Finite Dimensional Multiparameter Spectral Theory : The Nonderogatory Case

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Abstract

A completeness theorem for nonderogatory eigenvalues of multiparameter systems is proved in a finite-dimensional setting. Also a basis for the second root subspace of a simple eigenvalue is given. Weakly-elliptic multiparameter systems are introduced. It is shown that simple real eigenvalues of such systems are nonderogatory.

1 Introduction

When boundary value problems for partial differential equations are solved using the method of separation of variables the resulting system (called a multiparameter system) of boundary value problems for ordinary differential equations is linked linearly by separation constants (parameters). To give a solution in terms of Fourier-type series we need to solve completeness problems, i.e., to find a complete set of generalized eigenfunctions. These problems have been considered since the early days of multiparameter spectral theory late last century. For some recent presentations of multiparameter spectral theory and related boundary value problems we refer the interested reader to [6, 7, 14].

It was Atkinson [2] who revived the theory in 1960s, introducing an abstract setting for the study of multiparameter spectral problems. This setting involves a system of linear multiparameter pencils

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} V_{ij}\lambda_j - V_{i0},\ i = 1, 2, \ldots, n,\ (n \geq 2)\,, \tag{1}$$

called a multiparameter system, (here $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an n-tuple of parameters and V_{ij} are linear transformations on a vector space H_i), and a tensor product construction.

Namely, to a multiparameter system (1), which satisfies a certain regularity condition, an n-tuple of commuting linear transformations on a tensor product space is associated. This n-tuple of commuting linear transformations is called an associated system. We briefly introduce this construction in the next two sections.

Completeness problems in the abstract setting involve finding bases for the root subspaces of an associated system in terms of the underlying multiparameter system. Atkinson showed that the eigenspaces of the associated system are spanned by decomposable tensors which are easily described by the underlying multiparameter system. The structure of root vectors other than eigenvectors is yet to be described fully. This has been so far considered mostly for self-adjoint multiparameter systems. For example, Binding [3] solved it for real eigenvalues of uniformly elliptic multiparameter systems. Completeness problems for nonself-adjoint multiparameter systems have not been considered much in the literature (Isaev's paper [10] being an exception), even for finite-dimensional cases, though these problems were posed in the literature, for example by Atkinson [1] and Isaev [11].

In our presentation we study multiparameter systems on finite-dimensional vector spaces. We assume that the multiparameter system is nonsingular, i.e., a certain linear transformation on the tensor product space is invertible. Then we consider a class of eigenvalues for multiparameter systems called simple eigenvalues and a subclass of simple eigenvalues called nonderogatory eigenvalues.

The first important result is Theorem 5. It describes a basis for second root subspaces for simple eigenvalues defined in Section 4. In Section 5 we define nonderogatory eigenvalues for multiparameter systems. They correspond to the equivalent notion for commuting sets of matrices, of which associated systems are particular cases. In proving this we need a result from [13]. Our main result is the completeness theorem for nonderogatory eigenvalues of multiparameter systems, stated and proved in Section 7. From the discussion preceding the main theorem and from its proof we obtain a computational procedure to find a basis for the root subspace of a nonderogatory eigenvalue. We present this procedure in Section 8. We also include an example to illustrate the procedure. The structure of vectors in the basis is the same as the structure of root vectors in [3], although the methods used here are completely different. We briefly compare our methods with those of [3] at the end of this article.

In general self-adjoint multiparameter systems also possess nonreal and non semisimple eigenvalues. Hence our results are of interest for these systems as well. We consider such systems in the last section. There we introduce the notion of weakly-elliptic multiparameter systems. These include elliptic multiparameter systems as they are usually defined in the literature. We show that simple real eigenvalues of weakly-elliptic multiparameter systems are always nonderogatory and hence our previous results apply.

2 Multiparameter systems and their associated systems

Suppose H_i, $i=1,2,\ldots,n$, are finite-dimensional vector spaces over complex numbers. We write $\dim H_i = n_i$. A *multiparameter system* (1) is denoted by $\mathbf{W}$. We also write

$$U_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} V_{ij}\lambda_j.$$

Next we form a tensor product space $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Then it follows that $\dim H = \prod_{i=1}^{n} n_i$ and we write $\dim H = N$. A linear transformation V_{ij} induces a linear transformation $V_{ij}^{\dagger}$ on the vector space H. It is defined by

$$V_{ij}^{\dagger}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes V_{ij}x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$$

on a decomposable tensor $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in H$, where $x_i \in H_i$, and we extend this definition to H by linearity.

The *operator determinant* Δ_0 is defined as the determinant

$$\Delta_0 = \begin{vmatrix} V_{11}^{\dagger} & V_{12}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ V_{21}^{\dagger} & V_{22}^{\dagger} & \cdots & V_{2n}^{\dagger} \\ \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & V_{n2}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix}. \tag{2}$$

It is a linear transformation acting on H and it is well defined because two transformations from different rows in (2) commute. It can be written also as a 'tensor determinant'

$$\Delta_0 = \sum_{\sigma\in\Pi_n} (-1)^{\operatorname{sgn}\sigma} V_{1\sigma(1)} \otimes V_{2\sigma(2)} \otimes \cdots \otimes V_{n\sigma(n)},$$

where Π_n is a set of all permutations of the set $\{1,2,\ldots,n\}$ and $\operatorname{sgn}\sigma$ is the sign of a permutation $\sigma \in \Pi_n$. Similarly we define operator determinants

$$\Delta_i = \begin{vmatrix} V_{11}^{\dagger} & \cdots & V_{1,i-1}^{\dagger} & V_{10}^{\dagger} & V_{1,i+1}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ V_{21}^{\dagger} & \cdots & V_{2,i-1}^{\dagger} & V_{20}^{\dagger} & V_{2,i+1}^{\dagger} & \cdots & V_{2n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & \cdots & V_{n,i-1}^{\dagger} & V_{n0}^{\dagger} & V_{n,i+1}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix} \quad \text{for } i=1,2,\ldots,n.$$

If $z = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in H$ is a decomposable tensor then it follows that

$$\Delta_0 z = \sum_{\sigma\in\Pi_n} (-1)^{\operatorname{sgn}\sigma} V_{1\sigma(1)}x_1 \otimes V_{2\sigma(2)}x_2 \otimes \cdots \otimes V_{n\sigma(n)}x_n.$$

In determinantal form we have

$$\Delta_0 z = \begin{vmatrix} V_{11}x_1 & V_{12}x_1 & \cdots & V_{1n}x_1 \\ V_{21}x_2 & V_{22}x_2 & \cdots & V_{2n}x_2 \\ \vdots & \vdots & & \vdots \\ V_{n1}x_n & V_{n2}x_n & \cdots & V_{nn}x_n \end{vmatrix}. \tag{3}$$

If $w = y_1 \otimes y_2 \otimes \cdots \otimes y_n \in H$ is another decomposable tensor then we write

$$w^* z = \prod_{i=1}^{n} y_i^* x_i \tag{4}$$

and

$$w^* \Delta_0 z = \begin{vmatrix} y_{10}^* V_{11} x_{10} & y_{10}^* V_{12} x_{10} & \cdots & y_{10}^* V_{1n} x_{10} \\ y_{20}^* V_{21} x_{20} & y_{20}^* V_{22} x_{20} & \cdots & y_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n0} & y_{20}^* V_{n2} x_{n0} & \cdots & y_{n0}^* V_{nn} x_{n0} \end{vmatrix}.$$

Note that (4) is extended by linearity to a scalar product on H. Here $y_i^* x_i$ stands for a scalar product of vectors x_i and y_i in H_i.

It was proved by Atkinson in [2, Theorem 6.2.1] that the operator determinant (2) retains the usual properties of a scalar determinant when we perform column operations on it. It is an immediate consequence of this result that the same is true when column operations are performed on vector determinant (3). We state this result for future reference.

Lemma 1 *For an operator determinant* (2) *and a vector determinant* (3) *we have the following properties :*

(i) if two columns are interchanged the sign of a determinant changes,

(ii) if columns are linearly dependent the determinant vanishes,

(iii) the determinant remains unchanged if a scalar multiple of one column is added to another column,

(iv) the determinant is linear in every column.

Definition. A multiparameter system $\mathbf{W}$ is called *nonsingular* if the corresponding determinantal transformation Δ_0 is invertible.

We assume throughout our discussion that multiparameter systems considered are nonsingular. With a nonsingular multiparameter system $\mathbf{W}$ we associate an n-tuple of linear transformation $\Gamma_i = \Delta_0^{-1} \Delta_i$, $i = 1, 2, \ldots, n$. It is called the *associated system* (of a multiparameter system $\mathbf{W}$) and we write $\mathbf{\Gamma} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$. Atkinson proved in [2, Theorem 6.7.1] that transformations Γ_i commute. He also proved [2, Theorem 6.7.2] that if $\mathbf{W}$ is a multiparameter system and $\mathbf{\Gamma}$ is its associated system then

$$\sum_{j=1}^{n} V_{ij}^\dagger \Gamma_j = V_{i0}^\dagger, \quad \textit{for } i = 1, 2, \ldots, n. \tag{5}$$

Note that the system of relations (5) connecting the induced transformations $V_{ij}^\dagger$ and the associated transformations Γ_i can be considered as a generalization of the standard Cramer's rule for a system of scalar equations.

3 Spectrum, eigenspaces and second root subspaces

Definition. Suppose that $\mathbf{W}$ is a multiparameter system. Then an n-tuple $\boldsymbol{\lambda} \in \mathbb{C}^n$ is called an *eigenvalue* of $\mathbf{W}$ if all $W_i(\boldsymbol{\lambda})$ are singular. The set of all the eigenvalues of $\mathbf{W}$ is called the *spectrum* of $\mathbf{W}$ and it is denoted by $\sigma(\mathbf{W})$.

Definition. Suppose that $\boldsymbol{\Gamma} = \{\Gamma_i,\ i = 1, 2, \ldots, n\}$ is a set of commuting matrices. An n-tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ is called an *eigenvalue* of $\boldsymbol{\Gamma}$ if $\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i) \neq \{0\}$. The set of all the eigenvalues of $\boldsymbol{\Gamma}$ is called the *spectrum* of $\boldsymbol{\Gamma}$ and it is denoted by $\sigma(\boldsymbol{\Gamma})$. The subspace $\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i)$ is called the *eigenspace* of $\boldsymbol{\Gamma}$ at $\boldsymbol{\lambda}$ and we denote it by $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$. We also write

$$d_0 = \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}).$$

Before we continue our discussion we recall another result of Atkinson.

Theorem 2 [2, Theorem 6.9.1] *Suppose that* $\mathbf{W}$ *is a multiparameter system and* $\boldsymbol{\Gamma}$ *is its associated system. Then we have* $\sigma(\mathbf{W}) = \sigma(\boldsymbol{\Gamma})$. *Furthermore if* $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ *then*

$$\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}) = \ker W_1(\boldsymbol{\lambda}) \otimes \ker W_2(\boldsymbol{\lambda}) \otimes \cdots \otimes \ker W_n(\boldsymbol{\lambda}).$$

4 A basis for the second root subspace for simple eigenvalues

Definition. Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$. Then we call the subspace

$$\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 = \bigcap_{i=1}^{n} \bigcap_{j=i}^{n} \ker\left[(\lambda_i I - \Gamma_i)(\lambda_j I - \Gamma_j)\right]$$

the *second root subspace* of $\mathbf{W}$ at $\boldsymbol{\lambda}$. We write

$$d_1 = \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 - \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}).$$

Definition. An eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is called *simple* if $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for all i.

In this section we assume that an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is simple. Then it follows from Theorem 2 that $d_0 = 1$, and furthermore, if $x_{i0} \in H_i$, $i = 1, 2, \ldots, n$, are nonzero vectors such that $W_i(\boldsymbol{\lambda}) x_{i0} = 0$ then the vector $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ spans the eigenspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$.

Next we choose nonzero vectors $y_{i0} \in H_i,\ i = 1, 2, \ldots, n$, such that $W_i(\boldsymbol{\lambda})^* y_{i0} = 0$, where $W(\boldsymbol{\lambda})_i^*$ is the adjoint linear transformation of $W_i(\boldsymbol{\lambda})$, and we form an $n \times n$ matrix

$$B_0 = \begin{bmatrix} y_{10}^* V_{11} x_{10} & y_{10}^* V_{12} x_{10} & \cdots & y_{10}^* V_{1n} x_{10} \\ y_{20}^* V_{21} x_{20} & y_{20}^* V_{22} x_{20} & \cdots & y_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n0} & y_{n0}^* V_{n2} x_{n0} & \cdots & y_{n0}^* V_{nn} x_{n0} \end{bmatrix}.$$

Suppose that $z_1 \in \ker (\boldsymbol{\lambda} \mathbf{I} - \boldsymbol{\Gamma})^2 \setminus \ker (\boldsymbol{\lambda} \mathbf{I} - \boldsymbol{\Gamma})$. Then we have

$$(\lambda_i I - \Gamma_i) z_1 = a_i z_0, \ i = 1, 2, \ldots, n,$$

where a_i are complex numbers and not all of them are zero. We write

$$\mathbf{a} = [a_1, a_2, \ldots, a_n]^T .$$

Lemma 3 *Let* $\mathbf{a} \in \mathbb{C}^n$ *be as above. Then* $\mathbf{a} \in \ker B_0$.

Proof. It follows from relation (5) that

$$V_{i0}^{\dagger} z_1 = \sum_{j=1}^{n} V_{ij}^{\dagger} \Gamma_j z_1 = \sum_{j=1}^{n} V_{ij}^{\dagger} (\lambda_j z_1 - a_j z_0)$$

for $i = 1, 2, \ldots, n$. These relations can be written as

$$W_i (\boldsymbol{\lambda})^{\dagger} z_1 = U_i (\mathbf{a})^{\dagger} z_0. \tag{6}$$

Next we choose vectors $v_i \in H_i, \ i = 1, 2, \ldots, n$, such that $v_i^* x_{i0} = 1$. The vectors $w_i = v_1 \otimes \cdots \otimes v_{i-1} \otimes y_{i0} \otimes v_{i+1} \otimes \cdots \otimes v_n$ are such that $W_i (\boldsymbol{\lambda})^{*\dagger} w_i = 0$. Then it follows from (6) that $w_i^* U_i (\mathbf{a})^{\dagger} z_0 = 0$. Because z_0 and w_i are decomposable tensors and $v_i^* x_{i0} = 1$ it follows that $y_{i0}^* U_i (\mathbf{a}) x_{i0} = 0$ for all i. This is equivalent to $\mathbf{a} \in \ker B_0$. □

Lemma 4 *Suppose that* $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T \in \ker B_0$ *is not zero. Then there exist vectors* $x_{i1} \in H_i, \ i = 1, 2, \ldots, n$, *such that*

$$U_i (\mathbf{a}) x_{i0} = W_i (\boldsymbol{\lambda}) x_{i1}. \tag{7}$$

Furthermore, the vector

$$z_1 = \sum_{i=1}^{n} x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1} \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0} \tag{8}$$

is such that

$$(\lambda_i I - \Gamma_i) z_1 = a_i z_0.$$

Proof. For $\mathbf{a} \in \ker B_0$ it follows that $y_{i0}^* U_i (\mathbf{a}) x_{i0} = 0$ for all i. Then the vector $U_i (\mathbf{a}) x_{i0}$ is orthogonal to the kernel of $W_i (\boldsymbol{\lambda})^*$ and so it is in the range of $W_i (\boldsymbol{\lambda})$. Hence there exists a vector $x_{i1} \in H_i$ such that (7) holds.

It follows from Lemma 1 that

$$\lambda_i \Delta_0 - \Delta_i = \begin{vmatrix} V_{11}^{\dagger} & \cdots & V_{1,i-1}^{\dagger} & W_1 (\boldsymbol{\lambda})^{\dagger} & V_{1,i+1}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ V_{21}^{\dagger} & \cdots & V_{2,i-1}^{\dagger} & W_2 (\boldsymbol{\lambda})^{\dagger} & V_{2,i+1}^{\dagger} & \cdots & V_{2n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & \cdots & V_{n.i-1}^{\dagger} & W_n (\boldsymbol{\lambda})^{\dagger} & V_{n,i+1}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix} \tag{9}$$

for any index $i=1,2,\ldots,n$. Suppose now that $z_1 \in H$ is given by (8). Applying relations (7) and (9) we obtain

$$\left(\lambda_i \Delta_0 - \Delta_i\right) z_1 =$$

$$=\sum_{j=1}^{n} \begin{vmatrix} V_{11}x_{10} & \cdots & V_{1,i-1}x_{10} & 0 & V_{1,i+1}x_{10} & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{j-1,1}x_{j-1,0} & \cdots & V_{j-1,i-1}x_{j-1,0} & 0 & V_{j-1,i+1}x_{j-1,0} & \cdots & V_{j-1,n}x_{j-1,0} \\ V_{j1}x_{j1} & \cdots & V_{j,i-1}x_{j1} & U_j\left(\mathbf{a}\right)x_{j0} & V_{j,i+1}x_{j1} & \cdots & V_{jn}x_{j1} \\ V_{j+1,1}x_{j+1,0} & \cdots & V_{j+1,i-1}x_{j+1,0} & 0 & V_{j+1,i+1}x_{j+1,0} & \cdots & V_{j+1,n}x_{j+1,0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & V_{n,i-1}x_{n0} & 0 & V_{n,i+1}x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}. \tag{10}$$

The value of the determinants above remains the same if we replace the terms $V_{jk}x_{j1}$, $k=1,\ldots,i-1,i+1,\ldots,n$, by the terms $V_{jk}x_{j0}$. Then it follows from Lemma 1 that (10) is equal to

$$\begin{vmatrix} V_{11}^{\dagger} & \cdots & V_{1,i-1}^{\dagger} & U_1\left(\mathbf{a}\right)^{\dagger} & V_{1,i+1}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ V_{21}^{\dagger} & \cdots & V_{2,i-1}^{\dagger} & U_2\left(\mathbf{a}\right)^{\dagger} & V_{2,i+1}^{\dagger} & \cdots & V_{2n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & \cdots & V_{n,i-1}^{\dagger} & U_n\left(\mathbf{a}\right)^{\dagger} & V_{n,i+1}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix} z_0 = a_i \Delta_0 z_0.$$

Combining the last equality with (10) we have $\left(\lambda_i \Delta_0 - \Delta_i\right) z_1 = a_i \Delta_0 z_0$, or equivalently,

$$\left(\lambda_i I - \Gamma_i\right) z_1 = a_i z_0.$$

□

Now we state the main result of this section :

Theorem 5 *Assume that* $\boldsymbol{\lambda} \in \sigma\left(\mathbf{W}\right)$ *is a simple eigenvalue and that* $\left\{z_0;\, z_1^1, z_1^2, \ldots, z_1^{d_1}\right\}$ *is a basis for the second root subspace* $\ker\left(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}\right)^2$ *such that*

$$\left(\lambda_i I - \Gamma_i\right) z_1^k = a_i^k z_0 \tag{11}$$

for $i=1,2,\ldots,n$ *and* $k=1,2,\ldots,d_1$. *We write* $\mathbf{a}^k = \left[a_1^k, a_2^k, \ldots, a_n^k\right]^T$. *Then* $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{d_1}\right\}$ *is a basis for the kernel of* B_0.

Conversely, suppose that $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{d}\right\}$ *is a basis for the kernel of* B_0. *We write* $\mathbf{a}^k = \left[a_1^k, a_2^k, \ldots, a_n^k\right]^T$. *Then there exist vectors* $x_{i1}^k \in H_i$ *such that*

$$U_i\left(\mathbf{a}^k\right) x_{i0} = W_i\left(\boldsymbol{\lambda}\right) x_{i1}^k \tag{12}$$

for $i=1,2,\ldots,n$ *and* $k=1,2,\ldots,d$. *Furthermore, the vectors*

$$z_1^k = \sum_{i=1}^{n} x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1}^k \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0} \tag{13}$$

are such that

$$(\lambda_i I - \Gamma_i)\, z_1^k = a_i^k z_0 \tag{14}$$

for all i *and* k *and* $\left\{z_0;\ z_1^1, z_1^2, \ldots, z_1^d\right\}$ *is a basis for the second root subspace* $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$.

In particular we have :

Corollary 6 *Suppose that* $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ *is a simple eigenvalue. Then it follows that* $\dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 = 1 + \dim\ker B_0$*, i.e.,* $d_1 = \dim\ker B_0$.

Proof of Theorem 5. Suppose that $\left\{z_0;\ z_1^1, z_1^2, \ldots, z_1^{d_1}\right\}$ is a basis for the second root subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ such that (11) hold. Then Lemma 3 implies that $\mathbf{a}^k \in \ker B_0$. Because the vectors $z_1^1, z_1^2, \ldots, z_1^d$ are linearly independent and $z_0 \neq 0$ it follows that $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^{d_1}$ are linearly independent, hence

$$d_1 \leq \dim\ker B_0. \tag{15}$$

Conversely, suppose now that $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^d\right\}$ is a basis for the kernel of B_0. (Here we write $\dim\ker B_0 = d$.) Lemma 4 implies that there exist vectors $x_{i1}^k \in H_i$ such that relation (12) holds and the vector (13) is such that relation (14) holds for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, d$. Let us now show that vectors $z_0, z_1^1, z_1^2, \ldots, z_1^d$ are linearly independent. Suppose that $\alpha z_0 + \sum_{k=1}^{d} \beta_k z_1^k = 0$. Then it follows that

$$0 = \sum_{k=1}^{d} (\lambda_i I - \Gamma_i)\, \beta_k z_1^k = \sum_{k=1}^{d} \beta_k a_i^k z_0$$

for all i. Because $z_0 \neq 0$ and $\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^d$ are linearly independent it follows that $\beta_k = 0$ for all k and then also $\alpha = 0$. Hence the vectors $z_0, z_1^1, z_1^2, \ldots, z_1^d$ are linearly independent. This implies that $\dim\ker B_0 \leq d_1$ whence, together with (15), it follows that $d_1 = \dim\ker B_0$. □

5 Nonderogatory eigenvalues

In this section we first define nonderogatory eigenvalues for arbitrary n-tuples of commuting matrices and we consider some of their properties. Next we define nonderogatory eigenvalues for multiparameter systems so that the two notions of the nonderogatory eigenvalues coincide for associated systems of multiparameter systems.

Definition. Suppose that $\boldsymbol{\Gamma} = \{\Gamma_i,\ i = 1, 2, \ldots, n\}$ are commuting matrices. An eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(\boldsymbol{\Gamma})$ is called *nonderogatory* if there exists an integer $l \geq 1$ such that

$$\dim\left(\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i)^k\right) = k,\ \ for\ \ k = 1, 2, \ldots, l-1$$

and

$$\dim\left(\bigcap_{i=1}^{n}\ker\left(\lambda_i I-\Gamma_i\right)^k\right)=l,\ \ for\ k=l,l+1,l+2,\ldots$$

The integer l is called the *ascent* of the eigenvalue $\boldsymbol{\lambda}$.

Suppose that $\boldsymbol{\lambda}$ is an eigenvalue of an n-tuple of commuting matrices $\boldsymbol{\Gamma}$. As before we write

$$d_0=\dim\left(\bigcap_{i=1}^{n}\ker\left(\lambda_i I-\Gamma_i\right)\right)$$

and

$$d_1=\dim\left(\bigcap_{i=1}^{n}\bigcap_{j=i}^{n}\ker\left(\lambda_i I-\Gamma_i\right)\left(\lambda_j I-\Gamma_j\right)\right)-d_0.$$

Then the following results follow from Corollary 2 of [13].

Theorem 7 *Suppose that $\boldsymbol{\Gamma}$ is an n-tuple of commuting matrices and $\boldsymbol{\lambda}\in\sigma\left(\boldsymbol{\Gamma}\right)$. Then $\boldsymbol{\lambda}$ is nonderogatory if and only if $d_0=1$ and $d_1\leq 1$.*

Theorem 8 *An eigenvalue $\boldsymbol{\lambda}$ for an n-tuple of commuting matrices is nonderogatory if and only if at least one of λ_i is a nonderogatory eigenvalue for Γ_i.*

We denote the set of multiindices

$$\left\{\left(j_1,j_2,\ldots,j_n\right);\ 0\leq j_i,\ \sum_{i=1}^{n}j_i=k\right\} \tag{16}$$

by Ψ_k for $k=0,1,2,\ldots$ and we write $\mathbf{j}=\left(j_1,j_2,\ldots,j_n\right)$. Then we have as a consequence of the previous theorem :

Corollary 9 *Suppose that $\boldsymbol{\lambda}=\left(\lambda_1,\lambda_2,\ldots,\lambda_n\right)$ is nonderogatory. Then*

$$\bigcap_{i=1}^{n}\ker\left(\lambda_i I-\Gamma_i\right)^k=\bigcap_{\mathbf{j}\in\Psi_k}\ker\left[\left(\lambda_1 I-\Gamma_1\right)^{j_1}\left(\lambda_2 I-\Gamma_2\right)^{j_2}\cdots\left(\lambda_n I-\Gamma_n\right)^{j_n}\right]$$

for $k=1,2,3,\ldots$ □

Now we define a notion of a nonderogatory eigenvalue for a multiparameter system.

Definition. An eigenvalue $\boldsymbol{\lambda}\in\sigma\left(\mathbf{W}\right)$ is called *nonderogatory* if $\dim\ker W_i\left(\boldsymbol{\lambda}\right)=1$ for all i and $\dim\ker B_0\leq 1$.

The following result shows that the two notions of a nonderogatory eigenvalue are consistent for an associated system of a multiparameter system. The result follows from Theorems 5 and 7.

Corollary 10 *An eigenvalue $\boldsymbol{\lambda}$ of a multiparameter system $\mathbf{W}$ is nonderogatory if and only if it is a nonderogatory eigenvalue of the associated system $\boldsymbol{\Gamma}$.*

6 Auxiliary results

Definition. A Jordan chain $z_0, z_1, \ldots, z_p$ is called *maximal* for a linear transformation Γ at a nonderogatory eigenvalue λ_0 if the vector z_p does not belong to the range of the transformation $\lambda_0 I - \Gamma$.

The next result is a known biorthogonality property between right and left Jordan chains. We write it for an associated transformation Γ_h in the form we need later. We assume that the index h is fixed.

Lemma 11 *Suppose that $z_0, z_1, \ldots, z_p$ is a maximal Jordan chain for Γ_h at a nonderogatory eigenvalue λ_h and w_0 is a nonzero vector such that $(\lambda_h \Delta_0 - \Delta_h)^* w_0 = 0$. Then it follows that $w_0^* \Delta_0 z_k = 0$ for $k = 0, 1, \ldots, p-1$ and $w_0^* \Delta_0 z_p \neq 0$.*

Proof. Note that vector $\Delta_0^* w_0$ spans $\ker(\lambda_h I - \Gamma_h)^*$, the orthogonal complement of the range $\mathcal{R}_h$ of $\lambda_h I - \Gamma_h$. Because vectors $z_k \in \mathcal{R}_h$, $k = 0, 1, \ldots, p-1$ it follows that $w_0^* \Delta_0 z_k = 0$ and because $z_p \notin \mathcal{R}_h$ it follows that $w_0^* \Delta_0 z_p \neq 0$. □

Lemma 12 *Let $\left\{ B_l = \left[b_{ij}^l \right]_{i,j=1}^n, l = 0, 1, \ldots, k-1 \right\}$ be a set of $n \times n$ matrices and assume that $\operatorname{rank}(B_0) = n-1$. Choose a nonzero vector $\mathbf{a}_1 \in \ker(B_0)$. Then there exist vectors $\mathbf{a}_i, i = 2, 3, \ldots, k$ such that $\sum_{j=0}^{l-1} B_j \mathbf{a}_{l-j} = 0$ for $l = 1, 2, \ldots, k$ if and only if*

$$\sum_{\mathbf{j} \in \Psi_l} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_2} & b_{22}^{j_2} & \cdots & b_{2n}^{j_2} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_n} & b_{n2}^{j_n} & \cdots & b_{nn}^{j_n} \end{vmatrix} = 0 \tag{17}$$

for $l = 0, 1, \ldots, k-1$.

Proof. We construct a monic matrix polynomial

$$L(\mu) = I\mu^k + B_{k-1}\mu^{k-1} + B_{k-2}\mu^{k-2} + \cdots + B_0 = \left[b_{ij}(\mu) \right]_{i,j=1}^n .$$

Then

$$L^{(l)}(0) = l! \cdot B_l, \ l = 0, 1, \ldots, k-1 \tag{18}$$

and because $\dim(\ker L(0)) = 1$ the polynomial $L(\mu)$ has only one elementary divisor at $\mu = 0$. Then by [8, Corollary 1.14, p.35] it follows that $\mu = 0$ is a root of degree k for the scalar polynomial $d(\mu) = \det L(\mu)$ if and only if the matrix polynomial $L(\mu)$ has a Jordan chain $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ at $\mu = 0$. That is, by definition, if and only if the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$, $\mathbf{a}_1 \neq 0$, are such that

$$\sum_{j=0}^{l-1} \frac{1}{j!} L^{(j)}(0)\, \mathbf{a}_{l-j} = 0, \ \text{for } l = 1, 2, \ldots, k, \tag{19}$$

or, if we use (18), if and only if

$$\sum_{j=0}^{l-1} B_j \mathbf{a}_{l-j} = 0, \text{ for } l = 1, 2, \ldots, k.$$

The polynomial $d(\mu)$ has a root of degree (at least) k at $\mu = 0$ if and only if $d(0) = d'(0) = \ldots = d^{(k-1)}(0) = 0$. Finally the relations

$$d^{(l)}(0) = \sum_{\mathbf{j}\in\Psi_l} \frac{l!}{j_1!\, j_2!\, \cdots\, j_n!} \begin{vmatrix} b_{11}^{(j_1)}(0) & b_{12}^{(j_1)}(0) & \cdots & b_{1n}^{(j_1)}(0) \\ b_{21}^{(j_2)}(0) & b_{22}^{(j_2)}(0) & \cdots & b_{2n}^{(j_2)}(0) \\ \vdots & \vdots & & \vdots \\ b_{n1}^{(j_n)}(0) & b_{n2}^{(j_n)}(0) & \cdots & b_{nn}^{(j_n)}(0) \end{vmatrix} =$$

$$= l! \sum_{\mathbf{j}\in\Psi_l} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_2} & b_{22}^{j_2} & \cdots & b_{2n}^{j_2} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_n} & b_{n2}^{j_n} & \cdots & b_{nn}^{j_n} \end{vmatrix}$$

hold for $l = 0, 1, \ldots, k-1$ and the result follows. □

7 A basis for the root subspace of nonderogatory eigenvalues

Definition. Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is nonderogatory. Then we call the subspace

$$\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i)^N$$

the *root subspace* of $\mathbf{W}$ at $\boldsymbol{\lambda}$ and we write

$$\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N = \bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i)^N .$$

This definition of root subspace coincides with Atkinson's [2, (6.9.4)]. We recall that $N = \dim H$.

In this section we assume that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is nonderogatory. Our main objective is to construct a basis for the root subspace corresponding to $\boldsymbol{\lambda}$ in terms of $\mathbf{W}$. We will construct this basis by an inductive procedure. Theorem 5 gives the initial step in this procedure. It explains how to find a basis for the second root subspace. Now we proceed proving two lemmas yielding the general step of the inductive procedure.

Lemma 13 *Suppose that there exist vectors $x_{i0}, x_{i1}, \ldots, x_{i,p} \in H_i$ $(x_{i0} \neq 0)$ and n-tuples of complex numbers $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ $(\mathbf{a}_1 \neq 0)$ such that*

$$\sum_{j=1}^{k} U_i(\mathbf{a}_j)\, x_{i,k-j} = W_i(\boldsymbol{\lambda})\, x_{ik}, \ \ k = 0, 1, \ldots, p. \tag{20}$$

Then the vectors

$$z_k = \sum_{\mathbf{j} \in \Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n} \tag{21}$$

are such that

$$(\lambda_i I - \Gamma_i)\, z_k = \sum_{j=0}^{k-1} a_{k-j,i} z_j. \tag{22}$$

(Here $k = 0, 1, \ldots, p$, $i = 1, 2, \ldots, n$ and $\mathbf{a}_k = [a_{k1}, a_{k2}, \ldots, a_{kn}]^T$. We also assume that a sum with no terms is 0.) Furthermore, vectors $z_0, z_1, \ldots, z_p$ are linearly independent and they span the subspace

$$\bigcap_{i=1}^{n} \ker\,(\lambda_i I - \Gamma_i)^{p+1}. \tag{23}$$

Proof. Relations (9) and (20) imply that

$$(\lambda_i \Delta_0 - \Delta_i)\, z_k = \begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & W_1(\boldsymbol{\lambda})^\dagger & V_{1,i+1}^\dagger & \cdots & V_{1n}^\dagger \\ V_{21}^\dagger & \cdots & V_{2,i-1}^\dagger & W_2(\boldsymbol{\lambda})^\dagger & V_{2,i+1}^\dagger & \cdots & V_{2n}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & \cdots & V_{n,i-1}^\dagger & W_n(\boldsymbol{\lambda})^\dagger & V_{n,i+1}^\dagger & \cdots & V_{nn}^\dagger \end{vmatrix} z_k =$$

$$= \sum_{\mathbf{j} \in \Psi_k} \begin{vmatrix} V_{11} x_{1j_1} & \cdots & V_{1,i-1} x_{1j_1} & \sum_{l_1=0}^{j_1-1} U_1(\mathbf{a}_{j_1-l_1})\, x_{1l_1} & V_{1,i+1} x_{1j_1} & \cdots & V_{1n} x_{1j_1} \\ V_{21} x_{2j_2} & \cdots & V_{2,i-1} x_{2j_2} & \sum_{l_2=0}^{j_2-1} U_2(\mathbf{a}_{j_2-l_2})\, x_{2l_2} & V_{2,i+1} x_{2j_2} & \cdots & V_{2n} x_{2j_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} x_{nj_n} & \cdots & V_{n,i-1} x_{nj_n} & \sum_{l_n=0}^{j_n-1} U_n(\mathbf{a}_{j_n-l_n})\, x_{nl_n} & V_{n,i+1} x_{nj_n} & \cdots & V_{nn} x_{nj_n} \end{vmatrix}. \tag{24}$$

By Lemma 1 the above sum is equal to

$$\sum_{\mathbf{j} \in \Psi_k} \sum_{q,r=1}^{n} \sum_{l_q=0}^{j_q-1} a_{j_q-l_q,r} \begin{vmatrix} \cdots & V_{1,i-1} x_{1j_1} & 0 & V_{1,i+1} x_{1j_1} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{q-1,i-1} x_{q-1,j_{q-1}} & 0 & V_{q-1,i+1} x_{q-1,j_{q-1}} & \cdots \\ \cdots & V_{q,i-1} x_{qj_q} & V_{qr} x_{ql_q} & V_{q,i+1} x_{qj_q} & \cdots \\ \cdots & V_{q+1,i-1} x_{q+1,j_{q+1}} & 0 & V_{q+1,i+1} x_{q+1,j_{q+1}} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{n,i-1} x_{nj_n} & 0 & V_{n,i+1} x_{nj_n} & \cdots \end{vmatrix}. \tag{25}$$

In the displayed determinant (25) the first $i-1$ and the last $n-i-1$ columns are the same as in the determinant displayed in (24). The vectors $V_{qs}x_{qj_q}$, $s=1,\ldots,i-1,i+1,\ldots,n$, in (25) can be substituted for $V_{qs}x_{ql_q}$ without changing the determinant. The sum (25) is then equal to

$$\sum_{\mathbf{j}\in\Psi_k}\sum_{q,r=1}^{n}\sum_{l_q=0}^{j_q-1}a_{j_q-l_q,r}\begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & 0 & V_{1,i+1}^\dagger & \cdots & V_{1n}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{q-1,1}^\dagger & \cdots & V_{q-1,i-1}^\dagger & 0 & V_{q-1,i+1}^\dagger & \cdots & V_{q-1,n}^\dagger \\ V_{q1}^\dagger & \cdots & V_{q,i-1}^\dagger & V_{qr}^\dagger & V_{q,i+1}^\dagger & \cdots & V_{qn}^\dagger \\ V_{q+1,1}^\dagger & \cdots & V_{q+1,i-1}^\dagger & 0 & V_{q+1,i+1}^\dagger & \cdots & V_{q+1,n}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & \cdots & V_{n,i-1}^\dagger & 0 & V_{n,i+1}^\dagger & \cdots & V_{nn}^\dagger \end{vmatrix}.$$

$$\cdot x_{1j_1}\otimes\cdots\otimes x_{q-1,j_{q-1}}\otimes x_{ql_q}\otimes x_{q+1,j_{q+1}}\otimes\cdots\otimes x_{nj_n}. \tag{26}$$

For every multiindex $\mathbf{j}\in\Psi_l$, where $l<k$, the vector $x_{1j_1}\otimes x_{2j_2}\otimes\cdots\otimes x_{nj_n}$ appears exactly n^2 times in the summation (26), once for every $q,r=1,2,\ldots,n$, and we observe that the scalar coefficients it is multiplied by are $a_{k-l,r}$. Then we sum in (26) over q. Because a determinant with two equal columns is zero it follows that the sum (26) equals

$$\sum_{l=0}^{k-1}\sum_{\mathbf{j}\in\Psi_l}a_{k-l,i}\begin{vmatrix} V_{11}x_{1j_1} & V_{12}x_{1j_1} & \cdots & V_{1n}x_{1j_1} \\ V_{21}x_{2j_2} & V_{22}x_{2j_2} & \cdots & V_{2n}x_{2j_2} \\ \vdots & \vdots & & \vdots \\ V_{n1}x_{nj_n} & V_{n2}x_{nj_n} & \cdots & V_{nn}x_{nj_n} \end{vmatrix}=\sum_{l=0}^{k-1}a_{k-l,i}\Delta_0 z_l.$$

Thus relation (22) follows.

Suppose that $a_{1h}\neq 0$. (It exists because $\mathbf{a}_1\neq\mathbf{0}$.) Assume that $\sum_{k=0}^{p}\alpha_k z_k=0$. Then

$$0=(\lambda_h I-\Gamma_h)^{p-1}\sum_{k=0}^{p}\alpha_k z_k=a_{1h}^{p-1}\alpha_p z_0$$

and therefore $\alpha_p=0$. Next

$$0=(\lambda_h I-\Gamma_h)^{p-2}\sum_{k=0}^{p-1}\alpha_k z_k=a_{1h}^{p-2}\alpha_{p-1}z_0$$

and hence $\alpha_{p-1}=0$. Proceeding in the above manner we show that $\alpha_k=0$ for all k and thus vectors (21) are linearly independent. We denote their linear span by $\mathcal{N}$. In the basis $\{z_0,z_1,\ldots,z_p\}$ we have

$$\Gamma_i\,|_{\mathcal{N}}=\begin{bmatrix} \lambda_i & -a_{i1} & -a_{i2} & \cdots & -a_{ip} \\ 0 & \lambda_i & -a_{i1} & \ddots & -a_{i,p-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -a_{i1} \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

and because $a_{1h} \neq 0$ it follows by Theorem 8 that $\mathcal{N} = \bigcap_{i=1}^{n} \ker (\lambda_i I - \Gamma_i)^{p+1}$. $\square$

Lemma 14 *Suppose that λ_h is nonderogatory for Γ_h with the ascent at least $p+1$. Then there exist vectors $x_{i0}, x_{i1}, \ldots, x_{i,p} \in H_i$ $(x_{i0} \neq 0)$, $i = 1, 2, \ldots, n$, and n-tuples of complex numbers $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$ $(\mathbf{a}_1 \neq 0)$ such that*

$$\sum_{j=1}^{k} U_i(\mathbf{a}_j) x_{i,k-j} = W_i(\boldsymbol{\lambda}) x_{ik}, \ k = 0, 1, \ldots, p.$$

Proof. This result was proven in Theorem 5 for $p = 1$. Assume now that we have already found $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{k-1}$ and $x_{i1}, x_{i2}, \ldots, x_{i,k-1}$; $i = 1, 2, \ldots, n$ where $k \leq p$ such that

$$\sum_{j=0}^{l-1} U_i(\mathbf{a}_{l-j}) x_{ij} = W_i(\boldsymbol{\lambda}) x_{il}; \qquad l = 1, 2, \ldots, k-1. \tag{27}$$

It remains to show that we can also find $\mathbf{a}_k$ and $x_{i,k}$, $i = 1, 2, \ldots, n$ such that

$$\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j}) x_{ij} = W_i(\boldsymbol{\lambda}) x_{ik}. \tag{28}$$

To do so we first build vectors $z_l = \sum_{\mathbf{j} \in \Psi_l} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$, $l = 0, 1, \ldots, k-1$. By Lemma 13 it follows that $(\lambda_i I - \Gamma_i) z_l = \sum_{j=0}^{l-1} a_{l-j,i} z_j$, $l = 0, 1, \ldots, k-1$. Then the vectors $u_r = (\lambda_h I - \Gamma_h)^{k-1-r} z_{k-1}$, $r = 0, 1, \ldots, k-1$ form a Jordan chain for Γ_h of length k $(< p+1)$. Because λ_h is nonderogatory every corresponding Jordan chain can be extended to a maximal one (cf.[9, Theorem 2.9.2(b), p. 85]). Suppose that $y_{i0} \in H_i$ are nonzero vectors such that $W_i(\boldsymbol{\lambda})^* y_{i0} = 0$. Then vector $w_0 = y_{10} \otimes y_{20} \otimes \cdots \otimes y_{n0}$ is such that $(\lambda_h \Delta_0 - \Delta_h)^* w_0 = 0$ and Lemma 11 implies that $w_0^* \Delta_0 u_l = 0$ for $l = 0, 1, \ldots, k-1$. Because the sets $\{u_l;\ l = 0, 1, \ldots, k-1\}$ and $\{z_l;\ l = 0, 1, \ldots, k-1\}$ both span $\ker (\lambda_h I - \Gamma_h)^k$, it also follows that

$$w_0^* \Delta_0 z_l = 0, \ l = 0, 1, \ldots, k-1. \tag{29}$$

Next we form the $n \times n$ matrices $B_l = \left[b_{ij}^l\right]_{i,j=1}^n$, $l = 0, 1, \ldots, k-1$ where $b_{ij}^l = y_{i0}^* V_{ij} x_{il}$. Multiplying relations (27) by y_{i0}^* on the left-hand side yields

$$\sum_{j=0}^{l-1} y_{i0}^* U_i(\mathbf{a}_{l-j}) x_{ij} = 0, \ l = 1, 2, \ldots, k-1.$$

This is equivalent to $\sum_{j=0}^{l-1} B_j \mathbf{a}_{l-j} = 0$ for $l = 1, 2, \ldots, k-1$. Relations (29) are equivalent to

$$\sum_{\mathbf{j} \in \Psi_l} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_1} & b_{22}^{j_1} & \cdots & b_{2n}^{j_1} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_1} & b_{n2}^{j_1} & \cdots & b_{nn}^{j_1} \end{vmatrix} = 0, \qquad \text{for } l = 0, 1, \ldots, k-1. \tag{30}$$

Since $\boldsymbol{\lambda}$ is nonderogatory and rank $B_0 = n-1$, Lemma 12 implies that there exists an n-tuple $\mathbf{a}_k$ such that $\sum_{j=0}^{k-1} B_j \mathbf{a}_{k-j} = 0$ or, equivalently, such that

$$\sum_{j=0}^{k-1} y_{i0}^* U_i(\mathbf{a}_{k-j}) x_{ij} = 0$$

for $i = 1, 2, \ldots, n$. Then the vector $\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j}) x_{ij}$ is orthogonal to the kernel of $W_i(\boldsymbol{\lambda})^*$ and so there exists a vector $x_{ik} \in H_i$ such that (28) holds. We continue by induction until $k = p$. □

Now we summarize the preceding discussion into our main result, the completeness theorem for nonderogatory eigenvalues of a multiparameter system :

Theorem 15 *Suppose* $\boldsymbol{\lambda} \in \mathbb{C}^n$ *is a nonderogatory eigenvalue for a multiparameter system* $\mathbf{W}$ *and suppose that* $p+1$ *is the corresponding ascent. Then there exist* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p \in \mathbb{C}^n, \mathbf{a}_1 \neq 0,$ *and* $x_{i0}, x_{i1}, \ldots, x_{ip} \in H_i$, $(x_{i0} \neq 0)$ $i = 1, 2, \ldots, n$, *such that*

$$\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j}) x_{ij} = W_i(\boldsymbol{\lambda}) x_{ik} \qquad \text{for } k = 1, 2, \ldots, p; \; i = 1, 2, \ldots, n. \tag{31}$$

Moreover, the vectors

$$z_k = \sum_{\mathbf{j} \in \Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}, \; k = 0, 1, \ldots, p,$$

where Ψ_k *is defined in* (16), *are such that*

$$(\lambda_i I - \Gamma_i) z_k = \sum_{j=0}^{k-1} a_{k-j,i} z_j$$

and they form a basis for the root subspace $\bigcap_{i=1}^n \ker (\lambda_i I - \Gamma_i)^N$.

Proof. Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is nonderogatory. By Theorem 8 and Corollary 10 at least one of the eigenvalues, say λ_h, is nonderogatory for Γ_h. Lemma 14 then implies the existence of n-tuples $\mathbf{a}_k$ and vectors x_{ik} such that (31) holds. Finally, the result follows by Lemma 13. □

Let us mention that conditions (31) can be considered as generalized Jordan chain conditions.

8 An algorithm to construct a basis for a nonderogatory eigenvalue of a multiparameter system

In the proofs of Lemmas 13 and 14 we can find a procedure to construct a basis for the root subspace of a nonderogatory eigenvalue :

Algorithm 16 *Step I. For* $i = 1, 2, \ldots, n$ *find* $x_{i0} \neq 0$ *and* $y_{i0} \neq 0$ *such that*

$$W_i(\boldsymbol{\lambda})\, x_{i0} = 0 \text{ and } W_i(\boldsymbol{\lambda})^* \, y_{i0} = 0.$$

Form $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$*, a matrix* $B_0 = [y_{i0}^* V_{ij} x_{i0}]_{i,j=1}^n$ *and set* $k = 1$*.*

Step II. Find a matrix polynomial

$$L_k(\mu) = I\mu^k + B_{k-1}\mu^{k-1} + \cdots + B_0 \tag{32}$$

and its determinant $d_k(\mu) = \det L_k(\mu)$*. If*

$$d_k^{(k-1)}(0) = 0 \tag{33}$$

then go to Step III, otherwise quit the algorithm.

Step III. Find $\mathbf{a}_k \in \mathbb{C}^n, \mathbf{a}_1 \neq 0$*, such that*

$$\sum_{l=0}^{k-1} B_l \mathbf{a}_{k-l} = 0. \tag{34}$$

For $i = 1, 2, \ldots, n$ *find vectors* $x_{ik} \in H_i$ *such that*

$$\sum_{l=0}^{k-1} U_i(\mathbf{a}_{k-l})\, x_{il} = W_i(\boldsymbol{\lambda})\, x_{ik}.$$

Form $z_k = \sum_{\mathbf{j} \in \Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$ *and a matrix* $B_k = [y_{i0}^* V_{ij} x_{ik}]_{i,j=1}^n$*. Set* $k = k + 1$ *and repeat Step II.*

It follows from Theorem 15 that the vectors $z_0, z_1, \ldots, z_k$ obtained in the above algorithm form a basis for the root subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N$, and they satisfy the relations

$$(\lambda_i I - \Gamma_i)\, z_l = \sum_{j=0}^{l-1} \mathbf{a}_{l-j,i} z_j \text{ for } l = 0, 1, \ldots, k.$$

Because relations (29) and (30) are equivalent it follows as an immediate consequence of Lemmas 11 and 12 that :

Corollary 17 *The ascent of* $\boldsymbol{\Gamma}$ *at the eigenvalue* $\boldsymbol{\lambda}$ *is equal to the smallest integer* k *such that the condition* (33) *does not hold.*

Let us now demonstrate Algorithm 16 with an example :

Example 18 We consider a multiparameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}.$$

This system is nonsingular because Δ_0 is invertible. Then

$$\sigma(\mathbf{W}) = \left\{ (1,-1), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \right\}.$$

We consider the eigenvalue $\boldsymbol{\lambda}_0 = (1,-1)$. Then we have

$$W_1(\boldsymbol{\lambda}_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } W_2(\boldsymbol{\lambda}_0) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

We observe that $\dim\ker W_1(\boldsymbol{\lambda}_0) = \dim\ker W_2(\boldsymbol{\lambda}_0) = 1$. To complete *Step I* of Algorithm 16 we choose

$$x_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; x_{20} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; y_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } y_{20} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$$

The matrix $B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ has rank 1 and therefore $\boldsymbol{\lambda}_0$ is a nonderogatory eigenvalue. We have $z_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then we go to *Step III*. We choose $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$$x_{11} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } x_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that $V_{11}x_{10} = W_1(\boldsymbol{\lambda}_0)x_{11}$ and $V_{21}x_{20} = W_2(\boldsymbol{\lambda}_0)x_{21}$. Then it follows that $z_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $L_2(\mu) = \begin{bmatrix} \mu^2 & 0 \\ 0 & \mu^2+1 \end{bmatrix}$. We find that $d_2(\mu) = \mu^2(\mu^2+1)$. Because $d_2'(0) = 0$ we repeat *Step III*. Now we choose $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and vectors

$$x_{12} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ and } x_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that $V_{11}x_{11} = W_1(\boldsymbol{\lambda}_0)x_{12}$ and $V_{21}x_{21} = W_2(\boldsymbol{\lambda}_0)x_{22}$. We have $z_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$. The matrix polynomial $L_3(\mu) = \begin{bmatrix} \mu^3+2\mu^2 & \mu^2 \\ 0 & \mu^3+1 \end{bmatrix}$ has determinant

$d_3(\mu) = \mu^2(\mu+2)(\mu^3+1)$. Because $d_3''(0) \neq 0$ we quit the algorithm. The root subspace at the eigenvalue $\boldsymbol{\lambda}_0$ is three-dimensional and it has a basis $\{z_0, z_1, z_2\}$. □

9 Weakly-elliptic multiparameter systems

Definition. A multiparameter system $\mathbf{W}$ is called *self-adjoint* if all the transformations $V_{ij},\ i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, n$ are self-adjoint, i.e., $V_{ij} = V_{ij}^*$.

Now we denote a cofactor of $V_{ij}^\dagger$ in the operator determinant (2) by Δ_{0ij}. It is a linear transformation on H. We also assume that H is equipped with a scalar product that is induced by (4).

Definition. A self-adjoint multiparameter system is called *weakly-elliptic* if there exists a cofactor Δ_{0ij} of Δ_0 that is a positive definite operator on H.

A special case of weakly-elliptic multiparameter systems is an *elliptic* multiparameter system, i.e., if $\Delta_{0ij},\ i = 1, 2, \ldots, n$ are positive definite operators on H for some j, cf. [3, 4, 5]. Binding [3] proved a completeness theorem for real eigenvalues of elliptic multiparameter systems while with Browne in [4] he considered the multiplicities of general eigenvalues for two-parameter systems with two definite cofactors Δ_{0ii}. In the next result we consider simple real eigenvalues of weakly-elliptic multiparameter systems.

Theorem 19 *Assume that $\boldsymbol{\lambda}$ is a simple real eigenvalue for a weakly-elliptic multiparameter system $\mathbf{W}$. Then $\boldsymbol{\lambda}$ is nonderogatory.*

Proof. Suppose that $x_{i0} \in \ker W_i(\boldsymbol{\lambda})$ are nonzero vectors. Then we only need to show that $\operatorname{rank} B_0 \geq n-1$, where

$$B_0 = \begin{bmatrix} x_{10}^* V_{11} x_{10} & x_{10}^* V_{12} x_{10} & \cdots & x_{10}^* V_{1n} x_{10} \\ x_{20}^* V_{21} x_{20} & x_{20}^* V_{22} x_{20} & \cdots & x_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ x_{n0}^* V_{n1} x_{n0} & x_{n0}^* V_{n2} x_{n0} & \cdots & x_{n0}^* V_{nn} x_{n0} \end{bmatrix}.$$

The result then follows from Corollary 10. By definition of a weakly-elliptic multiparameter system it follows that $z_0^* \Delta_{0ij} z_0 \neq 0$ for some i and j. Since

$$z_0^* \Delta_{0ij} z_0 = x_{i0}^* x_{i0} \cdot \left(\widehat{z_0}^i\right)^* \Delta_{0ij} \widehat{z_0}^i,$$

where $\widehat{z_0}^i = x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}$, it follows that the cofactor of $x_{i0}^* V_{ij} x_{i0}$ in the matrix B_0 is nonzero and so rank $B_0 \geq n-1$. □

From the above theorem it follows that for simple real eigenvalues of a weakly-elliptic multiparameter system, Theorem 15 applies and we can find a basis for the corresponding root subspace by Algorithm 16.

A special case of Theorem 15 for elliptic multiparameter systems was proved by Binding [3, Theorem 3.1] in a more general setting with a different method. We do not, however, generalize his main result [3, Theorem 3.4]. We remark that the structure of vectors z_k of Theorem 15 is the same as the structure of vectors y_i of [3, Theorem 3.1]. The coefficients γ_n^{l-i} of [3, (3.6)] are a counter-part of the coefficients that form our n-tuples $\mathbf{a}_k$. While the coefficients γ_n^{l-i} are obtained via a differentiation process our n-tuples $\mathbf{a}_k$ form a Jordan chain of matrix polynomials (32) and are obtained by solving (34) for $\mathbf{a}_k$, $k = 1, 2, \ldots, p$. For comparison we consider the eigenvalue $\boldsymbol{\lambda} = \mathbf{0}$ of [3, Example 4.4] :

Example 20 We have a two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \lambda_1 + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \lambda_1 + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \lambda_2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalue $\mathbf{0}$ is simple and because the above two-parameter system is elliptic it is nonderogatory. As in [3] we choose $x_{10} = x_{20} = y_{10} = y_{20} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $B_0 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. Note that the vector $\begin{bmatrix} \gamma_1^1 \\ \gamma_2^1 \end{bmatrix} = \frac{1}{\delta_{022}(x^{\otimes})} \begin{bmatrix} \delta_{021}(x^{\otimes}) \\ \delta_{022}(x^{\otimes}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, in Binding's notation, is an element of $\ker B_0$. We write $\mathbf{a}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$. Then vectors $x_{11} = x_{21} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ are such that $U_i(\mathbf{a}_1) x_{i0} = W_i(\boldsymbol{\lambda}) x_{i1}$, $i = 1, 2$, and $B_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$. Because $(\det L_2(\mu))' \,|_{\mu=0} = \frac{3}{2} \neq 0$, it follows that vectors

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } z_1 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

form a basis for the root subspace of $\mathbf{W}$ at $\boldsymbol{\lambda} = \mathbf{0}$. □

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References

[1] F.V. Atkinson. Multiparameter Spectral Theory. *Bul. Amer. Math. Soc.*, 74:1–27, 1968.

[2] F.V. Atkinson. *Multiparameter Eigenvalue Problems.* Academic Press, 1972.

[3] P.A. Binding. Multiparameter Root Vectors. *Proc. Edin. Math. Soc.*, 32:19–29, 1989.

[4] P.A. Binding and P.J. Browne. Two Parameter Eigenvalue Problems for Matrices. *Lin. Alg. Appl.*, 113:139–157, 1989.

[5] P.A. Binding and K. Seddighi. Elliptic Multiparameter Eigenvalue Problems. *Proc. Edin. Math. Soc.*, 30:215–228, 1987.

[6] M. Faierman. *Two-parameter Eigenvalue Problems in Ordinary Differential Equations*, volume 205 of *Pitman Research Notes in Mathematics.* Longman Scientific and Technical, U.K., 1991.

[7] G.A. Gadzhiev. *Introduction to Multiparameter Spectral Theory* (in Russian). Azerbaijan State University, Baku, 1987.

[8] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials.* Academic Press, 1982.

[9] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* Wiley-Interscience, 1986.

[10] G.A. Isaev. On Root Elements of Multiparameter Spectral Problems. *Soviet. Math. Dokl.*, 21:127–130, 1980.

[11] H.(G.A.) Isaev. *Lectures on Multiparameter Spectral Theory.* Dept. of Math. and Stats., University of Calgary, 1985.

[12] T. Košir. *Commuting Matrices and Multiparameter Eigenvalue Problems.* PhD thesis, Dept. of Math. and Stats., University of Calgary, 1993.

[13] T. Košir. On the Structure of Commutative Matrices. *Lin. Alg. Appl.*, 187:163–182, 1993.

[14] H. Volkmer. *Multiparameter Eigenvalue Problems and Expansion Theorems*, volume 1356 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, New York, 1988.