On the Structure of Commutative Matrices II

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Abstract

A finite set $\mathbf{A}$ of $N \times N$ nilpotent commutative matrices that have one-dimensional joint kernel is considered. The theorem (due to Suprunenko and Tyshkevich) that the algebra $\mathcal{A}$, generated by $\mathbf{A}$ and the identity matrix, has the dimension equal to N is proved. A canonical basis for $\mathcal{A}$ is given, and related structure constants are discussed.

0 Introduction

In this article we continue to study the structure of commutative matrices that we began in [11]. Now, our main results are extensions of results of Kravchuk, Suprunenko, and Tyshkevich (see [18, §2.6–7]). Our motivation comes from multiparameter spectral theory [1]. In a similar way as results of [11, §2] are used to construct bases for root subspaces of nonderogatory eigenvalues in [12] the results of this paper are used to find the corresponding bases for simple eigenvalues (see [10]). We will present this application to multiparameter spectral theory separately.

In [11] we considered an n-tuple $\mathbf{A} = \{A_i,\ i = 1, 2, \ldots, n\}$ of commutative nilpotent $N \times N$ matrices over the complex numbers. Now we also consider the algebra $\mathcal{A}$ generated by $\mathbf{A}$ and the identity matrix. In the most part we make a further assumption that $\mathbf{A}$ is simple, i.e., that the joint kernel of matrices in $\mathbf{A}$ is one-dimensional. Then we show that the algebra

$\mathcal{A}$ has the (vector space) dimension equal to N. This result is found in [18, p. 62, Thm. 13]. We also describe a canonical basis $\mathcal{T}$ for the algebra $\mathcal{A}$. When $n = 2$ the basis $\mathcal{T}$ coincides (possibly after a change of basis for $\mathbb{C}^N$) with bases given in [2, 13, 20].

In [11] we viewed $\mathbf{A}$ also as a cubic array. The matrices in an array were brought by a simultaneous similarity to a special block upper-triangular form called the reduced form. The reduced form has two important properties : the column-cross sections of the blocks on the first upper diagonal are linearly independent and the products of row and column cross-sections are symmetric. (See Proposition 1 and Corollary 1 of [11].) The main result of [11] tells us how to reconstruct a commutative array from two sets of matrices, one of which is a set of symmetric matrices. Now we show, that when $\mathbf{A}$ is simple then the symmetric matrices are determined by the canonical basis and their entries are precisely the structure constants for multiplication in $\mathcal{A}$.

We proceed with a brief overview of the setup of the paper. In the next section we recall notations from [11] and in §2 we discuss some further properties of the general commutative array $\mathbf{A}$. We also obtain an upper bound for the dimension of the algebra $\mathcal{A}$ in terms of N and the dimension of the joint kernel of matrices in $\mathbf{A}$. In the remaining sections 3-5 we study the simple case. In §3 we show that the dimension of $\mathcal{A}$ is equal to N. Next, in §4, we introduce a canonical basis for algebra $\mathcal{A}$ and the associated set of structure constants. We show that a simple array $\mathbf{A}$ is determined by the structure constants and a set of coefficients that depend only on the joint kernel of A_i. This is a minimal set required to describe $\mathbf{A}$. In §5 we illustrate the discussion with two examples, and we consider the relation of our results with [2, 13, 20].

We conclude the introduction with a brief overview of some of the related literature. Finite sets of commutative matrices, algebras they generate, and their reduced forms under simultaneous similarity were studied, among others, by Trump [19] and Rutherford [17]. (See [14] for earlier references.) It was shown by Gel'fand and Ponomarev [5] that to find a canonical form for general n-tuple of commuting matrices is as hard as to find a canonical form for an arbitrary n-tuple of matrices. In §4 we briefly touch on this problem in the case when $\mathbf{A}$ is simple. While elementary properties of (nilpotent) commutative matrices are usually exhibited in monographs on linear algebra (e.g. [4, 6, 15]) our main reference is the book by Suprunenko and Tyshkevich [18].

It was pointed out by the referee that the results of Corollary 1 and

Theorem 2 are related to the problem of finding good bounds for the dimension of algebra $\mathcal{A}$. A satisfactory solution to this problem has not yet been found. Most authors have attempted to get a bound as a function of n and N. For instance, there are now several proofs (e.g. [2, 13, 20]) that if $n \leq 2$ the dimension of $\mathcal{A}$ is at most N and that, if the algebra $\mathcal{A}$ is maximal commutative subalgebra of the full matrix algebra, it has dimension exactly N. (This is the case in our setup when $\mathbf{A}$ is simple.) Our Corollary 1 provides a bound of a different type which involves N and the dimension d_1 of the joint kernel of $\mathbf{A}$; more precisely, we show that $\dim \mathcal{A} \leq 1 + d_1 (N - d_1)$. This is closer to a result of Gustafson [8] who used the joint cokernel (rather then the joint kernel) of matrices in $\mathbf{A}$. The approach in [8] is module theoretic; in the language of linear algebra the fact that θ in [8, §2] is a monomorphism implies that $\dim \mathcal{A} \leq 1 + r_1 (N - r_1)$, where r_1 (n in [8]) is the dimension of the joint cokernel.

After the paper had been submitted we came across another module theoretic paper [16] by Neubauer and Saltman, where the structure of two generated commutative matrix algebras was studied and several characterizations of algebras for which $\dim \mathcal{A} = N$ were given.

1 Commutative Arrays

We first recall notations and definitions from [11]. In addition, we now denote the set of integers $\{1, 2, \ldots, n\}$ by $\underline{n}$. A set of commutative nilpotent $N \times N$ matrices $\mathbf{A} = \{A_s,\ s \in \underline{n}\}$ is viewed also as a cubic array of dimensions $N \times N \times n$. Such an array is called commutative. For $i \geq 1$ we write

$$\ker \mathbf{A}^i = \bigcap_{k_1 + \cdots + k_n = i} \ker \left(A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} \right).$$

Suppose that $M = \min_i \left\{ \ker \mathbf{A}^i = \mathbb{C}^N \right\}$. Then

$$\{0\} \subset \ker \mathbf{A}^1 \subset \ker \mathbf{A}^2 \subset \cdots \subset \ker \mathbf{A}^M = \mathbb{C}^N \tag{1}$$

is a filtration of the vector space $\mathbb{C}^N$. Further we write

$$D_i = \dim \ker \mathbf{A}^i \quad \text{and} \quad d_i = D_i - D_{i-1} \tag{2}$$

for $i \in \underline{M}$. Here $D_0 = 0$. Then there exists a basis

$$\mathcal{B} = \left\{ z_1^1, z_2^1, \ldots, z_{d_1}^1;\quad z_1^2, z_2^2, \ldots, z_{d_2}^2;\quad \ldots\quad ;\quad z_1^M, z_2^M, \ldots, z_{d_m}^M \right\}$$

for $\mathbb{C}^N$ such that for every $i \in \underline{M}$ the set

$$\mathcal{B}_i = \left\{ z_1^1, z_2^1, \ldots, z_{d_1}^1; \quad z_1^2, z_2^2, \ldots, z_{d_2}^2; \quad \ldots \quad ; \quad z_1^i, z_2^i, \ldots, z_{d_i}^i \right\}$$

is a basis for $\ker \mathbf{A}^i$. Such a basis $\mathcal{B}$ is said to be *filtered.* A set of commutative nilpotent matrices $\mathbf{A}$ is then simultaneously reduced to a special upper-triangular form and viewed as a cubic array

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{12} & \mathbf{A}^{13} & \cdots & \mathbf{A}^{1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{23} & \cdots & \mathbf{A}^{2,M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \tag{3}$$

where

$$\mathbf{A}^{kl} = \begin{bmatrix} \mathbf{a}_{11}^{kl} & \mathbf{a}_{12}^{kl} & \cdots & \mathbf{a}_{1,d_l}^{kl} \\ \mathbf{a}_{21}^{kl} & \mathbf{a}_{22}^{kl} & \cdots & \mathbf{a}_{2,d_l}^{kl} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{d_k,1}^{kl} & \mathbf{a}_{d_k,2}^{kl} & \cdots & \mathbf{a}_{d_k,d_l}^{kl} \end{bmatrix} \tag{4}$$

is a cubic array of dimensions $d_k \times d_l \times n$ and $\mathbf{a}_{ij}^{kl} \in \mathbb{C}^n$. The *row* and *column cross-sections* of $\mathbf{A}^{kl}$ are

$$R_i^{kl} = [\, \mathbf{a}_{i1}^{kl} \quad \mathbf{a}_{i2}^{kl} \quad \cdots \quad \mathbf{a}_{i,d_l}^{kl} \,], \; i \in \underline{d_k}, \tag{5}$$

and

$$\left(C_j^{kl}\right)^T = [\, \mathbf{a}_{1j}^{kl} \quad \mathbf{a}_{2j}^{kl} \quad \cdots \quad \mathbf{a}_{d_k,j}^{kl} \,], \; j \in \underline{d_l}. \tag{6}$$

These are matrices of dimensions $n \times d_l$ and $n \times d_k$, respectively.

The array $\mathbf{A}$ in the form (3) is called *reduced* if the matrices $C_j^{k,k+1}$, $j \in \underline{d_{k+1}}$ are linearly independent for $k \in \underline{M-1}$.

By [11, Prop. 1] it follows that the array (3) is reduced. Furthermore, a commutative cubic array (3) is reduced if and only if it is written in a filtered basis.

We call a matrix A *symmetric* if $A = A^T$. In [11, Cor. 1] we observed that $\mathbf{A}$ is commutative if and only if certain products of row and column cross-section are symmetric. The main result of [11], Theorem 3, tells us how to construct the column cross-sections of $\mathbf{A}^{23}$ from the row cross-sections of $\mathbf{A}^{12}$ and a set of symmetric matrices.

2 Kravchuk Type Theorem for a Set of Commutative Matrices

For $k = 2, 3, \ldots, M$ we denote by $\mathcal{S}_k$ the linear span of the set

$$\left\{ \mathbf{a}_{ij}^{1l},\ l = 2, 3, \ldots k;\ i \in \underline{d_1};\ j \in \underline{d_l} \right\}.$$

Proposition 1 *For $k = 2, 3, \ldots, M-1,\ l = k+1, k+2, \ldots, M,\ i \in \underline{d_k},\ j \in \underline{d_l}$, it follows that $\mathbf{a}_{ij}^{kl} \in \mathcal{S}_{l-k+1}$.*

Proof. By the construction of column cross sections of the array $\mathbf{A}^{23}$ in the proof of [11, Thm. 3] (in particular see the first displayed formula in [11, p. 176]) it follows that $\mathbf{a}_{ij}^{23} \in \mathcal{S}_2$. In a similar way, we apply the construction of [11, Thm. 3] to the arrays $\mathbf{A}^{k-1,k}$ and $\mathbf{A}^{k,k+1}$, $k = 2, 3, \ldots, M-1$ to obtain that

$$\mathbf{a}_{ij}^{k,k+1} \in \mathcal{S}_{2k},$$

where $\mathcal{S}_{2k} = \mathrm{Span}\left\{ \mathbf{a}_{ij}^{k-1,k},\ i \in \underline{d_{k-1}},\ j \in \underline{d_{k+1}} \right\}$. Then it follows that

$$\mathbf{a}_{ij}^{k,k+1} \in \mathcal{S}_{2k} \subset \mathcal{S}_{2,k-1} \subset \cdots \subset \mathcal{S}_{21} = \mathcal{S}_2.$$

Next we apply the construction of [11, Thm. 3] to the arrays $\begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{13} \\ \mathbf{0} & \mathbf{A}^{23} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{A}^{24} \\ \mathbf{A}^{34} \end{pmatrix}$ (see [11, p. 177]). This shows that $\mathbf{a}_{ij}^{24} \in \mathcal{S}_3$. As in the case $\mathbf{a}_{ij}^{k,k+1}$ we show inductively that $\mathbf{a}_{ij}^{k,k+2} \in \mathcal{S}_3$ for $k \geq 2$. Proceeding in the above manner for $l - k + 1 = 3, 4, \ldots, M-1$ we obtain that $\mathbf{a}_{ij}^{kl} \in \mathcal{S}_{l-k+1}$ for all possible choices of i, j, k and l. □

Suppose that $M_N(\mathbb{C})$ is the algebra of all $N \times N$ matrices over $\mathbb{C}$ and that $\mathcal{A}$ is the subalgebra generated by the set of commutative matrices $\mathbf{A}$ and the identity matrix $I = I_N$. As a vector space, $\mathcal{A}$ is spanned by I and all the products of elements of $\mathbf{A}$, and in particular every element in $\mathcal{A}$ is of the form $A = \alpha I + N$, where $\alpha \in \mathbb{C}$ and N is nilpotent. Furthermore, A has a block upper triangular form $A = \left[A^{kl}\right]_{k,l=1}^{M}$, where A^{kl} is a $d_k \times d_l$ matrix block, $A^{kk} = \alpha I_{d_k}$, and $A^{kl} = 0$ for $k > l$.

The following is a version of Kravchuk's Theorem (see [18, p. 57]).

Theorem 1 *If* $A = \left[A^{kl}\right]_{k,l=1}^{M} \in \mathcal{A}$ *is such that* $A^{1l} = 0$ *for* $l \in \underline{M}$ *then* $A = 0$.

Proof. Since $A^{11} = 0$ it follows that $A^{kk} = 0$ for all $k \in \underline{M}$, and so A is nilpotent. Let $A_{n+1} = A$ and $\widehat{\mathbf{A}} = \{A_i,\ i \in \underline{n+1}\}$. Then $\widehat{\mathbf{A}}$ can be viewed as a commutative cubic array of dimensions $N \times N \times (n+1)$. Since $A_{n+1} \in \mathcal{A}$ it follows that $\widehat{\mathbf{A}} = \left[\widehat{\mathbf{A}}^{kl}\right]_{k,l=1}^{M}$ is in the reduced form (3). Proposition 1 applied to $\widehat{\mathbf{A}}$ implies that each entry of the block arrays $\widehat{\mathbf{A}}^{kl}$ is in the linear span of the entries of $\widehat{\mathbf{A}}^{1l}$. Since $A_{n+1}^{1l} = A^{1l} = 0$ it follows that $A_{n+1}^{kl} = 0$ for all k and l, and so $A_{n+1} = A = 0$. □

The next result follows immediately from Theorem 1.

Corollary 1 *Each element* $A = \left[A^{kl}\right]_{k,l=1}^{M}$ *in* $\mathcal{A}$ *is uniquely determined by its first (block) row, i.e. by the entries in* $A^{1,l}$, $l \in \underline{M}$. *Furthermore,* $\dim \mathcal{A} \leq 1 + d_1 (N - d_1)$.

3 The Simple Case

As we already mentioned in §1, we view $\mathbf{A}$ as a set of commutative matrices and also as a commutative array. A commutative array $\mathbf{A}$ is called *simple* if $d_1 = 1$, i.e., if $\dim \bigcap_{i=1}^{n} \ker A_i = 1$.

The results of this and the next section are a generalization of results in [18, §2.7]. The authors in [18] study maximal commutative algebras of nilpotent matrices, while we arrive at these results while studying n-tuples of nilpotent matrices. Also we work with the complete filtration (1).

Theorem 2 *If the array* $\mathbf{A}$ *is simple then* $\dim \mathcal{A} = N$.

Proof. Since $d_1 = 1$ it follows by Corollary 1 that

$$\dim \mathcal{A} \leq N. \tag{7}$$

To prove the converse inequality, we consider, for $j \in \underline{M-1}$, the set $\mathbf{A}_j$ of all products of j elements of $\mathbf{A}$ as a cubic array $\mathbf{A}_j = \left[\mathbf{A}_j^{kl}\right]_{k,l=1}^{M}$. Then it follows that $\mathbf{A}_j^{kl} = 0$ for $k > l - j$. Since $\dim \ker \mathbf{A}^j = D_j = \sum_{i=1}^{j} d_j$ it follows that the nonzero column cross-sections of $\mathbf{A}_j$ are linearly independent, in particular, the column cross-sections of $\mathbf{A}_j^{1,j+1}$ are linearly independent. Thus, it follows that we can find in $\mathcal{A}$ elements $T_h^j = \left[T_h^{jkl}\right]_{k,l=1}^M$, such that $T_h^{jkl} = 0$ for $k > l - j$ and

$$T_h^{j1,j+1} = \left[\begin{array}{ccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array}\right],$$

where 1 is in hth position. The elements T_h^j, $j \in \underline{M-1}$, $h \in \underline{d_{j+1}}$, together with the identity matrix I, are clearly linearly independent, and there are

$$1 + \sum_{j=1}^{M-1} d_{j+1} = N$$

of them. Therefore $\dim \mathcal{A} \geq N$, and so together with (7) we have that $\dim \mathcal{A} = N$. □

Corollary 2 *Algebra $\mathcal{A}$ is a maximal commutative subalgebra of $M_N(\mathbb{C})$.*

Proof. Suppose that $B \in M_N(\mathbb{C})$ is such that $AB = BA$ for all $A \in \mathcal{A}$. Write $B = [B_{ij}]_{i,j=1}^M$ and $B_{11} = [b_{11}]$. Let matrices T_h^j, $j \in \underline{M-1}$, $h \in \underline{d_{j+1}}$, be defined as in the proof of Theorem 2. Because $T_h^j B = B T_h^j$ for all j and h we first obtain that B is upper-triangular, and furthermore, we see that

$$B_{jj} = b_{11} I_{d_j}. \tag{8}$$

Now, let $A_{n+1} = B - b_{11} I$ and $\mathbf{A}' = \{A_s;\ s \in \underline{n+1}\}$. Then $\mathbf{A}'$ is a commutative array, and it is simple. Thus Theorem 2 implies that the algebra $\mathcal{A}'$ generated by $\mathbf{A}'$ and I has the dimension equal to N. Since $\mathcal{A} \subset \mathcal{A}'$ and $\dim \mathcal{A} = N$, it follows that $\mathcal{A} = \mathcal{A}'$. Then $B = A_{n+1} + b_{11} I$ is in $\mathcal{A}$, and hence $\mathcal{A}$ is maximal. □

Corollary 3 *If a set $\mathbf{A}$ of $N \times N$ commutative matrices is such that the eigenspace at each joint eigenvalue is one-dimensional, then the dimension of the algebra generated by $\mathbf{A}$ (and the identity matrix) is N.*

Proof. Since $\mathbb{C}^N$ is the direct sum of all joint spectral subspaces of matrices of $\mathbf{A}$, the result follows if we show it for each joint spectral subspace. For each joint eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $\mathbf{A}$ let $\mathcal{A}_{\boldsymbol{\lambda}}$ be the algebra generated by the restrictions of elements of $\mathbf{A}$ and the identity to the joint spectral subspace $V_{\boldsymbol{\lambda}}$ of $\mathbf{A}$ at $\boldsymbol{\lambda}$. The algebra $\mathcal{A}$ generated by $\mathbf{A}$ and I is a direct sum of the algebras $\mathcal{A}_{\boldsymbol{\lambda}}$ as $\boldsymbol{\lambda}$ ranges over all the joint eigenvalues of $\mathbf{A}$. But then it follows by Theorem 2 that $\dim \mathcal{A}_{\boldsymbol{\lambda}} = \dim V_{\boldsymbol{\lambda}}$, and thus $\dim \mathcal{A} = \sum_{\boldsymbol{\lambda}} \dim \mathcal{A}_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda}} \dim V_{\boldsymbol{\lambda}} = N$. □

4 Canonical Basis and Structure Constants for Algebra $\mathcal{A}$ in the Simple Case

Here we still assume that $\mathbf{A}$ is simple. Then Corollary 1 and Theorem 2 imply that for $g \in \underline{M-1}$ and $h \in \underline{d_{g+1}}$ there exist matrices $T_h^g = \left[T_h^{gkl}\right]_{k,l=1}^{M} \in \mathcal{A}$ such that

$$T_h^{gkl} = 0 \tag{9}$$

if either $k < l - g$ or $k = 1$ and $l \neq g + 1$, and

$$T_h^{g1,g+1} = \left[\begin{array}{cccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array} \right], \tag{10}$$

where 1 is in the hth position. Moreover, matrices T_h^g are uniquely determined by the conditions (9) and (10), and

$$\mathcal{T} = \{I\} \bigcup \left\{T_h^g;\ g \in \underline{M-1}, h \in \underline{d_{g+1}}\right\}$$

is a (canonical) basis for $\mathcal{A}$. We write $T_h^{gkl} = \left[t_{hij}^{gkl}\right]_{i=1,j=1}^{d_k \quad d_l}$. Then we have that

$$T_i^k T_j^l = \sum_{g=k+l}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{kl\,g} T_h^g = \sum_{g=k+l}^{M-1} \sum_{h=1}^{d_{g+1}} t_{jih}^{lkg} T_h^g. \tag{11}$$

Since T_h^g are linearly independent the relation (11) implies that $t_{ijh}^{kl\,g} = t_{jih}^{lkg}$. Note also that (11) implies that constants $t_{ijh}^{kl\,g}$ are the structure constants for multiplication in $\mathcal{A}$ expressed in basis $\mathcal{T}$.

Since $\mathcal{T}$ is a basis for $\mathcal{A}$ it follows that $A_i = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} a_{1hi}^{1g} T_h^g$. Then we obtain that

$$\mathbf{a}_{ij}^{kl} = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{kl\,g} \mathbf{a}_{1h}^{1g}.$$

Thus we proved the first of the following two theorems. The second then follows easily.

Theorem 3 *If $t_{ijh}^{kl\,g}$ are the structure constants for the multiplication in $\mathcal{A}$ expressed in basis $\mathcal{T}$ then $\mathbf{a}_{ij}^{kl} = \sum_{g=1}^{M-1} \sum_{h=1}^{d_{g+1}} t_{ijh}^{kl\,g} \mathbf{a}_{ij}^{1g}$.*

Theorem 4 *A simple commutative array* $\mathbf{A}$ *in the reduced form* (3) *is uniquely determined by the arrays* $\mathbf{A}^{1l}$, $l = 2, 3, \ldots, M$, *and structure constants for* $\mathcal{A}$, *the algebra generated by* $\mathbf{A}$.

Note that if we write $X_j = \left[t_{klj}^{112}\right]_{k,l=1}^{d_2}$, $j \in \underline{d_3}$, then X_j are symmetric and such that $C_j^{23} = R_1^{12} X_j$, where matrices R_1^{12} and C_j^{23} are defined in (5) and (6). Thus it follows that the entries of the symmetric matrices X_j in [11, Thm. 2] are precisely the structure constants for multiplication in $\mathcal{A}$. A similar construction can be obtained also for the column cross-sections of arrays

$$\begin{bmatrix} \mathbf{A}^{k2} \\ \mathbf{A}^{k3} \\ \vdots \\ \mathbf{A}^{k,k+1} \end{bmatrix} \quad \text{for } k \geq 3.$$

Because $t_{ijh}^{kl\,g}$ are the structure constants for multiplication in a commutative algebra $\mathcal{A}$ they satisfy higher order symmetries. These symmetries arise since the products of three or more matrices in $\mathcal{T}$ do not depend on the order of multiplication. We include the precise statement since it is needed in the application to multiparameter spectral theory. First we introduce some further notation.

For $m = 2, 3, \ldots, M$ and $2 \leq q \leq m$ we denote by $\Phi_{m,q}$ the set of multiindices $\{(k_1, k_2, \ldots, k_q)\,;\ k_i \geq 1,\ \sum_{i=1}^{q} k_i \leq m\}$. For $\mathbf{k} = (k_1, k_2, \ldots, k_q) \in \Phi_{m,q}$ we define a set $\chi_{\mathbf{k}} = \underline{d_{k_1}} \times \underline{d_{k_2}} \times \cdots \times \underline{d_{k_q}}$. The set of all permutations of the set $\underline{q}$ is denoted by Π_q. For a permutation $\sigma \in \Pi_q$ and multiindices $\mathbf{k} \in \Phi_{m,q}$ and $\mathbf{i} = (i_1, i_2, \ldots, i_q)$ we write $\mathbf{k}_\sigma = \left(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(q)}\right)$ and $\mathbf{i}_\sigma = \left(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(q)}\right)$. Then we define recursively numbers $s_{\mathbf{i}h}^{\mathbf{k}g}$: for $\mathbf{k} \in \Phi_{m,2}$ and $\mathbf{i} \in \chi_{\mathbf{k}}$ we write $s_{\mathbf{i}h}^{\mathbf{k}g} = t_{i_1 i_2 h}^{k_1 k_2 g}$ and for $q > 2$ and $\mathbf{k} \in \Phi_{m,q}$ and $\mathbf{i} \in \chi_{\mathbf{k}}$ we write

$$s_{\mathbf{i}h}^{\mathbf{k}g} = \sum_{l=k_1+k_2}^{m-k_3-\ldots-k_q} \sum_{j=1}^{d_l} t_{i_1 i_2 j}^{k_1 k_2 l}\; s_{(j,i_1,i_2,\ldots,i_q)h}^{(l,k_3,k_4,\ldots,k_q)g}.$$

Corollary 4 *For* $\mathbf{k} \in \Phi_{m,q}$ *and* $\mathbf{i} \in \chi_{\mathbf{k}}$ *the constants* $s_{\mathbf{i}h}^{\mathbf{k}g}$ *are symmetric in* $\mathbf{k}$ *and* $\mathbf{i}$, *i.e.*

$$s_{\mathbf{i}h}^{\mathbf{k}g} = s_{\mathbf{i}_\sigma h}^{\mathbf{k}_\sigma g} \tag{12}$$

for any permutation $\sigma \in \Pi_q$.

We remark that the relations (12) are the matching conditions (in the simple case) mentioned at the end of §4 in [11].

A canonical form for a simple commutative array would be obtained if we replaced the basis $\mathcal{B}$ by another filtered basis $\mathcal{B}'$, so that the matrix

$$R = \left[\begin{array}{cccc} R_1^{12} & R_1^{13} & \cdots & R_1^{1M} \end{array} \right]$$

is in a canonical form. This reduces to finding a canonical form for R when acting by permutation matrices on the left (if $\mathbf{A}$ is considered as a set only, i.e. the matrices A_i are not considered in any particular order) and by invertible block upper-triangular matrices on the right. The first immediate reduction we can achieve is that the nonzero columns in R are linearly independent.

Then in a particular case $d_1 = n$ we can assume that $R_1^{1l} = 0$ for $l \geq 3$. If we substitute vectors z_j^2 by vectors $\hat{z}_j^2 = \sum_{k=1}^{d_2} a_{1jk}^{12} z_k^2$ in basis $\mathcal{B}$ then $R_2^{12} = I$, and $A_h = T_h^1$ for $h \in \underline{n}$, is a canonical form for $\mathbf{A}$. In the general simple case a block version of the row reduced echelon form (see [9, §2.5] for the standard version and [3, §1] for some generalized versions) applied to R yields toward a canonical form for $\mathbf{A}$. However, this requires an extensive case by case analysis and we do not proceed with it. Rather we consider some examples.

5 Examples

Example 1 Suppose $n = 2$. Then sets of matrices that span the algebra $\mathcal{A}$, generated by a pair of matrices $\mathbf{A} = \{A_1, A_2\}$ and the identity matrix I, are described in [2, 13] (see also [7, 20]). In general the sets of matrices given there are not a basis; their elements may be linearly dependent. For example, if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

then neither $\{I, A_1;\ A_2, A_1A_2\}$ nor $\{I, A_1;\ A_2;\ A_1^2\}$ are linearly independent since $A_1A_2 = A_1^2 = 0$.

However, if $\mathbf{A}$ is simple then $\dim \mathcal{A} = N$ by Theorem 3, and so the sets given in [2, 13] are a basis. For instance, if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

then $\mathcal{A} = \mathrm{Sp}\{I, A_1, A_1^2, A_2\}$. Furthermore, if $\{e_i;\ i \in \underline{4}\}$ is the standard basis for $\mathbb{C}^4$ then in basis $\mathcal{B} = \{e_1;\ e_2, e_4;\ e_3\}$ the reduced form for the array $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

Next we find that $T_h^1 = A_h$, $h = 1, 2$ and $T_1^2 = A_1^2$, and so

$$\mathcal{T} = \left\{ I, A_1, A_1^2, A_2 \right\}$$

is a basis for $\mathcal{A}$. □

Example 2 We consider a commutative array $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}, \tag{15}$$

which is already in the reduced form (3). The columns of the first row of the array (15) are not linearly independent. To make them so, we substitute the vector $e_4 - \frac{1}{2}e_3$ for the vector e_4 in basis $\mathcal{B}$. (Here we assume that $\mathcal{B} = \{e_i,\ i \in \underline{4}\}$ is the standard basis of $\mathbb{C}^4$.) Note that the new basis is still filtered. The array $\mathbf{A}$ in the new basis is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

To find a canonical form for $\mathbf{A}$ we finally substitute vectors z_1^2 and z_2^2 by $z_1^2 + z_2^2$ and $2z_2^2$, respectively. The new basis is still filtered and we find that

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}. \tag{16}$$

So we have that $T_h^1 = A_h$, $h = 1, 2$, $T_1^2 = A_1^2$ and (16) is a canonical form for $\mathbf{A}$. $\square$

References

[1] F.V. Atkinson. *Multiparameter Eigenvalue Problems.* Academic Press, 1972.

[2] J. Barría and P.R. Halmos. Vector Bases for Two Commuting Matrices. *Lin. Multilin. Alg.*, 27:147–157, 1990.

[3] P. Gabriel and A.V. Roiter. Representations of Finite-Dimensional Algebras, in *Algebra VIII*, Encyclopaedia of Math. Sci. 73. Springer-Verlag, 1992.

[4] F.R. Gantmacher. *The Theory of Matrices*, volume 1 and 2. Chelsea Publishing, 1959.

[5] I.M. Gel'fand and V.A. Ponomarev. Remarks on the Classification of a Pair of Commuting Linear Transformations in a Finite-dimensional Space. *Funct. Anal. Appl.*, 3:325–326, 1969.

[6] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Wiley-Interscience, 1986.

[7] R.M. Guralnick. A Note on Commuting Pairs of Matrices. *Lin. Multilin. Alg.*, 31:71–75, 1992.

[8] W.H. Gustafson. On Maximal Commutative Algebras of Linear Transformations. *J. Alg.*, 42:557–563, 1976.

[9] K. Hoffman and R. Kunze. Linear Algebra. 2nd edition, Prentice Hall, 1971.

[10] T. Košir. *Commuting Matrices and Multiparameter Eigenvalue Problems.* PhD thesis, Dept. of Math. and Stats., University of Calgary, 1993.

[11] T. Košir. On the Structure of Commutative Matrices. *Lin. Alg. Appl.*, 187:163–182, 1993.

[12] T. Košir. The Finite-dimensional Multiparameter Spectral Theory : The Nonderogatory Case. *Lin. Alg. Appl.*, 212/213:45–70, 1994.

[13] T.J. Laffey and S. Lazarus. Two-Generated Commutative Matrix Subalgebras. *Lin. Alg. Appl.*, 147:249–273, 1991.

[14] C.C. MacDuffee. *The Theory of Matrices.* Springer-Verlag, 1933.

[15] A.I. Mal'cev. *Foundations of Linear Algebra.* W.H. Freeman and Co., 1963.

[16] M.G. Neubauer and D.J. Saltman. Two-Generated Commutative Subalgebras of $M_n(F)$. *J. Alg.*, 164:545–562, 1994.

[17] D.E. Rutherford. On Commuting Matrices and Commutative Algebras. *Proc. Roy. Soc. Edin. Sect. A*, 62:454–459, 1949.

[18] D.A. Suprunenko and R.I. Tyshkevich. *Commutative Matrices.* Academic Press, 1968.

[19] P.L. Trump. On a Reduction of a Matrix by the Group of Matrices Commutative with a Given Matrix. *Amer. Math. Soc. Bul. Ser. II*, 41:374–380, 1935.

[20] A.R. Wadsworth. The Algebra Generated by Two Commuting Matrices. *Lin. Multilin. Alg.*, 27:159–162, 1990.