# On the Structure of Commutative Matrices

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ABSTRACT

A finite set of commutative matrices is viewed as a cubic array. Its structure is considered via a collection of related symmetric matrices.

## 0. INTRODUCTION

Commutative matrices have been studied (at least) since late last century (see eg. [2,3,8]). They are not often topic of an independent study (as [9]), but usually we can find a chapter on commutative matrices in monographs on linear algebra (eg. [4,5,6]). The authors in [9, cf. pp. 66] noticed that commutativity implies certain symmetries in the structure of products of commutative matrices. We consider these symmetries in further detail. Our motivation for studying commutative matrices comes from multiparameter spectral theory [1].

A finite set of commutative matrices is considered as a cubic array. We restrict our interest to nilpotent commutative matrices. The general commutative case is easily deduced from the nilpotent one. In the first section we introduce some notation and define a basis in which the commutative matrices are simultaneously reduced to a special upper triangular form and so the corresponding cubic array is in a special upper triangular reduced form. In the non-derogatory case the cubic array can be reduced to upper Toeplitz form. This case is considered in section 2. Some auxiliary results concerning matrices whose products are symmetric are presented in the third section. The two consecutive blocks on the upper-diagonal of the cubic array can be reconstructed from a special triple. This main result is found in section 4. We illustrate the preceding discussion with an example in the last section.

## 1. NOTATION

Let $\mathbf{A} = \{A_s;\ s = 1, 2, \ldots, n\}$ be a set of $n$ commutative matrices. Each matrix $A_s$ is a $N \times N$ complex matrix. We also consider $\mathbf{A}$ as a cubic array of numbers of dimensions $N \times N \times n$. Such an array is called *commutative* (since $A_s$ pairwise commute). Two arrays (or two sets of commutative matrices) $\mathbf{A}$ and $\mathbf{A}'$ are called *similar* if there is an $N \times N$ invertible matrix $U$ such that $A_s = U^{-1} A'_s U$ for all $s$. For this collection of equations we also use the notation $\mathbf{A} = U^{-1}\mathbf{A}'U$.

The vector in $\mathbb{C}^n$ consisting of all the $(i,j)$-th entries of matrices in $\mathbf{A}$ is labelled

$$\mathbf{a}_{ij} = \begin{bmatrix} (A_1)_{ij} \\ (A_2)_{ij} \\ \vdots \\ (A_n)_{ij} \end{bmatrix}.$$

Then the row and column cross-sections of $\mathbf{A}$ are defined by

$$R_i = [\,\mathbf{a}_{i1} \quad \mathbf{a}_{i2} \quad \ldots \quad \mathbf{a}_{iN}\,]$$

and

$$C_j = \begin{bmatrix} \mathbf{a}_{1j}^T \\ \mathbf{a}_{2j}^T \\ \vdots \\ \mathbf{a}_{Nj}^T \end{bmatrix}.$$

These are $n \times N$ and $N \times n$ complex matrices, respectively, for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$.

**Definition.** *A complex $N \times N$ matrix is called symmetric if $A = A^T$, i.e. if it is equal to its transpose (without conjugation).*

**Lemma 1.** *The array $\mathbf{A}$ is commutative if and only if the products $R_i C_j$ are symmetric for all $i, j = 1, 2, \ldots, N$.*

*Proof.* The $(i,j)$-th entry of the product $A_r A_s$ $(r, s = 1, 2, \ldots, n)$ is

$$(A_r A_s)_{ij} = \sum_{k=1}^{N} (A_r)_{ik} (A_s)_{kj} = \sum_{k=1}^{N} (R_i)_{rk} (C_j)_{ks} = (R_i C_j)_{rs}$$

Thus $A_r A_s = A_s A_r$ if and only if $(R_i C_j)_{rs} = (R_i C_j)_{sr}$, that is, if and only if $R_i C_j$ are symmetric. $\diamondsuit$

It is well known (see eg. [5, pp.298]) that commutative linear transformations $A_s$ on $\mathbb{C}^n$ reduce the space into the direct sum of invariant subspaces for all $A_s$ such that every $A_s$ has on every direct summand exactly one eigenvalue $\lambda_s$. Replacing $A_s$ by $A_s - \lambda_s I$, restricted to a common invariant subspace, that eigenvalue is 0 for all $s$. Therefore we will assume in what follows that the commutative matrices $\mathbf{A}$ have only one eigenvalue 0, or equivalently that they are all nilpotent.

Let $m$ be the minimal number such that $A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} = 0$ for all collections of $k_j \geq 0$ such that $\sum_{j=1}^{n} k_j = m$. Since the product of $N$ upper triangular $N \times N$ matrices with zero diagonal is 0, it follows that $m \leq N$. (This idea can be found in the proof of Theorem 2 in [7] due to H. W. Lenstra Jr.) For $i = 1, 2, \ldots, m$ we write

$$\ker \mathbf{A}^i = \bigcap_{\sum_{j=1}^{n} k_j = i,\, k_j \geq 0} \ker \left( A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} \right) \qquad \text{and} \qquad D_i = \dim \ker \mathbf{A}^i$$

and $d_i = D_i - D_{i-1}$ for $i = 1, 2, \ldots, m$ where $D_0 = 0$. There exists a basis

$$\mathcal{B} = \left\{z_1^1, z_2^1, \ldots, z_{d_1}^1; \quad z_1^2, z_2^2, \ldots, z_{d_2}^2; \quad \ldots \quad ; \quad z_1^m, z_2^m, \ldots, z_{d_m}^m\right\}$$

for $\mathbb{C}^N$ such that for every $i = 1, 2, \ldots, m$ the set

$$\mathcal{B}_i = \left\{z_1^1, z_2^1, \ldots, z_{d_1}^1; \quad z_1^2, z_2^2, \ldots, z_{d_2}^2; \quad \ldots \quad ; \quad z_1^i, z_2^i, \ldots, z_{d_i}^i\right\}$$

is a basis for $\ker \mathbf{A}^i$.

If we now consider $\mathbf{A}$ as a cubic array with slices consisting of matrices $A_s$ $(s = 1, 2, \ldots, n)$ then $\mathbf{A}$ has the following representation on $\ker \mathbf{A}^m = \mathbb{C}^N$ in the basis $\mathcal{B}$ :

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{12} & \mathbf{A}^{13} & \cdots & \mathbf{A}^{1,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{23} & \cdots & \mathbf{A}^{2,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{m-1,m} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{1}$$

where $\mathbf{A}^{kl} = \begin{bmatrix} \mathbf{a}_{11}^{kl} & \mathbf{a}_{12}^{kl} & \cdots & \mathbf{a}_{1,d_l}^{kl} \\ \mathbf{a}_{21}^{kl} & \mathbf{a}_{22}^{kl} & \cdots & \mathbf{a}_{2,d_l}^{kl} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{d_k,1}^{kl} & \mathbf{a}_{2,d_k}^{kl} & \cdots & \mathbf{a}_{d_k,d_l}^{kl} \end{bmatrix}$ is a cubic array of dimensions $d_k \times d_l \times n$ and $\mathbf{a}_{ij}^{kl} \in \mathbb{C}^n$.
The array (1) is block upper tringular with zero diagonal since $A_s\left(\ker \mathbf{A}^i\right) \subset \ker \mathbf{A}^{i-1}$ for all $s$. The last relation follows from the definition of $\ker \mathbf{A}^i$. If we expand the vector $A_s z_j^l$ in the basis $\mathcal{B}$ then $\left(\mathbf{a}_{ij}^{kl}\right)_s$ is the coefficient of $z_i^k$ in this expansion. The row and column cross-sections of $\mathbf{a}^{kl}$ are

$$R_i^{kl} = [\,\mathbf{a}_{i1}^{kl} \quad \mathbf{a}_{i2}^{kl} \quad \cdots \quad \mathbf{a}_{i,d_l}^{kl}\,] \qquad\qquad i = 1, 2, \ldots, d_k$$

and

$$C_j^{kl} = \begin{bmatrix} \left(\mathbf{a}_{1j}^{kl}\right)^T \\ \left(\mathbf{a}_{2j}^{kl}\right)^T \\ \vdots \\ \left(\mathbf{a}_{d_k,j}^{kl}\right)^T \end{bmatrix} \qquad\qquad j = 1, 2, \ldots, d_l.$$

These are matrices of dimensions $n \times d_l$ and $d_k \times n$, respectively. In this setting we have

**Proposition 1.** *The basis $\mathcal{B}$ is chosen so that for $k = 1, 2, \ldots, m-1$ the matrices $C_j^{k,k+1}$, $j = 1, 2, \ldots, d_{k+1}$ are linearly independent.*

*Proof.* Let assume the contrary to obtain a contradiction. If the matrices $C_j^{k,k+1}$ are linearly dependent, i.e. $\sum_{j=1}^{d_{k+1}} \alpha_j C_j^{k,k+1} = 0$ and not all $\alpha_j$ equal 0, then there exists a vector $x \in \ker \mathbf{A}^{k+1} \setminus \ker \mathbf{A}^k$, i.e. $x = \sum_{j=1}^{d_{k+1}} \alpha_j z_j^{k+1}$, such that $A_s x \in \ker \mathbf{A}^{k-1}$ for all $s$. But this yields $x \in \ker \mathbf{A}^k$ which contradicts $x \notin \ker \mathbf{A}^k$. $\diamondsuit$

We will now restate Lemma 1 for the case when $\mathbf{A}$ is in the form (1).

**Corollary 1.** *An array* $\mathbf{A}$ *is commutative if and only if the matrices*

$$\sum_{h=k+1}^{l-1} R_i^{kh} C_j^{hl},$$

$k = 1, 2, \ldots, m-2; l = k+2, k+3, \ldots, m; i = 1, 2, \ldots, d_k; j = 1, 2, \ldots, d_l,$ *are symmetric.*

Note that there is no condition on $\mathbf{A}^{1m}$. So array $\mathbf{A}$ in form (1) for $m = 2$ is always commutative.

## 2. UPPER TOEPLITZ FORM

**Definition.** *Assume that* $d_1 = d_2 = \cdots = d_m = 1$. *Then* $\mathbf{A}$ *is in upper Toeplitz form if* $\mathbf{A}^{kl} = \mathbf{A}^{k-1,l-1}$ *for* $k = 2, 3, \ldots, m; l > k$, *and the other* $\mathbf{A}^{kl}$ *are* $\mathbf{0}$.

**Theorem 1.** *Assume that* $d_l = 1$ *for some* $l \geq 2$. *Then* $d_l = d_{l+1} = \cdots = d_m = 1$. *By a suitable change of basis we can assume*

$$\mathbf{A}^{l-1,l} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{a}_{d_{l-1}1}^{l-1,l} \end{bmatrix}$$

*and the bottom right* $(m-l+1) \times (m-l+1)$ *block of* $\mathbf{A}$ *can be written in upper Toeplitz form.*

*Proof.* By Corollary 1 the matrices

$$S_{i1} = \mathbf{a}_{i1}^{l-1,l} \cdot \left(\mathbf{a}_{11}^{l,l+1}\right)^T \qquad\qquad i = 1, 2, \ldots, d_{l-1}$$

are symmetric and by Proposition 1 they are not all 0. Thus there are complex numbers $\epsilon_{i1}$ not all 0 such that $\mathbf{a}_{i1}^{l-1,l} = \epsilon_{i1}\mathbf{a}_{11}^{l+1,l}$. If we replace $z_{d_{l-1}}^{l-1}$ in the basis $\mathcal{B}$ by the vector $\sum_{i=1}^{d_{l-1}} \epsilon_{i1} z_i^{l-1}$ we obtain a new basis in which the array $\mathbf{a}^{l-1,l}$ is of the required form

$$\mathbf{A}^{l-1,l} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{b} \end{bmatrix}$$

where $\mathbf{b} \neq \mathbf{0}$.

Now suppose that not all $d_j = 1$ for $j \geq l+1$. Then say that $h$ $(\geq l+1)$ is the smallest number such that $d_h > 1$. If $h \geq l+2$ then for $k = l, l+1, \ldots, h-2$ the arrays $\mathbf{A}^{k,k+1}$ are nonzero and of dimensions $1 \times 1 \times n$, so they can be considered as $n-$vectors. Thus we identify $\mathbf{A}^{k,k+1}$ with $\mathbf{a}_{11}^{k,k+1}$ and denote it $\mathbf{a}^{k,k+1}$. By Corollary 1 the matrices

$$S_{l-1} = \mathbf{b} \cdot \left(\mathbf{a}^{l,l+1}\right)^T \text{ and } S_k = \mathbf{a}^{k,k+1} \cdot \left(\mathbf{a}^{k+1,k+2}\right)^T ; \; k = l, l+1, \ldots, h-3$$

are symmetric and since the vectors $\mathbf{b}$ and $\mathbf{a}^{k,k+1}$ are nonzero the ranks of the $S_k$ are exactly 1. Therefore, there exist nonzero complex numbers $\epsilon_k$ such that $\mathbf{a}^{k,k+1} = \epsilon_k \mathbf{b}$ for $k = l, l+1, \ldots, h-3$. Further, if $h = l+1$ (resp. $h \geq l+2$) the matrices $S_{l-1}^j = \mathbf{b} \cdot \left(\mathbf{a}_{1j}^{l,l+1}\right)^T$ (resp. $S_{h-2}^j = \mathbf{a}^{h-2,h-1} \cdot \left(\mathbf{a}_{1j}^{h-1,h}\right)^T$) are symmetric and of rank exactly 1 for $j = 1, 2$. They are not zero since by Proposition 1 $\mathbf{a}_{1j}^{h-1,h}$ are linearly independent. Hence there exist nonzero numbers $\epsilon_{h-1}^j$ such that $\mathbf{a}_{1j}^{h-1,h} = \epsilon_{h-1}^j \mathbf{b}$. The vector $\epsilon_{h-1}^2 z_1^h - \epsilon_{h-1}^1 z_2^h$ is then in the subspace $\ker \mathbf{A}^{h-1}$. This contradicts the fact that the vectors $z_k^i$ with index $i \leq h-1$ form a basis for $\ker \mathbf{A}^{h-1}$ and $z_k^i$ with index $i \leq h$ basis for $\ker \mathbf{A}^h$. Thus $d_l = d_{l+1} = \cdots = d_m = 1$.

Now we restrict the matrices $A_i$ to the quotient $\mathcal{Q} = \mathbb{C}^N|_{\mathbb{C}^{l-1} \times \{0\}}$. To finish the proof it has to be shown that there is a basis for $\mathcal{Q}$ such that all the restricted matrices $A_s|_{\mathcal{Q}}$ are in upper Toeplitz form. In the first part of the proof we saw that for $k = l, l+1, \ldots, m-1$ all the $\mathbf{a}^{k,k+1}$ are nonzero multiples of $\mathbf{b}$. Therefore there is a number $r$ between 1 and $n$ such that $A_r|_{\mathcal{Q}}$ has a Jordan chain of length $m-l+1$. Then by [5, pp.296 or 6, pp.130] we can find a basis in which all $A_s|_{\mathcal{Q}}$ (and thus the bottom right $(m-l+1) \times (m-l+1)$ block of $\mathbf{A}$) are in the upper Toeplitz form. ♢

The following is a special case of Theorem 1

**Corollary 2.** *Assume that* $d_1 = d_2 = 1$. *Then for* $j = 1, 2, \ldots, m$ *each* $d_j = 1$ *and* $\mathbf{A}$ *has upper Toeplitz representation.*

This result is a generalization of results for the nonderogatory case in [5, pp.296, 6, pp.130].

## 3. MATRICES WHOSE PRODUCT IS SYMMETRIC

Before describing the structure of $\mathbf{A}$ further we will prove the following auxiliary results which are of interest in themselves.

**Lemma 2.** *Let* $R$ *and* $C^T$ *be* $p \times q$ *complex matrices where* $p \geq q$ *and assume that* $\operatorname{rank} R = q$. *Then* $RC$ *is symmetric if and only if there is a symmetric matrix* $X \in \mathbb{C}^{q \times q}$ *such that* $C^T = RX$. *The matrix* $X$ *is unique.*

*Proof.* Assume first that the product $RC$ is symmetric. Let $Y \in \mathbb{C}^{q \times p}$ be a left inverse for $R$. Then $C = YC^TR^T$ or $C^T = R\left(CY^T\right)$. Denoting $X = CY^T$, we have $X^T = YC^T = YRX = X$, thus $X$ is symmetric.

Conversely, let $C^T = RX$ and $X = X^T$. Then

$$RC = RX^TR^T = RXR^T = C^TR^T$$

and thus the product $RC$ is symmetric.

It remains to show that $X$ is unique. Suppose that $C^T = RX_1 = RX_2$. Then by left invertibility of $R$ it follows that $X_1 = X_2$. $\diamondsuit$

The next result will generalise Lemma 2 to the case where there are $k$ matrices $R_j;\quad j = 1, 2, \dots, k$ such that all the products $R_jC$ are symmetric. We assume that $kp \geq q$ and that

$$\operatorname{rank} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} = q. \tag{2}$$

Define $r = \operatorname{rank} [\, R_1 \quad R_2 \quad \cdots \quad R_k \,]$ and let the columns of the matrix $\widetilde{R} \in \mathbb{C}^{p\times r}$ form a basis for the space spanned by the columns of $[\, R_1 \quad R_2 \quad \cdots \quad R_k \,]$. Then for $j = 1, 2, \dots, k$ there is a matrix $S_j \in \mathbb{C}^{r\times q}$ such that $R_j = \widetilde{R}S_j$. Moreover (2) implies

$$\operatorname{rank} \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix} = q. \tag{3}$$

Since for every vector $x$ in the intersection of the kernels of $S_j$ it follows that $R_jx = \widetilde{R}S_jx = 0$ whence $x \in \cap_{j=1}^{k} \ker R_j = \{0\}$ and so $x = 0$. Property (3) implies that the matrix $\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix}$ has a left inverse $[\, Z_1 \quad Z_2 \quad \cdots \quad Z_k \,]$ where all $Z_j$ are $q \times r$ matrices. Using this notation we have

**Lemma 3.** *Assume that $C^T$ and $R_j;\quad j = 1, 2, \cdots, k$ are $p \times q$ matrices, that $kp \geq q$ and that* (2) *holds. Then the matrices $R_jC$ are all symmetric if and only if there exist $k$ symmetric matrices $X_j \in \mathbb{C}^{r\times r}$ such that*

$$C = \left( \sum_{j=1}^{k} Z_jX_j \right) \widetilde{R}^T \qquad \text{and} \qquad S_l \left( \sum_{j=1}^{k} Z_jX_j \right) = X_l; \quad l = 1, 2, \dots, k. \tag{4}$$

*Proof.* Let $R_jC$ be all symmetric. Then $R_jC = C^TR_j^T$ implies $\widetilde{R}(S_jC) = \left(C^TS_j^T\right)R^T$, so matrices $\widetilde{R}$ and $S_jC$ satisfy the conditions of Lemma 2. Then there are symmetric matrices $X_j \in \mathbb{C}^{r\times r}$ such that $S_jC = X_j\widetilde{R}^T$. From the proof of Lemma 2 we see that $X_j = S_jCY^T$ where $Y \in \mathbb{C}^{r\times p}$ is a left inverse of $\widetilde{R}$. The above equations can be put together as

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix} C = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \widetilde{R}^T$$

Multiplying on the left by $[\, Z_1 \quad Z_2 \quad \cdots \quad Z_k \,]$ we get

$$\left(\sum_{j=1}^{k} Z_j S_j\right) C = \left(\sum_{j=1}^{k} Z_j X_j\right) \widetilde{R}^T$$

and so

$$C = \left(\sum_{j=1}^{k} Z_j X_j\right) \widetilde{R}^T$$

Finally, a simple calculation gives the second part of (4), viz.

$$S_l \left(\sum_{j=1}^{k} Z_j X_j\right) = S_l \left(\sum_{j=1}^{k} Z_j S_j\right) CY^T = S_l CY^T = X_l$$

for all $l = 1, 2, \ldots, k$.

Let us now prove the converse. We have symmetric matrices $X_j$ which satisfy (4). Then $C^T R_l^T = \widetilde{R} \left(\sum_{j=1}^{k} X_j Z_j^T S_l^T\right) \widetilde{R}^T = \widetilde{R} X_l \widetilde{R}^T$ and $R_l C = \widetilde{R} S_l \left(\sum_{j=1}^{n} Z_j X_j\right) \widetilde{R}^T = \widetilde{R} X_l \widetilde{R}^T$. Hence the products $R_l C$ are all symmetric. ♢

## 4. STRUCTURE OF COMMUTATIVE MATRICES

**Proposition 2.** *Denote the dimension of the span of the set*

$$\left\{\mathbf{a}_{ij}^{l,l+1};\ i = 1, 2, \ldots, d_l;\ j = 1, 2, \ldots, d_{l+1}\right\}$$

*by $r_l$, for $l = 1, 2, \ldots, m-1$. Then*

$$\frac{d_{l+1}}{d_l} \le r_l \le \min\{n, d_l d_{l+1}\}$$

*for $l = 1, 2, \ldots, m-1$ and*

$$r_l \ge r_{l+1}$$

*for $l = 1, 2, \ldots, m-2$.*

*Proof.* The array $\mathbf{A}^{l,l+1}$ is constructed so that $r_l \le \min\{n, d_l d_{l+1}\}$. Furthermore, the rank of the matrix $\begin{bmatrix} R_1^{l,l+1} \\ R_2^{l,l+1} \\ \vdots \\ R_{d_l}^{l,l+1} \end{bmatrix} \in \mathbb{C}^{n d_l \times d_{l+1}}$ is $d_{l+1}$ (cf. Proposition 1). Since $r_{lj} = \operatorname{rank}\left(R_j^{l,l+1}\right) \le r_l$ for $j = 1, 2, \ldots, d_l$ and $\operatorname{rank}(\sum_{j=1}^{d} R_j) \le \sum_{j=1}^{d} \operatorname{rank} R_j$ for any matrices $R_j$ of the same sizes it follows

$$d_{l+1} \le \sum_{j=1}^{d_l} r_{lj} \le d_l r_l.$$

By Corollary 1 the matrices $R_i^{l,l+1}C_j^{l+1,l+2}$ are symmetric for $l = 1, 2, \ldots, m-2$ and by Proposition 1 the matrix $\begin{bmatrix} R_1^{l,l+1} \\ R_2^{l,l+1} \\ \vdots \\ R_{d_l}^{l,l+1} \end{bmatrix}$ has full rank. So for every $j$ the matrices $C_j^{l+1,l+2}$ and $R_i^{l,l+1}$, $i = 1, 2, \ldots, d_l$, satisfy the conditions of Lemma 3. Then by (4) the rows of $C_j^{l+1,l+2}$ are in the span of the columns of $R_i^{l,l+1}$ and so $r_l \geq r_{l+1}$. ♢

Let us now consider the case $m = 3$. Then

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{12} & \mathbf{A}^{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5}$$

Commutativity imposes conditions only on the arrays $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$. So we are only interested in these two arrays.

First we will discuss the special case when the row cross-sections of $\mathbf{A}^{12}$ span a one-dimensional subspace in $\mathbb{C}^{n\times d_2}$. By a suitable change of basis $\mathcal{B}_1$ we can assume that

$$\mathbf{A}^{12} = \begin{bmatrix} \mathbf{a}_{11}^{12} & \mathbf{a}_{12}^{12} & \cdots & \mathbf{a}_{1d_2}^{12} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{6}$$

Then we have a simpler version of the main result :

**Theorem 2.** *Assume that $\mathbf{A}$ is commutative with $m = 3$ and that $\mathbf{A}^{12}$ has the form (6). Then the array $\mathbf{A}^{23}$ is generated by a set of $d_3$ symmetric matrices of sizes $d_2 \times d_2$.*

*Proof.* By Corollary 1 the products $R_1^{12}C_j^{23}$ are symmetric and by Proposition 1 the matrix $R_1^{12}$ has full rank. Thus by Lemma 2 there exist symmetric matrices $X_j$ such that $\left(C_j^{23}\right)^T = R_1^{12}X_j$ for all $j$. ♢

The above special case has some significance in the application to multiparameter spectral theory. This will be discussed elsewhere.

Before we state the main result for the general case $m = 3$ let us introduce some new notions. Proposition 2 makes the following definition sensible.

**Definition.** *The set of integers $\mathcal{D} = \{d_1, d_2, d_3;\ r\}$, where all $d_j$ and $r$ are positive, is called an admissible set if*

$$\sum_{j=1}^{3} d_j = N, \qquad r \leq n \qquad \text{and} \qquad \frac{d_{l+1}}{d_l} \leq r \leq d_l d_{l+1} \quad \text{for } l = 1, 2.$$

For the set of matrices $X_{ij} \in \mathbb{C}^{r\times s}$; $i = 1, 2, \ldots, d_1$; $j = 1, 2, \ldots, d_2$ we introduce the matrix

$$\mathbf{X}_j = \begin{bmatrix} X_{1j} \\ X_{2j} \\ \vdots \\ X_{d_1 j} \end{bmatrix}$$

and we denote by $\mathcal{S}$ the subspace in $\mathbb{C}^{d_1 r}$ spanned by the union of the ranges of $\mathbf{X}_j$ for all $j$. Similarly for a set of matrices $\{S_i \in \mathbb{C}^{r\times s};\ i = 1, 2, \ldots, d_1\}$ we write

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_1} \end{bmatrix}$$

and

$$\mathbf{S}^T = [\, S_1 \quad S_2 \quad \cdots \quad S_{d_1} \,] .$$

**Definition.** *For a given admissible set $\mathcal{D}$ the triple $(\widetilde{R}, \mathcal{X}, P)$, where $\widetilde{R}$ is a full rank $n \times r$ matrix, $\mathcal{X} = \{X_{ij};\ i = 1, 2, \ldots, d_1;\ j = 1, 2, \ldots, d_3\}$ is a set of $r \times r$ symmetric matrices and $P$ is a projection in $\mathbb{C}^{d_1 r \times d_1 r}$, is a structure triple (for $\mathcal{D}$) if it satisfies the conditions :*

*(i)* $\mathbf{X}_j$, $j = 1, 2, \ldots, d_3$ *are linearly independent*

*(ii) the rank of $P$ is $d_2$*

*(iii) $\mathcal{S}$ is a subspace of $\mathcal{R} = \mathrm{Im}\ P$.*

**Theorem 3.** *Given a structure triple we can describe (to within similarity) the arrays $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$ of a commutative cubic array $\mathbf{A}$. (Commutativity does not depend on the choise of the array $\mathbf{A}^{13}$.)*

*Conversely, for a given commutative array $\mathbf{A}$ with $m = 3$ we can find a structure triple which generates the arrays $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$ of $\mathbf{A}$.*

*Proof.* Suppose we are given a structure triple $(\widetilde{R}, \mathcal{X}, P)$. Let $\ker P = \mathcal{K}$ and $\mathrm{Im}\ P = \mathcal{R}$. The projection $P$ can be written in the form

$$P = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_1} \end{bmatrix} [\, Z_1 \quad Z_2 \quad \cdots \quad Z_{d_1} \,] \tag{7}$$

where $S_j, Z_k^T \in \mathbb{C}^{r\times d_2}$, $\sum_{j=1}^{d_1} Z_j S_j = I$ and $\mathrm{Im}\ \mathbf{S} = \mathcal{R}$. The decomposition (7) can be obtained for example from the matrix $T = \begin{bmatrix} S_1 & S_{11} \\ S_2 & S_{21} \\ \vdots & \vdots \\ S_{d_1} & S_{d_1 1} \end{bmatrix}$, $S_{j1} \in \mathbb{C}^{r \times r d_1 - d_2}$ where the first $d_2$ columns form a basis for $\mathcal{R}$ and the rest form basis for $\mathcal{K}$. Then we choose $[\, Z_1 \quad Z_2 \quad \cdots \quad Z_{d_1} \,]$ to be the first $d_2$ rows of the inverse $T^{-1}$. Any other decomposition of $P$ as in (7) is given by

$$P = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_1} \end{bmatrix} UU^{-1} [\, Z_1 \quad Z_2 \quad \cdots \quad Z_{d_1} \,]$$

for some invertible matrix $U \in \mathbb{C}^{d_2 \times d_2}$. Then an array $\mathbf{A}$ is generated as follows. The rows of $\mathbf{A}^{12}$ are given by

$$R_i = \widetilde{R} S_i \qquad\qquad i = 1, 2, \ldots, d_1$$

and the columns of $\mathbf{A}^{23}$ are given by

$$C_j^T = \widetilde{R} \sum_{i=1}^{d_1} X_{ij} Z_i^T \qquad\qquad j = 1, 2, \cdots, d_3.$$

First, the columns of $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$ are linearly independent. The columns of $\mathbf{A}^{23}$ are linearly independent since $\mathbf{X}_j$ are linearly independent and the columns of $\mathbf{A}^{12}$ are linearly independent since the columns of $\mathbf{S}$ are linearly independent. In order to prove that $\mathbf{A}$ is commutative it remains to show by Corollary 1 and Lemma 3 that $S_l \left( \sum_{i=1}^{d_1} Z_i X_{ij} \right) = X_{lj}$ for all $l$ and $j$. Since $\mathcal{S} \subset \mathcal{R}$ we have $P\mathbf{X}_j = \mathbf{X}_j$ or written by blocks $\sum_{i=1}^{d_1} S_l Z_i X_{ij} = X_{lj}$ for all $l$ and $j$, which proves commutativity.

If we take another decomposition

$$P = \begin{bmatrix} S_1 U \\ S_2 U \\ \vdots \\ S_{d_1} U \end{bmatrix} [\, U^{-1} Z_1 \quad U^{-1} Z_2 \quad \cdots \quad U^{-1} Z_{d_1} \,]$$

we will get a similar array $\mathbf{A}_U$. The similarity transformation between $\mathbf{A}$ and $\mathbf{A}_U$ is given by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Let us now explain how to obtain the structure triple from a commutative array $\mathbf{A}$. Since $\mathbf{A}$ is commutative the products $R_i^{12} C_j^{23}$ are symmetric for $i = 1, 2, \ldots, d_1$; $j = 1, 2, \ldots, d_3$ by Corollary 1. For every $j$ the matrices $R_i^{12}$, $i = 1, 2, \ldots, d_1$ and $C_j^{23}$ satisfy the conditions of Lemma 3. So there exist matrices $\widetilde{R}, X_{ij}, S_l$ and $Z_l$ as in Lemma 3. We can choose the matrices $\widetilde{R}$, $S_l$ and $Z_l$ to be the same for all $j$ since they depend only on $R_i^{12}$. Then the triple $(\widetilde{R}, \mathcal{X}, P)$ is a structure triple where $\mathcal{X} = \{X_{ij};\ i = 1, 2, \ldots, d_1;\ j = 1, 2, \ldots, d_3\}$ and $P = \mathbf{S}\mathbf{Z}^T$. We need to check conditions $(i)$-$(iii)$. Condition $(i)$ holds since $C_j$ are linearly independent. By the construction of $\mathbf{S}$ and $\mathbf{Z}$ the rank of $P$ is equal to $\operatorname{rank} \mathbf{S} = d_2$ and by the right-hand equations in (4) the span of the ranges of $\mathbf{X}_j$ is a subspace of $\operatorname{Im} P$. ◊

Theorem 3 tells us that in the case $m = 3$ the array (1) is commutative if the arrays $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$ are given through a structure triple. So we ensure that the row-column products of Corollary 1 are symmetric. If $m \geq 4$ we can consider the array (1) as a collection of $\frac{(m-1)(m-2)}{2}$ cases with

$m = 3$. Namely, for every pair of integers $(k, l)$; $1 \le k \le l-2 \le m-2$ we have the problem

$$\left[\begin{array}{ccc} \mathbf{0} & \begin{pmatrix} \mathbf{A}^{k,k+1} & \mathbf{A}^{k,k+2} & \cdots & \mathbf{A}^{k,l-1} \\ \mathbf{0} & \mathbf{A}^{k+1,k+2} & \cdots & \mathbf{A}^{k+1,l-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{l-2,l-1} \end{pmatrix} & * \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} \mathbf{A}^{k+1,l} \\ \mathbf{A}^{k+2,l} \\ \vdots \\ \mathbf{A}^{l-1,l} \end{pmatrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{8}$$

with

$$\mathcal{D}_{kl} = \left\{ \sum_{i=k}^{l-2} d_i, \sum_{i=k+1}^{l-1} d_i, d_l;\, r_{kl} \right\}.$$

The number $r_{kl}$ is the dimension of the span of

$$\left\{ \mathbf{a}_{ij}^{kh},\, h = k+1, k+2, \ldots, l-1;\, i = 1, 2, \ldots, d_k;\, j = 1, 2, \ldots, d_h \right\}.$$

The array

$$\begin{pmatrix} \mathbf{A}^{k,k+1} & \mathbf{A}^{k,k+2} & \cdots & \mathbf{A}^{k,l-1} \\ \mathbf{0} & \mathbf{A}^{k+1,k+2} & \cdots & \mathbf{A}^{k+1,l-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{l-2,l-1} \end{pmatrix} \tag{9}$$

is acting as the array $\mathbf{A}^{12}$ in the case $m = 3$ and the array

$$\begin{pmatrix} \mathbf{A}^{k+1,l} \\ \mathbf{A}^{k+2,l} \\ \vdots \\ \mathbf{A}^{l-1,l} \end{pmatrix} \tag{10}$$

as the array $\mathbf{A}^{23}$ in the case $m = 3$. The sizes of $\mathbf{0}$ and $*$ in (8) are not important when we generate the arrays (9) and (10) from a structure triple as described in Theorem 3 for $\mathbf{A}^{12}$ and $\mathbf{A}^{23}$. The row-column products of the arrays (9) and (10) are exactly the products in Corollary 1. So $\mathbf{A}$ is commutative if and only if these products are symmetric. Then the structure triples of the above problems (8) (subject to appropriate matching conditions), together with an array $\mathbf{A}^{1,m}$, describe $\mathbf{A}$.

## 5. EXAMPLE

Let us consider an example similar to the one in [6 pp.130-131]. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the (nilpotent) matrices that commute with $A_1$ have the form

$$A_2 = \begin{bmatrix} 0 & a_{11} & a_{12} & a_{21} & a_{22} \\ 0 & 0 & a_{11} & 0 & a_{21} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{31} & a_{32} & 0 & a_{41} \\ 0 & 0 & a_{31} & 0 & 0 \end{bmatrix}.$$

where all $a_{ij}$ are arbitrary. In order to construct the array $\mathbf{A}$ in the form (1) we need to look at different cases depending if some of $a_{ij}$ are 0. There are two choices for $m$, 3 and 5. In the case $m = 3$ there are two choices for admissible sets : $\{1, 2, 2;\ 2\}$ and $\{2, 2, 1;\ 2\}$. If $m = 5$ then $d_i = 1$ for all $i$. We here present the cubic array $\mathbf{A}$ as a two-dimensional array of column vectors.

($i$) Let all $a_{ij}$ in $A_2$ be nonzero. Then $m = 5$ and $d_1 = d_2 = d_3 = d_4 = d_5 = 1$. In the basis $\mathcal{B}_1 = \{e_1, e_4, e_2, e_5, e_3\}$ the array $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix} & \begin{pmatrix}0\\a_{22}\end{pmatrix} & \begin{pmatrix}0\\a_{12}\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{31}\end{pmatrix} & \begin{pmatrix}1\\a_{41}\end{pmatrix} & \begin{pmatrix}0\\a_{32}\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{31}\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \end{bmatrix}$$

and in the basis $\mathcal{B}_1' = \left\{e_1, e_4, \alpha e_2, \alpha e_5 + \beta e_2, \alpha^2 e_3 + \beta e_5 + \gamma e_2\right\}$ where $\alpha = \frac{a_{21}}{a_{31}}$, $\beta = \frac{a_{21}}{a_{31}^2}(a_{11} - a_{41})$, $\gamma = \frac{a_{21}}{a_{31}^3}\left((a_{11} - a_{41})^2 + a_{22}a_{31} - a_{21}a_{32}\right)$ the array $\mathbf{A}$ is in the upper Toeplitz form

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}\alpha\\\delta\end{pmatrix} & \begin{pmatrix}\beta\\\eta\end{pmatrix} & \begin{pmatrix}\gamma\\\phi\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}\alpha\\\delta\end{pmatrix} & \begin{pmatrix}\beta\\\eta\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}\alpha\\\delta\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \end{bmatrix}$$

where

$$\delta = \frac{a_{11}a_{21}}{a_{31}}, \ \eta = \frac{a_{21}}{a_{31}^2}\left(a_{11}^2 + a_{22}a_{31} - a_{11}a_{41}\right)$$

and

$$\phi = \frac{a_{21}}{a_{31}^2}\left(a_{21}a_{12} + a_{11}a_{22} - a_{22}a_{41} + \frac{a_{11}}{a_{31}}\left((a_{11} - a_{41})^2 + a_{22}a_{31} - a_{21}a_{32}\right)\right).$$

(*ii*) Suppose now that $a_{21}=0$ and the other $a_{ij}$ are nonzero. Then $m=3$ and $d_1=2$, $d_2=2$ and $d_3=1$. In the basis $\mathcal{B}_2=\{e_1,e_4;\ e_2,e_5;\ e_3\}$ we have

$$\mathbf{A}=\begin{bmatrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix} & \begin{pmatrix}0\\a_{22}\end{pmatrix} & \begin{pmatrix}0\\a_{12}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{31}\end{pmatrix} & \begin{pmatrix}1\\a_{41}\end{pmatrix} & \begin{pmatrix}0\\a_{32}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{31}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \end{bmatrix}.$$

We can choose the structure triple of $\mathbf{A}$ to be

$$\widetilde{R}=\begin{bmatrix}1&0\\0&1\end{bmatrix};\ X_{11}=\begin{bmatrix}1&a_{11}\\a_{11}&a_{11}^2+a_{22}a_{31}\end{bmatrix},\ X_{21}=\begin{bmatrix}0&a_{31}\\a_{31}&a_{31}(a_{11}+a_{41})\end{bmatrix}$$

and

$$P=\mathbf{SZ}^T=\begin{bmatrix}1&0\\a_{11}&a_{22}\\0&1\\a_{31}&a_{41}\end{bmatrix}\begin{bmatrix}1&0&0&0\\-\frac{a_{11}}{a_{22}}&\frac{1}{a_{22}}&0&0\end{bmatrix}.$$

The array $\mathbf{A}^{13}$ is

$$\mathbf{A}^{13}=\begin{bmatrix}\begin{pmatrix}0\\a_{12}\end{pmatrix}\\\begin{pmatrix}0\\a_{32}\end{pmatrix}\end{bmatrix}.$$

(*iii*) The last case we will consider is $a_{31}=0$ while the other $a_{ij}\neq 0$. Then $m=3$ and $d_1=1$, $d_2=2$ and $d_3=2$. In the basis $\mathcal{B}_3=\{e_1;\ e_2,e_4;\ e_3,e_5\}$ we find

$$\mathbf{A}=\begin{bmatrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix} & \begin{pmatrix}0\\a_{12}\end{pmatrix} & \begin{pmatrix}0\\a_{22}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\a_{11}\end{pmatrix} & \begin{pmatrix}0\\a_{21}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\a_{32}\end{pmatrix} & \begin{pmatrix}1\\a_{41}\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \end{bmatrix}.$$

One possible choice for the structure triple is

$$\widetilde{R}=\begin{bmatrix}1&0\\0&1\end{bmatrix};\ X_{11}=\begin{bmatrix}1&a_{11}\\a_{11}&a_{11}^2+a_{21}a_{32}\end{bmatrix},\ X_{12}=\begin{bmatrix}0&a_{21}\\a_{21}&a_{21}(a_{11}-a_{41})\end{bmatrix},\ P=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

The decomposition (7) for $P$ is

$$P = \begin{bmatrix} 1 & 0 \\ a_{11} & a_{21} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{a_{11}}{a_{21}} & \frac{1}{a_{21}} \end{bmatrix}$$

and the array $\mathbf{A}^{13}$ is

$$\mathbf{A}^{13} = \left[ \begin{pmatrix} 0 \\ a_{12} \end{pmatrix} \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} \right].$$

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