On stability of invariant subspaces of commuting matrices

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Abstract

We study stability of (joint) invariant subspaces of a finite set of commuting matrices. We generalize some of the results of Gohberg, Lancaster, and Rodman for the single matrix case. For sets of two or more commuting matrices we exhibit some phenomena different from the single matrix case. We show that each root subspace is a stable invariant subspace, that each invariant subspace of a root subspace of a nonderogatory eigenvalue is stable, and that, even in the derogatory case, the eigenspace is stable if it is one-dimensional. We prove that a pair of commuting matrices has only finitely many stable invariant subspaces. At the end, we discuss the stability of invariant subspaces of an algebraic multiparameter eigenvalue problem.

1 Introduction

In the paper we study stability of invariant subspaces of k-tuples ($k \geq 2$) of commuting matrices. The problem of stability arose in applications to multiparameter eigenvalue problems [1]. The stability is crucial when numerical calculations are performed to find a basis of an invariant subspace [16]. In this paper, an invariant (resp. root) subspace of a k-tuple of commuting matrices always refers to a joint invariant (resp. root) subspace of the k-tuple.

In the single matrix case (i.e., if $k = 1$) Gohberg, Lancaster, and Rodman [4] characterized all stable invariant subspaces. They showed that each root subspace is stable, each invariant subspace of a root subspace of a nonderogatory eigenvalue is stable, and that direct sums of these two types of subspaces are the only stable invariant subspaces. We generalize most of these results. We show that each root subspace is stable, and that each invariant subspace of a root subspace of a nonderogatory eigenvalue is stable. Moreover, if there is only one invariant subspace of a root subspace of a given dimension then it is stable. In particular, if the eigenspace is one-dimensional then it is stable. We show that also direct sums of these types of subspaces are stable. However, we do not know if these are the only possible stable invariant subspaces of a k-tuple of commuting matrices.

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We obtain some further results for pairs of commuting matrices. We show that a pair of commuting matrices has only finitely many stable invariant subspaces. We consider a few examples and state a number of open problems. We conclude with some results on the stability of invariant subspaces of an algebraic multiparameter eigenvalue problem. Such a problem has an associated k-tuple of commuting matrices. (See §5 for a brief introduction and [1] for details.) Plestenjak [16] studied a numerical algorithm for computing a basis of a root subspace at a nonderogatory eigenvalue of an associated k-tuple of commuting matrices. Since each invariant subspace of a root subspace of a nonderogatory eigenvalue is stable there is no problem of stability in the algorithm presented in [16].

2 Preliminaries

Let $\mathbf{A} = (A_1, \ldots, A_k)$, $(k \geq 2)$, be a set of commuting $n \times n$ matrices over $\mathbb{C}$. We say that a subspace $\mathcal{N}$ of $\mathbb{C}^n$ is $\mathbf{A}$–*invariant* if

$$A_l \mathcal{N} \subset \mathcal{N}, \quad l = 1, \ldots, k.$$

The set of all $\mathbf{A}$–invariant subspaces is denoted by $\mathrm{Inv}(\mathbf{A})$.

A k-tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ is an *eigenvalue* of a set of commuting matrices $\mathbf{A}$ if

$$\mathrm{Ker}(\mathbf{A} - \boldsymbol{\lambda I}) := \bigcap_{l=1}^{k} \mathrm{Ker}(A_l - \lambda_l I) \neq \{0\}.$$

A nonzero vector $z \in \mathrm{Ker}(\mathbf{A} - \boldsymbol{\lambda I})$ is an eigenvector for $\boldsymbol{\lambda}$ and $\mathbf{A}$. The *root subspace* for an eigenvalue $\boldsymbol{\lambda}$ is denoted by $\mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$ and is equal to

$$\bigcap_{l_1 + \cdots + l_k = n} \mathrm{Ker}\left[(A_1 - \lambda_1 I)^{l_1} \cdots (A_k - \lambda_k I)^{l_k}\right]$$

An eigenvalue $\boldsymbol{\lambda}$ is called *geometrically simple* if $\dim \mathrm{Ker}(\mathbf{A} - \boldsymbol{\lambda I}) = 1$. It is called *nonderogatory* if

$$\dim \bigcap_{l=1}^{k} \mathrm{Ker}(A_l - \lambda_l I)^j = j,$$

for $j = 1, 2, \ldots, \dim \mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$. We say that an eigenvalue is *derogatory* if it is not nonderogatory. We remark that an eigenvalue is nonderogatory if it is geometrically simple and $\dim \bigcap_{l=1}^{k} \mathrm{Ker}(A_l - \lambda_l I)^2 \leq 2$ (see [12, Cor. 2] and [13, Thm. 7]). If the eigenvalue is nonderogatory then, in $\mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$, there is exactly one $\mathbf{A}$-invariant subspace of dimension j for each $j = 0, 1, \ldots, \dim \mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$. This follows from the definition of a nonderogatory eigenvalue.

If a simple rectifiable contour γ_l splits the spectrum of A_l for $l = 1, \ldots, k$, then the Riesz projectors are defined by

$$P(A_l, \gamma_l) := \frac{1}{2\pi i} \int_{\gamma_l} (\lambda I - A_l)^{-1} d\lambda,$$

$l = 1, \ldots, k$. They commute and we define

$$P(\mathbf{A}; \boldsymbol{\gamma}) := P(A_1; \gamma_1) \cdots P(A_k; \gamma_k).$$

The *gap* between the subspaces $\mathcal{L}$ and $\mathcal{M}$ in $\mathbb{C}^n$ is defined by

$$\theta(\mathcal{L}, \mathcal{M}) = \|P_{\mathcal{L}} - P_{\mathcal{M}}\|,$$

where $P_{\mathcal{L}}$ and $P_{\mathcal{M}}$ are the orthogonal projectors on $\mathcal{L}$ and $\mathcal{M}$, respectively. If $\mathcal{L}, \mathcal{M} \neq \{0\}$ then

$$\theta(\mathcal{L}, \mathcal{M}) = \max \left\{ \sup_{\substack{x \in \mathcal{M} \\ \|x\|=1}} d(x, \mathcal{L}), \sup_{\substack{x \in \mathcal{L} \\ \|x\|=1}} d(x, \mathcal{M}) \right\} \tag{1}$$

(see Theorem 13.1.1 in [4, p. 388]).

We say that an $\mathbf{A}$–invariant subspace $\mathcal{N}$ is *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{B} = (B_1, \ldots, B_k)$ is a set of commuting matrices with $\|A_l - B_l\| < \delta$ for $l = 1, \ldots, k$, then there exists a $\mathbf{B}$–invariant subspace $\mathcal{M}$ such that

$$\theta(\mathcal{N}, \mathcal{M}) < \epsilon.$$

For comparison with our results we state Theorem 15.2.1 of [4, p. 448] that characterizes stable invariant subspaces for a single matrix.

Theorem 2.1 (Gohberg, Lancaster, and Rodman) *Suppose that $\lambda_1, \ldots, \lambda_r$ are all the distinct eigenvalues of an $n \times n$ matrix A over $\mathbb{C}$. A subspace $\mathcal{N}$ of $\mathbb{C}^n$ is A–invariant and stable if and only if $\mathcal{N} = \mathcal{N}_1 \dot{+} \cdots \dot{+} \mathcal{N}_r$, where for each j the subspace $\mathcal{N}_j$ is an arbitrary A–invariant subspace of $\mathcal{R}_{\lambda_j}(A)$ if $\dim \operatorname{Ker}(\lambda_j I - A) = 1$, and either $\mathcal{N}_j = \{0\}$ or $\mathcal{N}_j = \mathcal{R}_{\lambda_j}(A)$ if $\dim \operatorname{Ker}(\lambda_j I - A) \geq 2$.*

3 Stability and root subspaces

In this section we show that it suffices to study the stability of invariant subspaces of root subspaces of $\mathbf{A}$. The main result is that an $\mathbf{A}$–invariant subspace $\mathcal{N}$ of $\mathbb{C}^n$ is stable if and only if $\mathcal{N}$ is a direct sum $\mathcal{N}_1 \dot{+} \cdots \dot{+} \mathcal{N}_r$, where each $\mathcal{N}_j$ is a stable $\mathbf{A}$–invariant subspace of a root subspace of $\boldsymbol{A}$. The following two lemmas are generalizations of Lemmas 15.3.2 and 15.3.3 of [4, pp. 452-454]. The proofs are almost identical and therefore omitted.

Lemma 3.1 *Let $\gamma_i \subset \mathbb{C}$ be a simple rectifiable contour that splits the spectrum of A_i for $i = 1, \ldots, k$. Let*

$$P(\mathbf{A}; \boldsymbol{\gamma}) = P(A_1; \gamma_1) \cdots P(A_k; \gamma_k)$$

be the Riesz projector for $\boldsymbol{A}$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$ and let $\mathbf{A}_0 = (A_{10}, \ldots, A_{k0})$ be the restriction of $\mathbf{A}$ to $\operatorname{Im} P(\mathbf{A}; \boldsymbol{\gamma})$. Let $\mathcal{N}$ be a subspace of $\operatorname{Im} P(\mathbf{A}; \boldsymbol{\gamma})$. Then $\mathcal{N}$ is a stable invariant subspace for $\mathbf{A}$ if and only if $\mathcal{N}$ is a stable invariant subspace for $\mathbf{A}_0$.

Lemma 3.2 *Let $\mathcal{N} \subset \mathbb{C}^n$ be an invariant subspace of $\mathbf{A} = (A_1, \ldots, A_k)$ and assume that the contour $\gamma_i \subset \mathbb{C}$ splits the spectrum of A_i for $i = 1, \ldots, k$. If $\mathcal{N}$ is stable for $\mathbf{A}$ then $P(\mathbf{A}; \boldsymbol{\gamma})\mathcal{N}$ is a stable invariant subspace for the restriction $\mathbf{A}_0 = (A_{10}, \ldots, A_{k0})$ of $\mathbf{A}$ to* $\operatorname{Im} P(\mathbf{A}; \boldsymbol{\gamma})$.

Lemmas 3.1 and 3.2 imply the following theorem.

Theorem 3.3 *Let $\boldsymbol{\lambda_1} = (\lambda_{11}, \ldots, \lambda_{1k}), \ldots, \boldsymbol{\lambda_r} = (\lambda_{r1}, \ldots, \lambda_{rk})$ be all the different eigenvalues of a set of commuting matrices $\mathbf{A} = (A_1, \ldots, A_k)$. A subspace $\mathcal{N}$ of $\mathbb{C}^n$ is $\mathbf{A}$–invariant and stable if and only if $\mathcal{N} = \mathcal{N}_1 \dot{+} \cdots \dot{+} \mathcal{N}_r$, where $\mathcal{N}_j$ is a stable $\mathbf{A}_j$–invariant subspace of the restriction $\mathbf{A}_j = (A_{j1}, \ldots, A_{jk})$ of $\boldsymbol{A}$ to $\mathcal{R}_{\boldsymbol{\lambda_j}}(\mathbf{A})$ for $j = 1, \ldots, r$.*

Proof. Suppose that $\mathcal{N}$ is a stable $\mathbf{A}$–invariant subspace. It is easy to see that $\mathcal{N} = \mathcal{N}_1 \dot{+} \cdots \dot{+} \mathcal{N}_r$, where $\mathcal{N}_j = \mathcal{N} \cap \mathcal{R}_{\boldsymbol{\lambda_j}}(\mathbf{A})$ for $j = 1, \ldots, r$. It follows from Lemma 3.2 that $\mathcal{N}_j$ is a stable invariant subspace of the restriction $\mathbf{A}_j$ for $j = 1, \ldots, r$.

Next assume that each $\mathcal{N}_j$ is a stable $\mathbf{A}_j$–invariant subspace. Lemma 3.1 implies that $\mathcal{N}_j$ is a stable invariant subspace for $\mathbf{A}$ and therefore the direct sum $\mathcal{N} = \mathcal{N}_1 \dot{+} \cdots \dot{+} \mathcal{N}_r$ is a stable invariant subspace for $\mathbf{A}$. ∎

Theorem 3.3 is similar to but weaker than Theorem 2.1 as it does not characterize the stable invariant subspaces. In particular, it is not yet clear which invariant subspaces of a root subspace at a derogatory eigenvalue are stable. Nevertheless, it enables us to study only the restriction of a set of commuting matrices to a root subspace.

Now we are able to show that as it is the case for a single matrix a root subspace is a stable invariant subspace for a set of commuting matrices.

Theorem 3.4 *If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is an eigenvalue of a set of commuting matrices $\mathbf{A} = (A_1, \ldots, A_k)$ then the root subspace $\mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$ is a stable invariant subspace.*

Proof. Let $\gamma_i \subset \mathbb{C}$ be such closed contour that λ_i lies inside γ_i and all the other eigenvalues of A_i lie outside γ_i for $i = 1, \ldots, k$. It follows that the root subspace $\mathcal{R}_{\boldsymbol{\lambda}}(\mathbf{A})$ is equal to the image of the Riesz projector $P(\mathbf{A}; \boldsymbol{\gamma}) = P(A_1; \gamma_1) \cdots P(A_k; \gamma_k)$.

Let $\mathbf{B} = (B_1, \ldots, B_k)$ be a set of commuting matrices. If $\|B_i - A_i\|$ is sufficiently small then the matrix $\lambda I - B_i$ is invertible for every $\lambda \in \gamma_i$ and the Riesz projector $P(B_i; \gamma_i)$ is well defined. For each $\epsilon > 0$ there exists $\delta > 0$ such that if $\|B_i - A_i\| < \delta$ then it follows $\|P(B_i; \gamma_i) - P(A_i; \gamma_i)\| < \epsilon$ (see [4, p. 448] for details).

The subspace $\operatorname{Im} P(\boldsymbol{B}; \boldsymbol{\gamma})$, where $P(\mathbf{B}; \boldsymbol{\gamma}) = P(B_1; \gamma_1) \cdots P(B_k; \gamma_k)$, is invariant for $\mathbf{B}$. It is easy to see that for each $\eta > 0$ there exists $\epsilon > 0$ such that if $\|P(B_i; \gamma_i) - P(A_i; \gamma_i)\| < \epsilon$ for $i = 1, \ldots, k$ then it follows that

$$\theta(\operatorname{Im} P(\boldsymbol{B}; \boldsymbol{\gamma}), \operatorname{Im} P(\boldsymbol{A}; \boldsymbol{\gamma})) < \eta.$$

As a consequence $\operatorname{Im} P(\boldsymbol{A}; \boldsymbol{\gamma})$ is a stable invariant subspace. ∎

Theorem 3.3 implies that it is enough to treat only sets of nilpotent commuting matrices. First we show that invariant subspaces of root subspaces of nonderogatory eigenvalues are stable. This also coincides with the theory for the single matrix case.

A chain of subspaces

$$\{0\} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n = \mathbb{C}^n$$

is called *complete* if $\dim \mathcal{M}_i = i$ for $i = 0, 1, \ldots, n$. It is well known fact that a set of commuting matrices is simultaneously similar to a set of upper-triangular commuting matrices. It follows then that for every set of commuting matrices there exists a complete chain of invariant subspaces. Furthermore, we claim that if $\mathcal{M}$ is $\mathbf{A}$-invariant subspace then there exists a complete chain of invariant subspaces that contains $\mathcal{M}$. Suppose that $\mathbb{C}^n = \mathcal{M} \oplus \mathcal{N}$ is a direct sum decomposition and that with respect to this decomposition $A_i = \begin{bmatrix} B_i & C_i \\ 0 & D_i \end{bmatrix}$, $i = 1, \ldots, k$. Then $(B_1, \ldots, B_k)$ and $(D_1, \ldots, D_k)$ are k-tuples of commuting matrices and they are simultaneously similar to upper-triangular matrices. The claim now follows easily.

The following theorem is a generalization of Theorem 15.2.3 of [4, p. 449]. The proof is very similar and it is omitted.

Theorem 3.5 *Let $\mathbf{A} = (A_1, \ldots, A_k)$ be a set of commuting matrices. For a given $\epsilon > 0$, there exists $\delta > 0$ such that the following holds: if $\mathbf{B} = (B_1, \ldots, B_k)$ is such a set of commuting matrices that $\|A_i - B_i\| < \delta$ for $i = 1, \ldots, k$ and $\{\mathcal{M}_j\}$ is a complete chain of $\mathbf{B}$–invariant subspaces, then there exists a complete chain $\{\mathcal{N}_j\}$ of $\mathbf{A}$–invariant subspaces such that $\theta(\mathcal{N}_j, \mathcal{M}_j) < \epsilon$ for $j = 1, \ldots, n-1$.*

Corollary 3.6 *If $\mathbf{0} = (0, \ldots, 0)$ is a nonderogatory eigenvalue of a set of nilpotent commuting matrices $\mathbf{A} = (A_1, \ldots, A_k)$ then each $\mathbf{A}$–invariant subspace is stable.*

Proof. Since the eigenvalue $\mathbf{0}$ is nonderogatory the set $\mathbf{A}$ has only one j-dimensional invariant subspace $\mathcal{N}_j$ for $j = 0, 1, \ldots, n$. (See the definition of a nonderogatory eigenvalue and the remark following it.) Subspaces $\mathcal{N}_0, \ldots, \mathcal{N}_n$ form a complete chain and we can apply Theorem 3.5. ∎

Corollary 3.7 *Let $\mathbf{A} = (A_1, \ldots, A_k)$ be a set of nilpotent commuting matrices. If $\mathcal{N}$ is the only $\mathbf{A}$–invariant subspace of the dimension $\dim \mathcal{N}$, then $\mathcal{N}$ is a stable $\mathbf{A}$–invariant subspace.*

Proof. Recall that there always exists a complete chain of invariant subspaces for $\mathbf{A}$. Suppose that $\mathbf{A}$ has only one invariant subspace $\mathcal{N}$ of the dimension $\dim \mathcal{N}$. It follows then that the subspace $\mathcal{N}$ is a part of all complete chains of invariant subspaces. The result now follows from Theorem 3.5. ∎

A simple consequence of Corollary 3.7 is stability of the eigensubspace of a geometrically simple eigenvalue. The eigenvalue need not be nonderogatory and this result differs from the single matrix case. Namely, in the single matrix case, it follows that if an eigenspace is one-dimensional then the eigenvalue is nonderogatory and the stability follows by Theorem 2.1. On the other hand, in the case of a set of commuting matrices there exist eigenvalues that are geometrically simple and derogatory (see Example 3.9).

Corollary 3.8 *If $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ is a geometrically simple eigenvalue of a set of commuting matrices $\mathbf{A} = (A_1, \ldots, A_k)$ then the eigenspace $\operatorname{Ker}(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})$ is a stable invariant subspace.*

Example 3.9 *Suppose that $n = 3$ and that e_i, $i = 1, 2, 3$, are the standard basis vectors for $\mathbb{C}^3$. Then*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a pair of nilpotent commuting matrices for which $\mathbf{0}$ is a geometrically simple and derogatory eigenvalue. By Corollary 3.8 the eigenspace $\mathcal{L}(e_1)$ is a stable invariant subspace. Here $\mathcal{L}(X)$ is the linear span of the set of vectors X.

Assume that ϵ is a small positive number. Consider now two commuting perturbations:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \epsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & \epsilon & 0 \end{bmatrix}.$$

It is easy to observe that $\mathcal{L}(e_1, e_2)$ is the only two-dimensional invariant subspace for the first perturbation and that $\mathcal{L}(e_1, e_3)$ is the only two-dimensional invariant subspace for the second perturbation. Therefore the pair (A, B) has no stable invariant subspace of dimension 2.

If we take the transposed matrices

$$A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

then $\mathcal{L}(\alpha e_2 + \beta e_3)$, for $(\alpha, \beta) \in \mathbb{C}^2 \setminus (0, 0)$, are all the one-dimensional invariant subspaces of $\left(A^T, B^T\right)$, while $\mathcal{L}(e_2, e_3)$ is the only two-dimensional invariant subspace. It follows by Corollary 3.6 that $\mathcal{L}(e_2, e_3)$ is stable. The above analysis of pair (A, B) also shows that there is no stable one-dimensional invariant subspace for $\left(A^T, B^T\right)$.

The example $\mathcal{L}(e_2, e_3)$, which is a two-dimensional eigenspace for $\left(A^T, B^T\right)$, shows that eigenspaces of dimension more than one can be stable invariant subspaces for sets of two or more commuting matrices. This differs from a single matrix case where it follows from Theorem 2.1 that all the eigenspaces of dimension two or more that are proper subspaces of a root subspace are unstable invariant subspaces. ■

Problem 3.10 The main problem that remains open is to characterize all stable invariant subspaces of a k-tuple of nilpotent commuting matrices.

Question 3.11 *It is known that for a fixed dimension d the variety of d-dimensional invariant subspaces of a single nilpotent matrix is connected [8, 17]. Is the variety of d-dimensional invariant subspaces of a k-tuple of nilpotent commuting matrices still connected?*

4 A pair of commuting matrices

If the set contains only two commuting matrices, then we are able to show some additional results. First we show that although a pair of commuting matrices A and B may have infinitely many invariant subspaces, it has only finitely many stable invariant subspaces. We use the fact that the set of pairs of commuting matrices where one of the matrices is nonderogatory is dense in the set of all pairs of commuting matrices. It was pointed out to us by one of the referees that this was an old result proved first by Motzkin and Taussky [15] and rediscovered several times. (See [6].) We reproduce here a proof given by Guralnick [6]. We do so for the convenience of the reader and to facilitate the discussion on commuting triples of matrices.

We say an $n \times n$ matrix is *generic* if it has n distinct eigenvalues.

Theorem 4.1 *If (A, B) is a pair of commuting $n \times n$ matrices over $\mathbb{C}$ then it has finitely many stable invariant subspaces. More precisely, it has at most $2^n - 1$ nonzero stable invariant subspaces.*

Proof. It follows from Theorem 3.3 that it is enough to consider only a commuting pair of nilpotent matrices. If $(0, 0)$ is a nonderogatory eigenvalue for (A, B), then there are only finitely many invariant subspaces which are all stable as a result of Corollary 3.6. Thus we assume that $(0, 0)$ is a derogatory eigenvalue.

Let $A = XJX^{-1}$, where

$$J = \operatorname{diag}(J_{n_1}, \ldots, J_{n_r})$$

is the Jordan canonical form for A. Since $(0, 0)$ is a derogatory eigenvalue for (A, B), 0 is a derogatory eigenvalue for A and $r \geq 2$. For distinct $\lambda_1, \ldots, \lambda_r$ the matrix

$$R = X \operatorname{diag}(\lambda_1 I_{n_1} + J_{n_1}, \ldots, \lambda_r I_{n_r} + J_{n_r}) X^{-1}$$

is nonderogatory and commutes with matrix A.

The matrix

$$B_\epsilon = B + \epsilon R \tag{2}$$

commutes with A for arbitrary $\epsilon \in \mathbb{C}$. Matrix B_ϵ is nonderogatory except for finitely many values of ϵ. Therefore it is possible to choose arbitrary small $\epsilon > 0$ such that B_ϵ is nonderogatory.

Assume now that B_ϵ is nonderogatory. Then there exists a polynomial p such that $A = p(B_\epsilon)$. For an arbitrary $\delta > 0$ we can approximate B_ϵ with a generic matrix G such that $\|B_\epsilon - G\| < \delta$. Since $A = p(B_\epsilon)$, there exists $\delta > 0$ such that $\|A - p(G)\| < \eta$ for $\|B - G\| < \delta$, i.e. pair $(p(G), G)$ is close to pair (A, B).

Since G is a generic matrix, it has only finitely many invariant subspaces and it follows that the pair (A, B) has only finitely many stable invariant subspaces. Namely, if G is generic then pair $(G, p(G))$ has $2^n - 1$ nonzero invariant subspaces. For $\epsilon > 0$ but small, these subspaces can be close to at most $2^n - 1$ invariant subspaces of (A, B). ∎

Observe that in the above proof the polynomial p can be chosen so that both G and $p(G)$ are generic. Also note that, in general, the bound $2^n - 1$ is best possible. If A (or B) is generic then pair (A, B) has precisely $2^n - 1$ nonzero stable invariant subspaces.

The following lemma shows that for a pair of commuting matrices stable invariant subspaces are determined by invariant subspaces of nearby generic commuting pairs.

Lemma 4.2 *Let (A, B) be a pair of commuting nilpotent matrices and let $\mathcal{N}$ be an (A, B)–invariant subspace. Then $\mathcal{N}$ is stable if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\left(\widetilde{A}, \widetilde{B}\right)$ is a pair of generic commuting matrices with $\|\widetilde{A} - A\|, \|\widetilde{B} - B\| < \delta$ then there exists a $\left(\widetilde{A}, \widetilde{B}\right)$–invariant subspace $\mathcal{M}$ such that $\theta(\mathcal{N}, \mathcal{M}) < \epsilon$.*

Proof. We only need to show that the condition is sufficient for the stability of $\mathcal{N}$. Suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\left(\widetilde{A}, \widetilde{B}\right)$ is a pair of generic commuting matrices with $\|\widetilde{A} - A\|, \|\widetilde{B} - B\| < \delta$ then there exists a $\left(\widetilde{A}, \widetilde{B}\right)$–invariant subspace $\mathcal{M}$ such that $\theta(\mathcal{N}, \mathcal{M}) < \epsilon$.

Let $\mathcal{N}$ be an unstable invariant subspace for (A, B). Then for each $m = 1, 2, \ldots$ there exist commuting pair (A_m, B_m) and $\eta_m > 0$ such that $\|A_m - A\|, \|B_m - B\| < 1/m$ and that $\theta(\mathcal{N}, \mathcal{M}_m) \geq \epsilon + \eta_m$ for all invariant subspaces $\mathcal{M}_m$ of (A_m, B_m). It follows from Theorem 3.5 that there exists $\vartheta_m > 0$ such that $\vartheta_m < 1/m$ and that if $\left(\widetilde{A}_m, \widetilde{B}_m\right)$ is a commuting pair that satisfies $\|\widetilde{A}_m - A_m\|, \|\widetilde{B}_m - B_m\| < \vartheta_m$ then for each $\left(\widetilde{A}_m, \widetilde{B}_m\right)$–invariant subspace $\widetilde{\mathcal{P}}$ there exist (A_m, B_m)–invariant subspace $\mathcal{P}$ such that $\theta(\mathcal{P}, \widetilde{\mathcal{P}}) < \eta_m/2$.

Since it is possible to find a generic commuting pair arbitrarily close to the original commuting pair (see the proof of Theorem 4.1), for each $m = 1, 2, \ldots$ this implies the existence of a generic commuting pair $\left(\widetilde{A}_m, \widetilde{B}_m\right)$ such that $\|\widetilde{A}_m - A\|, \|\widetilde{B}_m - B\| < 2/m$ and that $\theta(\mathcal{N}, \mathcal{M}_m) > \epsilon$ for all invariant subspaces $\mathcal{M}_m$ of $\left(\widetilde{A}_m, \widetilde{B}_m\right)$. This contradicts the initial assumption and thus it follows that $\mathcal{N}$ has to be a stable invariant subspace. ■

Question 4.3 *Is the set of stable invariant subspaces of any k-tuple ($k \geq 3$) of commuting matrices finite?*

Question 4.4 *For a single matrix an invariant subspace is stable if and only if it corresponds to an isolated point of the variety of invariant subspaces. Is this the case also for a pair (or more generally for a k-tuple, $k \geq 3$) of commuting matrices? (See also Example 4.6.)*

Remark 4.5 *If a set contains three or more commuting matrices then, in general, it is not possible to construct a nearby generic commutative set as it is done for pairs in the proof of Theorem 4.1. Suppose that we have a set of commuting matrices (A, B, C). If we follow the proof of Theorem 4.1 then it fails in the moment when we want to use the matrix B_ϵ. This matrix commutes with A but not necessarily with C. Guralnick [6] has even shown that in the general case of three or more commuting matrices it is not possible to approximate the set with a set of generic commuting matrices (see also [7, 9, 10]). For this reason it is not possible, in general, to extend the proof of Theorem 4.1 to the commutative sets with more than two matrices. It follows from the results of Guralnick [6]* *that the approximation for commuting k-tuples, $k \geq 4$, is possible if the size n of matrices is at most 3 and is not possible in general if $n \geq 4$. For triples of commuting matrices, it follows by results of Holbrook and Omladič in [10] that the approximation is possible if the size n is at most 5 and is not possible if $n \geq 30$. For the remaining n, it is not known if the approximation is possible. The bounds for n in [10] are an improvement of bounds given earlier by Guralnick [6] and Guralnick and Sethuraman [7]. We conclude that the same arguments as in the proof of Theorem 4.1 show that if $k = 3$ and $n \leq 5$ or $k \geq 4$ and $n \leq 3$ then a k-tuple of commuting $n \times n$ matrices has only finitely many stable invariant subspaces.*

Note that Lemma 4.2 can not be generalized to the arbitrary sets of three or more commuting matrices for the same reasons as Theorem 4.1. ■

Example 4.6 *Suppose that $n = 4$ and that e_i, $i = 1, 2, 3, 4$, are the standard basis vectors for $\mathbb{C}^4$. Then*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a pair of nilpotent commuting matrices. The eigenvalue $\mathbf{0}$ is geometrically simple and derogatory. By Corollary 3.8 the eigenspace $\mathcal{L}(e_1)$ is a stable invariant subspace. Recall that $\mathcal{L}(X)$ is the linear span of the set of vectors X. It is easy to show that two-dimensional invariant subspaces form the family $\mathcal{L}(e_1, \alpha e_2 + \beta e_3)$ and three-dimensional invariant subspaces form the family $\mathcal{L}\left(e_1, e_2 + e_3, \alpha e_4 + \beta(e_2 - e_3)\right)$, where $(\alpha, \beta) \in \mathbb{C}^2 \setminus ((0, 0))$.

Assume that ϵ is a small positive number. Consider now two commuting perturbations of (A, B):

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & \epsilon & -\epsilon & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & \epsilon & -\epsilon & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon^8 & 0 & 0 & 1 \\ 0 & -\epsilon^4 & \epsilon^4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ \epsilon^8 & 0 & 0 & 1 \\ 2\epsilon^8 & 0 & 0 & 1+\epsilon^4 \\ 0 & -\epsilon^4+\epsilon^8 & \epsilon^4 & 0 \end{bmatrix}.$$

The first perturbed pair is nilpotent and nonderogatory. Its complete chain of invariant subspaces is $\{0\} \subset \mathcal{L}(e_1) \subset \mathcal{L}(e_1, e_2 + e_3) \subset \mathcal{L}(e_1, e_2 + e_3, e_4) \subset \mathbb{C}^4$. The second perturbed pair has four distinct eigenvalues. Corresponding joint eigenvectors are:

$$\begin{bmatrix} 1 \\ -\epsilon^3 \\ -\epsilon^3 - \epsilon^5 \\ \epsilon^6 \end{bmatrix}, \begin{bmatrix} 1 \\ -i\epsilon^3 \\ -i\epsilon^3 + i\epsilon^5 \\ -\epsilon^6 \end{bmatrix}, \begin{bmatrix} 1 \\ i\epsilon^3 \\ i\epsilon^3 - i\epsilon^5 \\ -\epsilon^6 \end{bmatrix}, \begin{bmatrix} 1 \\ \epsilon^3 \\ \epsilon^3 + \epsilon^5 \\ \epsilon^6 \end{bmatrix}.$$

All its two-dimensional invariant subspaces are near the subspace $\mathcal{L}(e_1, e_2 + e_3)$ and all its three-dimensional subspaces are near the subspace $\mathcal{L}(e_1, e_2, e_3)$. These perturbations show *that there is no three-dimensional stable invariant subspace for the pair* (A, B). *It also follows that two-dimensional invariant subspaces other than* $\mathcal{L}(e_1, e_2 + e_3)$ *are not stable. However, neither were we able to find a commuting perturbation that would show that* $\mathcal{L}(e_1, e_2 + e_3)$ *is not stable, nor were we able to show that it is a stable invariant subspace.* ■

Remark 4.7 *We observe that the subspace* $\mathcal{L}(e_1, e_2 + e_3)$ *in the above example is the only joint marked two-dimensional invariant subspace [4, p. 83] for matrices* A *and* B. *Before we discuss this statement we give the definition of a marked invariant subspace.*

Let A *be a* $n \times n$ *matrix over* $\mathbb{C}$. *The sequence of vectors* $x_1, \ldots, x_k$, $x_k \neq 0$, *such that*

$$(A - \lambda I)x_i = \begin{cases} x_{i+1} & , \quad i = 1, 2, \ldots, k-1 \\ 0 & , \quad i = k, \end{cases}$$

is a Jordan chain of matrix A *for the eigenvalue* λ. *Let* $\mathcal{N} \subset \mathbb{C}^n$ *be an invariant subspace of* $A \in \mathbb{C}^{n\times n}$. *We say that* $\mathcal{N}$ *is* marked *if there is a basis*

$$\mathcal{B} = \{x_{11}, x_{12}, \ldots, x_{1n_1}; x_{21}, \ldots, x_{2n_2}; \ldots; x_{r1}, \ldots, x_{rn_r}\} \tag{3}$$

for $\mathbb{C}^n$, *such that it consists of Jordan chains of* A *and some subset of it is a basis for* $\mathcal{N}$.

In other words, $\mathcal{N}$ *is a marked invariant subspace of* A *if it is possible to choose its basis in such a way that it is extendable to a basis for* $\mathbb{C}^n$ *consisting of Jordan chains of* A. *The notion of marked invariant subspace was first defined by Gohberg, Lancaster, and Rodman [4, p. 83]. See [3] for an interesting characterization of marked invariant subspaces.*

Now we return to Example 4.6. Observe that the subspaces $\mathcal{L}(e_1, e_3)$ *and* $\mathcal{L}(e_1, e_2 + e_3)$ *are the only two-dimensional marked invariant subspaces of* A *and that* $\mathcal{L}(e_1, e_2)$ *and* $\mathcal{L}(e_1, e_2 + e_3)$ *are the only two-dimensional marked invariant subspaces of* B. *(See also [4, Example 2.9.1, pp. 83-84].)*

More generally, suppose that (A, B) *is a pair of commuting nilpotent matrices. Suppose further that* $\mathcal{B}$ *is a Jordan basis for* A *given in (3) and that* n_j *are chosen so that* $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. *The Jordan basis* $\mathcal{B}$ *for* A *can be further chosen in such a way that vector* Bx_{ij} *is in the span of vectors* x_{kl} *with either* $l > j$ *or* $l = j$ *and* $k > i$ *(see e.g. the proof of Lemma 3 in [2]). Suppose that* $\mathcal{B}'$ *is the basis obtained from* $\mathcal{B}$ *when we use the lexicographic ordering on* x_{ij} *with* $j > i$ *instead of* $i > j$, *i.e.,*

$$\mathcal{B}' = \{x_{11}, x_{21}, \ldots, x_{r1}, \ \ldots \ , x_{1n_1}, x_{2n_1}, \ldots, x_{sn_1}\},$$

where $s = \max\{i;\ n_s = n_1\}$. *Then it follows that the matrices for* A *and* B *with respect to* $\mathcal{B}'$ *are both lower-triangular. Let* B_ϵ *be the matrix (2) defined in the same way as in the proof of Theorem 4.1. Recall from the proof there that for* $\epsilon > 0$ *but small enough the matrix* B_ϵ *is nonderogatory. It follows from our particular choice of the Jordan basis* $\mathcal{B}$ *that the spectrum of* B_ϵ *is equal to* $\{\epsilon\lambda_1, \epsilon\lambda_2, \ldots, \epsilon\lambda_r\}$ *and that the multiplicity of the eigenvalue* $\epsilon\lambda_j$ *is equal to* n_j. *This is easily observed from the fact that the matrices for* A *and* B *with respect to the basis* $\mathcal{B}'$ *are both lower-triangular.*

Recall from the proof of Theorem 4.1 that $A = p(B_\epsilon)$ *for some polynomial* p. *Then it follows that each invariant subspace of* B_ϵ, *and therefore also of the commuting pair*

(A, B_ϵ), is a marked invariant subspace of A. Thus, better understanding of the set of joint marked invariant subspaces of A and B might shed some light on problem of characterization of the set of stable invariant subspaces for pair (A, B). ■

The above observation leads us to pose the following question.

Question 4.8 *Is a stable invariant subspace of a pair of commuting matrices marked invariant subspace for each of the matrices?*

5 Connection to algebraic multiparameter spectral theory

In this section we study the stability of invariant subspaces of an algebraic multiparameter eigenvalue problem. We consider an algebraic multiparameter system $\mathbf{W}$:

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{k} V_{ij}\lambda_j - V_{i0}, \quad i = 1, 2, \ldots, k, \ (k \geq 2),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ are parameters and V_{ij} are $n_i \times n_i$ matrices over $\mathbb{C}$.

The tensor product space $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_k}$ is isomorphic to $\mathbb{C}^N$, where $N = n_1 n_2 \cdots n_k$. Linear transformations $V_{ij}^\dagger$ on $\mathbb{C}^N$ are induced by V_{ij}, $i = 1, 2, \ldots, k; j = 0, 1, \ldots, k$, and defined by

$$V_{ij}^\dagger(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = x_1 \otimes \cdots \otimes V_{ij}x_i \otimes \cdots \otimes x_k$$

and linearity. On $\mathbb{C}^N$ we also define operator determinants

$$\Delta_0 = \begin{vmatrix} V_{11}^\dagger & V_{12}^\dagger & \cdots & V_{1k}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger & \cdots & V_{2k}^\dagger \\ \vdots & \vdots & & \vdots \\ V_{k1}^\dagger & V_{k2}^\dagger & \cdots & V_{kk}^\dagger \end{vmatrix}$$

and

$$\Delta_i = \begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & V_{10}^\dagger & V_{1,i+1}^\dagger & \cdots & V_{1k}^\dagger \\ V_{21}^\dagger & \cdots & V_{2,i-1}^\dagger & V_{20}^\dagger & V_{2,i+1}^\dagger & \cdots & V_{2k}^\dagger \\ \vdots & & \vdots & \vdots & & \vdots & \\ V_{k1}^\dagger & \cdots & V_{k,i-1}^\dagger & V_{k0}^\dagger & V_{k,i+1}^\dagger & \cdots & V_{kk}^\dagger \end{vmatrix}$$

for $i = 1, \ldots, k$.

A multiparameter system $\mathbf{W}$ is called nonsingular if the corresponding operator determinant Δ_0 is invertible. In the case of a nonsingular multiparameter system $\mathbf{W}$, we associate with $\mathbf{W}$ a k-tuple of commuting linear transformations $\boldsymbol{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$, where $\Gamma_i = \Delta_0^{-1}\Delta_i$, $i = 1, \ldots, k$ (see [1, Thm. 6.7.1]).

An k-tuple $\boldsymbol{\lambda} \in \mathbb{C}^k$ is called an eigenvalue of the multiparameter system $\mathbf{W}$ if all $W_i(\boldsymbol{\lambda})$ are singular. If

$$\mathrm{Ker}(\boldsymbol{\Gamma} - \boldsymbol{\lambda I}) := \bigcap_{i=1}^{k} \mathrm{Ker}(\Gamma_i - \lambda_i I) \neq \{0\},$$

then $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{\Gamma}$. Let $\sigma(\mathbf{W})$ and $\sigma(\boldsymbol{\Gamma})$ denote the set of all the eigenvalues of $\mathbf{W}$ and $\boldsymbol{\Gamma}$, respectively. It was shown by Atkinson [1, Thm. 6.9.1] that $\sigma(\mathbf{W}) = \sigma(\boldsymbol{\Gamma})$ and that

$$\mathrm{Ker}(\boldsymbol{\Gamma} - \boldsymbol{\lambda I}) = \mathrm{Ker}\, W_1(\boldsymbol{\lambda}) \otimes \mathrm{Ker}\, W_2(\boldsymbol{\lambda}) \otimes \cdots \otimes \mathrm{Ker}\, W_k(\boldsymbol{\lambda}).$$

An eigenvalue $\boldsymbol{\lambda}$ of a multiparameter system $\mathbf{W}$ is called nonderogatory [13] if $\boldsymbol{\lambda}$ is a nonderogatory eigenvalue of the associated system $\boldsymbol{\Gamma}$.

We say that $\mathcal{M} \subset \mathbb{C}^N$ is an invariant subspace for $\boldsymbol{W}$ if

$$\Gamma_i \mathcal{M} \subset \mathcal{M}, \quad i = 1, \ldots, k.$$

We say that an invariant subspace $\mathcal{N}$ of the multiparameter system (5) is *stable* if for a given $\epsilon > 0$ there exists $\delta > 0$ such that the following holds: if a nonsingular multiparameter system $\mathbf{W}'$:

$$W_i'(\boldsymbol{\lambda}) = \sum_{j=1}^{k} V_{ij}' \lambda_j - V_{i0}', \quad i = 1, 2, \ldots, k, \tag{4}$$

is such that

$$\|V_{ij} - V_{ij}'\| < \delta$$

for all (i, j) then there exists an invariant subspace $\mathcal{M}$ of $\mathbf{W}'$ such that

$$\theta(\mathcal{N}, \mathcal{M}) < \epsilon.$$

The stability is very important for the numerical calculation, for example, for the calculation of a basis for the root subspace of a nonderogatory eigenvalue [13, 16]. If the invariant subspace is not stable then we can not expect stable numerical calculation.

Since Γ_i for $i = 1, \ldots, k$ commute the stability of invariant subspaces for the algebraic multiparameter problem is closely related to the stability of invariant subspaces for commuting matrices. Multiparameter system $\mathbf{W}'$ is equivalent to the associated system

$$\Gamma_i' x = \lambda_i x, \quad x \neq 0, \quad i = 1, \ldots, k. \tag{5}$$

It is obvious that for each $\eta > 0$ there exists $\delta > 0$ such that if

$$\|V_{ij} - V_{ij}'\| < \delta$$

for all (i, j) then

$$\|\Gamma_i - \Gamma_i'\| < \eta, \quad i = 1, \ldots, k.$$

As a result we can apply a part of the theory on the stability of invariant subspaces of commuting matrices to the stability of invariant subspaces of multiparameter systems. The problems of stability are connected but not identical since in the study of stability for multiparameter eigenvalue problems we have to restrict the set of commuting matrices only to the matrices that form associated systems of multiparameter systems.

For instance, let $\mathcal{N}$ be an invariant subspace of a multiparameter system $\mathbf{W}$. If $\mathcal{N}$ is a stable invariant subspace for the commuting set $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$, then $\mathcal{N}$ is also a stable invariant subspace of $\mathbf{W}$. The converse is not necessarily true since an arbitrary set of commuting matrices is not necessarily an associated system of a multiparameter system. If we take for example matrices

$$\Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then Γ_1 and Γ_2 are not associated with any multiparameter system (see [11, Example 2.13]).

Summary of results that can be applied to the multiparameter eigenvalue problems is as follows. It follows from Theorem 3.4 that the complete root subspace is a stable invariant subspace. Corollary 3.6 yields that all invariant subspaces of root subspace of a nonderogatory eigenvalue are stable. This means that it is possible to numerically stable compute the basis for the root subspace of a nonderogatory eigenvalue [13, 16]. It also follows from Corollary 3.8 that the eigenspace of a geometrically simple eigenvalue is stable.

New answers on the stability of invariant subspaces of multiparameter systems are connected with a study of conditions a set of commuting matrices $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ must satisfy in order that there exists a multiparameter system $\mathbf{W}$ such that $\mathbf{\Gamma}$ is its associated system. Some of the conditions are given in the preprint [14].

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