IRREDUCIBLE OPERATOR SEMIGROUPS SUCH THAT AB AND BA ARE PROPORTIONAL

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Dedicated to Prof. Heydar Radjavi on the occasion of his seventieth birthday.

Abstract. Let $\mathcal{S}$ be an irreducible semigroup of compact linear operators on a Banach space of dimension at least 2 with the property that the products AB and BA are proportional for each pair of elements A and B in $\mathcal{S}$. We show that $\mathcal{S}$ is necessarily a nilpotent matrix group of nilpotency class 2 with possible addition of the zero matrix. We also study generalizations of the property.

1. Introduction

Several authors have studied semigroups of linear operators satisfying polynomial identities and considered the problem of irreducibility of these semigroups (see e.g. [1, 2, 3]). Then one assumes that every sequence of elements satisfies the given identities. Here we consider a weaker problem when the coefficients of the identities can depend on the chosen elements. We first study the reducibility of an operator semigroup $\mathcal{S}$ such that for each pair of elements A and B in $\mathcal{S}$ the products AB and BA are proportional.

It is easy to find examples of irreducible matrix groups having this property. Indeed, let U be the permutation $n \times n$ ($n \geq 2$) matrix defined on the standard basis vectors $\{e_i\}_{i=1}^n$ by $Ue_i = e_{i+1}$ for $1 \leq i < n$ and $Ue_n = e_1$, and let D be the diagonal matrix $\mathrm{diag}(1, w, w^2, \ldots, w^{n-1})$, where w is the primitive n-th root of unity. Then the group $\mathcal{G}$ generated by U and D is irreducible and for each pair $A, B \in \mathcal{G}$ there exists $j \in \{0, 1, 2, \ldots, n-1\}$ such that $AB = w^j BA$. These matrix groups are often minimal counterexamples for reducibility when one studies various reducibility questions (see e.g. Sections 3.3, 4.2 and 4.3 in [3]). Most of these reducibility questions were considered in the infinite-dimensional setting as well. One may also ask whether there are infinite-dimensional examples of irreducible operator semigroups $\mathcal{S}$ such that AB and BA are proportional for all $A, B \in \mathcal{S}$.

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In this note we show that the only such irreducible semigroups of compact linear operators on a Banach space of dimension at least 2 are irreducible nilpotent matrix groups of class 2 with possible addition of the zero matrix. In these groups for each pair A, B the commutator $ABA^{-1}B^{-1}$ is a scalar matrix, say γI, and so $AB = \gamma BA$.

We also obtain weaker results in the case when the pair $\{AB, BA\}$ in the condition is replaced by pairs $\{ABA^k, BA^{k+1}\}$ or $\{A^{k+1}B, A^kBA\}$ for some $k \geq 1$ depending on A, B.

In our results the compactness assumption can not be omitted: there exists a bounded operator on l^1 without nontrivial invariant subspace (see e.g. [4]) so that the semigroup it generates is irreducible and commutative.

2. Results

Throughout the paper we assume that the dimension of the underlying Banach space is at least 2. We denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. The closure of a set S in a topological space is denoted by $\overline{S}$.

In the proof of our first result we use several times the well-known fact that $r(AB) = r(BA)$ for all bounded operators A and B, where r denotes the spectral radius.

Theorem 2.1. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on a complex Banach space X with the property that for each pair of elements A and B in $\mathcal{S}$ the products AB and BA are proportional. Then*

(a) the dimension n of X is finite.

(b) the set $\mathcal{G} = \overline{\mathbb{R}^+\mathcal{S}} \setminus \{0\}$ is a nilpotent group of class 2, and its subgroup $\mathcal{G}_0 = \{G \in \mathcal{G} : \det(G) = 1\}$ is finite.

(c) there exists a basis of X in which operators of the group $\mathcal{G}_0$ are represented by monomial unitary matrices.

(d) for each pair $A, B \in \mathcal{G}$ there is a (unique) n-th root $\gamma_{A,B}$ of unity such that $AB = \gamma_{A,B}BA$.

(e) for every $A \in \mathcal{G}$ the operator A^n is a multiple of the identity.

Proof. With no loss of generality we may assume that $\mathbb{R}^+\mathcal{S} = \mathcal{S}$. Denote by $\mathcal{I}$ the set of all quasinilpotents in $\mathcal{S}$. We claim that $\mathcal{I}$ is an ideal of $\mathcal{S}$. Assume on the contrary that $r(AS) > 0$ for some $A \in \mathcal{I}$ and $S \in \mathcal{S}$. By the assumption, there exist complex numbers α and β, not both equal to zero, such that $\alpha AS = \beta SA$. Since $r(AS) = r(SA) > 0$, we conclude that $|\alpha| = |\beta| > 0$. It follows that

$$\|(AS)^m\| = \|A^m S^m\| \leq \|A^m\| \, \|S^m\|$$

for all positive integers m, and so $r(AS) \leq r(A)\, r(S) = 0$. This contradiction shows that $\mathcal{I}$ is an ideal of $\mathcal{S}$.

By the famous Turovskii's theorem (see [6] or [3]), the ideal $\mathcal{I}$ is reducible. Since every non-zero ideal of an irreducible semigroup is also irreducible we obtain that $\mathcal{I} = \{0\}$. This implies that for $A, B \in \mathcal{S}$ we have $AB = 0$ if and only if $BA = 0$. Consequently, we may assume for each pair $A, B \in \mathcal{S}$ there exists a nonzero complex number $\gamma_{A,B}$ such that $AB = \gamma_{A,B} BA$. Furthermore, we may assume that $\gamma_{A,B}$ is on the unit circle. Indeed, for each pair $A, B \in \mathcal{S}$ with $AB \neq 0$ we have $r(AB) > 0$, and so it follows from $r(AB) = |\gamma_{A,B}|\, r(BA)$ that $|\gamma_{A,B}| = 1$.

Now, it is easy to verify that there is no loss of generality in assuming that $\mathcal{S}$ is closed. By [3, Theorem 7.4.5], $\mathcal{S}$ contains non-zero finite rank operators. Denote by n the minimal rank of non-zero members of $\mathcal{S}$ and pick $E \in \mathcal{S}$ of rank n. Since $AE = \gamma_{A,E} EA$ for all $A \in \mathcal{S}$, the range $\operatorname{im} E$ of E is invariant under every member of $\mathcal{S}$. In view of irreducibility of $\mathcal{S}$ we obtain that $\operatorname{im} E = X$ so that (a) holds and every member of $\mathcal{G}$ is invertible. Moreover, $\mathcal{G}$ is a group by [3, Lemma 3.1.6]. The property $AB = \gamma_{A,B} BA$ implies that $ABA^{-1}B^{-1}$ belongs to the center of $\mathcal{G}$ for all $A, B \in \mathcal{G}$. Since $\mathcal{G}$ is not commutative (otherwise it was reducible), $\mathcal{G}$ is a nilpotent group of class 2. Since its subgroup $\mathcal{G}_0 = \{G \in \mathcal{G} : \det(G) = 1\}$ is finite by [5, Theorem 1, p. 208], the proof of (b) is finished.

By [5, Lemma 6, p. 207] there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of X in which operators of $\mathcal{G}$ are represented by monomial matrices. We recall a well known argument to prove (c). Denote by $(\cdot, \cdot)$ the standard inner product on X with respect to the basis $\{e_1, e_2, \ldots, e_n\}$. If G_1, G_2, $\ldots$, G_m are all members of $\mathcal{G}_0$, then we define a new inner product on X as follows

$$\langle x, y \rangle = \frac{1}{m} \sum_{i=1}^{m} (G_i x, G_i y).$$

It is easily verified that every member $G \in \mathcal{G}_0$ has a monomial unitary matrix with respect to the basis $\{f_1, \ldots, f_n\}$ defined by $f_i = e_i / \sqrt{\langle e_i, e_i \rangle}$, $i = 1, \ldots, n$. This completes the proof of (c).

For every pair $A, B \in \mathcal{G}$ we have $\det(AB) = \gamma_{A,B}^n \det(BA)$ which yields that $\gamma_{A,B}^n = 1$ proving (d). To show (e), take $A \in \mathcal{G}$. It follows from (d) that the operator A^n commutes with any $B \in \mathcal{G}$. Since $\mathcal{G}$ is irreducible, we conclude that A^n must be a multiple of the identity. □

Corollary 2.2. *Let $\mathcal{S}$ be an irreducible semigroup of $n \times n$ complex matrices. Then for all pairs $A, B \in \mathcal{S}$ there exists a complex number $\gamma_{A,B}$ such that $AB = \gamma_{A,B} BA$ if and only if $\overline{\mathbb{R}^+ \mathcal{S}} \setminus \{0\}$ is a nilpotent group of class 2.*

Proof. Only the converse implication needs a proof. If $\mathcal{G} = \overline{\mathbb{R}^+ \mathcal{S}} \setminus \{0\}$ is a nilpotent group of class 2, then for any pair $A, B \in \mathcal{G}$ the group commutator $ABA^{-1}B^{-1}$ belongs to the center of $\mathcal{G}$. Since $\mathcal{G}$ is irreducible, the center consists of scalar matrices only, and so there is a complex number $\gamma_{A,B}$ such that $AB = \gamma_{A,B} BA$. □

We denote by $\mathcal{G}(k)$ the irreducible subgroup of $\mathrm{GL}_k(\mathbb{C})$ generated by the cyclic permutation matrix U defined on the standard basis vectors $\{e_i\}_{i=1}^k$ by $Ue_i = e_{i+1}$ for $1 \le i < k$ and $Ue_k = e_1$, and by the diagonal matrix $\mathrm{diag}(1, w, w^2, \ldots, w^{k-1})$, where w is a primitive k-th root of unity.

Corollary 2.3. *A semigroup $\mathcal{S}$ of $n \times n$ complex matrices is a maximal irreducible semigroup such that for all pairs $A, B \in \mathcal{S}$ there exists a complex number $\gamma_{A,B}$ with $AB = \gamma_{A,B} BA$ if and only is there exists a factorization $n = k_1 k_2 \cdots k_t$ (all $k_j \ge 2$ and $k_j | k_i$ for $j > i$) such that $\mathcal{S}$ is conjugate to the semigroup $\mathbb{C} \cdot \mathcal{G}(k_1) \otimes \mathcal{G}(k_2) \otimes \cdots \otimes \mathcal{G}(k_t)$.*

Proof. By Corollary 2.2 we know that $\overline{\mathbb{R}^+ \mathcal{S}} \setminus \{0\}$ is an irreducible nilpotent matrix group of class 2. It is contained in a maximal nilpotent matrix group of class 2 which we denote by $\mathcal{G}$. Another application of Corollary 2.2 implies that for all pairs $A, B \in \mathcal{G}$ there exists a complex number $\gamma_{A,B}$ such that $AB = \gamma_{A,B} BA$. By maximality it follows that $\mathcal{S} = \overline{\mathbb{R}^+ \mathcal{S}} = \mathcal{G} \cup \{0\}$. By [5, Theorem 7, pp. 210-211] we know that $\mathcal{G}$ is conjugate to the matrix group $\mathbb{C}^* \cdot \mathcal{G}(k_1) \otimes \mathcal{G}(k_2) \otimes \cdots \otimes \mathcal{G}(k_t)$ for a factorization $n = k_1 k_2 \cdots k_t$ such that all $k_j \ge 2$ and $k_j | k_i$ for $j > i$. Here $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then it follows that $\mathcal{S}$ is of the required form. □

Theorem 2.4. *Let Γ be a non-void compact subset of $\mathbb{C} \setminus \{0\}$, and let $\mathcal{S}$ be an irreducible semigroup of compact operators on a complex Banach space X with the property that for each pair $A, B \in \mathcal{S} \setminus \{0\}$ there exist $\gamma = \gamma_{A,B} \in \Gamma$ and $k = k_{A,B} \in \mathbb{N}$ such that $(AB - \gamma BA)A^k = 0$. Then the conclusions of Theorem 2.1 hold.*

Proof. With no loss of generality we may assume again that $\mathbb{R}^+ \mathcal{S} = \mathcal{S}$. By Turovskii's theorem the semigroup $\mathcal{S}$ contains an operator that is not quasinilpotent. Since $\mathbb{R}^+ \mathcal{S} = \mathcal{S}$, there is an operator $K \in \mathcal{S}$ of spectral radius 1. As in the proof of [3, Lemma 7.4.5] we obtain an unbounded sequence $\{m_i\}$ in $\mathbb{N}$ and a sequence $\{t_i\}$ in $(0,\infty)$ such that the sequence $\{t_i K^{m_i}\}$ converges to a non-zero finite-rank operator F that is either idempotent or nilpotent with $F^2 = 0$. (If F is idempotent, we can even take $t_i = 1$.) By the assumption for any $B \in \mathcal{S} \setminus \{0\}$ there exist $\gamma_i = \gamma_{K^{m_i},B} \in \Gamma$ and $k_i = k_{K^{m_i},B} \in \mathbb{N}$ such that

$$(K^{m_i}B - \gamma_i BK^{m_i})(K^{m_i})^{k_i} = 0.$$

We can find a subsequence $\{m_{i_j}\}$ of $\{m_i\}$ such that

$$m_{i_{j+1}} \geq m_{i_j} \cdot k_{i_j}$$

for all j, and so

$$(K^{m_{i_j}}B - \gamma_{i_j} BK^{m_{i_j}})K^{m_{i_{j+1}}} = 0.$$

Since Γ is a compact set, by passing to a subsequence if necessary we may assume that the sequence $\{\gamma_{i_j}\}$ converges to $\gamma \in \Gamma$. Multiplying the last equation by $t_{i_j}\, t_{i_{j+1}}$ and taking the limit we obtain $(FB - \gamma BF)F = 0$. If F is nilpotent of index 2, we have $FBF = 0$ so that $B(\operatorname{im} F) \subseteq \ker F$ for all $B \in \mathcal{S}$. Then for any non-zero vector $x \in \operatorname{im} F$ the closed linear span of the set $\{x\} \cup \{Sx : S \in \mathcal{S}\}$ is contained in $\ker F \neq X$ and is invariant under each member of $\mathcal{S}$ which is a contradiction with the irreducibility of $\mathcal{S}$. So, F has to be idempotent, and it follows from $FBF = \gamma BF$ that the range $\operatorname{im} F$ is invariant under every member of the irreducible semigroup $\mathcal{S}$, so that $\operatorname{im} F = X$. Now note that with $n = \dim X$ it holds that $\operatorname{im}(A^k) = \operatorname{im}(A^n)$ for all $A \in \mathcal{S}$ and all $k > n$, and so the semigroup $\mathcal{S}$ has the property that for each pair $A, B \in \mathcal{S} \setminus \{0\}$ there exists $\gamma_{A,B} \in \Gamma$ such that $(AB - \gamma_{A,B}BA)A^n = 0$. By a simple compactness argument we may assume that $\mathcal{S}$ is closed. Then by [3, Lemma 3.1.6] $\mathcal{S}$ contains an idempotent E of minimal rank in $\mathcal{S} \setminus \{0\}$. Since $EBE = \gamma_{E,B}BE$ for all $B \in \mathcal{S}$, the range $\operatorname{im} E$ is invariant under every member of $\mathcal{S}$, so that E is the identity operator. It follows that every non-zero member of $\mathcal{S}$ is invertible, and so $\mathcal{S}$ has the property that for each pair $A, B \in \mathcal{S} \setminus \{0\}$ there exists $\gamma_{A,B} \in \Gamma$ such that $AB = \gamma_{A,B}BA$. Now Theorem 2.1 can be applied to complete the proof. □

The following example shows that the condition $0 \notin \Gamma$ in Theorem 2.4 is necessary.

Example 2.5. *Let $n \geq 2$. We denote by E_{ij}, $i, j = 1, 2, \ldots, n$ the standard linear basis for $M_n(\mathbb{C})$, i.e., E_{ij} is the matrix with the only nonzero entry (i, j) equal to 1. Then $\mathcal{S} = \{E_{ij},\ i, j = 1, 2, \ldots, n\} \cup \{0\}$ is a matrix semigroup. It is easy to check that for each* *pair* $A, B \in \mathcal{S}$*,* $A \neq B$ *we have* $ABA^2 = 0$*, and so for each pair* $A, B \in \mathcal{S}$ *there exists a complex number* γ *such that* $(AB - \gamma BA)A^2 = 0$*.*

The following result is a 'dual' version of Theorem 2.4. We omit its proof, since it is a slight modification of the proof of Theorem 2.4.

Theorem 2.6. *Let* Γ *be a non-void compact set of* $\mathbb{C} \setminus \{0\}$*, and let* $\mathcal{S}$ *be an irreducible semigroup of compact operators on a complex Banach space* X *with the property that for each pair* $A, B \in \mathcal{S} \setminus \{0\}$ *there exist* $\gamma = \gamma_{A,B} \in \Gamma$ *and* $k = k_{A,B} \in \mathbb{N}$ *such that* $A^k(AB - \gamma BA) = 0$*. Then the conclusions of Theorem 2.1 hold.*

The semigroup $\mathcal{S}$ in Example 2.5 satisfies the condition that $A^2(AB - BA)B^2 = 0$ for each pair $A, B \in \mathcal{S}$. Therefore further weakening of conditions in Theorems 2.4 and 2.6 to the condition that for each pair $A, B \in \mathcal{S} \setminus \{0\}$ there exist $\gamma_{A,B} \neq 0$ and $k = k_{A,B} \in \mathbb{N}$ such that $A^k(AB - \gamma BA)B^k = 0$ will no more yield the conclusions of Theorem 2.1.

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