On commuting compact self-adjoint operators on a Pontryagin space*

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Abstract

Suppose that $A_1, A_2, \ldots, A_n$ are compact commuting self-adjoint linear maps on a Pontryagin space K of index k and that their joint root subspace $M_{\mathbf{0}}$ at the zero eigenvalue in $\mathbb{C}^n$ is a nondegenerate subspace. Then there exist joint invariant subspaces H and F in K such that $K = F \oplus H$, H is a Hilbert space and F is finite-dimensional space with $k \le \dim F \le (n+2)k$. We also consider the structure of restrictions $A_j|_F$ in the case $k = 1$.

1 Introduction

Let K be a Pontryagin space whose index of negativity (henceforward called *index*) is k and A be a compact self-adjoint operator on K with non-degenarate root subspace at the eigenvalue 0. Then K can be decomposed into an orthogonal direct sum of a Hilbert subspace and a Pontryagin subspace both of which are invariant under A and this Pontryagin subspace has dimension at most $3k$. This has many applications among which we mention the study of elliptic multiparameter problems [2]. Binding and Seddighi gave a complete proof of this decomposition in [3] and in fact proved that non-degenaracy of the root subspace at 0 is necessary and sufficient for such a decomposition. They show

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that the bound $3k$ can be attained. We refer to the books [1, 4] for general theory of operators on a Pontryagin space.

We present a generalization of the decomposition to encompass a tuple of commuting compact operators on a Pontryagin space. Such tuples occur naturally in applications to boundary value problems for partial differential equation, say of Sturm-Liouville type, that are coupled by several parameters. When such multiparameter boundary value problems are of so-called elliptic type, their analysis leads to an n-tuple of commuting compact self-adjoint operators on a Pontryagin space that is not a Hilbert space. We shall not elaborate on the multiparameter aspects here. They can be found, for example, in [2] and [5]. In this paper, our aim is to obtain a decomposition of K and also classify tuples of commuting compact operators when $k = 1$. A compact normal operator can be thought of as a pair of commuting compact self-adjoint operators. In the case of finite-dimensional Pontryagin space of index 1, normal operators are completely classified in [8]. Thus the classification of n-tuples of commuting compact self-adjoint operators on a Pontryagin space of index 1 is a natural question.

There are two main results in this paper. The first one of them, Theorem 2.7 gives a decomposition of the entire space into joint invariant subspaces one of which is a Hilbert space H and the other, say F, is a Pontryagin space of index k and its dimension is at most $(n+2)k$. We give an example to show that this bound is indeed sharp. The structure of F in the decomposition of Theorem 2.7 is described in further detail. The subspace F is equal to a direct sum $F_1 \oplus F_2$, where F_1 is spanned by all joint root subspaces at nonreal eigenvalues and the spectra of restrictions to F_2 are real. Furthermore, the dimension of F_1 is exactly twice the index of F_1, while the dimension of F_2 is bounded below by the index of F_2 and above by $n + 2$ times the index of F_2. In particular, the bound $(n + 2)k$ above can be achieved only if all the eigenvalues are real. In Theorem 3.1 we classify the n-tuples of commuting compact self-adjoint operators on a Pontryagin space of index 1.

2 Splitting of an invariant finite-dimensional subspace with an invariant complement that is a Hilbert space

Let $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of commuting compact self-adjoint linear maps on K. If L is a subspace of K then $L^{[\perp]} = \{u \in K;\ [u, v] = 0 \text{ for all } v \in L\}$ is its *orthogonal complement*. Here $[\cdot, \cdot]$ is the inner product on K. The subspace L is called *nondegenerate* if $L \cap L^{[\perp]} = 0$. Any nondegenerate subspace of K is a Pontryagin space in its own right and we denote by $\kappa(L)$ its index. A subspace L is called *ortho-complemented* if $L + L^{[\perp]}$ is equal to K. If $A : K \to K$ is a linear operator then we denote by $A^{[*]}$ its adjoint. One has $[Au, v] = [u, A^{[*]}v]$ for all $u, v \in K$.

If $X \subset K$ is a nonempty set we denote by $\mathcal{L}(X)$ the closed linear span of X in K.

For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ we write

$$L_{\boldsymbol{\alpha}}(\mathbf{A}, j) = \bigcap_{\sum_{s=1}^{n} t_s = j,\ t_s \geq 0} \mathcal{N}\left((A_1 - \alpha_1 I)^{t_1} \cdots (A_n - \alpha_n I)^{t_n}\right)$$

and

$$L_{\boldsymbol{\alpha}}(\mathbf{A}) = \bigcup_{j=1}^{\infty} L_{\boldsymbol{\alpha}}(\mathbf{A}, j).$$

Here $\mathcal{N}(A)$ is the nullspace of a linear map A. Note that $L_{\boldsymbol{\alpha}}(\mathbf{A})$ is the joint root subspace of $\mathbf{A}$ at $\boldsymbol{\alpha}$. We use the notation $L_\alpha(A, j)$ and $L_\alpha(A)$ if $\mathbf{A} = \{A\}$. The tuple $\overline{\boldsymbol{\alpha}}$ denotes the n-tuple obtained from $\boldsymbol{\alpha} \in \mathbb{C}^n$ by conjugating all its components. We define

$$M_\alpha(A) = \begin{cases} L_\alpha(A) \text{ if } \alpha \in \mathbb{R} \\ L_\alpha(A) + L_{\overline{\alpha}}(A) \text{ if } \alpha \notin \mathbb{R}. \end{cases}$$

For $\boldsymbol{\alpha} \in \mathbb{C}^n$ we define $M_{\boldsymbol{\alpha}}(\mathbf{A}) = \cap_{j=1}^n M_{\alpha_j}(A_j)$. We remark that for $\boldsymbol{\alpha} \in \mathbb{R}^n$ we have $L_{\boldsymbol{\alpha}}(\mathbf{A}) = M_{\boldsymbol{\alpha}}(\mathbf{A})$. If $\boldsymbol{\alpha}, \boldsymbol{\beta} \notin \mathbb{R}^n$ then it follows from Lemma 2.1 below that $M_{\boldsymbol{\alpha}}(\mathbf{A}) = M_{\boldsymbol{\beta}}(\mathbf{A})$ if and only if $\alpha_j \in \{\beta_j, \overline{\beta}_j\}$ for all j. To avoid duplication when considering the subspaces $M_{\boldsymbol{\alpha}}(\mathbf{A})$, we assume that the imaginary parts of components α_j, $j = 1, \ldots, n$, of $\boldsymbol{\alpha}$ are nonnegative.

In this paper we assume that the joint root subspace $M_{\mathbf{0}} = M_{\mathbf{0}}(\mathbf{A})$ *at* $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{C}^n$ *is a nondegenerate subspace.*

We say that an eigenvalue $\boldsymbol{\alpha}$ of $\mathbf{A}$ is *normal* if $L_{\boldsymbol{\alpha}}(\mathbf{A})$ is finite-dimensional and it has a closed complement that is invariant for $\mathbf{A}$. If $\alpha \in \mathbb{C}$ is a nonzero eigenvalue of a compact linear map then it is a normal eigenvalue (see e.g. [10, p.190]). It follows that a nonzero eigenvalue $\boldsymbol{\alpha} \in \mathbb{C}^n$ of $\mathbf{A}$ is a normal eigenvalue.

A subspace $L \subset K$ is invariant for $\mathbf{A}$ if $A_i L \subset L$ for all i. A closed invariant subspace L for $\mathbf{A}$ is called *decomposable* if there exist nonzero closed subspaces L_1 and L_2, invariant for $\mathbf{A}$, such that $L = L_1 \oplus L_2$. If such L_1 and L_2 do not exist we call L an *indecomposable subspace* for $\mathbf{A}$. Observe that subspaces $L_{\boldsymbol{\alpha}}(\mathbf{A})$ for $\boldsymbol{\alpha} \neq \mathbf{0}$, and the closure $\overline{M}_{\mathbf{0}}$ of $M_{\mathbf{0}}$ are closed and invariant for $\mathbf{A}$. Then it follows that if L is an indecomposable subspace for $\mathbf{A}$ the restrictions $A_i|_L$ have only one eigenvalue. Moreover, each subspace $L_{\boldsymbol{\alpha}}(\mathbf{A})$ and $\overline{M}_{\mathbf{0}}$ are direct sums of indecomposable subspaces for $\mathbf{A}$. If L is an indecomposable subspace for $\mathbf{A}$ and $L \subset L_{\boldsymbol{\alpha}}(\mathbf{A})$ then we say that L is an indecomposable subspace for $\mathbf{A}$ at $\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}$ is the eigenvalue corresponding to L. If $n = 1$ then an invariant subspace is indecomposable if and only if it is a linear span of a single Jordan chain.

Now we prove a few auxiliary results that lead to the proof of our main results.

Lemma 2.1 *If* $\alpha \neq \mathbf{0}$ *is an eigenvalue of* $\mathbf{A}$ *then* $M_{\boldsymbol{\alpha}}(\mathbf{A})$ *is nondegenerate.*

Proof. Since $\boldsymbol{\alpha} \neq \mathbf{0}$ there is an index l such that $\alpha_l \neq 0$. We may assume without loss of generality that $l = 1$. Then $M_{\alpha_1}(A_1) = \oplus_{\boldsymbol{\beta} \in \Sigma_1} M_{\boldsymbol{\beta}}(\mathbf{A})$, where Σ_1 is the set of all the eigenvalues $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ of $\mathbf{A}$ such that imaginary parts of β_j are nonnegative and $\beta_1 = \alpha_1$. We know by [3, Lemma 1] that $M_{\alpha_1}(A_1)$ is nondegenerate. For each eigenvalue $\boldsymbol{\beta} \in \Sigma_1$ and $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$ there exists an index j such that $\alpha_j \neq \beta_j$. Then $M_{\beta_j}(A_j) \subset M_{\alpha_j}(A_j)^{[\perp]}$. By the fact that $L_2^{[\perp]} \subset L_1^{[\perp]}$ if $L_1 \subset L_2$, we have $M_{\boldsymbol{\beta}}(\mathbf{A}) \subset M_{\beta_j}(A_j) \subset M_{\alpha_j}(A_j)^{[\perp]} \subset M_{\boldsymbol{\alpha}}(\mathbf{A})^{[\perp]}$. Therefore $M_{\boldsymbol{\alpha}}(\mathbf{A})$ is an ortho-complemented subspace of $M_{\alpha_1}(A_1)$. An ortho-complemented subspace is nondegenerate (see [4, Cor. I.9.5]). □

The following result is a consequence of the assumption that $M_{\mathbf{0}}$ is nondegenarate.

Corollary 2.2 *For each j the subspace $M_0(A_j)$ is nondegenarate and the closure of the linear span of Jordan chains of A_j is equal to K.*

Proof. Observe that $M_0(A_j)$ is a direct sum of $M_{\mathbf{0}}$ and the subspaces $M_{\boldsymbol{\alpha}}(\mathbf{A})$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, such that $\alpha_j = 0$. By [1, Cor. 3.14] it follows that $M_{\boldsymbol{\alpha}}(\mathbf{A}) \subset M_{\mathbf{0}}^{[\perp]}$. By our assumptions and Lemma 2.1 all the subspaces $M_{\boldsymbol{\alpha}}(\mathbf{A})$ are nondegenerate. Therefore $M_0(A_j)$ is nondegenerate. Theorem 1 of [3] implies the second part of the statement. □

Theorem 2.3 *If $\boldsymbol{\alpha} \notin \mathbb{R}^n$ is an eigenvalue of $\mathbf{A}$ then $\kappa(M_{\boldsymbol{\alpha}}(\mathbf{A})) = \frac{1}{2} \dim M_{\boldsymbol{\alpha}}(\mathbf{A})$.*

Proof. Since $\boldsymbol{\alpha}$ is normal it follows that $M_{\boldsymbol{\alpha}}(\mathbf{A})$ is finite-dimensional. By Lemma 2.1 it is also nondegenerate. Suppose α_j is a nonreal component of $\boldsymbol{\alpha}$. Then the restriction of A_j to $M_{\boldsymbol{\alpha}}(\mathbf{A})$ has a conjugate pair $\alpha_j, \overline{\alpha}_j$ for its spectrum. The lemma then follows by [6, Thm. I.3.3]. □

Let α be an eigenvalue for a compact self-adjoint operator A on a Pontryagin space and let $J = \{v_0, v_1, \ldots, v_l\}$ be a Jordan chain at α. Then we follow [3] and call J *negative* if $[v_0, v_0] \leq 0$ and *positive* if $[v_0, v_0] > 0$. Note that if $\alpha \notin \mathbb{R}$ or if $l \geq 1$ then $[v_0, v_0] = 0$ and J is negative.

If L is an indecomposable subspace for a set $\mathbf{A} = \{A_1, A_2, \ldots, A_n\}$ of compact commuting self-adjoint operators then L is called *positive* if it contains only positive Jordan chains for each A_i, otherwise it is called *negative*. Observe that if L is a positive indecomposable subspace for $\mathbf{A}$ then it is one-dimensional, spanned by a joint eigenvector of A_i, and the corresponding eigenvalue is real.

Lemma 2.4 *Let K be a Pontryagin space of index k. Given a compact self-adjoint operator A on K with nondegenerate root subspace $M_0(A)$, the whole space K can be written as an orthogonal direct sum $K = H \oplus F$ where H is a Hilbert space and F is a finite-dimensional Pontryagin space of index k such that both F and H are invariant for A and $k \leq \dim F \leq 3k$. Moreover, the Jordan canonical form of the restriction of A to F has at most k blocks. In particular, there are at most k negative Jordan chains in a Jordan basis for A.*

Proof. A maximal nonpositive subspace of K has dimension equal to k (see [1, 4]). Since we assume that $M_0(A)$ is nondegenerate it follows by [3, Thm. 1] that root vectors of A are complete. By [1, Thms. 2.26 and 3.4] or [4, Thms. 4.6 and 4.9] it follows that a Jordan chain J of A at a real eigenvalue has length $l \leq 2k + 1$ and the dimension of a maximal nonpositive subspace in $\mathcal{L}(J)$ is equal to $\frac{l}{2}$ if l is even and to $[\frac{l}{2}]$ or $[\frac{l}{2}] + 1$ if l is odd. A Jordan chain J of A at a nonreal eigenvalue α has length $l \leq k$. For J there is a chain $\overline{J}$ for A at $\overline{\alpha}$ such that J and $\overline{J}$ are of equal length and $\mathcal{L}(J \cup \overline{J})$ is a nondegenerate subspace (see [6, Thm. I.3.3]). Moreover, a maximal nonpositive subspace of $\mathcal{L}(J \cup \overline{J})$ has the dimension equal to l. It follows now that the subspace F spanned by the union of all negative Jordan chains is a Pontryagin space. The subspace H spanned by the remaining Jordan chains is a Hilbert space. Since Jordan chains are complete it follows that $K = F \oplus H$ and therefore F has index k. Since the linear span of each chain in F always contains a one-dimensional nonpositive subspace, it follows that the restriction A to F has at most k Jordan blocks. It also follows by the above discussion that $k \leq \dim F \leq 3k$. The former inequality holds since F contains a k dimensional nonpositive subspace. The latter inequality is an equality if and only if $A|_F$ has real spectrum and each corresponding Jordan chain J is of length equal to 3 and such that a maximal nonpositive subspace of $\mathcal{L}(J)$ has dimension equal to 1. $\square$

Lemma 2.5 *Suppose that* $\mathbf{A} = \{A_1, A_2\}$ *is a pair of commuting compact self-adjoint nilpotent operators on a Pontryagin space* K *of index* k *and that* $M_{\mathbf{0}}(\mathbf{A}) = K$. *Further, suppose a finite-dimensional Pontryagin space* F_1 *of index* k *and a Hilbert space* H_1 *are such that both* F_1 *and* H_1 *are invariant for* A_1, $K = F_1 \oplus H_1$, *and the restriction of* A_1 *to* F_1 *has* l *negative Jordan blocks. Then there exist a finite-dimensional Pontryagin space* F *of index* k *and a Hilbert space* H *such that both* F *and* H *are invariant for* $\mathbf{A}$, $K = F \oplus H$, $F_1 \subset F$, $H \subset H_1$ *and* $\dim F \leq \dim F_1 + l$. *The restrictions of* A_1 *and* A_2 *to* H *are equal to* 0.

Proof. By [6, Thm. I.3.3] it follows that in an appropriate basis for F_1 we have

$$A_1|_{F_1} = \begin{bmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_l & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where J_j are nilpotent Jordan blocks, and the inner product is given by the matrix

$$\begin{bmatrix} P_1 & 0 & \cdots & 0 & 0 \\ 0 & P_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & P_l & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix},$$

where

$$P_j = \pm \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Note that $A_1|_{H_1} = 0$. By analogy, if we replace A_1 by A_2, there are a finite-dimensional Pontryagin space F_2 of index k and a Hilbert space H_2 such that both F_2 and H_2 are invariant for A_2 and $K = F_2 \oplus H_2$. Next we denote by E the subspace of K spanned by F_1 and F_2. It follows that E is a finite-dimensional subspace for $\mathbf{A}$. It is invariant for A_j, $j = 1, 2$, since for $v \in E$ there exist $f_j \in F_j$ and $h_j \in H_j$ such that $v = f_j + h_j$ and $A_j v = A_j f_j \in F_j$. Observe that there is a complement G of E in K such that $K = G \oplus E$

and $G \subset H_1$ is a Hilbert space. Since A_2 commutes with A_1 the restriction $A_2|_E$ is of the form

$$A_2|_E = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1l} & W_1 \\ B_{21} & B_{22} & \cdots & B_{2l} & W_2 \\ \vdots & \vdots & & \vdots & \vdots \\ B_{l1} & B_{l2} & \cdots & B_{ll} & W_l \\ U_1 & U_2 & \cdots & U_l & C \end{bmatrix},$$

where each block B_{ij} is an upper-triangular Toeplitz matrix, $W_j = \begin{bmatrix} w_j^* \\ 0 \end{bmatrix}$, $U_j = \begin{bmatrix} 0 & u_1 \end{bmatrix}$ and w_j, u_j are column vectors (see [7, Thm. 9.1.1]). Blocks B_{ij}, $i, j = 1, 2, \ldots, l$, correspond to the negative Jordan chains of A_1. Since A_2 is self-adjoint in K it follows that $B_{ij}^* P_i = P_j B_{ji}$, $u_j = \pm w_j$, $j = 1, 2, \ldots, l$ and $C = C^{[\perp]}$. Note that C is an operator on a Hilbert space, therefore $C = C^*$. Then we can assume that C is a diagonal matrix without changing the structure of other blocks of A_2. Next let E_1 be the linear span of the set

$$\left\{ \begin{bmatrix} 0 \\ u_1 \end{bmatrix}, \begin{bmatrix} 0 \\ u_2 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ u_l \end{bmatrix} \right\}$$

and F the linear span of F_1 and U. It is clear that $\dim U \leq l$ and thus $\dim F \leq \dim F_1 + l$. Since C is diagonal there is a complement E_2 of F in E that is spanned by eigenvectors of A_2. With respect to the decomposition $E = F_1 \oplus U \oplus E_2$ the matrix for $A_2|_E$ is of the form

$$A_2|_E = \begin{bmatrix} B & W & 0 \\ U & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}.$$

But A_2 is nilpotent and C_2 diagonal, hence it follows that $C_2 = 0$. Furthermore, F is invariant for $\mathbf{A}$ and since $F_1 \subset F$ it follows that F is a Pontryagin space of index k. Observe that we can now choose a complement E_2' of F in E so that $E_2' \subset H_1$. Finaly, we conclude that $H = G \oplus E_2'$ is a Hilbert space such that H is invariant for $\mathbf{A}$ and $H \subset H_1$. □

Theorem 2.6 *If $\alpha \in \mathbb{R}^n$ is an eigenvalue of $\mathbf{A}$ then there exist a finite-dimensional subspace F_α and a Hilbert space H_α such that both F_α and H_α are invariant for $\mathbf{A}$, $M_\alpha(\mathbf{A}) = F_\alpha \oplus H_\alpha$ and $\kappa(M_\alpha(\mathbf{A})) \leq \dim F_\alpha \leq (n+2)\,\kappa\,(M_\alpha(\mathbf{A}))$.*

Proof. We prove the theorem by induction on n. For brevity we write $M_\alpha = M_\alpha(\mathbf{A})$ and $k_\alpha = \kappa\,(M_\alpha)$. Assume first that $n = 1$. Applying Lemma 2.4 to $A|_{M_\alpha}$, we get a finite-dimensional Pontryagin space F_1 of index k_α and a Hilbert space H_1 satisfying $M_\alpha = F_1 \oplus H_1$, both F_1 and H_1 are invariant for A_1 and $k_\alpha \leq \dim F_1 \leq 3k_\alpha$. The Jordan canonical form of the restriction of A_1 to F_1 has at most k_α negative blocks. The restriction of A_1 to H_1 is equal to $\alpha_1 I_{H_1}$, where I_{H_1} is the identity operator on H_1.

Assume for $n \geq 2$ that we already have a finite-dimensional subspace F_{n-1} and a Hilbert space H_{n-1} satisfying $M_\alpha = F_{n-1} \oplus H_{n-1}$, both F_{n-1} and H_{n-1} are invariant for $A_1, A_2, \ldots, A_{n-1}$ and $k_\alpha \leq \dim F_{n-1} \leq (n+1)k_\alpha$. Moreover, the restriction of A_1 to

F_{n-1} has at most $k_{\boldsymbol{\alpha}}$ negative Jordan chains and $A_j|_{H_{n-1}} = \alpha_j I_{H_{n-1}}$, $j = 1, 2, \ldots, n-1$. Applying Lemma 2.5 to the pair $A_1 - \alpha_1 I, A_n - \alpha_n I$ we get a finite-dimensional Pontryagin space F_n of index $k_{\boldsymbol{\alpha}}$ and a Hilbert space H_n such that both F_n and H_n are invariant for A_1 and A_n, $M_{\boldsymbol{\alpha}} = F_n \oplus H_n$, $F_{n-1} \subset F_n$, $H_n \subset H_{n-1}$, $A_n|_{H_n} = \alpha_n I_{H_n}$ and $k_{\boldsymbol{\alpha}} \leq \dim F_n \leq \dim F_{n-1} + k_{\boldsymbol{\alpha}} \leq (n+2)k_{\boldsymbol{\alpha}}$. It is clear that H_n is invariant for $\mathbf{A}$, moreover $A_j|_{H_n} = \alpha_j I_{H_n}$, $j = 1, 2, \ldots, n$. Since the intersection $F_n \cap H_{n-1}$ is a subspace in H_{n-1} it is also invarint for $A_1, A_2, \ldots, A_{n-1}$. Then it follows that F_n is invariant for $\mathbf{A}$. This concludes the proof. □

Theorem 2.7 *Suppose that* $\mathbf{A}$ *is an* n*-tuple of commuting compact self-adjoint operators on a Pontryagin space* K *of index* k *such that* $M_{\mathbf{0}}(\mathbf{A})$ *is nondegenerate. Then there exist a finite-dimensional subspace* F *and a Hilbert space* H *such that both* F *and* H *are invariant for* $\mathbf{A}$, $K = F \oplus H$ *and*

$$k \leq \dim F \leq (n+2)k.$$

This F *need not be unique. But if* $K = \oplus_{j \in J} K_j$ *is a decomposition of* K *into a direct sum of indecomposable subspaces* K_j *for* $\mathbf{A}$ *and* F_1 *is spanned by all subspaces* K_j *at nonreal eigenvalues and* F_2 *is spanned by the all remaining negative subspaces at real eigenvalues then a minimal one among such* F *is given by* $F_1 \oplus F_2$. *Moreover,* $\dim F_1 = 2\,\kappa(F_1)$ *and* $\kappa(F_2) \leq \dim F_2 \leq (n+2)\,\kappa(F_2)$.

Proof. Since $M_{\mathbf{0}}(\mathbf{A})$ is nondegenerate it follows that $K = \oplus_{\boldsymbol{\alpha}} M_{\boldsymbol{\alpha}}(\mathbf{A})$, where the direct sum is over all eigenvalues of $\mathbf{A}$ with nonnegative imaginary parts. There are at most k eigenvalues with a negative Jordan chain since a maximal nonpositive subspace has dimension equal to k. The theorem then follows by Theorems 2.3 and 2.6, and remarks on indecomposable subspaces for $\mathbf{A}$ preceeding Lemmas 2.1 and 2.4. □

The bounds in Theorem 2.7 coincide for $n = 1$ with those in [3]. For $n = 2$ observe that $A = A_1 + iA_2$ is a normal operator on K. The bounds in Theorem 2.7 then coincide with those given for a normal operator in [9, Thm. 1]. A normal operator A on a Pontryagin space can be considered as a pair $A_1 = \frac{1}{2}\left(A + A^{[*]}\right)$ and $A_2 = \frac{1}{2i}\left(A - A^{[*]}\right)$ of commuting self-adjoint operators. It is obvious that pairs $A, A^{[*]}$ and A_1, A_2 have the same joint invariant subspaces. Moreover, a subspace is invariant for the pair $A, A^{[*]}$ if and only if it is a sum of indecomposable invariant subspaces for A (see [8, 9]). Compare also the case $n = 2$ in Theorem 3.1 below and [8, Thm. 1].

Example 2.8 *The bounds* $(n+2)k_{\boldsymbol{\alpha}}$ *and* $(n+2)k$ *in Theorems 2.6 and 2.7 cannot, in general, be improved. The proof of Lemma 2.5 suggests how to find examples where the bound is achieved. It is clear that the bound* $n+2$ *for the dimension of* F *is attained if and only if there is no non-real eigenvalues. For example, if* $k = 1$ *and* $n = 3$ *then the* *matrices*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

commute and are self-adjoint with respect to the inner product $[u, v] = \langle Pu, v\rangle$*, where* $\langle u, v\rangle$ *is the standard scalar product in* $\mathbb{C}^6$ *and*

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the linear span of the first 5 *coordinate vectors is the minimal invariant subspace of the matrices that contains all their negative Jordan chains. Observe that* $5 = (n+2)k$*.* □

3 Reduced form for commuting compact self-adjoint operators on a Pontryagin space of index 1

If K is a Potryagin space of index 1 and $\mathbf{A}$ is an n-tuple of commuting compact self-adjoint operators on K such that $M_{\mathbf{0}}(\mathbf{A})$ is nondegenerate, then Theorem 2.7 gives the existence of a finite-dimensional Pontryagin subspace F of index 1 and a Hilbert space H satisfying $K = F \oplus H$, both F and H are invariant for $\mathbf{A}$ and $1 \le \dim F \le n+2$. Assume that F is a minimal subspace with the required properties.

The restrictions $A_j|_H$ are compact commuting self-adjoint operators on a Hilbert space and thus by the spectral theorem, they can be simultaneously diagonalised. We are interested in structure of restrictions $A_j|_F$. In the following theorem, $\langle\cdot,\cdot\rangle$ denotes a definite inner product.

Theorem 3.1 *Suppose that* $\mathbf{A}$ *and* F *are as above. Then the spectrum of the restrictions of* $\mathbf{A}$ *to* F *contains a single real eigenvalue or a pair of complex conjugate eigenvalues. Assume that* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is an eigenvalue. Then one and only one of the following is true:*

1. *If* $\boldsymbol{\alpha} \notin \mathbb{R}^n$ *then*

$$A_j|_F = \begin{bmatrix} \alpha_j & 0 \\ 0 & \overline{\alpha}_j \end{bmatrix} \tag{1}$$

and the inner product on $F \cong \mathbb{C}^2$ *is given by* $[u, v] = \langle Pu, v\rangle$*, where* $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2. *If* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *and* $\dim F = 1$ *then* $A_j|_F = \begin{bmatrix} \alpha_j \end{bmatrix}$ *and the inner product is given by* $[u, u] = -|u|^2$ *for* $u \in F \cong \mathbb{C}$.

3. *If* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *and* $f = \dim F \geq 2$ *then*

$$A_j|_F = \begin{bmatrix} \alpha_j & a_j^* & x_j \\ 0 & \alpha_j I & a_j \\ 0 & 0 & \alpha_j \end{bmatrix},$$

where I *is the identity matrix of order* $f - 2$, $a_j \in \mathbb{C}^{f-2}$, $\{x_j, a_i^* a_j\} \subset \mathbb{R}$ *for* $i, j = 1, 2, \ldots, n$ *and* $a_1, a_2, \ldots, a_n$ *are linearly independent. The inner product on* $F \cong \mathbb{C}^f$ *is given by* $[u, v] = \langle Pu, v \rangle$, *where* $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Proof. Note that the linear span of each negative Jordan chain contains at least one nonpositive subspace of dimension one. It follows by minimality of F that the spectrum of the restrictions of $\mathbf{A}$ to F contains a single real eigenvalue $\boldsymbol{\alpha}$ or a pair of complex conjugate eigenvalues $\boldsymbol{\alpha}, \overline{\boldsymbol{\alpha}}$. If $\boldsymbol{\alpha} \notin \mathbb{R}^n$ then one of its components is nonreal, without loss we may assume that $\alpha_1 \notin \mathbb{R}$. Since F has index 1 it follows that

$$A_1|_F = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \overline{\alpha}_1 \end{bmatrix}$$

and the inner product on $F \cong \mathbb{C}^2$ is given by $[u, v] = \langle Pu, v \rangle$, where $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since A_j commute it follows that they are all diagonal, and thus of the form (1) (where some of α_j may be real).

Next assume that $\boldsymbol{\alpha} \in \mathbb{R}^n$. If $\dim F = 1$ then we obtain case 2. So suppose that $f = \dim F \geq 2$. By the minimality of F it follows that at least one of the operators $A_j|_F$ is not diagonalizable. Without loss we may assume that A_1 is such. Then there are nonzero vectors v_0, v_1 such that $A_j v_0 = \alpha_j v_0$, $j = 1, 2, \ldots, n$ and $A_1 v_1 = \alpha_1 v_1 + v_0$. Then it follows that $[v_0, v_0] = 0$. Since F is nondegenerate there exists a vector $u \in F$ such that $[u, v_0] \neq 0$. Then $w = \frac{1}{[u,v_0]} u - \frac{[u,u]}{2|[u,v_0]|^2} v_0$ is such that $[w, v_0] = 1$ and $[w, w,] = 0$. Now let $V = \mathcal{L}(v_0, w)^{[\perp]}$. Here and later in the proof the orthogonal complement is taken in F. We write $W = \mathcal{L}(w)$ and $V_i = \mathcal{L}(v_i)$ for $i = 0, 1$. It is easy to verify that $V_0^{[\perp]} = V_0 \oplus V$ and $F = V_0 \oplus V \oplus W$. We want to show that $V_0 \oplus V$ is an invariant subspace for all A_j. To do so choose $z \in V_0 \oplus V$. Then $[A_j z, v_0] = [z, A_j v_0] = \alpha_j [z, v_0] = 0$ and thus $A_j z \in V_0^{[\perp]} = V_0 \oplus V$. Then it follows that with respect to the decomposition $F = V_0 \oplus V \oplus W$ we have

$$A_j|_F = \begin{bmatrix} \alpha_j & b_j^* & x_j \\ 0 & B_j & a_j \\ 0 & 0 & \alpha_j \end{bmatrix}, \quad j = 1, 2, \ldots, n,$$

and that the inner product on $F \cong \mathbb{C}^f$ is given by $[y, z] = \langle Py, z \rangle$, where

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & Q & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and Q is a positive definite matrix. Since A_j commute and are self-adjoint it follows that B_j are commuting linear maps on a Hilbert space. Thus $B_j = \alpha_j I$ and we can assume that $Q = I$. The conditions $b_j = a_j$ and $x_j, a_j^* a_i \in \mathbb{R}$ hold because A_j are commuting and self-adjoint. □

In the paragraph preceding Example 2.8 we explained how the case $n = 2$ is related to a single normal operator on a Pontryagin space. Then an improvement of Theorem 3.1 for $n = 2$ can be deduced from the canonical form for a normal operator on a Pontryagin space of index 1 given by Gohberg and Reichstein [8, Thm. 1].

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