COALGEBRAS AND SPECTRAL THEORY IN ONE AND SEVERAL PARAMETERS

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Abstract

The coalgebraic versions of the primary decomposition theorem for a single linear map and for several commuting linear maps are proved. They lead to a description of the primary decomposition for multiparameter eigenvalue problems in terms of the underlying multiparameter system. Also the coalgebraic version of the primary decomposition theorem for a monic matrix polynomial is discussed.

1 Introduction

The main goal of our paper is to give an elementary and self-contained introduction to coalgebras and their use in spectral theory. We first introduce basic properties of coalgebras and then we prove the primary decomposition theorem for a linear map. When the underlying field is algebraically closed this result reduces to the theorem on the decomposition of a vector space into spectral subspaces. We also give versions of the primary decomposition theorem for a monic matrix polynomial and for several commuting linear maps. These results are well-known and coalgebraic techniques provide an alternative point of view. However, the coalgebraic versions of these results together with other coalgebraic techniques are essential for a solution of the root subspace problem for multiparameter systems. This problem was posed by Atkinson in [2, 3]. Several authors gave partial solutions of the problem using various methods in [4, 6, 8, 18, 20, 21]. A general solution using coalgebraic techniques was presented in [13]. We describe the problem and its solution in §7.

The structure of a coalgebra is dual to the structure of an algebra. Moreover, an algebra structure can be 'dualized' to give a structure of a coalgebra on a dual. The precise construction is discussed in §2. The particular case of the coalgebra structure on the 'dual' of a polynomial algebra is presented in §3. This case is used in the core of our paper, i.e. for §4-§7. There the primary decomposition theorem for a single linear map, for a matrix polynomial and for several commuting linear maps are proved. The result for several commuting linear maps is then applied to multiparameter systems. The 'dual' of a polynomial algebra carries, not only the structure of a coalgebra, but that of a (commutative and cocommutative) Hopf algebra [23] (see also [1, 22, 25]). However, for our paper the coalgebra structure is the important part.

2 Coalgebras and Comodules

The structure of a coalgebra is dual to the structure of an algebra. Let us first consider the structure of an algebra in a slightly different way than it is customary in linear algebra or operator theory. Suppose that $\mathcal{A}$ is an algebra over a field F. (In this paper F is the fixed underlying field for all vector spaces, algebras and coalgebras under consideration. We further assume that the characteristic of F is 0, although most of our discussion remains valid for more general fields.) The multiplication and the unit of $\mathcal{A}$ are linear maps $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $e : F \to \mathcal{A}$, respectively. They are given by $\mu(a \otimes b) = ab$ and $e(\alpha) = \alpha 1$, where 1 is the unit element in $\mathcal{A}$. The associativity is given by the equality

(i) $\mu(\mu \otimes I_{\mathcal{A}}) = \mu(I_{\mathcal{A}} \otimes \mu)$, i.e. by the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \stackrel{\mu \otimes I_{\mathcal{A}}}{\longrightarrow} & \mathcal{A} \otimes \mathcal{A} \\ \downarrow^{I_{\mathcal{A}} \otimes \mu} & & \downarrow^{\mu} \\ \mathcal{A} \otimes \mathcal{A} & \stackrel{\mu}{\longrightarrow} & \mathcal{A} \end{array},$$

and the unit law is given by the equality

(ii) $\mu(e \otimes I_{\mathcal{A}}) = I_{\mathcal{A}} = \mu(I_{\mathcal{A}} \otimes e)$, i.e. by the commutative diagram

$$\begin{array}{ccccc} F \otimes \mathcal{A} & \cong & \mathcal{A} & \cong & \mathcal{A} \otimes F \\ \downarrow^{e \otimes I_{\mathcal{A}}} & & \|^{I_{\mathcal{A}}} & & \downarrow^{I_{\mathcal{A}} \otimes e} \\ \mathcal{A} \otimes \mathcal{A} & \stackrel{\mu}{\longrightarrow} & \mathcal{A} & \stackrel{\mu}{\longleftarrow} & \mathcal{A} \otimes \mathcal{A} \end{array}.$$

Here $I_{\mathcal{A}}$ is the identity map on $\mathcal{A}$.

Dualizing the above structure, we say that a vector space $\mathcal{C}$ is a coalgebra if there exist linear maps $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to F$ such that

(i') $(\delta \otimes I_{\mathcal{C}})\delta = (I_{\mathcal{C}} \otimes \delta)\delta$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{\delta}{\longrightarrow} & \mathcal{C} \otimes \mathcal{C} \\ \downarrow^{\delta} & & \downarrow^{I_{\mathcal{C}} \otimes \delta} \\ \mathcal{C} \otimes \mathcal{C} & \stackrel{\delta \otimes I_{\mathcal{C}}}{\longrightarrow} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \end{array}$$

commutes, and

(ii') $(\varepsilon \otimes I_{\mathcal{C}})\,\delta = I_{\mathcal{C}} = (I_{\mathcal{C}} \otimes \varepsilon)\,\delta$, i.e. the diagram

$$\begin{array}{ccccc} \mathcal{C}\otimes\mathcal{C} & \xleftarrow{\delta} & \mathcal{C} & \xrightarrow{\delta} & \mathcal{C}\otimes\mathcal{C} \\ \downarrow{\scriptstyle \varepsilon\otimes I_{\mathcal{C}}} & & \|{\scriptstyle I_{\mathcal{C}}} & & \downarrow{\scriptstyle I_{\mathcal{C}}\otimes\varepsilon} \\ F\otimes\mathcal{C} & \cong & \mathcal{C} & \cong & \mathcal{C}\otimes F \end{array}$$

commutes.

The maps δ and ε are called the *comultiplication* and the *counit*, respectively; and the properties (i') and (ii') are the *coassociativity* and the *counit law*, respectively.

If a vector subspace $\mathcal{B} \subset \mathcal{C}$ is a coalgebra for the restricted maps of δ and ε then we call $\mathcal{B}$ a *subcoalgebra* of $\mathcal{C}$. Note that for $\mathcal{B} \subset \mathcal{C}$ to be a subcoalgebra it suffices to require that $\delta(\mathcal{B}) \subset \mathcal{B} \otimes \mathcal{B}$.

If $\mathcal{A}$ is a finite-dimensional algebra then the canonical linear map $\varphi : \mathcal{A}^* \otimes \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$, determined by $\varphi(f \otimes g)(a \otimes b) = f(a)\,g(b)$ for $f, g \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$, is an isomorphism. We identify the vector spaces $\mathcal{A}^* \otimes \mathcal{A}^*$ and $(\mathcal{A} \otimes \mathcal{A})^*$ (via φ). Thus $\mathcal{A}^*$ becomes a coalgebra with comultiplication μ^* and counit e^*. If $\mathcal{A}$ is not finite-dimensional then $\varphi : \mathcal{A}^* \otimes \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$ is not an isomorphism. In this case $\mathcal{A}^*$ is replaced by the subspace $\mathcal{A}^{\mathrm{o}}$ of all representative functionals. A functional $f \in \mathcal{A}^*$ is called *representative* if its kernel contains a *cofinite* ideal (i.e. an ideal $\mathcal{I} \subset \mathcal{A}$ such that $\mathcal{A}/\mathcal{I}$ is a finite dimensional algebra). The map φ restricts to an isomorphism $\varphi : \mathcal{A}^{\mathrm{o}} \otimes \mathcal{A}^{\mathrm{o}} \to (\mathcal{A} \otimes \mathcal{A})^{\mathrm{o}}$ (cf. [1, 22, 25] and also Proposition 2.1 below). If we identify $(\mathcal{A} \otimes \mathcal{A})^{\mathrm{o}}$ and $\mathcal{A}^{\mathrm{o}} \otimes \mathcal{A}^{\mathrm{o}}$ via φ, then the vector space $\mathcal{A}^{\mathrm{o}}$ is a coalgebra with comultiplication and counit

$$\delta = \mu^{\mathrm{o}} : \mathcal{A}^{\mathrm{o}} \to \mathcal{A}^{\mathrm{o}} \otimes \mathcal{A}^{\mathrm{o}} \text{ and } \varepsilon = e^{\mathrm{o}} : \mathcal{A}^{\mathrm{o}} \to F, \tag{1}$$

given by the restrictions of μ^* and e^* to $\mathcal{A}^{\mathrm{o}}$. The coalgebra $\mathcal{A}^{\mathrm{o}}$ is called the *coalgebra dual* of $\mathcal{A}$. Note that if $\mathcal{A}$ is finite-dimensional then $\mathcal{A}^{\mathrm{o}} = \mathcal{A}^*$. The following result is found in [25, Lemma 6.0.1(b)].

PROPOSITION 2.1 *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two algebras then $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}} \cong \mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}}$.*

PROOF. Suppose that $f \in (\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}}$ and that $\mathcal{K} \subset \ker f$ is a cofinite ideal. We imbed the algebra $\mathcal{A}_1$ in $\mathcal{A}_1 \otimes \mathcal{A}_2$ by mapping $a \mapsto a \otimes 1$ and the algebra $\mathcal{A}_2$ by $a \mapsto 1 \otimes a$. Since the inverse image of a cofinite ideal is a cofinite ideal it follows that

$$\mathcal{I}_1 = \{a \in \mathcal{A}_1;\ a \otimes 1 \in \mathcal{K}\} \text{ and } \mathcal{I}_2 = \{a \in \mathcal{A}_2;\ 1 \otimes a \in \mathcal{K}\}$$

are cofinite ideals in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then $\mathcal{I} = \mathcal{A}_1 \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes \mathcal{A}_2$ is an ideal in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Note that $\mathcal{I} \subset \mathcal{K}$. Let $\pi_i : \mathcal{A}_i \to \mathcal{A}_i/\mathcal{I}_i$ $(i = 1, 2)$ be the quotient map.

Then $\mathcal{I}$ is the kernel of the quotient map $\pi_1 \otimes \pi_2$. Because $\mathcal{A}_1/\mathcal{I}_1 \otimes \mathcal{A}_2/\mathcal{I}_2$ is finite dimensional it follows that $\mathcal{I}$ is cofinite. Since $\mathcal{I} \subset \mathcal{K}$ there exists a unique functional $\overline{f} : \mathcal{A}_1/\mathcal{I}_1 \otimes \mathcal{A}_2/\mathcal{I}_2 \to F$ such that $f = \overline{f} \circ (\pi_1 \otimes \pi_2)$, and since the algebra $\mathcal{A}_1/\mathcal{I}_1 \otimes \mathcal{A}_2/\mathcal{I}_2$ is finite dimensional it follows that $(\mathcal{A}_1/\mathcal{I}_1 \otimes \mathcal{A}_2/\mathcal{I}_2)^* \cong (\mathcal{A}_1/\mathcal{I}_1)^* \otimes (\mathcal{A}_2/\mathcal{I}_2)^*$. In view of this identification the functional f may be written as a finite sum $\overline{f} = \sum_j \overline{f}_{1j} \otimes \overline{f}_{2j}$, where $\overline{f}_{ij} \in (\mathcal{A}_i/\mathcal{I}_i)^*$ $(i = 1, 2)$. Since $f = \sum_j \overline{f}_{1j}\pi_1 \otimes \overline{f}_{2j}\pi_2 \in \mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}}$ it follows that $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}} \subset \mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}}$.

To prove the opposite inclusion assume that $f_i \in \mathcal{A}_i^{\mathrm{o}}$ $(i = 1, 2)$ and $\mathcal{I}_i \subset \ker f_i$ is a cofinite ideal. Then $f_1 \otimes f_2$ vanishes on the cofinite ideal $\mathcal{A}_1 \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes \mathcal{A}_2$ and so $f_1 \otimes f_2 \in (\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}}$. Hence also $\mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}} \subset (\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}}$. □

The structure of a comodule over a coalgebra is dual to the structure of a module over an algebra. If $\mathcal{A}$ is an algebra and $\mathcal{M}$ a vector space then $\mathcal{M}$ is an $\mathcal{A}$-module if there is a linear map $\omega : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$, called the *action* of $\mathcal{A}$ on $\mathcal{M}$, such that

(iii) $\omega\,(e \otimes I_{\mathcal{M}}) = I_{\mathcal{M}}$, i.e. the diagram

$$\begin{array}{ccc} F \otimes \mathcal{M} & \cong & \mathcal{M} \\ \downarrow e\otimes I_{\mathcal{M}} & & \| \, I_{\mathcal{M}} \\ \mathcal{A} \otimes \mathcal{M} & \xrightarrow{\omega} & \mathcal{M} \end{array}$$

commutes, and

(iv) $\omega\,(\mu \otimes I_{\mathcal{M}}) = \omega\,(I_{\mathcal{A}} \otimes \omega)$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{M} & \xrightarrow{\mu\otimes I_{\mathcal{M}}} & \mathcal{A} \otimes \mathcal{M} \\ \downarrow I_{\mathcal{A}}\otimes\omega & & \downarrow \omega \\ \mathcal{A} \otimes \mathcal{M} & \xrightarrow{\omega} & \mathcal{M} \end{array}$$

commutes.

By duality, if $\mathcal{C}$ is a coalgebra and $\mathcal{R}$ a vector space then $\mathcal{R}$ is a $\mathcal{C}$-comodule if there is a linear map $\alpha : \mathcal{R} \to \mathcal{C} \otimes \mathcal{R}$, called the *coaction* of $\mathcal{C}$ on $\mathcal{R}$, such that

(iii') $(\varepsilon \otimes I_{\mathcal{R}})\,\alpha = I_{\mathcal{R}}$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{R} & \xrightarrow{\alpha} & \mathcal{C} \otimes \mathcal{R} \\ \| \, I_{\mathcal{R}} & & \downarrow \varepsilon\otimes I_{\mathcal{R}} \\ \mathcal{R} & \cong & F \otimes \mathcal{R} \end{array}$$

commutes, and

(iv') $(\delta \otimes I_{\mathcal{R}})\,\alpha = (I_{\mathcal{C}} \otimes \alpha)\,\alpha$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{R} & \stackrel{\alpha}{\longrightarrow} & \mathcal{C} \otimes \mathcal{R} \\ \downarrow \alpha & & \downarrow \delta \otimes I_{\mathcal{R}} \\ \mathcal{C} \otimes \mathcal{R} & \stackrel{I_{\mathcal{C}} \otimes \alpha}{\longrightarrow} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{R} \end{array}$$

commutes.

If a vector subspace $\mathcal{S} \subset \mathcal{R}$ is a comodule for the restricted map of α, i.e. if $\alpha(\mathcal{S}) \subset \mathcal{C} \otimes \mathcal{S}$, then we call $\mathcal{S}$ a *subcomodule* of $\mathcal{R}$.

If V is a vector space then $\alpha = \delta \otimes I_V$ is a coaction on $\mathcal{C} \otimes V$. This follows directly from the properties (i') and (ii') for the comultiplication δ. The comodule $\mathcal{C} \otimes V$ is called *cofree*.

Suppose that $\mathcal{R}$ and $\mathcal{S}$ are $\mathcal{C}$-comodules with coactions $\alpha_{\mathcal{R}}$ and $\alpha_{\mathcal{S}}$, respectively. A linear map $\varphi : \mathcal{R} \to \mathcal{S}$ is called a *comodule map* if

(v') $\alpha_{\mathcal{S}}\varphi = (I_{\mathcal{C}} \otimes \varphi)\,\alpha_{\mathcal{R}}$, i.e. the diagram

$$\begin{array}{ccc} \mathcal{R} & \stackrel{\varphi}{\longrightarrow} & \mathcal{S} \\ \downarrow \alpha_{\mathcal{R}} & & \downarrow \alpha_{\mathcal{S}} \\ \mathcal{C} \otimes \mathcal{R} & \stackrel{I_{\mathcal{C}} \otimes \varphi}{\longrightarrow} & \mathcal{C} \otimes \mathcal{S} \end{array}$$

commutes.

Note that the kernel $\mathcal{K}$ of a comodule map φ is a subcomodule of $\mathcal{R}$ since the relation (v') implies that $\alpha_{\mathcal{R}}(\mathcal{K}) \subset \mathcal{C} \otimes \mathcal{K}$.

The following is usually called the *fundamental theorem for comodules* [1, 11, 25]. To keep our presentation elementary and complete we provide a simple proof. Note that the dimension of a coalgebra or a comodule is its dimension as a vector space over F.

LEMMA 2.2 *Suppose $\mathcal{R}$ is a $\mathcal{C}$-comodule. If V is a finite-dimensional (vector) subspace of $\mathcal{R}$ then V is contained in a finite-dimensional subcomodule of $\mathcal{R}$.*

PROOF. Since V is finite-dimensional there exists a finite-dimensional vector subspace $\mathcal{W} \subset \mathcal{R}$ such that

$$\alpha(V) \subseteq \mathcal{C} \otimes \mathcal{W}. \tag{2}$$

This is the case because $\alpha(v) = \sum_i c_i(v) \otimes u_i(v)$, where $c_i(v) \in \mathcal{C}$ and $u_i(v) \in \mathcal{R}$, is a finite sum for every $v \in V$. Then it suffices for $\mathcal{W}$ to be the span of all $u_i(v)$, where v ranges over a basis of V.

Next we want to show that $\mathcal{W}$ is a subcomodule. Since $(\varepsilon \otimes I_{\mathcal{R}})\,\alpha = I_{\mathcal{R}}$ (cf. (iii')) it follows that $\alpha^{-1}(\varepsilon \otimes I_{\mathcal{R}})^{-1}(\mathcal{W}) = \mathcal{W}$. We have also that $(\varepsilon \otimes I_{\mathcal{W}})^{-1}(\mathcal{W}) = \mathcal{C} \otimes \mathcal{W}$, and hence

$$\alpha^{-1}(\mathcal{C} \otimes \mathcal{W}) = \mathcal{W}. \tag{3}$$

Then the restriction of the coaction α to $\mathcal{W}$ maps into $\mathcal{C} \otimes \mathcal{W}$, and therefore $\mathcal{W}$ is a subcomodule. From the relations (2) and (3) it follows also that $V \subseteq \mathcal{W}$. □

Next we state the corresponding result for coalgebras.

LEMMA 2.3 *Suppose $\mathcal{C}$ is a coalgebra. If V is a finite-dimensional (vector) subspace of $\mathcal{C}$ then V is contained in a finite-dimensional subcoalgebra of $\mathcal{C}$.*

PROOF. By Lemma 2.2 applied to the cofree comodule $\mathcal{C} \otimes F \cong \mathcal{C}$ it follows that there is a finite-dimensional subcomodule $\mathcal{W}$ such that $V \subseteq \mathcal{W}$. Since the coaction on $\mathcal{C}$ is given by the comultiplication it follows that $\delta(\mathcal{W}) \subset \mathcal{C} \otimes \mathcal{W}$. The second equality in (ii') implies that also $\delta(\mathcal{W}) \subset \mathcal{W} \otimes \mathcal{C}$, and therefore $\delta(\mathcal{W}) \subset \mathcal{W} \otimes \mathcal{W}$, i.e. $\mathcal{W}$ is a subcoalgebra of $\mathcal{C}$. □

The *cotensor product* of two $\mathcal{C}$-comodules $\mathcal{R}$ and $\mathcal{S}$ is defined by the equalizer diagram

$$\mathcal{R} \otimes^{\mathcal{C}} \mathcal{S} \to \mathcal{R} \otimes \mathcal{S} \underset{\sigma_{12}(\alpha_{\mathcal{R}} \otimes I_{\mathcal{S}})}{\overset{I_{\mathcal{R}} \otimes \alpha_{\mathcal{S}}}{\rightrightarrows}} \mathcal{R} \otimes \mathcal{C} \otimes \mathcal{S},$$

i.e. $\mathcal{R} \otimes^{\mathcal{C}} \mathcal{S} = \ker(I_{\mathcal{R}} \otimes \alpha_{\mathcal{S}} - \sigma_{12}(\alpha_{\mathcal{R}} \otimes I_{\mathcal{S}}))$, where σ_{ij} switches the ith and jth tensor factor.

The following property of the cotensor product $\otimes^{\mathcal{C}}$ is used in §7. It is a special case of Proposition 2.1.1 of [16].

LEMMA 2.4 *If $\mathcal{K}$ is the kernel of a comodule map $\varphi : \mathcal{R} \to \mathcal{S}$ and if $\mathcal{L}$ is another comodule then $\mathcal{K} \otimes^{\mathcal{C}} \mathcal{L}$ is the kernel of the induced map $\varphi \otimes^{\mathcal{C}} I_{\mathcal{L}} : \mathcal{R} \otimes^{\mathcal{C}} \mathcal{L} \to \mathcal{S} \otimes^{\mathcal{C}} \mathcal{L}$.*

PROOF. Let $\tau_{\mathcal{P}\mathcal{L}} = I_{\mathcal{P}} \otimes \alpha_{\mathcal{L}} - \sigma_{12}(\alpha_{\mathcal{P}} \otimes I_{\mathcal{L}})$, where $\mathcal{P}$ is either $\mathcal{K}$, $\mathcal{R}$ or $\mathcal{S}$. By definition we have that $\mathcal{P} \otimes^{\mathcal{C}} \mathcal{L} = \ker \tau_{\mathcal{P}\mathcal{L}}$. Then the diagram

$$\begin{array}{ccccc}
\mathcal{K} \otimes^{\mathcal{C}} \mathcal{L} & \to & \mathcal{R} \otimes^{\mathcal{C}} \mathcal{L} & \overset{\varphi \otimes^{\mathcal{C}} I_{\mathcal{L}}}{\longrightarrow} & \mathcal{S} \otimes^{\mathcal{C}} \mathcal{L} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K} \otimes \mathcal{L} & \to & \mathcal{R} \otimes \mathcal{L} & \overset{\varphi \otimes I_{\mathcal{L}}}{\longrightarrow} & \mathcal{S} \otimes \mathcal{L} \\
\downarrow^{\tau_{\mathcal{K}\mathcal{L}}} & & \downarrow^{\tau_{\mathcal{R}\mathcal{L}}} & & \downarrow^{\tau_{\mathcal{S}\mathcal{L}}} \\
\mathcal{K} \otimes \mathcal{C} \otimes \mathcal{L} & \to & \mathcal{R} \otimes \mathcal{C} \otimes \mathcal{L} & \overset{\varphi \otimes I_{\mathcal{C}} \otimes I_{\mathcal{L}}}{\longrightarrow} & \mathcal{S} \otimes \mathcal{C} \otimes \mathcal{L}
\end{array}$$

commutes. (Here an arrow without a label indicates an inclusion.) Since also $\mathcal{K}\otimes\mathcal{L}=\ker(\varphi\otimes I_{\mathcal{L}})$ and $\mathcal{K}\otimes\mathcal{C}\otimes\mathcal{L}=\ker(\varphi\otimes I_{\mathcal{C}}\otimes I_{\mathcal{L}})$ it follows that

$$\mathcal{K}\otimes^{\mathcal{C}}\mathcal{L}\subset\ker\left(\varphi\otimes^{\mathcal{C}}I_{\mathcal{L}}\right). \tag{4}$$

To prove the opposite inclusion suppose that $x\in\ker\left(\varphi\otimes^{\mathcal{C}}I_{\mathcal{L}}\right)$. Then it follows that $x\in\ker(\varphi\otimes I_{\mathcal{L}})$, and by definition of the cotensor that $x\in\ker\tau_{\mathcal{RL}}$. Hence $x\in\ker\tau_{\mathcal{KL}}=\mathcal{K}\otimes^{\mathcal{C}}\mathcal{L}$, and so $\ker\left(\varphi\otimes^{\mathcal{C}}I_{\mathcal{L}}\right)\subset\mathcal{K}\otimes^{\mathcal{C}}\mathcal{L}$. This relation together with (4) proves the lemma.
□

3 The Coalgebra Dual of a Polynomial Algebra

The coalgebras we use in our paper are derived from the coalgebra dual of a polynomial algebra $F[x]$, where F is the underlying field of characteristic 0. A functional $f\in F[x]^*$ is determined by the values $f_k=f\left(x^k\right)$, $(k\geq 0)$. Thus we may identify $F[x]^*$ with the vector space of all infinite sequences $f=(f_k)_{k=0}^{\infty}$. If $f\in F[x]^{\mathrm{o}}$ then $f(\mathcal{J})=0$ for some cofinite ideal $\mathcal{J}$. Since $F[x]$ is a principal ideal domain there exists a monic polynomial, say $p(x)=x^l-b_{l-1}x^{l-1}-\cdots-b_0$, such that $\mathcal{J}$ is the ideal generated by p. Then $f\left(x^kp(x)\right)=0$ for all $k\geq 0$, i.e.

$$f_{k+l}=\sum_{i=0}^{l-1}b_if_{k+i},\ (k\geq 0). \tag{5}$$

Hence $(f_k)_{k=0}^{\infty}$ is a linearly recursive sequence. Conversely, if $f=(f_k)_{k=0}^{\infty}$ is a linearly recursive sequence then it satisfies (5) for some b_i and so $f(q)=0$, for all q in $\mathcal{J}$, the ideal generated by $p(x)=x^l-b_{l-1}x^{l-1}-\cdots-b_0$. Thereafter we identify $F[x]^{\mathrm{o}}$ with the space of all linearly recursive sequences.

If $m_x:F[x]\to F[x]$ is the map of multiplication by x then its dual $D:F[x]^{\mathrm{o}}\to F[x]^{\mathrm{o}}$ is given by

$$Df\left(x^k\right)=f\cdot\left(m_x\left(x^k\right)\right)=f\left(x^{k+1}\right). \tag{6}$$

From (6) it follows that $D(f_k)_{k=0}^{\infty}=(f_{k+1})_{k=0}^{\infty}$ and so D is the *shift operator* on $F[x]^{\mathrm{o}}$.

The counit ε and the comultiplication δ of the coalgebra dual $\mathcal{A}^{\mathrm{o}}=F[x]^{\mathrm{o}}$ are given by (1). If $f=(f_k)_{k=0}^{\infty}\in F[x]^{\mathrm{o}}$ then

$$\varepsilon(f)(1)=f(e(1))=f_0 \tag{7}$$

and

$$\delta(f)(x^r\otimes x^s)=f(\mu(x^r\otimes x^s))=f\left(x^{r+s}\right)=f_{r+s}. \tag{8}$$

The relation (7) implies that $\varepsilon(f) = f(1)$. To find a description of δ some additional notation is needed. Assume that $\mathcal{J}$ is an ideal, which is maximal among ideals contained in $\ker f$. Then we can assume that $\mathcal{J} = \langle p \rangle$ is a principal ideal generated by a monic polynomial p. Suppose $l = \deg p$ and consider the Hankel matrix $H = [f_{i+j-2}]_{i,j=1}^{l}$. Since $\mathcal{J}$ is maximal in $\ker f$ it follows that there is no linearly recursive relation for the sequence $(f_k)_{k=0}^{\infty}$ of degree less than l. A nonzero linear combination of columns of H would give a linear recursive relation for the sequence $(f_k)_{k=0}^{\infty}$ of degree strictly less than l. So, it follows that H is an invertible matrix. Let $H^{-1} = [g_{ij}]_{i,j=1}^{l}$. Then

$$\delta(f) = \sum_{i,j=1}^{l} g_{ij} D^{i-1} f \otimes D^{j-1} f. \tag{9}$$

To verify (9) observe that

$$\sum_{i,j=1}^{l} g_{ij} \left(D^{i-1} f \otimes D^{j-1} f \right) (x^r \otimes x^s) = \sum_{j=1}^{l} \sum_{i=1}^{l} g_{ij} f_{i-1+r} f_{j-1+s} = \sum_{j=1}^{l} \delta_{j-1,r} f_{j-1+s} = f_{r+s},$$

which coincides with (8).

For a monic polynomial $p \in F[x]$ we define $\mathcal{C}_p = \{f \in F[x]^{\circ};\ f(p) = 0\}$. By the definition of D (see (6)) it follows that

$$\mathcal{C}_p = \{f \in F[x]^{\circ};\ p(D) f = 0\}. \tag{10}$$

If we choose arbitrarily the values for $f \in \mathcal{C}_p$ at x^k for $k < \deg p$ then the values $f\left(x^k\right)$ for $k \geq \deg p$ are determined by (5). Thus it follows that

$$\dim \mathcal{C}_p = \deg p. \tag{11}$$

If p is a power of a monic irreducible polynomial q, i.e. $p(x) = q(x)^{l+1}$ then we write $\mathcal{C}_p = \mathcal{B}_q^{(l)}$ and

$$\mathcal{B}_q = \bigcup_{l=0}^{\infty} \mathcal{B}_q^{(l)}.$$

If q is of degree one, i.e. $q(x) = x - \lambda$ for some $\lambda \in F$, then we write $\mathcal{B}_q = \mathcal{B}_\lambda$. Note that a consequence of (10) is that D leaves each $\mathcal{C}_p$, and also each $\mathcal{B}_q$, invariant. By (9) it follows that $\delta(\mathcal{C}_p) \subset \mathcal{C}_p \otimes \mathcal{C}_p$ and $\delta(\mathcal{B}_q) \subset \mathcal{B}_q \otimes \mathcal{B}_q$, and so $\mathcal{C}_p$ and $\mathcal{B}_q$ are subcoalgebras of $F[x]^{\circ}$. If $\mathcal{J} = \langle p \rangle$ is maximal among the ideals contained in the kernel of f, then $\mathcal{C}_p$ is the minimal (finite-dimensional) coalgebra containing f (cf. Lemma 2.3).

LEMMA 3.1 *If $p(x) = q_1(x)^{l_1+1} q_2(x)^{l_2+1} \cdots q_s(x)^{l_s+1}$ is the factorization of the monic polynomial p into distinct monic irreducible factors q_i then $\mathcal{C}_p \cong \oplus_{i=1}^s \mathcal{B}_{q_i}^{(l_i)}$. Furthermore, as a coalgebra*

$$F[x]^{\mathrm{o}} = \bigoplus_{q \in Q} \mathcal{B}_q, \tag{12}$$

where Q is the set of all monic irreducible polynomials in $F[x]$. If F is algebraically closed then $F[x]^{\mathrm{o}} = \bigoplus_{\lambda \in F} \mathcal{B}_\lambda$.

PROOF. Let $\langle p \rangle$ denote the ideal generated by the polynomial p. Since $\left\langle \ p \ \right\rangle \subset \left\langle q_i^{l_i+1} \right\rangle$ for all i, it follows that

$$\mathcal{B}_{q_i}^{(l_i)} \subset \mathcal{C}_p. \tag{13}$$

Because q_i and q_j, $i \neq j$, are relatively prime there exist polynomials r_i and r_j such that $q_i^{l_i+1} r_i + q_j^{l_j+1} r_j = 1$. For $f \in \mathcal{B}_{q_i}^{(l_i)} \cap \mathcal{B}_{q_j}^{(l_j)}$ it follows then that

$$f\left(x^k\right) = f\left(x^k q_i^{l_i+1} r_i + x^k q_j^{l_j+1} r_j\right) = 0.$$

Therefore $f = 0$ and $\oplus_{i=1}^s \mathcal{B}_{q_i}^{(l_i)}$ is a direct sum. We use the equality (11) to obtain

$$\dim \mathcal{C}_p = \deg p = \sum_{i=1}^s (l_i + 1) \deg q_i = \sum_{i=1}^s \dim \mathcal{B}_{q_i}^{(l_i)}.$$

Because the relation (13) holds for all i it follows then that $\mathcal{C}_p = \oplus_{i=1}^s \mathcal{B}_{q_i}^{(l_i)}$.

We have seen above that $\mathcal{B}_{q_i}^{(l_i)} \cap \mathcal{B}_{q_j}^{(l_j)} = 0$ for two distinct irreducible polynomials q_i and q_j, and for all l_i and l_j. Hence $\mathcal{B}_{q_i} \cap \mathcal{B}_{q_j} = 0$. Then $\sum_{q \in Q} \mathcal{B}_q$, where Q is the set of all monic irreducible polynomials in $F[x]$, is a direct sum. For each $f \in F[x]^{\mathrm{o}}$ there is a (nonzero) polynomial p such that $f \in \mathcal{C}_p$. Then the first part of this lemma implies that $f \in \bigoplus_{q \in Q} \mathcal{B}_q$, and hence the relation (12) follows. □

We refer to [23] for a more complete presentation of the structure of commutative and cocommutative Hopf algebra on $F[x]^{\mathrm{o}}$.

4 The Primary Decomposition Theorem

The direct sum decomposition (12) of the coalgebra dual $F[x]^{\mathrm{o}}$ is essential for the further discussion. It induces a decomposition of a cofree comodule

$$F[x]^{\mathrm{o}} \otimes V = \bigoplus_{q \in Q} \mathcal{B}_q \otimes V. \tag{14}$$

Assume now that V is a finite-dimensional vector space. Let $A : V \to V$ be a linear map and let m be its (monic) minimal polynomial. As in §3 the map D is dual to the multiplication by x on $F[x]$. We denote the induced maps $I \otimes A$ and $D \otimes I$ on the comodule $F[x]^{\mathrm{o}} \otimes V$ again by A and D, respectively. Clearly, A and D are comodule maps. Therefore the kernel $\mathcal{R}$ of $A - D : F[x]^{\mathrm{o}} \otimes V \to F[x]^{\mathrm{o}} \otimes V$ is a subcomodule of $F[x]^{\mathrm{o}} \otimes V$. Then (14) induces a direct sum decomposition of $\mathcal{R}$, which is, as we show later in the section, the coalgebraic version of the primary decomposition theorem for A. First we prove an auxiliary result.

PROPOSITION 4.1 *If $\underline{v} \in F[x]^{\mathrm{o}} \otimes V$ is such that $(\varepsilon \otimes I_V) D^i \underline{v} = 0$ for all i then $\underline{v} = 0$.*

PROOF. Suppose that $\underline{v} = \sum_{j=1}^{k} f_j \otimes v_j$, where $f_j \in F[x]^{\mathrm{o}}$ and $v_j \in V$, and that v_j are linearly independent. Then $0 = (\varepsilon \otimes I_V) D^i \underline{v} = \sum_{j=1}^{k} f_j (x^i) v_j$, and so $f_j (x^i) = 0$ for all i and j. This implies that $f_j = 0$ and hence $\underline{v} = 0$. □

We denote by $\mathcal{R}_p$ the kernel of the restricted map $A - D : \mathcal{C}_p \otimes V \to \mathcal{C}_p \otimes V$.

LEMMA 4.2 *The kernel $\mathcal{R}_p$ is a subcomodule of $\mathcal{C}_p \otimes \ker p(A)$ and the composite $\mathcal{R}_p \hookrightarrow \mathcal{C}_p \otimes \ker p(A) \xrightarrow{\varepsilon \otimes I_V} \ker p(A)$ is bijective. In particular, if $p = m$ is the (monic) minimal polynomial of A then $\mathcal{R} \subset \mathcal{C}_m \otimes V$ and $\mathcal{R} \hookrightarrow \mathcal{C} \otimes V \xrightarrow{\varepsilon \otimes I_V} \ker m(A)$ is bijective.*

PROOF. Suppose that $p(x) = x^l + a_{l-1} x^{l-1} + \cdots + a_0$. Since A and D commute it follows that

$$p(A) - p(D) = r(A, D)(A - D),$$

where

$$r(A, D) = \sum_{i=0}^{l-1} a_{i+1} \sum_{j=0}^{i} A^j D^{i-j}$$

and $a_l = 1$. Because $p(D) f = 0$ for $f \in \mathcal{C}_p$ (see (10)) we have that $p(D) \underline{v} = 0$ for $\underline{v} \in \mathcal{C}_p \otimes V$. If also $\underline{v} \in \mathcal{R}_p$ then

$$p(A) \underline{v} = (p(A) - p(D)) \underline{v} = r(A, D)(A - D) \underline{v} = 0.$$

This implies that

$$\mathcal{R}_p \subset \mathcal{C}_p \otimes \ker p(A),$$

and thus also that

$$(\varepsilon \otimes I_V) \mathcal{R}_p \subset \ker p(A). \tag{15}$$

Suppose that $g \in \mathcal{C}_p$ is given by

$$g\left(x^i\right) = \begin{cases} 0 & \text{if } i \neq l-1, \\ 1 & \text{if } i = l-1. \end{cases}$$

For $v \in \ker p(A)$ we define $\eta(v) = r(A, D)\, g \otimes v$. Then it follows that

$$(A - D)\, \eta(v) = (p(A) - p(D))\, g \otimes v = 0$$

and

$$(\varepsilon \otimes I_V)\, \eta(v) = \sum_{i=0}^{l-1} a_{i+1} \sum_{j=0}^{i} g\left(x^{i-j}\right) A^j v = v.$$

Hence $\ker p(A) \subset (\varepsilon \otimes I_V)\, \mathcal{R}_p$. The latter inclusion together with (15) implies that

$$(\varepsilon \otimes I_V)\, \mathcal{R}_p = \ker p(A).$$

It remains to show that $\varepsilon \otimes I_V : \mathcal{R}_p \to \ker p(A)$ is one-to-one. Suppose that $(\varepsilon \otimes I_V)\, \underline{v} = 0$ for $\underline{v} \in \mathcal{R}_p$. Then it follows that $A\underline{v} = D\underline{v}$ and therefore $(\varepsilon \otimes I_V)\, D^i \underline{v} = (\varepsilon \otimes I_V)\, A^i \underline{v} = A^i (\varepsilon \otimes I_V)\, \underline{v} = 0$ for all i. Then $\underline{v} = 0$ by Proposition 4.1, and hence $\varepsilon \otimes I_V$ is one-to-one.

The second part of the lemma follows from the first one if we set $p = m$. □

Note that if $p(x) = (x - \lambda)^k$ in Lemma 4.2 then $V_p = (\varepsilon \otimes I_V)\, \mathcal{R}_p = \ker (A - \lambda)^k$ is the kth root subspace of A at the eigenvalue λ.

Now we are ready to prove the coalgebraic version of the primary decomposition theorem [17, Thm. 12, p. 220].

THEOREM 4.3 *Suppose that $m = q_1^{l_1+1} q_2^{l_2+1} \cdots q_s^{l_s+1}$ is the factorization of the minimal polynomial m of A into distinct irreducible polynomials. Then*

$$\mathcal{R} = \bigoplus_{i=1}^{s} \mathcal{R}_i, \tag{16}$$

where $\mathcal{R}_i$ is the kernel of $A - D : \mathcal{B}_{q_i}^{(l_i)} \otimes V \to \mathcal{B}_{q_i}^{(l_i)} \otimes V$. Moreover, if $V_i = (\varepsilon \otimes I_V)\, \mathcal{R}_i$ then

$$V = \bigoplus_{i=1}^{s} V_i \tag{17}$$

and $V_i = \ker q_i^{l_i+1}(A)$.

PROOF. By Lemma 3.1 applied to the polynomial m we have that $\mathcal{C}_m = \bigoplus_{i=1}^{s} \mathcal{B}_{q_i}^{(l_i)}$. Since $\mathcal{B}_{q_i}^{(l_i)}$ is invariant for D it follows that $\mathcal{B}_{q_i}^{(l_i)} \otimes V$ is invariant for $A - D$. Then it follows that $\mathcal{R}_m = \bigoplus_{i=1}^{s} \mathcal{R}_i$. By Lemma 4.2 we have that $V_i = \ker q_i^{l_i+1}(A)$ and that $\varepsilon \otimes I_V : \mathcal{R} \to V$ is bijective. Thus $\varepsilon \otimes I_V$ carries the direct sum (16) to the direct sum (17). □

COROLLARY 4.4 *If F is algebraically closed and $m = (x - \lambda_1)^{l_1+1} \cdots (x - \lambda_s)^{l_s+1}$ then $\mathcal{R} = \bigoplus_{i=1}^{s} \mathcal{R}_i$ and $V = \bigoplus_{i=1}^{s} V_i$, where $\mathcal{R}_i$ is the kernel of $A - D : \mathcal{B}_{\lambda_i}^{(l_i)} \otimes V \to \mathcal{B}_{\lambda_i}^{(l_i)} \otimes V$ and $V_i = (\varepsilon \otimes I_V)\, \mathcal{R}_i = \ker (A - \lambda_i I)^{l_i+1}$.*

5 Monic Matrix Polynomials

We consider a monic matrix polynomial

$$L(x) = Ix^l + A_{l-1}x^{l-1} + \cdots + A_0,$$

where A_i are linear maps on a finite-dimensional vector space V. The matrix

$$C_L = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_{l-1} \end{bmatrix}$$

acting on V^l is called the *companion matrix of* L. Then there exist matrix polynomials $E(x)$ and $F(x)$ such that their inverses are again matrix polynomials and such that

$$E(x)(xI - C_L)F(x) = L(x) \oplus I_{l-1}. \tag{18}$$

Here I_{l-1} is the identity map on V^{l-1}. We refer to [10, pp. 13–14] for details.

With L we associate a comodule map $L(D) : F[x]^{\mathrm{o}} \otimes V \to F[x]^{\mathrm{o}} \otimes V$ by substituting D for x, i.e. $L(D) = D^l + A_{l-1}D^{l-1} + \cdots + A_1D + A_0$.

LEMMA 5.1 *If $\mathcal{R}$ is the kernel of $L(D)$ then the composite map $\mathcal{R} \hookrightarrow F[x]^{\mathrm{o}} \otimes V \overset{\varepsilon^{(l)} \otimes I_V}{\longrightarrow} V^l$, where*

$$\varepsilon^{(l)} \otimes I_V = \begin{bmatrix} \varepsilon \otimes I_V & \varepsilon D \otimes I_V & \cdots & \varepsilon D^{l-1} \otimes I_V \end{bmatrix}^T : F[x]^{\mathrm{o}} \otimes V \to V^l,$$

is bijective.

PROOF. Note that in (18) we can replace x by D since D commutes with maps acting on V^l. Because $E(D)$ and $F(D)$ are invertible linear maps it follows from (18) that

$$\dim \mathcal{R} = \dim \ker (D - C_L) = l \dim V. \tag{19}$$

Here the latter equality follows by Lemma 4.2. Suppose next that $\underline{v} \in \mathcal{R}$ is such that $\left(\varepsilon^{(l)} \otimes I_V\right)\underline{v} = 0$, i.e. $(\varepsilon D^i \otimes I_V)\underline{v} = 0$ for $i = 0, 1, \ldots, l-1$. Now assume that for some $k\,(\geq l)$ we have that $(\varepsilon D^i \otimes I_V)\underline{v} = 0$ for all $i < k$. Because $\underline{v} \in \mathcal{R}$ it follows that

$$\left(\varepsilon D^k \otimes I_V\right)\underline{v} = -\sum_{i=0}^{l-1}\left(A_i \varepsilon D^{k-l+i} \otimes I_V\right)\underline{v} = 0.$$

By induction we have then that $(\varepsilon D^i \otimes I_V)\underline{v} = 0$ for all i, and so Proposition 4.1 implies that $\underline{v} = 0$. Thus $\varepsilon^{(l)} \otimes I_V : \mathcal{R} \to V^l$ is injective and, since $\dim \mathcal{R} = \dim V^l$ by (19), it is bijective. □

The following is a version of the primary decomposition theorem for monic matrix polynomials. (See also [15, Thm. 5.3].) It follows directly using the decomposition (14) of $F[x]^{\mathrm{o}} \otimes V$ and Lemma 5.1.

THEOREM 5.2 *Suppose that $\mathcal{R} = \ker L(D)$ and $m(x) = \det L(x)$. Then $\mathcal{R} = \oplus_q \mathcal{R}_q$, where the direct sum is over all monic irreducible divisors q of m and $\mathcal{R}_q$ is the kernel of $L(D) : \mathcal{B}_q \otimes V \to \mathcal{B}_q \otimes V$. Furthermore, $V^l = \oplus_q V_q$, where $V_q = \left(\varepsilon^{(l)} \otimes I_V\right) \mathcal{R}_q$.*

We remark that coalgebraic techniques are used to study regular matrix polynomials in [15], and for the general (including singular) one and several parameter matrix polynomials in [14].

6 Several Commuting Maps

In this section we generalize the results of §4 to the case of several commuting linear maps. The discussion follows steps similar to those in §4. The coalgebra used now is the coalgebra dual of a polynomial algebra in several variables.

If δ_i and ε_i $(i = 1, 2)$ are the comultiplication and the counit of the coalgebra dual $\mathcal{A}_i^{\mathrm{o}}$ then $\delta = \sigma_{23}(\delta_1 \otimes \delta_2)$ and $\varepsilon = \varepsilon_1 \otimes \varepsilon_2$ are the comultiplication and the counit, respectively, for the coalgebra $\mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}}$. Here we identify $(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{o}}$ with $\mathcal{A}_1^{\mathrm{o}} \otimes \mathcal{A}_2^{\mathrm{o}}$ (see Proposition 2.1), and we recall that σ_{ij} switches the ith and jth tensor factor. If $F[\mathbf{x}]$ is the polynomial algebra in n variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then $F[\mathbf{x}] \cong F[x_1] \otimes F[x_2] \otimes \cdots \otimes F[x_n]$. So it follows that $F[\mathbf{x}]^{\mathrm{o}} \cong F[x_1]^{\mathrm{o}} \otimes F[x_2]^{\mathrm{o}} \otimes \cdots \otimes F[x_n]^{\mathrm{o}}$. The decomposition (12) applied to each tensor factor $F[x_i]^{\mathrm{o}}$ yields the decomposition

$$F[\mathbf{x}]^{\mathrm{o}} \cong \bigoplus_{\mathbf{q} \in \mathbf{Q}} \mathcal{B}_{\mathbf{q}}, \tag{20}$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ is an n-tuple of monic irreducible polynomials (in one variable), $\mathbf{Q}$ the set of all such n-tuples, and $\mathcal{B}_{\mathbf{q}} = \mathcal{B}_{q_1} \otimes \mathcal{B}_{q_2} \otimes \cdots \otimes \mathcal{B}_{q_n}$. If q_i are linear, i.e. if $q_i(x) = x - \lambda_i$ for all i, then we write $\mathcal{B}_{\boldsymbol{\lambda}} = \mathcal{B}_{\lambda_1} \otimes \mathcal{B}_{\lambda_2} \otimes \cdots \otimes \mathcal{B}_{\lambda_n}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in F^n$. We denote by D_i the map on $F[\mathbf{x}]^{\mathrm{o}}$ induced by the map D on $F[x_i]^{\mathrm{o}}$. If V is a finite-dimensional vector space then the map induced by D_i on a cofree comodule $F[\mathbf{x}]^{\mathrm{o}} \otimes V$ is also denoted by D_i.

The following is an auxiliary result. It is an analogue of Proposition 4.1.

PROPOSITION 6.1 *If $\underline{v} \in F[\mathbf{x}]^{\mathrm{o}} \otimes V$ is such that $(\varepsilon \otimes I_V) D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} \underline{v} = 0$ for all choices of indices $i_1, i_2, \ldots, i_n$ then $\underline{v} = 0$.*

PROOF. Suppose that $\underline{v} = \sum_{j=1}^{k} f_j \otimes v_j$, where $f_j \in F[\mathbf{x}]^{\mathrm{o}}$ and $v_j \in V$, and that the v_j are linearly independent. Then

$$0 = (\varepsilon \otimes I_V) D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} \underline{v} = \sum_{j=1}^{k} f_j \left(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \right) v_j,$$

and so $f_j \left(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \right) = 0$ for all choices of indices $i_1, i_2, \ldots, i_n$ and j. This implies that $f_j = 0$ and hence $\underline{v} = 0$. □

Consider commuting linear maps $A_i : V \to V$ $(i = 1, 2, \ldots, n)$. They induce commuting comodule maps $I \otimes A_i$ on the cofree comodule $F[\mathbf{x}]^{\mathrm{o}} \otimes V$, which we denote again by A_i. The comodule maps $A_i - D_i$ on $F[\mathbf{x}]^{\mathrm{o}} \otimes V$ have a joint kernel

$$\mathcal{R} = \bigcap_{i=1}^{n} \ker (A_i - D_i),$$

which is a subcomodule of $F[\mathbf{x}]^{\mathrm{o}} \otimes V$. Note that (20) induces the direct sum decomposition

$$F[\mathbf{x}]^{\mathrm{o}} \otimes V = \bigoplus_{\mathbf{q} \in \mathbf{Q}} \mathcal{B}_{\mathbf{q}} \otimes V, \tag{21}$$

and then also a direct sum decomposition of $\mathcal{R}$. The latter gives the coalgebraic version of the primary decomposition theorem for several commuting maps (see Theorem 6.3).

If $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is an n-tuple of monic polynomials then we write $\mathcal{C}_{\mathbf{p}} = \mathcal{C}_{p_1} \otimes \mathcal{C}_{p_2} \otimes \cdots \otimes \mathcal{C}_{p_n}$. The restrictions of the comodule maps $A_i - D_i$ $(i = 1, 2, \ldots n)$ to $\mathcal{C}_{\mathbf{p}} \otimes V$ have a joint kernel $\mathcal{R}_{\mathbf{p}} = \mathcal{R} \bigcap (\mathcal{C}_{\mathbf{p}} \otimes V)$. Now we are set to prove the analogues of Lemma 4.2 and Theorem 4.3 for several commuting maps.

LEMMA 6.2 *The joint kernel $\mathcal{R}_{\mathbf{p}}$ is a subcomodule of $\mathcal{C}_{\mathbf{p}} \otimes \bigcap_{i=1}^{n} \ker p_i(A_i)$ and the composite $\mathcal{R}_{\mathbf{p}} \hookrightarrow \mathcal{C}_{\mathbf{p}} \otimes \bigcap_{i=1}^{n} \ker p_i(A_i) \xrightarrow{\varepsilon \otimes I_V} \bigcap_{i=1}^{n} \ker p_i(A_i)$ is bijective. In particular, if $p_i = m_i$ is the monic minimal polynomial of A_i then $\mathcal{R} \subset \mathcal{C}_{\mathbf{m}} \otimes V$, where $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, and $\mathcal{R}_{\mathbf{p}} \hookrightarrow \mathcal{C}_{\mathbf{p}} \otimes \bigcap_{i=1}^{n} \ker p_i(A_i) \xrightarrow{\varepsilon \otimes I_V} V$ is bijective.*

PROOF. By Lemma 4.2 the kernel $\mathcal{R}_{ip_i}$ of $A_i - D_i : \mathcal{C}_{p_i} \otimes V \to \mathcal{C}_{p_i} \otimes V$ is a subcomodule of $\mathcal{C}_{p_i} \otimes \ker p_i(A_i)$. Then it follows that the kernel of $A_i - D_i : \mathcal{C}_{\mathbf{p}} \otimes V \to \mathcal{C}_{\mathbf{p}} \otimes V$ is a subcomodule of $\mathcal{C}_{\mathbf{p}} \otimes \ker p_i(A_i)$, and so $\mathcal{R}_{\mathbf{p}} \subset \mathcal{C}_{\mathbf{p}} \otimes \bigcap_{i=1}^{n} \ker p_i(A_i)$.

By Lemma 4.2 each of the maps $\varepsilon_i \otimes I_V : \mathcal{R}_i \to \ker p_i(A_i)$ is onto, so it follows that $\varepsilon \otimes I_V : \mathcal{R}_{\mathbf{p}} \to \bigcap_{i=1}^{n} \ker p_i(A_i)$ is also onto. To complete the proof we show that $\varepsilon \otimes I_V : \mathcal{R}_{\mathbf{p}} \to \bigcap_{i=1}^{n} \ker p_i(A_i)$ is one-to-one. Suppose that $\underline{v} \in \mathcal{R}_{\mathbf{p}}$ is such that $(\varepsilon \otimes I_V) \underline{v} = 0$. Then it follows that

$$(\varepsilon \otimes I_V) D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} \underline{v} = (\varepsilon \otimes I_V) A_1^{i_1} A_2^{i_2} \cdots A_n^{i_n} \underline{v} = A_1^{i_1} A_2^{i_2} \cdots A_n^{i_n} (\varepsilon \otimes I_V) \underline{v} = 0$$

for all choices of indices $i_1, i_2, \ldots, i_n$, and so $\underline{v} = 0$ by Proposition 6.1. □

THEOREM 6.3 *Suppose that m_i is the minimal polynomial of A_i. Then*

$$\mathcal{R} = \bigoplus_{\mathbf{q}} \mathcal{R}_{\mathbf{q}}, \tag{22}$$

where the sum is over all the n-tuples of monic irreducible polynomials $(q_1, q_2, \ldots, q_n)$ such that
$\bigcap_{i=1}^{n} \ker q_i(A_i) \neq 0$. *Moreover, if $V_{\mathbf{q}} = (\varepsilon \otimes I_V)\mathcal{R}_{\mathbf{q}}$ then*

$$V = \bigoplus_{\mathbf{q}} V_{\mathbf{q}} \tag{23}$$

and $V_{\mathbf{q}} = \bigcap_{i=1}^{n} \ker q_i^{l_i}(A_i)$, where l_i is the multiplicity of q_i in m_i.

PROOF. Since $\mathcal{B}_{\mathbf{q}}$ are invariant for all D_i it follows that $\mathcal{B}_{\mathbf{q}} \otimes V$ is invariant for all $A_i - D_i$. Then the direct sum $\mathcal{R} = \bigoplus_{\mathbf{q}} \mathcal{R}_{\mathbf{q}}$, where the sum is over all n-tuples of irreducible monic polynomials, is induced by (21). However, $\varepsilon \otimes I_V : \mathcal{R}_{\mathbf{q}} \to \bigcap_{i=1}^{n} \ker q_i(A_i)$ is bijective by Lemma 6.2, and hence $\mathcal{R}_{\mathbf{q}} = 0$ if $V_{\mathbf{q}} = \bigcap_{i=1}^{n} \ker q_i(A_i) = 0$. Now the result follows easily. $\square$

COROLLARY 6.4 *If F is algebraically closed then*

$$\mathcal{R} = \bigoplus_{\boldsymbol{\lambda}} \mathcal{R}_{\boldsymbol{\lambda}} \quad \text{and} \quad V = \bigoplus_{\boldsymbol{\lambda}} V_{\boldsymbol{\lambda}},$$

where the sums are over all joint eigenvalues of $\{A_i\}_{i=1}^{n}$, $\mathcal{R}_{\boldsymbol{\lambda}}$ is the joint kernel of the $A_i - D_i : \mathcal{B}_{\boldsymbol{\lambda}} \otimes V \to \mathcal{B}_{\boldsymbol{\lambda}} \otimes V$, and $V_{\boldsymbol{\lambda}} = (\varepsilon \otimes I_V)\mathcal{R}_{\boldsymbol{\lambda}}$.

To conclude the section we remark that the problem posed by Davis [5] on minimal representations of commuting linear maps by tensor products was solved using the above coalgebraic construction in [12]. This shows that, in general, the maps D_i provide a minimal model for n-tuples of commuting maps (cf. [7]).

7 Multiparameter Systems

We consider a system of n-parameter linear pencils

$$W_i(\mathbf{x}) = \sum_{j=1}^{n} A_{ij} x_j - A_{i0}, \quad (i = 1, 2, \ldots, n). \tag{24}$$

For each i the A_{ij} are linear maps on a finite-dimensional vector space V_i. If $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ then A_{ij} induces a linear map $A_{ij}^{\dagger}$ on V by acting on the ith tensor factor of V. The determinant $\Delta_0 : V \to V$ of the matrix

$$A = \begin{bmatrix} A_{11}^{\dagger} & A_{12}^{\dagger} & \cdots & A_{1n}^{\dagger} \\ A_{21}^{\dagger} & A_{22}^{\dagger} & \cdots & A_{2n}^{\dagger} \\ \vdots & \vdots & & \vdots \\ A_{n1}^{\dagger} & A_{n2}^{\dagger} & \cdots & A_{nn}^{\dagger} \end{bmatrix} \tag{25}$$

is a linear map. It is well-defined because any two entries from distinct rows commute. We define determinants Δ_j by replacing the jth column in (25) by the column $\left[A_{i0}^{\dagger}\right]_{i=1}^{n}$. We assume that the multiparameter system considered is *regular*, i.e. that Δ_0 is invertible. Then the linear maps $\Gamma_j = \Delta_0^{-1}\Delta_j$ are called the *associated maps* of the multiparameter system (24). Atkinson [3, Thm. 6.7.1–2] shows that the Γ_j commute, and that

$$\sum_{j=1}^{n} A_{ij}^{\dagger}\Gamma_j = A_{i0}^{\dagger}, \quad (i = 1, 2, \ldots, n)\,. \tag{26}$$

The proof of the latter statement is similar to the scalar case via the adjoint matrix whose entries are the cofactors of determinants Δ_j. Note that the relations (26) are a generalization of *Cramer's Rule*. In matrix form they become $A\left[\Gamma_i\right]_{i=1}^{n} = \left[A_{i0}^{\dagger}\right]_{i=1}^{n}$. For further studies of multiparameter systems in the finite-dimensional and in the general Hilbert space setting we refer to [3, 6, 9, 19, 24, 26].

Atkinson [2, 3] asked for a description of the joint spectral subspace

$$\bigcap_{i=1}^{n} \ker\left(\Gamma_i - \lambda_i I\right)^N,$$

where N is large enough, e.g. $N \geq \dim V$, in terms of the original linear maps A_{ij} without constructing Γ_i explicitly. To answer this problem we now combine Atkinson's approach with the coalgebraic techniques outlined in the preceding sections.

For $i = 1, 2, \ldots, n$ we define comodule maps $W_i(\mathbf{D}) = A_{i0} - \sum_{j=1}^{n} A_{ij} D_j$ which act on the cofree comodules $F[\mathbf{x}]^{\mathrm{o}} \otimes V_i$, and the induced maps $W_i(\mathbf{D})^{\dagger} = A_{i0}^{\dagger} - \sum_{j=1}^{n} A_{ij}^{\dagger} D_j$ which act on $F[\mathbf{x}]^{\mathrm{o}} \otimes V$. From (26) it follows that

$$\sum_{j=1}^{n} A_{ij}^{\dagger}\left(\Gamma_j - D_j\right) = W_i(\mathbf{D})^{\dagger},$$

or, written in matrix form, that

$$A\left[\Gamma_j - D_j\right]_{j=1}^{n} = \left[W_i(\mathbf{D})^{\dagger}\right]_{i=1}^{n}. \tag{27}$$

Next consider the joint kernels $\mathcal{R}_{\mathbf{\Gamma}} = \bigcap_{j=1}^{n} \ker\left(\Gamma_j - D_j\right)$ and $\mathcal{R}_{\mathbf{W}} = \bigcap_{i=1}^{n} \ker W_i(\mathbf{D})^{\dagger}$. Because A is invertible it follows by (27) that

$$\mathcal{R}_{\mathbf{\Gamma}} = \mathcal{R}_{\mathbf{W}}, \tag{28}$$

and by Theorem 6.3 we have that $\mathcal{R}_{\mathbf{\Gamma}} = \bigoplus_{\mathbf{q}} \mathcal{R}_{\mathbf{q}}$, where $\mathbf{q}$ ranges over all the n-tuples of irreducible polynomials such that $\bigcap_{i=1}^{n} \ker q_i(\Gamma_i) \neq 0$.

The following result answers Atkinson's question in full generality. It provides a remarkable example of coalgebraic techniques yielding new results in spectral theory. In the theorem we describe the comodule $\mathcal{R}_{\mathbf{q}}$ in terms of the kernels of $W_i(\mathbf{D})$. Then via $V_{\mathbf{q}} = (\varepsilon \otimes I_V)\mathcal{R}_{\mathbf{q}}$ we get a description of the joint spectral subspace for Γ_i in terms of the original linear maps A_{ij}. In particular, if all the components of $\mathbf{q}$ are linear (always the case if F is algebraically closed) then we answer to Atkinson's question. To simplify the notation we drop the index $\mathbf{q}$.

THEOREM 7.1 *If $\mathcal{R}_i$ is the kernel of $W_i(\mathbf{D}) : \mathcal{B} \otimes V_i \to \mathcal{B} \otimes V_i$ then*

$$\mathcal{R} = \mathcal{R}_1 \otimes^{\mathcal{B}} \mathcal{R}_2 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_n.$$

PROOF. We write $\mathcal{V}_i = \mathcal{B} \otimes V_i$. Because the cotensor product preserves kernels by Lemma 2.4 it follows that $W_1(\mathbf{D})^{\dagger} = \mathcal{R}_1 \otimes^{\mathcal{B}} \mathcal{V}_2 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n$. Now suppose that

$$\bigcap_{i=1}^{k} \ker W_i(\mathbf{D})^{\dagger} = \mathcal{R}_1 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_k \otimes^{\mathcal{B}} \mathcal{V}_{k+1} \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n \tag{29}$$

for some $k < n$. We want to show that (29) holds when k is replaced by $k+1$ and thus prove the theorem by induction on k. If we apply Lemma 2.4 twice then we get that $\mathcal{R}_k \otimes^{\mathcal{B}} \mathcal{R}_{k+1}$ is the intersection of the kernels of

$$W_i(\mathbf{D})^{\dagger} : \mathcal{V}_k \otimes^{\mathcal{B}} \mathcal{V}_{k+1} \to \mathcal{V}_k \otimes^{\mathcal{B}} \mathcal{V}_{k+1} \quad (i = k, k+1).$$

Next we cotensor the comodule $\mathcal{V}_k \otimes^{\mathcal{B}} \mathcal{V}_{k+1}$ by $\mathcal{R}_1 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_{k-1}$ on the left-hand side and by $\mathcal{V}_{k+2} \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n$ on the right-hand side. Then it follows by applying Lemma 2.4 again that $\mathcal{R}_1 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_{k+1} \otimes^{\mathcal{B}} \mathcal{V}_{k+2} \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n$ is the intersection of the kernels of $W_i(\mathbf{D})^{\dagger}$ $(i = k, k+1)$ considered as comodule maps on $\mathcal{R}_1 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_{k-1} \otimes^{\mathcal{B}} \mathcal{V}_k \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n$. This, together with the inductive hypothesis, implies that

$$\bigcap_{i=1}^{k+1} \ker W_i(\mathbf{D})^{\dagger} = \mathcal{R}_1 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_{k+1} \otimes^{\mathcal{B}} \mathcal{V}_{k+2} \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{V}_n. \tag{30}$$

When $k+1 = n$ the relation (30) is $\mathcal{R}_{\mathbf{W}} = \mathcal{R}_1 \alpha^{\mathcal{B}} \mathcal{R}_2 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_n$, and because $\mathcal{R}_{\mathbf{\Gamma}} = \mathcal{R}_{\mathbf{W}}$ it follows that $\mathcal{R}_{\mathbf{\Gamma}} = \mathcal{R}_1 \otimes^{\mathcal{B}} \mathcal{R}_2 \otimes^{\mathcal{B}} \cdots \otimes^{\mathcal{B}} \mathcal{R}_n$. □

Theorem 7.1 is proved in [13, Thm. 5.1]. There, the structure of elements of $\mathcal{R}$ is studied closely and an algorithm to construct a 'canonical' basis for V in the case $n = 2$ and F algebraically closed is discussed. A generalization to the case of Fredholm operators in a Hilbert space is considered as well. For further applications of coalgebraic techniques to multiparameter spectral theory see also [14].

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