The following is our main result:

Theorem 5.4. *The set*

$$\mathcal{B}_2 \cup \left\{ z_3^f;\ f \in \underline{d} \right\}$$

is a filtered basis for $\mathcal{R}_3$.

Conversely, if $z_3^1, z_3^2, \ldots, z_3^{d_3}$ *are such that* $\mathcal{B}_2 \cup \left\{ z_3^f;\ f \in \underline{d_3} \right\}$ *is a filtered basis for* $\mathcal{R}_3$ *and* $T_{31}, T_{32}, \ldots, T_{3d_3}$ *are the associated symmetric matrices (3.8) then*

$$\left\{ \psi_{32}(T_{31}), \psi_{32}(T_{32}), \ldots, \psi_{32}(T_{3d_3}) \right\}$$

is a basis for $\mathcal{N}(\mathcal{S}_3)$ *and* $d = d_3$.

Proof. If $\mathcal{T}$ is a basis for $\mathcal{N}(\mathcal{S}_3)$ then by Lemma 5.2 it follows that we can find required vectors x_{i3}^f, symmetric matrices T_{3f} and n-tuples $\mathbf{a}_f^{03}$. Lemma 5.3 implies that the associated vectors z_3^f are in $\mathcal{R}_3 \backslash \mathcal{R}_2$. They are linearly independent since T_{3f} are linearly independent. It follows that $d \le d_3$ and, because $d \ge d_3$ by Lemma 5.1, we have $d = d_3$.

To prove the converse assume that z_3^f, $f \in \underline{d_3}$, are such that $\mathcal{B}_3 = \mathcal{B}_2 \cup \left\{ z_3^f;\ f \in \underline{d_3} \right\}$ is a filtered basis for $\mathcal{R}_3$ and T_{3f}, $f \in \underline{d_3}$, are the associated symmetric matrices (3.8). By Theorem 3.2*(i)* it follows that matrices T_{3f}^2, $f \in \underline{d_3}$ are linearly independent. Note that then also the images $\psi_{32}(T_{3f})$, $f \in \underline{d_3}$ are linearly independent. If $d_3 < d$ we complete the set $\left\{ \psi_{32}(T_{3f});\ f \in \underline{d_3} \right\}$ to a basis $\mathcal{T}$ of $\mathcal{N}(\mathcal{S}_3)$. Then we proceed as in the first part of the proof. We construct vectors $\widetilde{z}_3^f$, $f \in \underline{d}$, as described in Lemmas 5.1 – 5.3. By the same arguments as above it follows that $\mathcal{B}_2 \cup \left\{ \widetilde{z}_3^f;\ f \in \underline{d} \right\}$ is a basis for $\mathcal{R}_3$ and therefore that $d = d_3$. Hence $\left\{ \psi_{32}(T_{31}), \psi_{32}(T_{32}), \ldots, \psi_{32}(T_{3d_3}) \right\}$ is a basis for $\mathcal{N}(\mathcal{S}_3)$. □

6. Two Examples

In this section we consider two examples of computations of bases for root subspaces at geometrically simple eigenvalues that illustrate our methods. The first computation is at an eigenvalue (with Fredholm index 0) of a finite dimensional three-parameter system and the second at an eigenvalue with nonzero Fredholm index of an infinite dimensional two-parameter system. We wish to note that the examples are chosen so that they are not covered by the methods of [4, 5, 15, 16].

Example 6.1. *Let us consider the* 3*-parameter system* $\mathbf{W}$, *where the matrices* $W_1(\boldsymbol{\lambda})$, $W_2(\boldsymbol{\lambda})$ *and* $W_3(\boldsymbol{\lambda})$ *are given by*

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_0 + \lambda_1 + 1 & 0 & 0 & 0 \\ 2 & 2\lambda_0 + \lambda_1 + 1 & -\lambda_1 & 2 \\ 0 & \lambda_0 + \lambda_1 & \lambda_0 + \lambda_1 & \lambda_1 - 1 \\ 0 & 0 & 0 & \lambda_0 + 2\lambda_1 \end{bmatrix},$$

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_1 - 1 & \lambda_0 & 0 \\ \lambda_0 & \lambda_0 & 0 \\ 0 & 1 & \lambda_1 \end{bmatrix} \text{ and } W_3(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}.$$

Their determinants are

$$\det W_1(\boldsymbol{\lambda}) = (\lambda_0 + 2\lambda_1)(\lambda_0 + \lambda_1)(2\lambda_0 + 1)(\lambda_0 + \lambda_1 + 1),$$

$$\det W_2(\boldsymbol{\lambda}) = \lambda_0\lambda_1(\lambda_1 - \lambda_0 - 1) \text{ and } \det W_3(\boldsymbol{\lambda}) = \lambda_2^2.$$

We apply [11, Thm. 5.2]. Since in each of the three determinants has exactly two irreducible factors (counting multiplicities) that are zero at $(0,0,0)$, and all the factors are linear, it follows that the dimension of the root subspace $\mathcal{R}$ at the eigenvalue $(0,0,0)$ is equal to $2^3 = 8$.

It is easy to check that $(0,0,0)$ is a geometrically simple eigenvalue. We apply our construction to form the corresponding root vectors. First we choose vectors

$$x_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_{20} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_{30} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$y_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, y_{20} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, y_{30} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrix B_0 is zero, hence $d_1 = 3$. We choose

$$\mathbf{a}_1^{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{a}_2^{01} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{a}_3^{01} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Vectors x_{i1}^k, $i, k = 1, 2, 3$, have to solve the equations

$$A_{i,k-1}x_{i0} + A_{i3}x_{i1}^k = 0.$$

Possible solution is

$$x_{11}^1 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, x_{11}^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, x_{21}^2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, x_{31}^3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and the remaining vectors x_{i1}^k are zero. We choose $\mathbf{b}_0$ to be the identity matrix. Then

$$\mathcal{S}_2 = \begin{bmatrix} 1 & 3 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since $\mathcal{S}_2$ has rank equal to 3 it follows that $d_3 = 3$. The elements

$$\mathbf{t}_1^2 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{t}_2^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{t}_3^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the nullspace $\mathcal{N}(\mathcal{S}_2)$. The corresponding vectors x_{i2}^k, $i, k = 1, 2, 3$, have to solve equations

$$2A_{i0}x_{i1}^1 - A_{i1}x_{i1}^2 + A_{i3}x_{i2}^1 = 0, \;\; A_{i2}x_{i1}^1 + A_{i0}x_{i1}^3 + A_{i3}x_{i2}^2 = 0$$

and

$$A_{i2}x_{i2}^2 + A_{i1}x_{i1}^3 + A_{i3}x_{i2}^3 = 0.$$

Zero vectors solve all the equations except the first one for $i = 1$. We choose $x_{12}^1 = \begin{bmatrix} 0 \\ 15 \\ 0 \\ -4 \end{bmatrix}$ for a solution. Next we find the matrices S^{ij}, $i, j = 1, 2$: $S^{11} = \mathcal{S}_2$, $S^{21} = 0$,

$$S^{12} = \begin{bmatrix} -4 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$S^{22} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since S^{11} has full rank and $S^{21} = 0$ it follows that $\mathcal{S}_3 = S^{22}$. Its rank is equal to 8 and thus it follows that $d_3 = 1$. The corresponding elements $\mathbf{t}_1^{3j}$, $j = 1, 2$, are chosen to be equal to

$$\mathbf{t}_1^{32} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \text{ and } \mathbf{t}_1^{31} = \begin{pmatrix} 16 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Vectors x_{i3}^1, $i = 1, 2, 3$, have to solve equations

$$16A_{i0}x_{i1}^1 + 2A_{i1}x_{i2}^1 - A_{i2}x_{i2}^2 + A_{i0}x_{i2}^3 + A_{i3}x_{i3}^1 = 0.$$

A solution is $x_{13}^1 = \begin{bmatrix} 0 \\ 54 \\ 0 \\ -10 \end{bmatrix}$, $x_{23}^1 = 0$ *and* $x_{33}^1 = 0$. *Using the above vectors we construct the eigenvector*

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the root vectors

$$z_1^1 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; z_1^2 = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$z_1^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; z_2^1 = \left(\begin{bmatrix} 0 \\ -15 \\ 0 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$z_2^2 = \begin{bmatrix} 0 \\ -4 \\ 0 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; z_2^3 = 2 \left(\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$z_3^1 = \left(\begin{bmatrix} 0 \\ 54 \\ 0 \\ -10 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 30 \\ 0 \\ -8 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \\ 0 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $d_0 + d_1 + d_2 + d_3 = 8$ *it follows that the root subspace* $\mathcal{R}$ *is equal to* $\mathcal{R}_3$, *i.e., the ascent is* $M = 3$, *and that the above vectors* $z_0, z_1^1, z_1^2, z_1^3, z_2^1, z_2^2, z_2^3, z_3^1$ *form a basis for the root subspace at the eigenvalue* $(0, 0, 0)$. □

Example 6.2. *Consider a separable Hilbert space* H *with an orthonormal basis* $\{e_m\}_{m=0}^{\infty}$. *We define a two-parameter system on* $H_1 = H_2 = H$ *by:*

$$\frac{1}{2} A_{10} e_m = A_{21} e_m = \begin{cases} e_m, & m \text{ even}, \\ -e_m, & m \text{ odd}, \end{cases} \qquad A_{11} e_m = A_{20} e_m = \begin{cases} 0, & m = 0, 1, \\ e_m, & m \geq 2, \end{cases}$$

$$A_{12} e_m = A_{22} e_m = \begin{cases} e_0 - e_1, & m = 0, \\ -e_0 + e_1, & m = 1, \\ e_2 - e_3, & m = 2, \\ e_{m+1}, & m \geq 3. \end{cases}$$

Then the operators $\Delta_2 = A_{10} \otimes A_{21} - A_{11} \otimes A_{20} : H \to H$ *and*

$$\mathcal{A}_2 = \begin{bmatrix} A_{10}^\dagger & A_{11}^\dagger \\ A_{20}^\dagger & A_{21}^\dagger \end{bmatrix} : H^2 \to H^2$$

are given by

$$\Delta_2 e_m \otimes e_k = \begin{cases} (-1)^{m+k} 2e_m \otimes e_k, & \text{if } m \le 1 \text{ or } k \le 1, \\ \left((-1)^{m+k} 2 - 1\right) e_m \otimes e_k, & \text{if } m, k \ge 2, \end{cases}$$

and

$$\mathcal{A}_2 \begin{bmatrix} e_r \otimes e_s \\ e_m \otimes e_n \end{bmatrix} = \begin{cases} \begin{bmatrix} (-1)^r 2e_r \otimes e_s \\ (-1)^n e_m \otimes e_n \end{bmatrix}, & \text{if } m, s \le 1 \text{ and } n, r \ge 0, \\ \begin{bmatrix} (-1)^r 2e_r \otimes e_s \\ (-1)^n e_m \otimes e_n + e_r \otimes e_s \end{bmatrix}, & \text{if } m \le 1,\ s \ge 2 \text{ and } n, r \ge 0, \\ \begin{bmatrix} e_m \otimes e_n + (-1)^r 2e_r \otimes e_s \\ (-1)^n e_m \otimes e_n \end{bmatrix}, & \text{if } s \le 1,\ m \ge 2 \text{ and } n, r \ge 0, \\ \begin{bmatrix} e_m \otimes e_k + (-1)^r 2e_r \otimes e_s \\ (-1)^n e_m \otimes e_n + e_r \otimes e_s \end{bmatrix}, & \text{if } m, s \ge 2 \text{ and } n, r \ge 0. \end{cases}$$

It is easy to verify that Δ_2 is bounded, one-to-one and onto, and that $\mathcal{A}_2$ is onto. Hence Assumptions I and III follow. We choose $\boldsymbol{\lambda} = (0, 0, 1)$. Then $W_i(\boldsymbol{\lambda}) = A_{i2}$ for $i = 1, 2$. Since $\mathcal{N}(A_{i2}) = \mathcal{L}(e_0 + e_1)$ and $\mathcal{N}(A_{i2}^*) = \mathcal{L}(e_0 + e_1, e_2 + e_3)$, where $\mathcal{L}(S)$ is the linear span of the set S, it follows that $\boldsymbol{\lambda}$ is a geometrically simple eigenvalue and that both $W_i(\boldsymbol{\lambda})$ are Fredholm operators (of index -1). So Assumption II follows as well. For $i = 1, 2$ we choose $x_{i0} = y_{i0}^1 = e_0 + e_1$ and $y_{i0}^2 = e_2 + e_3$. Then $A_{ij}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $i = 1, 2$ and $j = 0, 1$, so B_0 is a 4×2 zero matrix and $d_1 = 2$. We choose $\mathbf{a}^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a}^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. To find the corresponding root vectors z_1^1 and z_1^2 we need to find vectors x_{i1}^k, $i, k = 1, 2$, such that

$$A_{i2} x_{i1}^1 + A_{i0} x_{i0} = 0 \text{ and } A_{i2} x_{i1}^2 + A_{i1} x_{i0} = 0.$$

We choose $x_{11}^1 = 2e_1$, $x_{11}^2 = x_{21}^1 = 0$ and $x_{21}^2 = e_1$. To find a basis for the root subspace $\mathcal{R}_2$ we follow the construction in §4 and we find that

$$\mathcal{S}_2 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $d_2 = 1$ and we choose $\mathbf{t} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$ to span the kernel of $\mathcal{S}_2$. Then

$$T = \psi^{-1}(\mathbf{t}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A possible choice for $\mathbf{a}^{02} \in \mathbb{C}^2$ and $x_{12}, x_{22} \in H$ that solve the corresponding equations (4.11) and (4.3) are $\mathbf{a}^{02} = \mathbf{0}$ and $x_{12} = x_{22} = 0$. To continue with the computations of possible basis vectors for the root subspace $\mathcal{R}_3$ observe that the corresponding matrix T_{3f}^2 is a 2×1 matrix, *say* $T^2_{3f} = \begin{bmatrix} t \\ z \end{bmatrix}$. *Then relation (5.3) with* $h_1 = h_3 = 1$ *and* $h_2 = 2$ *implies* $t = 0$ *and relation (5.3) with* $h_1 = h_3 = 2$ *and* $h_2 = 1$ *implies* $z = 0$. *Hence there is no nonzero solution for the matrix* T^2_{3f}. *So* $d_3 = 0$ *and* $\mathcal{R} = \mathcal{R}_2$. *From the previous discussion it follows that*

$$\mathcal{B} = \{(e_0 + e_1) \otimes (e_0 + e_1), 2e_1 \otimes (e_0 + e_1), (e_0 + e_1) \otimes e_1, 2e_1 \otimes e_1\}$$

is a basis of $\mathcal{R}$. □

Appendix: Proof of Lemma 5.3

The proof is similar to the proof of Proposition 4.2. We use a direct calculation to show (5.14). We set $i = 0$. The calculation for other i is similar.

First we have

$$(-1)^n (\Delta_0 - \lambda_0 \Delta_n) z_3^1 = \sum_{j=1}^{n} \begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j-1,1}x_{j-1,0} & \cdots & A_{j-1,n-1}x_{j-1,0} \\ -W_j(\boldsymbol{\lambda}) x_{j3}^1 & A_{j1}x_{j3}^1 & \cdots & A_{j,n-1}x_{j3}^1 \\ 0 & A_{j+1,1}x_{j+1,0} & \cdots & A_{j+1,n-1}x_{j+1,0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+ \sum_{\substack{j,k=1 \\ j<k}}^{n} \sum_{h_1,h_2=1}^{d_1} t^{31}_{1h_1h_2} \begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ -W_j(\boldsymbol{\lambda}) x_{j1}^{h_1} & A_{j1}x_{j1}^{h_1} & \cdots & A_{j,n-1}x_{j1}^{h_1} \\ \vdots & \vdots & & \vdots \\ -W_k(\boldsymbol{\lambda}) x_{k1}^{h_2} & A_{k1}x_{k1}^{h_2} & \cdots & A_{k,n-1}x_{k1}^{h_2} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+ \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \sum_{h_1=1}^{d_1} \sum_{h_2=1}^{d_2} t^{32}_{1h_1h_2} \begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ -W_j(\boldsymbol{\lambda}) x_{j1}^{h_1} & A_{j1}x_{j1}^{h_1} & \cdots & A_{j,n-1}x_{j1}^{h_1} \\ \vdots & \vdots & & \vdots \\ -W_k(\boldsymbol{\lambda}) x_{k2}^{h_2} & A_{k1}x_{k2}^{h_2} & \cdots & A_{k,n-1}x_{k2}^{h_2} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+\sum_{\mathbf{j}\in\underline{n}_3}\sum_{\mathbf{h}\in\underline{d_1}^3}\left(\sum_{g=1}^{d_2} t^2_{gh_1h_2}t^{32}_{1h_3g}\right)\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ -W_{j_1}(\boldsymbol{\lambda})\,x^{h_1}_{j_11} & A_{j_11}x^{h_1}_{j_11} & \cdots & A_{j_1,n-1}x^{h_1}_{j_11} \\ \vdots & \vdots & & \vdots \\ -W_{j_2}(\boldsymbol{\lambda})\,x^{h_2}_{j_21} & A_{j_21}x^{h_2}_{j_21} & \cdots & A_{j_2,n-1}x^{h_2}_{j_21} \\ \vdots & \vdots & & \vdots \\ -W_{j_3}(\boldsymbol{\lambda})\,x_{j_31} & A_{j_31}x^{h_3}_{j_31} & \cdots & A_{j_3,n-1}x^{h_3}_{j_31} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes}. \tag{A.1}$$

By virtue of (3.1), (4.3), (5.13) and column operations it follows that (A.1) is equal to

$$\sum_{j=1}^{n}\left(\sum_{g=1}^{d_2}\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j-1,1}x_{j-1,0} & \cdots & A_{j-1,n-1}x_{j-1,0} \\ U_j\left(\mathbf{a}^{23}_{g1}\right)x^g_{j2} & A_{j1}x^g_{j2} & \cdots & A_{j,n-1}x^g_{j2} \\ 0 & A_{j+1,1}x_{j+1,0} & \cdots & A_{j+1,n-1}x_{j+1,0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} + \right.$$

$$\left. + \sum_{g=1}^{d_1}\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j-1,1}x_{j-1,0} & \cdots & A_{j-1,n-1}x_{j-1,0} \\ U_j\left(\mathbf{a}^{13}_{g1}\right)x^g_{j1} & A_{j1}x^g_{j1} & \cdots & A_{j,n-1}x^g_{j1} \\ 0 & A_{j+1,1}x_{j+1,0} & \cdots & A_{j+1,n-1}x_{j+1,0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes}\right) +$$

$$+\begin{vmatrix} U_1\left(\mathbf{a}^{03}_1\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ U_2\left(\mathbf{a}^{03}_1\right)x_{20} & A_{21}x_{20} & \cdots & A_{2,n-1}x_{20} \\ \vdots & \vdots & & \vdots \\ U_n\left(\mathbf{a}^{03}_1\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n-1,n}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+\sum_{\substack{j,k=1\\ j<k}}^{n}\sum_{h_1,h_2=1}^{d_1} t^{31}_{1h_1h_2}\left(\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ U_j\left(\mathbf{a}^{01}_{h_1}\right)x_{j0} & A_{j1}x_{j0} & \cdots & A_{j,n-1}x_{j0} \\ \vdots & \vdots & & \vdots \\ 0 & A_{k1}x^{h_2}_{k1} & \cdots & A_{k,n-1}x^{h_2}_{k1} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes} + \right.$$

$$
+\begin{vmatrix}
0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\
\vdots & \vdots & & \vdots \\
0 & A_{j1}x_{j1}^{h_1} & \cdots & A_{j,n-1}x_{j1}^{h_1} \\
\vdots & \vdots & & \vdots \\
U_k\left(\mathbf{a}_{h_2}^{01}\right)x_{k0} & A_{k1}x_{k0} & \cdots & A_{k,n-1}x_{k0} \\
\vdots & \vdots & & \vdots \\
0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}
\end{vmatrix}^{\otimes}\Bigg) +
$$

$$
+\sum_{\substack{j,k=1\\ j\neq k}}^{n}\sum_{h_1=1}^{d_1}\sum_{h_2=1}^{d_2} t_{1h_1h_2}^{32}\left(\begin{vmatrix}
0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\
\vdots & \vdots & & \vdots \\
U_j\left(\mathbf{a}_{h_1}^{01}\right)x_{j0} & A_{j1}x_{j0} & \cdots & A_{j,n-1}x_{j0} \\
\vdots & \vdots & & \vdots \\
0 & A_{k1}x_{k2}^{h_2} & \cdots & A_{k,n-1}x_{k2}^{h_2} \\
\vdots & \vdots & & \vdots \\
0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}
\end{vmatrix}^{\otimes} + \right.
$$

$$
+\sum_{g=1}^{d_1}\begin{vmatrix}
0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\
\vdots & \vdots & & \vdots \\
0 & A_{j1}x_{j1}^{h_1} & \cdots & A_{j,n-1}x_{j1}^{h_1} \\
\vdots & \vdots & & \vdots \\
U_k\left(\mathbf{a}_{gh_2}^{12}\right)x_{k1}^{g} & A_{k1}x_{k1}^{g} & \cdots & A_{k,n-1}x_{k1}^{g} \\
\vdots & \vdots & & \vdots \\
0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}
\end{vmatrix}^{\otimes} +
$$

$$
+\begin{vmatrix}
0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\
\vdots & \vdots & & \vdots \\
0 & A_{j1}x_{j1}^{h_1} & \cdots & A_{j,n-1}x_{j1}^{h_1} \\
\vdots & \vdots & & \vdots \\
U_k\left(\mathbf{a}_{h_2}^{02}\right)x_{k0} & A_{k1}x_{k0} & \cdots & A_{k,n-1}x_{k0} \\
\vdots & \vdots & & \vdots \\
0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}
\end{vmatrix}^{\otimes}\Bigg) +
$$

$$+\sum_{\mathbf{j}\in\underline{n}_3}\sum_{\mathbf{h}\in\underline{d_1}^3}^{d_1}\left(\sum_{g=1}^{d_2} t^2_{gh_1h_2}t^{32}_{1h_3g}\right)\left(\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ U_{j_1}\left(\mathbf{a}^{01}_{h_1}\right)x_{j_10} & A_{j_11}x_{j_10} & \cdots & A_{j_1,n-1}x_{j_10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_21}x^{h_2}_{j_21} & \cdots & A_{j_2,n-1}x^{h_2}_{j_21} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_31}x^{h_3}_{j_31} & \cdots & A_{j_3,n-1}x^{h_3}_{j_31} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes}+\right.$$

$$+\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_11}x^{h_1}_{j_11} & \cdots & A_{j_1,n-1}x^{h_1}_{j_11} \\ \vdots & \vdots & & \vdots \\ U_{j_1}\left(\mathbf{a}^{01}_{h_2}\right)x_{j_20} & A_{j_21}x_{j_20} & \cdots & A_{j_2,n-1}x_{j_20} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_31}x^{h_3}_{j_31} & \cdots & A_{j_3,n-1}x^{h_3}_{j_31} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes}+$$

$$\left.+\begin{vmatrix} 0 & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_11}x^{h_1}_{j_11} & \cdots & A_{j_1,n-1}x^{h_1}_{j_11} \\ \vdots & \vdots & & \vdots \\ 0 & A_{j_21}x^{h_2}_{j_21} & \cdots & A_{j_2,n-1}x^{h_2}_{j_21} \\ \vdots & \vdots & & \vdots \\ U_{j_3}\left(\mathbf{a}^{01}_{h_3}\right)x_{j_30} & A_{j_31}x_{j_30} & \cdots & A_{j_3,n-1}x_{j_30} \\ \vdots & \vdots & & \vdots \\ 0 & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0} \end{vmatrix}^{\otimes}\right). \tag{A.2}$$

Next we apply all the conclusions of Theorem 3.2. In addition, note that the symmetry of matrices T_{2g} and Theorem 3.2*(iii)* imply that

$$\sum_{g=1}^{d_2} t^2_{gh_1h_2}t^{32}_{1h_3g}=\sum_{g=1}^{d_2} t^2_{gh_3h_1}t^{32}_{1h_2g}=\sum_{g=1}^{d_2} t^2_{gh_2h_3}t^{32}_{1h_1g}$$

for any $\mathbf{h}=(h_1,h_2,h_3)\in\underline{d_1}^3$. Then (A.2) is equal to

$$\begin{vmatrix} U_1\left(\mathbf{a}^{03}_1\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10} \\ U_2\left(\mathbf{a}^{03}_1\right)x_{20} & A_{21}x_{20} & \cdots & A_{2,n-1}x_{20} \\ \vdots & \vdots & & \vdots \\ U_n\left(\mathbf{a}^{03}_1\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n-1,n}x_{n0} \end{vmatrix}^{\otimes}+$$

$$+\sum_{j=1}^{n}\sum_{h_1,h_2=1}^{d_1} t^{31}_{1h_1h_2}\begin{vmatrix} U_1\left(\mathbf{a}^{01}_{h_1}\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10}\\ \vdots & \vdots & & \vdots\\ U_j\left(\mathbf{a}^{01}_{h_1}\right)x^{h_2}_{j1} & A_{j1}x^{h_2}_{j1} & \cdots & A_{j,n-1}x^{h_2}_{j1}\\ \vdots & \vdots & & \vdots\\ U_n\left(\mathbf{a}^{01}_{h_1}\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}\end{vmatrix}^{\otimes} +$$

$$+\sum_{j=1}^{n}\sum_{h_1=1}^{d_1}\sum_{h_2=1}^{d_2} t^{32}_{1h_1h_2}\begin{vmatrix} U_1\left(\mathbf{a}^{01}_{h_1}\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10}\\ \vdots & \vdots & & \vdots\\ U_j\left(\mathbf{a}^{01}_{h_1}\right)x^{h_2}_{j2} & A_{j1}x^{h_2}_{j2} & \cdots & A_{j,n-1}x^{h_2}_{j2}\\ \vdots & \vdots & & \vdots\\ U_n\left(\mathbf{a}^{01}_{h_1}\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}\end{vmatrix}^{\otimes} +$$

$$+\sum_{j=1}^{n}\sum_{h_1=1}^{d_1}\sum_{h_2=1}^{d_2} t^{32}_{1h_1h_2}\begin{vmatrix} U_1\left(\mathbf{a}^{02}_{h_2}\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10}\\ \vdots & \vdots & & \vdots\\ U_j\left(\mathbf{a}^{02}_{h_2}\right)x^{h_1}_{j1} & A_{j1}x^{h_1}_{j1} & \cdots & A_{j,n-1}x^{h_1}_{j1}\\ \vdots & \vdots & & \vdots\\ U_n\left(\mathbf{a}^{02}_{h_2}\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}\end{vmatrix}^{\otimes} +$$

$$+\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n}\sum_{\mathbf{h}\in\underline{d_1}^3}\sum_{g=1}^{d_2} t^{2}_{gh_1h_2}t^{32}_{1h_3g}\begin{vmatrix} U_1\left(\mathbf{a}^{01}_{h_3}\right)x_{10} & A_{11}x_{10} & \cdots & A_{1,n-1}x_{10}\\ \vdots & \vdots & & \vdots\\ U_{j_1}\left(\mathbf{a}^{01}_{h_3}\right)x^{h_1}_{j_11} & A_{j_11}x^{h_1}_{j_11} & \cdots & A_{j_1,n-1}x^{h_1}_{j_11}\\ \vdots & \vdots & & \vdots\\ U_{j_2}\left(\mathbf{a}^{01}_{h_3}\right)x^{h_2}_{j_21} & A_{j_21}x^{h_2}_{j_21} & \cdots & A_{j_2,n-1}x^{h_2}_{j_21}\\ \vdots & \vdots & & \vdots\\ U_n\left(\mathbf{a}^{01}_{h_3}\right)x_{n0} & A_{n1}x_{n0} & \cdots & A_{n,n-1}x_{n0}\end{vmatrix}^{\otimes}. \tag{A.3}$$

Finally, by Theorem 3.2*(ii)* and column operations, it follows that (A.3) is equal to

$$(-1)^n\,\Delta_n\Bigg(a^{03}_{10}x_{10}\otimes x_{20}\otimes\cdots\otimes x_{n0}+\sum_{h_1=1}^{d_1}a^{13}_{h_110}\sum_{j=1}^{n}x_{10}\otimes\cdots\otimes x^{h_1}_{j1}\otimes\cdots\otimes x_{n0}+$$

$$+\sum_{h_2=1}^{d_2}a^{23}_{h_210}\sum_{j=1}^{n}x_{10}\otimes\cdots\otimes x^{h_2}_{j2}\otimes\cdots\otimes x_{n0}+$$

$$+\sum_{g=1}^{d_2}a^{23}_{g10}\sum_{h_1,h_2=1}^{d_1}t^{2}_{gh_1h_2}\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{n}x_{10}\otimes\cdots\otimes x^{h_1}_{j_11}\otimes\cdots\otimes x^{h_2}_{j_21}\otimes\cdots\otimes x_{n0}\Bigg)=$$

$$= (-1)^n \, \Delta_n \left(a_{10}^{03} z_0 + \sum_{h=1}^{d_1} a_{h10}^{13} \, z_1^h + \sum_{h=1}^{d_2} a_{h10}^{23} z_2^h \right).$$

□

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