PRODUCTS OF COMMUTING NILPOTENT OPERATORS*

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Abstract. We characterize the matrices that are products of two (or more) commuting square-zero matrices and matrices that are products of two commuting nilpotent matrices. We also give a characterization of operators on an infinite dimensional Hilbert space that are products of two (or more) commuting square-zero operators and operators on an infinite-dimensional vector space that are products of two commuting nilpotent operators.

1. Introduction. Is every complex singular square matrix a product of two nilpotent matrices? Laffey [5] and Sourour [8] proved that the answer is positive: any complex singular square matrix A (which is not 2×2 nilpotent with rank 1) is a product of two nilpotent matrices with ranks both equal to the rank of A. Earlier Wu [9] studied the problem. (Note that [9, Lem. 3] holds but the decomposition given in its proof on [9, p. 229] is not correct since the latter matrix given for the odd case is not always nilpotent.) Novak [6] characterized all singular matrices in $\mathcal{M}_n(\mathbb{F})$, where $\mathbb{F}$ is a field, which are a product of two square-zero matrices. Related problem of existence of k-th root of a nilpotent matrix was studied by Psarrakos in [7].

Similar results were proved for the set $\mathcal{B}(\mathcal{H})$ of all bounded (linear) operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. Fong and Sourour [3] proved that every compact operator is a product of two quasinilpotent operators and that a normal operator is a product of two quasinilpotent operators if and only if 0 is in its essential spectrum. Drnovšek, Müller, and Novak [2] proved that an operator is a product of two quasinilpotent operators if and only if it is not semi-Fredholm. Novak [6] characterized operators that are products of two and of three square-zero operators.

Here we consider similar questions for products of commuting square-zero or commuting nilpotent operators on a finite dimensional vector space or on a infinite-dimensional Hilbert or vector space. The commutativity condition considerably restricts the set of operators that are such products. Namely, if $A = BC$ and B, C are commuting nilpotent operators then A is nilpotent as well and it commutes with both B and C. If in addition B and C are square-zero then so is A.

In the paper we characterize the following sets of matrices and operators:

- Matrices that are products of k commuting square-zero matrices for each $k \geq 2$.
- Matrices that are products of two commuting nilpotent matrices.
- Operators on a Hilbert space that are products of k commuting square-zero operators for each $k \geq 2$.

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- Operators on an infinite-dimensional vector space that are products of two commuting nilpotent operators.

2. When is a matrix a product of commuting square-zero matrices?. First we consider the following question:

QUESTION 1. *Which matrices $A \in \mathcal{M}_n(\mathbb{F})$ can be written as a product $A = BC$, where $B^2 = C^2 = 0$ and $BC = CB$?*

Observe that if A, B and C are as above then B and C commute with A.

EXAMPLE 2.1. *It can be easily seen that*

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

but E_{13} cannot be written as a product of two commuting square-zero matrices. Therefore the set of matrices that can be written as a product of two commuting square-zero matrices is not the same as the set of matrices that are products of two square-zero matrices.

Next, we have that

$$E_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = 0.$$

Thus E_{14} is a product of two commuting square-zero matrices. □

We denote by $J_{\underline{\mu}} = J_{(\mu_1,\mu_2,\ldots,\mu_t)} = J_{\mu_1} \oplus J_{\mu_2} \oplus \ldots \oplus J_{\mu_t}$ the upper triangular nilpotent matrix in its Jordan canonical form with blocks of order $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t > 0$. If A is similar to $J_{\underline{\mu}}$ then we call $\underline{\mu}$ the *partition* corresponding to A. We also say that $\underline{\mu}$ is the Jordan canonical form of A. For a finite sequence of natural numbers $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ we denote by $\operatorname{ord}(\underline{\lambda}) = \underline{\mu}$ the ordered sequence $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t$.

Let $\iota(A)$ denote the index of nilpotency of matrix A. For a nilpotent matrix A, define a sequence

$$\mathcal{J}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_{\iota(A)}) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where α_i is the number of Jordan blocks of the size i and $\alpha_j = 0$ for $j > \iota(A)$. Note that $\sum_{j=1}^{n} j\alpha_j = n$.

If C commutes with $J_{\underline{\mu}}$ it is of the form $C = [C_{ij}]$, where $C_{ij} \in \mathcal{M}_{\mu_i \times \mu_j}$ and C_{ij} are all upper triangular Toeplitz matrices (see e. g. [4, p. 297]), i.e. for $1 \le i \le j \le t$ we have

$$C_{ij} = \begin{bmatrix} 0 & \ldots & 0 & c_{ij}^0 & c_{ij}^1 & \ldots & c_{ij}^{\mu_i - 1} \\ \vdots & & \ddots & 0 & c_{ij}^0 & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \ddots & c_{ij}^1 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & c_{ij}^0 \end{bmatrix} \quad \text{and} \quad C_{ji} = \begin{bmatrix} c_{ji}^0 & c_{ji}^1 & \ldots & c_{ji}^{\mu_i - 1} \\ 0 & c_{ji}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{ji}^1 \\ \vdots & & 0 & c_{ji}^0 \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}. \tag{2.1}$$

If $\mu_i = \mu_j$ then we omit the rows or columns of zeros in C_{ji} or C_{ij} above.

Proposition 2.2. *A matrix A is a product of two commuting square-zero matrices if and only if it has a Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{4}$, i.e. if and only if $\mathcal{J}(A) = (n - 2x, x)$ for some $x \le \frac{n}{4}$.*

Proof. Since $A^2 = B^2C^2 = 0$, it follows that also A is a square-zero matrix. Since $B^2 = 0$, the Jordan canonical form of matrix B is equal to $\underline{\mu} = (2^a, 1^{n-2a})$ for some $0 \le a \le \frac{n}{2}$. Suppose that $B = J_{\underline{\mu}}$ is in its Jordan canonical form. Since C commutes with B it is of the form $C = [C_{ij}]$, where C_{ij} are given in (2.1). Following Basili [1, p. 60, Lemma 2.3], the matrix C is similar to

$$\begin{bmatrix} U_1 & X & Y \\ 0 & U_1 & 0 \\ 0 & W & U_2 \end{bmatrix},$$

where $U_1, X \in \mathcal{M}_{a \times a}$, $Y \in \mathcal{M}_{a \times (n-2a)}$, $W \in \mathcal{M}_{(n-2a) \times a}$, $U_2 \in \mathcal{M}_{(n-2a) \times (n-2a)}$ and U_1 and U_2 are strictly upper triangular matrices. Note that B is transformed by the same similarity to

$$\tilde{B} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Take an invertible matrix P_1 such that $P_1 U_1 P_1^{-1} = J_{\underline{\lambda}}$ and denote

$$\tilde{C} = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} U_1 & X & Y \\ 0 & U_1 & 0 \\ 0 & W & U_2 \end{bmatrix} \begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_1^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} J_{\underline{\lambda}} & X' & Y' \\ 0 & J_{\underline{\lambda}} & 0 \\ 0 & W' & U_2 \end{bmatrix}.$$

Note that $\tilde{B}$ does not change under the above similarity. Since $C^2 = 0$, also $\tilde{C}^2 = 0$ and thus $J_{\underline{\lambda}}^2 = 0$. Therefore, $\underline{\lambda} = (2^x, 1^{a-2x})$, where $0 \le x \le \frac{a}{2} \le \frac{n}{4}$. We see that

$$\tilde{B}\tilde{C} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_{\underline{\lambda}} & X' & Y' \\ 0 & J_{\underline{\lambda}} & 0 \\ 0 & W' & U_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & J_{\underline{\lambda}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, it easily follows that $\operatorname{rk}(A) = \operatorname{rk}(\tilde{B}\tilde{C}) = x$. Since $A^2 = 0$, we see that A must have Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{4}$.

Now, take a nilpotent matrix A with its Jordan canonical form $(2^x, 1^{n-2x})$, where $x \le \frac{n}{4}$. Then there exists an invertible matrix Q such that

$$QAQ^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{x} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{n-4x}.$$

In Example 2.1 we observed that the matrix E_{14} is a product of two commuting square-zero matrices. Then it follows that QAQ^{-1} and A are also products of two commuting square-zero matrices. We have proved the proposition. □

Theorem 2.3. *A matrix A is a product of k pairwise commuting square-zero matrices if and only if it has a Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{2^k}$, i.e. if and only if $\mathcal{J}(A) = (n-2x, x)$ for some $x \le \frac{n}{2^k}$.*

Proof. Let A be a matrix with Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{2^k}$. Then it is similar to a matrix

$$A' = \underbrace{E_{1\,2^k} \oplus E_{1\,2^k} \oplus \ldots \oplus E_{1\,2^k}}_{x} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{n-2^k x},$$

where $E_{1\,2^k} \in \mathcal{M}_{2^k}(\mathbb{C})$ is a matrix with only nonzero element (equal to 1) in the upper-right corner. To prove that A is a product of k pairwise commuting square-zero matrices it is sufficient to show, that $E_{1\,2^k}$ is a product of k pairwise commuting square-zero matrices.

We define matrices

$$C_i = \begin{bmatrix} 0_{2^{i-1}} & I_{2^{i-1}} \\ 0_{2^{i-1}} & 0_{2^{i-1}} \end{bmatrix} \in \mathcal{M}_{2^i}(\mathbb{C})$$

for every $i = 1, 2, ..., k$ and let

$$B_i = \underbrace{C_i \oplus C_i \oplus \ldots \oplus C_i}_{2^{k-i}} \in \mathcal{M}_{2^k}(\mathbb{C}).$$

It is easy to check that $B_i^2 = 0$ and $B_i B_j = B_j B_i$ for every i, j and that

$$E_{1\,2^k} = B_1 B_2 \ldots B_k.$$

To prove the converse we have to show that every product of k pairwise commuting square-zero matrices has rank at most $\frac{n}{2^k}$. We will show this by induction. The assertion is true for $k = 2$ by the previous proposition. Suppose that every product of k pairwise commuting square-zero matrices has rank at most $\frac{n}{2^k}$ and let

$$A = B_1 B_2 \ldots B_{k+1}$$

where $B_1, B_2, \ldots B_{k+1}$ are pairwise commuting square-zero matrices. Denote by m the rank of B_1. Since $B_1^2 = 0$ we have that $m \leq \frac{n}{2}$. Now the matrix B_1 is similar to a matrix

$$B_1' = \begin{bmatrix} 0_m & I_m & 0 \\ 0_m & 0_m & 0 \\ 0 & 0 & 0_{2n-m} \end{bmatrix}$$

Again following Basili [1, p. 60, Lemma 2.3], we transform the matrices B_i simultaneously by similarity to the matrices

$$B_i' = \begin{bmatrix} X_i & Y_i & Z_i \\ 0 & X_i & 0 \\ 0 & U_i & V_i \end{bmatrix}.$$

Here matrices X_i are square-zero and they pairwise commute. Now

$$A' = B_1' B_2' \ldots B_{k+1}' =$$

$$= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_2 & Y_2 & Z_2 \\ 0 & X_2 & 0 \\ 0 & U_2 & V_2 \end{bmatrix} \ldots \begin{bmatrix} X_{k+1} & Y_{k+1} & Z_{k+1} \\ 0 & X_{k+1} & 0 \\ 0 & U_{k+1} & V_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & X_2 \ldots X_{k+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the matrix $X_2 \ldots X_{k+1}$ is a product of k pairwise commuting square-zero matrices, so it has the rank at most $\frac{m}{2^k}$ and thus the rank of A is at most $\frac{n}{2^{k+1}}$. $\square$

3. When is a matrix a product of two commuting nilpotent matrices?. In this section we study the following question:

QUESTION 2. *Which matrices $A \in \mathcal{M}_n(\mathbb{F})$ can be written as $A = BC = CB$, where B and C are nilpotent matrices?*

Clearly, A must be nilpotent. Thus, not every singular matrix is a product of two commuting nilpotent matrices.

Moreover, suppose that $\mathrm{rk}(A) = n-1$ and $A = BC = CB$ with B and C nilpotent. Then also $\mathrm{rk}(B) = \mathrm{rk}(C) = n-1$ and thus $B = PJ_nP^{-1}$ and $P^{-1}CP = p(J_n)$, where p is a polynomial such that $p(0) = 0$. Then $A = BC = PJ_np(J_n)P^{-1}$ and thus $\mathrm{rk}(A) < n-1$, which is a contradiction. Hence not every nilpotent matrix is a product of two commuting nilpotent matrices (for example J_n is not).

Example 3.1. *Suppose* $A = \begin{bmatrix} J_m & 0 \\ 0 & 0 \end{bmatrix}$, *where* $m \geq 3$. *Can A be written as a product of two commuting nilpotent matrices? Assume that $A = BC$ is such a product. Since B and C commute with A it follows that* $B = \begin{bmatrix} T_B & W_B \\ V_B & U_B \end{bmatrix}$ *and* $C = \begin{bmatrix} T_C & W_C \\ V_C & U_C \end{bmatrix}$, *where $T_B, T_C \in \mathcal{M}_m(\mathbb{F})$ are (strictly) upper triangular Toeplitz matrices, $U_B, U_C \in \mathcal{M}_k(\mathbb{F})$ are nilpotent matrices (see Basili [1]), $W_B, W_C \in \mathcal{M}_{m\times k}(\mathbb{F})$ have the only nonzero entries in the first row and $V_B, V_C \in \mathcal{M}_{k\times m}(\mathbb{F})$ have the only nonzero entries in the last column.*

Since $A = BC$ it follows that $J_m = T_BT_C + W_BV_C$. The product W_BV_C has the only nonzero entry in the first row and the last column, and T_B and T_C are strictly upper-triangular. The assumption that $m \geq 3$ is needed to conclude that $T_BT_C + W_BV_C$ is upper triangular Toeplitz matrix with zero superdiagonal. This contradicts the fact that J_m has nonzero superdiagonal and implies that A is not a product of two commuting nilpotent matrices. □

What is the Jordan canonical form of J_n^t for $t \geq 2$? It is an easy observation that the partition of n corresponding to J_n^t is equal to $(\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $\lambda_1 - \lambda_t \leq 1$. We denote this partition by $r(n,t)$. If $n = kt + r$, where $0 \leq r < t$, then $r(n,t) = ((k+1)^r, k^{t-r})$. Note that $k = \lfloor \frac{n}{t} \rfloor$. It follows that $\mathcal{J}(J_n^t) = (\underbrace{0, \ldots, 0}_{\lfloor \frac{n}{t} \rfloor - 1}, t-r, r)$.

Proposition 3.2. *If a nilpotent matrix A has a Jordan canonical form*

$$\mathrm{ord}(r(n_1,t_1), r(n_2,t_2), \ldots, r(n_m,t_m), 1^k),$$

where $n = k + \sum_{i=1}^m n_i$ and $t_i \geq 2$ for all i, then A can be written as a product of two commuting nilpotent matrices.

Proof. Since the Jordan canonical form of $J_{n_i}^{t_i}$ is $r(n_i,t_i)$, matrix A is similar to $J_{n_1}^{t_1} \oplus J_{n_2}^{t_2} \oplus \ldots \oplus J_{n_m}^{t_m} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k}$, which is obviously equal to the product of two commuting nilpotent matrices

$$\left(J_{n_1} \oplus J_{n_2} \oplus \ldots \oplus J_{n_m} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k} \right)$$

and

$$\left(J_{n_1}^{t_1-1} \oplus J_{n_2}^{t_2-1} \oplus \ldots \oplus J_{n_m}^{t_m-1} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k}\right).$$

Thus also A can be written as a product of two commuting nilpotent matrices. □

In the following we show that the converse is true as well.

THEOREM 3.3. *For a nilpotent matrix A the following are equivalent:*
(a) A can be written as a product of two commuting nilpotent matrices,
(b) A has a Jordan canonical form

$$\operatorname{ord}(r(n_1,t_1), r(n_2,t_2), \ldots, r(n_m,t_m), 1^k),$$

where $n = k + \sum_{i=1}^m n_i$ and $t_i \geq 2$ for all i,
(c) $\mathcal{J}(A)$ does **not** *include a subsequence $(0, \underbrace{1,1,\ldots,1}_{2l-1}, 0)$ for any $l \geq 1$.*

We first prove the following lemma and propositions.

LEMMA 3.4. *If $\mathcal{J}(A) = (\underbrace{0,\ldots,0}_{m-2l+1}, \underbrace{1,\ldots,1}_{2l-1})$, where $l \geq 1$ and $m \geq 2l$, then A is* **not** *a product of two commuting nilpotent matrices.*

Proof. Suppose that $A = BC = CB$ is nilpotent matrix with $\mathcal{J}(A)$ as in the statement of the lemma. Let us denote $s = 2l-1$ and let us assume that $A = J_{(m,m-1,\ldots,m-s+1)}$. Then

$$B = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1s} \\ B_{21} & B_{22} & \ldots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \ldots & B_{ss} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1s} \\ C_{21} & C_{22} & \ldots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{s1} & C_{s2} & \ldots & C_{ss} \end{bmatrix},$$

where all B_{ij} and C_{ij} are upper triangular Toeplitz and we use the notation introduced in (2.1).

Since $J_m = B_{11}C_{11} + B_{12}C_{21} + \ldots + B_{1s}C_{s1} = C_{11}B_{11} + C_{12}B_{21} + \ldots + C_{1s}B_{s1}$ and the only possible summands with nonzero superdiagonal are $B_{12}C_{21}$ and $C_{12}B_{21}$, it follows that $b_{12}^0 c_{21}^0 = c_{12}^0 b_{21}^0 = 1$. Since $J_{m-1} = B_{21}C_{12} + B_{22}C_{22} + \ldots + B_{2s}C_{s2} = C_{21}B_{12} + C_{22}B_{22} + \ldots + C_{2s}B_{s2}$ and the only possible summands with nonzero superdiagonals are $B_{21}C_{12} + B_{23}C_{31}$ and $C_{21}B_{12} + C_{23}B_{31}$, it follows that $b_{21}^0 c_{12}^0 + b_{23}^0 c_{32}^0 = c_{21}^0 b_{12}^0 + c_{23}^0 b_{32}^0 = 1$ and therefore $b_{23}^0 c_{32}^0 = c_{23}^0 b_{32}^0 = 0$.

Similarly, we show by induction, that $b_{i,i+1}^0 c_{i+1,i}^0 = c_{i,i+1}^0 b_{i+1,i}^0 = 0$ for all even i and $b_{i,i+1}^0 c_{i+1,i}^0 = c_{i,i+1}^0 b_{i+1,i}^0 = 1$ for all odd i. In particular, it follows that $b_{s-1,s}^0 c_{s,s-1}^0 = c_{s-1,s}^0 b_{s,s-1}^0 = 0$.

Furthermore, $J_{m-s+1} = B_{s1}C_{1s} + B_{s2}C_{2s} + \ldots + B_{ss}C_{ss} = C_{s1}B_{1s} + C_{s2}B_{2s} + \ldots + C_{ss}B_{ss}$ and the only possible summands with nonzero superdiagonals are $B_{s,s-1}C_{s-1,s}$

and $C_{s,s-1}B_{s-1,s}$. It follows that the superdiagonal of J_{m-s+1} is equal to $1 = b^0_{s,s-1}c^0_{s-1,s} = c^0_{s,s-1}b^0_{s-1,s} = 0$, which is a contradiction. □

Proposition 3.5. *If $\mathcal{J}(A) = (\alpha_1, \ldots, \alpha_{m-2l}, 0, \underbrace{1, \ldots, 1}_{2l-1}, 0, \alpha_{m+2}, \ldots, \alpha_n)$, where $l \geq 1$, then A is* **not** *a product of two commuting nilpotent matrices.*

Proof. Denote $\underline{\mu} = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, (m+2)^{\alpha_{m+2}})$, $\underline{\lambda} = (m, m-1, \ldots, m-2l+2)$ and $\underline{\mu}' = ((m-2l)^{\alpha_{m-2l}}, (m-2l-1)^{\alpha_{m-2l-1}}, \ldots, 1^{\alpha_1})$.

Suppose that $A = BC = CB$ with $\mathcal{J}(A)$ as in the statement. Then we can assume that $A = J_{\underline{\mu}} \oplus J_{\underline{\lambda}} \oplus J_{\underline{\mu}'}$, $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$ and $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$ in the same block partition, where all B_{ij}, C_{ij} are block upper triangular Toeplitz.

We compute that $J_{\underline{\lambda}} = B_{21}C_{12} + B_{22}C_{22} + B_{23}C_{32} = C_{21}B_{12} + C_{22}B_{22} + C_{23}B_{32}$. Since $m-2l+2 > m-2l+1$, it follows that superdiagonals of all blocks of $J_{\underline{\lambda}}$ must be equal to superdiagonals of $B_{22}C_{22}$ and by symmetry to superdiagonals of $C_{22}B_{22}$. We have already seen in the proof of Lemma 3.4 that this is not possible. □

Proposition 3.6. *If $\mathcal{J}(A)$ does* **not** *include a subsequence $(0, \underbrace{1, 1, \ldots, 1}_{2l-1}, 0)$ for any $l \geq 1$, then the Jordan canonical form of a matrix A is equal to*

$$\mathrm{ord}(r(n_1, t_1), r(n_2, t_2), \ldots, r(n_m, t_m), 1^k),$$

where $n = k + \sum_{i=1}^{m} n_i$ and $t_i \geq 2$ for all i.

Proof. If $\mathcal{J}(A)$ does not include a subsequence of the form $(0, \underbrace{1, 1, \ldots, 1}_{2l-1}, 0)$, then for any subsequence of $\mathcal{J}(A)$ of the form

$$(0, \alpha_t, \alpha_{t+1}, \ldots, \alpha_s, 0), \tag{3.1}$$

for some $2 \leq t \leq s \leq n$, where $\alpha_i \neq 0$ for $i = t, t+1, \ldots, s$ holds either

(a) $s - t + 1$ is even or

(b) $s - t + 1$ is odd and there exists j, $t \leq j \leq s$, such that $\alpha_j \geq 2$.

So, the matrix A can be written as a direct sum $A_1 \oplus A_2 \oplus \ldots \oplus A_r$, where each A_i has one of the following forms:

(i) $\mathcal{J}(A_i) = (\alpha_1)$ and the Jordan canonical form of A_i is equal to (1^{α_1}).

(ii) $\mathcal{J}(A_i) = (\underbrace{0, \ldots, 0}_{q_i-1}, \alpha_{q_i})$, where $\alpha_{q_i} \geq 2$ and the Jordan canonical form of A_i is equal to $r(q_i\alpha_{q_i}, \alpha_{q_i})$.

(iii) $\mathcal{J}(A_i) = (\underbrace{0, \ldots, 0}_{q_i-1}, \alpha_{q_i}, \alpha_{q_i+1})$ and the Jordan canonical form of A_i is equal to $r(q_i\alpha_{q_i} + (q_i+1)\alpha_{q_i+1}, \alpha_{q_i} + \alpha_{q_i+1})$.

(iv) $\mathcal{J}(A_i) = (\underbrace{0, \dots, 0}_{q_i-2}, \alpha_{q_i-1}, \alpha_{q_i}, \alpha_{q_i+1})$, where $\alpha_{q_i} \geq 2$ and the Jordan canonical form of A_i is equal to

$$\operatorname{ord}(r((q_i-1)\alpha_{q_i-1}+q_i(\alpha_{q_i}-1), \alpha_{q_i-1}+\alpha_{q_i}-1), r(q_i+(q_i+1)\alpha_{q_i+1}, 1+\alpha_{q_i+1})).$$

In case (a) we can write each block corresponding to subsequences of the form (3.1) as a direct sum of blocks of type (iii). In the case (b) we use types (ii) and (iii) if there is an odd $i \geq 1$ such that $\alpha_{t-1+i} \geq 2$ and types (iii) and (iv) otherwise. If $\alpha_1 \geq 1$ then the block corresponding to the subsequence $(\alpha_1, \alpha_2, \dots, \alpha_s, 0)$, $\alpha_i \geq 1$, is decomposed as a direct sum of blocks of type (iii) if s is even and a combination of types (i) and (iii) if s is odd. This proves the proposition. □

Proof. (of Theorem 3.3) Since Proposition 3.5 holds also for $m = n$, the implication (a) ⇒ (c) follows from Proposition 3.5. The implication (c) ⇒ (b) is the statement of Proposition 3.6 and the implication (b) ⇒ (a) is the statement of Proposition 3.2. □

4. When is an operator a product of commuting square-zero operators?. In this section we assume that $\mathcal{H}$ is an infinite-dimensional, separable, real or complex Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all operators (i.e., bounded linear transformations) on $\mathcal{H}$.

QUESTION 3. *Which operators $A \in \mathcal{B}(\mathcal{H})$ can be written as a product of two commuting square-zero operators?*

Similarly as in the finite-dimensional case we notice that also A is square-zero. Therefore $\overline{\operatorname{im} A} \subseteq \ker A$. So the space $\overline{\operatorname{im} A} + \ker A$ is closed.

THEOREM 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $A = BC = CB$, where $B^2 = C^2 = 0$, if and only if $\dim(\ker A \cap \ker A^*) = \infty$ and $A^2 = 0$.*

Proof. If A is a product of two square-zero operators it follows by [6] that $\dim(\ker A \ominus \overline{\operatorname{im} A}) = \infty$. Since $\ker A \ominus \overline{\operatorname{im} A} = \ker A \cap \left(\overline{\operatorname{im} A}\right)^{\perp} = \ker A \cap \ker A^*$ we have that $\dim(\ker A \cap \ker A^*) = \infty$ and $A^2 = 0$.

It remains to prove the converse. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_2 \subseteq \ker A \cap \ker A^*$. The matrix of A relative to this decomposition is of the form $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. Since $A^2 = 0$ also $D^2 = 0$. Therefore we can find a decomposition of space $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$, where both subspaces are infinite-dimensional and $\overline{\operatorname{im} D} \subseteq \mathcal{H}_{11}$. The matrix of D relative to this decomposition is $\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix}$. Since $\mathcal{H}_2$ is infinite-dimensional space, we can write it as a direct sum of two infinite-dimensional subspaces $\mathcal{H}_{21}$ and $\mathcal{H}_{22}$. The form of A relative to the decomposition $\mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22}$ is

$$\begin{bmatrix} 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define operators B and C on $\mathcal{H}$ by

$$B = \begin{bmatrix} 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is evident that $A = BC = CB$ and $B^2 = C^2 = 0$. □

The factorization in the proof above is based on the factorization in the finite-dimensional case. Since $\mathcal{H}_2$ is an infinite-dimensional space, we can write it as a direct sum of k infinite-dimensional subspaces. Using the factorization in the proof of Theorem 2.3 we get the following result.

Corollary 4.2. *An operator A is a product of two commuting square-zero operators if and only if A is a product of k square-zero operators.*

5. When is an operator a product of two commuting nilpotent operators?. Let V be an infinite-dimensional vector space and $A : V \to V$ a nilpotent operator with index of nilpotency n. We proceed to define the sequence

$$\mathcal{J}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where now $\alpha_i \in \mathbb{N} \cup \{0, \infty\}$.

For $k = 0, 1, \ldots, n-1$ we choose subspaces V_{n-k} such that:

1. For $k = 0$ we have $V = \ker A^n = \ker A^{n-1} \oplus V_n$.
2. For $k > 0$ we have $\ker A^{n-k} = \ker A^{n-k-1} \oplus AW_{n-k+1} \oplus V_{n-k}$, where $W_{n-k+1} = AW_{n-k+2} \oplus V_{n-k+1}$ and $W_{n+1} = 0$.

Then we define $\alpha_i = \dim V_i$, $i = 1, 2, \ldots, n$. Observe that if $\dim V < \infty$ then this definition of $\mathcal{J}(A)$ coincides with the one given in §2.

Observe that if an operator A is a product of two commuting nilpotent operators, then A is also a nilpotent operator.

Theorem 5.1. *A nilpotent operator A with $\mathcal{J}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a product of two commuting nilpotent operators if and only if a matrix B with $\mathcal{J}(B) = (\beta_1, \beta_2, \ldots, \beta_n)$, where $\beta_i = \min\{\alpha_i, 2\}$, is a product of two commuting nilpotent matrices.*

Before we prove the theorem let us show the following two lemmas.

LEMMA 5.2. *If $\mathcal{J}(A) = (0, \dots, 0, \infty)$, then A is a product of two commuting nilpotent operators.*

Proof. Since $\alpha_i = 0$ for $i \neq n$, it follows that all the indecomposable blocks are of size n. Then A is similar to $\oplus_{k=1}^{\infty} J_{k,n} = \oplus_{k=1}^{\infty}(J_{2k-1,n} \oplus J_{2k,n})$, where $J_{k,n}$, $k \in \mathbb{N}$, are indecomposable blocks of A each of them similar to J_n. Since $J_{2k-1,n} \oplus J_{2k,n}$ is a product of two commuting nilpotent matrices by Theorem 3.3, the assertion follows. □

LEMMA 5.3. *Let A be a nilpotent matrix with $\mathcal{J}(A) = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where there exists an index j such that $\alpha_j \geq 2$. Suppose that B is a matrix with $\mathcal{J}(B) = (\beta_1, \beta_2, \dots, \beta_n)$, where $\beta_i = \alpha_i$ for $i \neq j$ and $\beta_j = \alpha_j + 1$. Then A is a product of two commuting nilpotent matrices if and only if B is a product of two commuting nilpotent matrices.*

Proof. By Theorem 3.3 a matrix A is a product of two commuting nilpotent matrices if and only if $\mathcal{J}(A)$ is not of the form

$$(\alpha_1, \dots, \alpha_i, 0, \underbrace{1, \dots, 1}_{2l-1}, 0, \alpha_k, \dots, \alpha_n). \tag{5.1}$$

It follows easily that matrices A and B are simultaneously the products of two commuting nilpotent matrices. □

Proof. (of Theorem 5.1) Suppose that A is a product of two commuting nilpotent operators. Similarly as in the finite-dimensional case we can show that $\mathcal{J}(A)$ can not be of the form (5.1). Hence also $\mathcal{J}(B)$ is not of that form and therefore B is a product of two commuting nilpotent matrices.

To prove the converse write A as a direct sum of A_1 and A_2 with

$$\mathcal{J}(A_j) = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{nj})$$

so that $\alpha_{i1} = \alpha_i$ and $\alpha_{i2} = 0$ if α_i is finite, and $\alpha_{i1} = 2$ and $\alpha_{i2} = \infty$ otherwise. It suffices to show that A_1 and A_2 are the products of commuting nilpotent operators, which follows from the lemmas above. □

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