On a Representation of Commuting Maps by Tensor Products

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Abstract

It is shown that a representation by tensor products of degree n^2 exists for every pair of commuting linear maps on an n-dimensional vector space, but in general, not a representation of degree less than n^2.

1 Introduction

In his paper [4] Chandler Davis introduced a representation of a pair of commuting linear operators by tensor products. When the underlying Hilbert space has finite dimension, say n, then the tensor product operators constructed in [4] act on a vector space of dimension n^3. Davis asked whether a representation by tensor products in dimension less than n^3 is possible. In general we cannot expect that this dimension, called the degree of representation, is less than n^2 (see [7, Example 1.28] and our Example below). Here we show that a pair of commuting linear maps on an n-dimensional vector space can be represented by tensor products on a vector space of dimension n^2. Thus it follows that, in general, n^2 is the minimal possible degree for such a representation which answers Davis' question. Our construction is similar to the one given originally in [4]. However coalgebraic techniques enable us to reduce the degree of the representation. We remark that this construction has a direct generalizations for a k-tuple of commuting linear maps on V. The representation we obtain then has degree less than or equal to n^k, also best possible in general.

We introduce the notion of a representation by tensor products and discuss the above mentioned Example in §2. In §3 we introduce coalgebras and comodules and in §4 we present some of the properties of the coalgebra of representative functionals on polynomial algebras. Comodules associated to a pair of commuting linear maps are studied in §5. The main result is proved in §6. We only outline the properties of coalgebras and comodules needed. For details we refer to [1, 6, 8, 9].

Representations by tensor products for commuting operators on Hilbert spaces were also studied by Fong and Sourour [5, Theorem 3.2], while De Boor and Rice [2] considered special pairs of commuting matrices that have a representation by tensor products on the original vector space.

2 Preliminaries

We assume throughout that V is a vector space of dimension n over a field F. We consider a pair A_1, A_2 of commuting linear maps on V. Suppose that there exist vector spaces W_1 and W_2, an injective linear map $T : V \to W$, where $W = W_1 \otimes W_2$, and linear maps $B_i : W_i \to W_i$ $(i = 1, 2)$ such that the diagram

$$\begin{array}{ccc} V & \overset{A_i}{\to} & V \\ {}^{T}\downarrow & & \downarrow^{T} \\ W_1 \otimes W_2 & \overset{B_i^\dagger}{\to} & W_1 \otimes W_2 \end{array} \tag{1}$$

commutes. Here

$$B_1^\dagger = B_1 \otimes I \text{ and } B_2^\dagger = I \otimes B_2. \tag{2}$$

Such a construction is called a *representation of the pair A_1, A_2 by tensor products* and the dimension of the vector space W is *the degree of the representation.*

If we already have $A_i = B_i^\dagger$ $(i = 1, 2)$ for some $B_i^\dagger$ as in (2) then the pair A_1, A_2 has a representation by tensor products already on the original space and hence the smallest degree of a representation by tensor products for A_1, A_2 is n. In general this is not the case. We consider next an example of a pair of commuting matrices for which the smallest degree of a representation by tensor products is n^2. It is a special case of [7, Example 1.28].

Example Suppose that $V = F^n$ and that

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now choose the vector space $W = F^n \otimes F^n$ and the subspace $\mathcal{M} \subset W$ spanned by the set $\mathcal{B} = \left\{ x_j = \sum_{i=0}^{j-1} v_{i+1} \otimes v_{j-i} \right\}_{j=1}^{n}$ where v_i $(i = 1, 2, \ldots, n)$, denote the standard basis vectors in F^n, i.e., $v_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ where 1 is in the i-th position. For $j = 1, 2, \ldots, n$ we have $B_i^\dagger x_j = x_{j-1}$, where $x_0 = 0$, $B_1^\dagger = A_1 \otimes I$ and $B_2^\dagger = I \otimes A_2$. Next define a linear map $T : F^n \longrightarrow W$ by $T(v_j) = x_j$. Then $TA_i = B_i^\dagger T$ is a representation by tensor products for A_i. It is of the smallest degree possible because for the tensor product space $F^p \otimes F^q$ where $p < n$ or $q < n$ there do not exist two $n \times n$ matrices C_1 and C_2 such that both $C_1 \otimes I$ and $I \otimes C_2$ have a Jordan chain of length n. (This is a consequence of the Aitken-Roth theorem, see e.g. [3, Theorem 4.6].) □

3 Coalgebras and Comodules

A *coalgebra* $\mathcal{C}$ is a vector space with a structure dual to that of an algebra, i.e., with a *counit* $\varepsilon : \mathcal{C} \to F$ and a *comultiplication* $\delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, which are linear maps such that the diagrams

$$\begin{array}{ccccc} F \otimes \mathcal{C} & \cong & \mathcal{C} & \cong & \mathcal{C} \otimes F \\ & \nwarrow^{\varepsilon \otimes I_{\mathcal{C}}} & {}^{\delta}\downarrow & {}^{I_{\mathcal{C}} \otimes \varepsilon}\nearrow & \\ & & \mathcal{C} \otimes \mathcal{C} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{C} & \stackrel{\delta}{\to} & \mathcal{C} \otimes \mathcal{C} \\ {}^{\delta}\downarrow & & \downarrow^{I_{\mathcal{C}} \otimes \delta} \\ \mathcal{C} \otimes \mathcal{C} & \stackrel{\delta \otimes I_{\mathcal{C}}}{\to} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \end{array}$$

commute. We call the pair of maps ε and δ also the structure maps of $\mathcal{C}$. The first of the above diagrams is the counit law and the second is coassociativity. Here we use the symbol $I_{\mathcal{C}}$ to denote the identity map of $\mathcal{C}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two coalgebras with structure maps ε_1, δ_1 and ε_2, δ_2, respectively, then $\mathcal{C}_1 \otimes \mathcal{C}_2$ is a coalgebra with structure maps $\varepsilon_1 \otimes \varepsilon_2$ and $\sigma_{23} (\delta_1 \otimes \delta_2)$, where σ_{ij} switches the i-th and the j-th tensor factor. All the coalgebras considered in this paper are cocommutative, i.e., $\delta = \sigma_{12}\delta$.

A notion dual to the notion of a module over an algebra is the notion of a comodule over a coalgebra. Suppose that $\mathcal{C}$ is a coalgebra. Then a vector space $\mathcal{N}$ is a $\mathcal{C}$-*comodule* if there is a linear map $\alpha : \mathcal{N} \to \mathcal{N} \otimes \mathcal{C}$, called a *coaction* of $\mathcal{C}$ on $\mathcal{N}$, such that

$$\begin{array}{ccc} \mathcal{N} & \cong & \mathcal{N} \otimes F \\ {}^{\alpha}\downarrow & {}^{I_{\mathcal{N}} \otimes \varepsilon}\nearrow & \\ \mathcal{N} \otimes \mathcal{C} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{N} & \stackrel{\alpha}{\to} & \mathcal{N} \otimes \mathcal{C} \\ {}^{\alpha}\downarrow & & \downarrow^{I_{\mathcal{N}} \otimes \delta} \\ \mathcal{N} \otimes \mathcal{C} & \stackrel{\alpha \otimes I_{\mathcal{C}}}{\to} & \mathcal{N} \otimes \mathcal{C} \otimes \mathcal{C} \end{array} \tag{3}$$

commute. If V is a vector space and $\mathcal{C}$ a coalgebra then the comultiplication δ of $\mathcal{C}$ induces a coaction $\alpha = I_V \otimes \delta$ on $V \otimes \mathcal{C}$. Such a comodule $V \otimes \mathcal{C}$ is called *cofree*. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-comodules then a linear map $\varphi : \mathcal{M} \to \mathcal{N}$ is a comodule homomorphism if the diagram

$$\begin{array}{ccc} \mathcal{M} & \stackrel{\varphi}{\to} & \mathcal{N} \\ {}^{\alpha_{\mathcal{M}}}\downarrow & & \downarrow^{\alpha_{\mathcal{N}}} \\ \mathcal{M} \otimes \mathcal{C} & \stackrel{\varphi \otimes I_{\mathcal{C}}}{\to} & \mathcal{N} \otimes \mathcal{C} \end{array}$$

commutes. For further details on the theory of coalgebras, comodules and Hopf algebras we refer to the books of Abe [1] and Sweedler [9].

4 The Coalgebra of Functionals on Polynomials

A linear functional $f : F[x] \to F$ is called a *representative functional* if its kernel contains an ideal of finite codimension. The vector space $F[x]^{\mathrm{o}}$ of all representative functionals on the polynomial ring $F[x]$ has a (topological) basis $\{e_m\}_{m=0}^{\infty}$, where $e_m(x^n) = \delta_{mn}$ and δ_{mn} is the Kronecker symbol. An element $f \in F[x]^{\mathrm{o}}$ has an infinite series representation $f = \sum_{m=0}^{\infty} \alpha_m e_m$, where $\{\alpha_m\}_{m=0}^{\infty}$ forms a linearly recursive sequence. The canonical coalgebra structure on $F[x]^{\mathrm{o}}$ is defined by

$$\varepsilon(e_m) = \delta_{0m} \quad \text{and} \quad \delta(e_m) = \sum_{r+s=m} e_r \otimes e_s. \tag{4}$$

The structure maps (4) are extended on the whole of $F[x]^{\mathrm{o}}$ by (infinite) linearity, for example $\delta\left(\sum_{m=0}^{\infty}\alpha_m e_m\right)=\sum_{m=0}^{\infty}\alpha_m\delta(e_m)$.

The linear map $D: F[x]^{\mathrm{o}}\to F[x]^{\mathrm{o}}$, defined by

$$Df(p)=f(xp) \tag{5}$$

for $p\in F[x]$ is dual to the map $M_x: F[x]\to F[x]$ given by $M_x p = xp$.

Next suppose next that $p(x)=x^d-a_{d-1}x^{d-1}-a_{d-2}x^{d-2}-\ldots-a_0$ is an arbitrary monic polynomial and that (p) is the ideal generated by p in $F[x]$. The dual space

$$\mathcal{C}_p=\left(F[x]\,/\,(p)\right)^* \tag{6}$$

is a subcoalgebra of $F[x]^{\mathrm{o}}$, called the subcoalgebra associated with p. If $f_r\in\mathcal{C}_p$ is defined by $f_r(x^s)=\delta_{rs}\ (r,s=0,1,\ldots,d-1)$, where $d=\deg p$, then

$$\{f_r\}_{r=0}^{d-1} \tag{7}$$

is a basis for $\mathcal{C}_p$. Furthermore we have

$$Df_0=a_0f_{d-1}\quad\text{and}\quad Df_r=f_{r-1}+a_rf_{d-1}\ \ (r=1,2,\ldots,d-1)\,. \tag{8}$$

The interested reader will find more information about the structure of $F[x]^{\mathrm{o}}$ in [6, 8].

5 Comodules Associated with Linear Maps

Now we turn our attention to a pair of commuting linear maps $A_1,\ A_2$. Suppose that p_i is the minimal polynomial of $A_i\ (i=1,2)$. Let $\mathcal{C}_i$ be the coalgebra associated with the polynomial p_i, of degree d_i, as in (6) and $\mathcal{C}=\mathcal{C}_1\otimes\mathcal{C}_2$. Then we have

$$\dim\mathcal{C}_i=d_i\le n. \tag{9}$$

The restriction of the map (5) to the coalgebra $\mathcal{C}_i$ is denoted by D_i. We write $D_1^\dagger=D_1\otimes I_{\mathcal{C}_2}$ and $D_2^\dagger=I_{\mathcal{C}_1}\otimes D_2$ for the induced maps on $\mathcal{C}$. But for the map induced by D_i (resp. $D_i^\dagger$) on the cofree comodules $V\otimes\mathcal{C}_i$ and $V\otimes\mathcal{C}$ we use the same symbol D_i (resp. $D_i^\dagger$). Similarly the map induced by A_i on $V\otimes\mathcal{C}_i$ and $V\otimes\mathcal{C}$ is still denoted by A_i.

Because $(A_i-D_i): V\otimes\mathcal{C}_i\to V\otimes\mathcal{C}_i$ and $\left(A_i-D_i^\dagger\right): V\otimes\mathcal{C}\to V\otimes\mathcal{C}$ are comodule homomorphisms it follows that their kernels r_i and $\mathcal{R}_i$ are subcomodules of $V\otimes\mathcal{C}_i$ and $V\otimes\mathcal{C}$, respectively. Moreover,

$$\mathcal{R}_1=\mathrm{r}_1\otimes\mathcal{C}_2 \text{ and } \mathcal{R}_2=\mathcal{C}_1\otimes\mathrm{r}_2. \tag{10}$$

If $\mathbf{v}$ is in $\mathcal{R}_{12}=\mathcal{R}_1\cap\mathcal{R}_2$ then $A_i\mathbf{v}=D_i^\dagger\mathbf{v}\ \ (i=1,2)$, and so the diagram

$$\begin{array}{ccc} \mathcal{R}_{12} & \overset{A_i}{\underset{D_i^\dagger}{\rightrightarrows}} & \mathcal{R}_{12} \\ {\scriptstyle I_V\otimes\varepsilon}\downarrow & & \downarrow{\scriptstyle I_V\otimes\varepsilon} \\ V & \overset{A_i}{\rightarrow} & V \end{array} \tag{11}$$

commutes.

Lemma *Let $\mathcal{R}$ be either r_1, r_2 or $\mathcal{R}_{12}$, and let $\overline{\varepsilon} : \mathcal{R} \to V$ be the restriction of $I_V \otimes \varepsilon$ to $\mathcal{R}$ in each case. Then $\overline{\varepsilon}$ is invertible. Moreover,* $\dim \mathcal{R} = n$ *and* $\dim \mathcal{R}_i = nd_i$ for $i = 1, 2$.

Proof. The second part of the lemma follows from the first part and (10). We prove the first part of the lemma for $\mathcal{R} = \mathcal{R}_{12}$. The proof for r_1 and r_2 is similar.

Suppose that $\overline{\varepsilon}\mathbf{v} = 0$. Because (11) commutes it follows that $0 = A_1^r A_2^s \overline{\varepsilon}\mathbf{v} = \overline{\varepsilon} D_1^{\dagger r} D_2^{\dagger s} \mathbf{v}$ ($r = 0, 1, \ldots, d_1 - 1$ and $s = 0, 1, \ldots, d_2 - 1$). By the induction on r and s one shows that $\mathbf{v} = 0$, and so $\overline{\varepsilon}$ is one-to-one.

Now choose $v \in V$ and consider $\mathbf{v} = \sum_{r=0}^{d_1-1} \sum_{s=0}^{d_2-1} A_1^r A_2^s v \otimes f_r^1 \otimes f_s^2 \in V \otimes \mathcal{C}$, where $\{f_r^i\}_{r=0}^{d_i-1}$ is the basis for $\mathcal{C}_i$ given by (7). We write $p_i(x) = x^{d_i} - a_{i,d_i-1}x^{d_i-1} - \ldots - a_{i,0}$, and then using (8), we see that

$$D_1^{\dagger}\mathbf{v} = \sum_{r=1}^{d_1-1} \sum_{s=0}^{d_2-1} A_1^r A_2^s v \otimes f_{r-1}^1 \otimes f_s^2 + \sum_{r=0}^{d_1-1} \sum_{s=0}^{d_2-1} A_1^r A_2^s v \otimes a_{ir} f_{d_1-1}^1 \otimes f_s^2. \qquad (12)$$

Since $p_i(A_i) = 0$, the right-hand side of (12) is equal to

$$\sum_{r=0}^{d_1-2} \sum_{s=0}^{d_2-1} A_1^{r+1} A_2^s v \otimes f_r^1 \otimes f_s^2 + \sum_{s=0}^{d_2-1} A_1^{d_1} A_2^s v \otimes f_{d_1-1}^1 \otimes f_s^2 = A_1\mathbf{v}.$$

Hence $\mathbf{v} \in \mathcal{R}_1$. Similarly we show that $\mathbf{v} \in \mathcal{R}_2$. Because $\overline{\varepsilon}\mathbf{v} = v$ it follows that $\overline{\varepsilon}$ is also surjective. □

6 Representation by Tensor Products

The following is our main theorem.

Theorem *Every pair of commuting linear maps A_1 and A_2 on V has a representation by tensor products of degree less than or equal to n^2. In particular, there exists such a representation of degree $n \cdot \min\{d_1, d_2\}$.*

Proof. By the lemma $\overline{\varepsilon} : \mathcal{R}_{12} \to V$ is invertible. Define $T : V \to \mathrm{r}_1 \otimes \mathcal{C}_2$ by $T = \iota \circ \overline{\varepsilon}^{-1}$, where $\iota : \mathcal{R}_{12} \hookrightarrow \mathrm{r}_1 \otimes \mathcal{C}_2$ is the inclusion. The diagram

$$\begin{array}{ccc} V & \xrightarrow{A_i} & V \\ {}^{T}\downarrow & & \downarrow^{T} \\ \mathrm{r}_1 \otimes \mathcal{C}_2 & \xrightarrow{D_i^{\dagger}} & \mathrm{r}_1 \otimes \mathcal{C}_2 \end{array} \qquad (13)$$

commutes for $i = 1, 2$ because diagram (11) commutes. Then (13) is a representation by tensor products for A_1, A_2 as defined in (1). The analog of diagram (13), where $\mathrm{r}_1 \otimes \mathcal{C}_2$ is replaced by $\mathcal{C}_1 \otimes \mathrm{r}_2$ also commutes. By our Lemma it follows then that there exists a representation of the required degree. □

We remark that T is as in [4]. However our approach reveals further structure of Davis' construction and this enables us to find a representation of best possible degree.

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