Alternating sign matrices and discrete imprecise probability[1]

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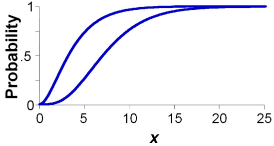
Outline

- Traditional probability uses single values to represent the chance of an event.
- **Imprecise probability** represents probabilities as intervals $[\underline{P}(A), \overline{P}(A)]$.
- The **lower probability** $\underline{P}(A)$ captures the evidence supporting A, while the **upper probability** $\overline{P}(A)$ expresses the lack of evidence against A.
- $\underline{P}$ and $\overline{P}$ need not be additive measures but are assumed to be monotone in sets.
- This point of view is useful when exact probability is not available or not desired, such as in finance, game theory, and reliability. It also has several applications in social sciences.
- The foundation of imprecise probability was set by Peter Walley in his 1991 monograph *Statistical Reasoning with Imprecise Probabilities*.

- Modeling imprecision of random variables is based on the notion of a *p*-**box** (a probability box).

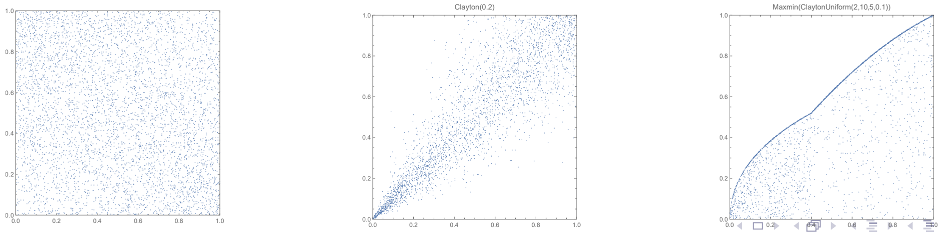


- For a single random variable X, a *p*-box is a pair of cumulative distribution functions $[\underline{F}(x), \overline{F}(x)]$ such that $\underline{F}(x) \leq \overline{F}(x)$ for all $x \in \mathbb{R}$, where $\underline{F}(x)$ and $\overline{F}(x)$ represent the lower and the upper probability of the event $[X \leq x]$.
- The **point-wise infimum and supremum** (the envelopes) of any *p*-box are its lower and upper bound. They remain within the class of cumulative distribution functions.
- The situation changes for distribution functions of two or more variables. There the envelopes of a *p*-box may not be cumulative distribution functions.
- The set of all distribution functions of two (or more) variables is not equal to its **Dedekind-MacNeille completion**.

- Copulas are distribution function of random vectors that have uniform marginal distributions on $[0,1]$. Their importance is emphasized by **Sklar's Theorem**.

Theorem (Sklar, 1959)

Suppose that $F(x,y)$ is the joint distribution function of a random vector (X,Y), and functions F_X and F_Y are the marginal distribution functions of X and Y, respectively. Then there exist a copula C such that $F(x,y) = C(F_X(x), F_Y(y))$. If (X,Y) is a continuous random vector then C is unique.



- Consider a uniform partition of the unit interval $I_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ for $n \in \mathbb{N}$ and define discrete (quasi-) copulas on the square grid I_n^2.
- A **discrete copula** is a function $C : I_n^2 \to [0,1]$ that satisfies certain conditions (C1) and (C2).
- A **discrete quasi-copula** is a function $Q : I_n^2 \to [0,1]$ satisfying certain conditions (Q1)–(Q3).
- **To avoid fractions we consider transformed maps that we still call discrete copulas or discrete quasi-copulas:** A discrete (quasi-)copula $Q : I_n^2 \to [0,1]$ can be expressed as a map $Q' : {L_n}^2 \to [0,n]$, where $L_n = \{0,1,\ldots,n\}$, by $Q'(r,s) = n \cdot Q\left(\frac{r}{n}, \frac{s}{n}\right)$ for $r, s \in L_n$.

Discrete Copulas

Let us consider formal definitions:

Definition

A function $C : L_n^2 \to [0, n]$ is a **discrete copula** if and only if

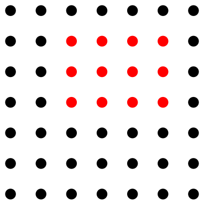
(C1') C is **grounded** $C(r, 0) = C(0, s) = 0$ and it **has a unit** $C(r, n) = r$, $C(n, s) = s$ for every $r, s \in L_n$;

(C2') C is **2-increasing**: $C(r_1, s_1) + C(r_2, s_2) \geq C(r_1, s_2) + C(r_2, s_1)$ for every $r_1, r_2, s_1, s_2 \in L_n$ such that $r_1 \leq r_2, s_1 \leq s_2$.

- (C2') implies that the **C-volume**

$$V_C(R) = C(r_1, s_1) + C(r_2, s_2) - C(r_1, s_2) - C(r_2, s_1)$$

of any (discrete) rectangle $R = [r_1, s_1] \times [r_2, s_2]$ is **nonnegative**.



Definition

A function $Q : L_n^2 \to [0, n]$ is a **discrete quasi-copula** if and only if it satisfies the following conditions:

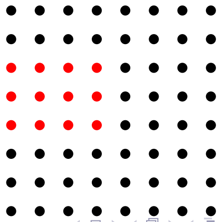
(Q1') Q is **grounded** $Q(r, 0) = Q(0, s) = 0$ and it **has a unit** $Q(r, n) = r$, $Q(n, s) = s$ for every $r, s \in L_n$,

(Q2b') $Q : L_n^2 \to [0, n]$ satisfies $Q(r_1, s_1) + Q(r_2, s_2) \geq Q(r_1, s_2) + Q(r_2, s_1)$ whenever $r_1 \leq r_2$, $s_1 \leq s_2$ and at least one of r_1, r_2, s_1, s_2 is either equal to 0 or to n.

(Q2b') implies that the **Q-volume**

$$V_Q(R) = Q(r_1, s_1) + Q(r_2, s_2) - Q(r_1, s_2) - Q(r_2, s_1)$$

of any rectangle $R = [r_1, s_1] \times [r_2, s_2]$ with at least one edge on the boundary of L_2^n is **nonnegative**.



- Note that (C1') and (Q1') are equal. Condition (Q2b') is equivalent to conditions (Q2') and (Q3') where

Definition

(Q2') Q is **increasing in each component**.

(Q3') Q satisfies the **1-Lipschitz condition**, i.e., for every $r_1, r_2, s_1, s_2 \in L_n$, $|Q(r_2, s_2) - Q(r_1, s_1)| \leq |r_2 - r_1| + |s_2 - s_1|$.

- Quasi-copulas and copulas on L_n can be identified with square matrices of size n.
- We denote the matrix of a (quasi-)copula Q by the same letter Q and write

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}, \quad \text{where } q_{rs} = Q\,(r,s)\,.$$

- We omit the values at $r = 0$ and $s = 0$ that are all 0.
- Observe that $q_{rn} = q_{nr} = r$ for all r.
- The value q_{rs} is **the probability** that Q assigns to the **rectangle** $[0,r] \times [0,s]$. Hence, $q_{rs} \geq 0$.

- The probability distributions assigned by Q is then given by the values

$$a_{rs} = Q(r,s) + Q(r-1,s-1) - Q(r,s-1) - Q(r-1,s) \text{ for any } r \text{ and } s.$$

- Condition (C2') imples that if Q is a copula then $a_{rs} \geq 0$ for all r and s.
- This is no longer the case for a general quasi-copula.
- We say that a quasi-copula is **proper** if $a_{rs} < 0$ for at least one pair (r,s).
- We write $A = A(Q)$ for the matrix A corresponding to a quasi-copula Q, and $Q = Q(A)$ for the inverse correspondence given by

$$Q(r,s) = \sum_{i=1}^{r} \sum_{j=1}^{s} a_{i,j}, \text{ for } r,s \geq 1.$$

- Discrete copulas correspond to the points of the **Birkhoff polytope**, which is the set of all **bistochastic matrices**.
- The extreme points of this polytope are **permutation matrices**.

- Discrete quasi-copulas correspond to points of the **alternating sign matrix (ASM) Polytope**.
- This is the set of all **alternating bistochastic matrices (ABM)**.
- The extreme points are the **alternating sign matrices**.
- We say that a quasi-copula is **of minimal range** if all its values are in L_n. Condistions $Q(r, n) = r$ imply that L_n is always in the range of Q.
- Quasi-copulas of minimal range are exactly all the quasi-copulas Q such that the corresponding matrix $A(Q)$ is an alternating sign matrix.
- So, the simplest proper discrete quasi-copula Q of minimal range and its corresponding ASM are

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } A(Q) = F_3^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- **Defects of quasi-copulas** were introduced by Dibala, Saminger-Platz, Mesiar, and Klement in 2016.
- To introduce defects we first define **four sets of rectangles** for each point (r, s) on the square grid $L_n \times L_n$.

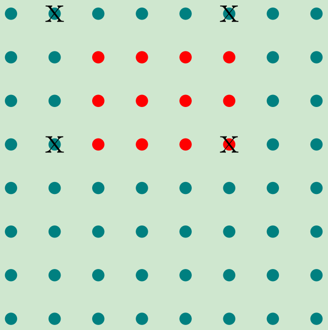
$$
\begin{aligned}
\mathcal{R}_{\searrow}(r,s) &= \{[r+1, r+i] \times [s+1, s+j];\ 1 \le i \le n-r, 1 \le j \le n-s\}\,, \\
\mathcal{R}_{\swarrow}(r,s) &= \{[r+1, r+i] \times [j, s];\ 1 \le i \le n-r, 1 \le j \le s\}\,, \\
\mathcal{R}_{\nwarrow}(r,s) &= \{[i, r] \times [j, s];\ 1 \le i \le r, 1 \le j \le s\}\,, \\
\mathcal{R}_{\nearrow}(r,s) &= \{[i, r] \times [s+1, s+j];\ 1 \le i \le r, 1 \le j \le n-s\}\,.
\end{aligned}
$$

- For a rectangle $R = [i, r] \times [j, s]$ in L_n^2 its **volume (mass)** with respect to a quasi-copula Q is

$$V_Q(R) = \sum_{k=i}^{r} \sum_{l=j}^{s} a_{k,l}. \tag{1}$$

Example

Rectangle $R = [2,4] \times [3,6]$ belongs to $\mathcal{R}_{\searrow}(1,2)$, $\mathcal{R}_{\swarrow}(1,6)$, $\mathcal{R}_{\nwarrow}(4,6)$ and $\mathcal{R}_{\nearrow}(4,2)$.



Its Q-volume is the sum of all entries of $A(Q)$ with indices in R.

- Now, for each discrete quasi-copula Q we define four **directional defect matrices** $D^Q_{\nearrow}$, $D^Q_{\nwarrow}$, $D^Q_{\swarrow}$ and $D^Q_{\searrow}$.
- Their entries are numbered by the grid points in the set L_n^2. We omit the bottom row and left column of zeros and consider them as square matrices of size n.
- Their entries are given by

$$
\begin{aligned}
D^Q_{\searrow}(r,s) &= \min\{0, V_Q(R);\ R \in \mathcal{R}_{\searrow}(r,s)\}, \\
D^Q_{\swarrow}(r,s) &= \min\{0, V_Q(R);\ R \in \mathcal{R}_{\swarrow}(r,s)\}, \\
D^Q_{\nwarrow}(r,s) &= \min\{0, V_Q(R);\ R \in \mathcal{R}_{\nwarrow}(r,s)\}, \\
D^Q_{\nearrow}(r,s) &= \min\{0, V_Q(R);\ R \in \mathcal{R}_{\nearrow}(r,s)\}.
\end{aligned}
$$

- Observe that if Q is a copula then all the defect matrices are equal to 0.
- For a proper quasi-copula all its defect matrices have some nonzero entries.

- Two additional defect matrices are important. They are called **the main and the opposite defect matrices** and denoted by D_M^Q and D_O^Q, respectively. Their entries are given by

$$\begin{aligned} D_M^Q(r,s) &= \min\left\{D_{\searrow}^Q(r,s), D_{\nwarrow}^Q(r,s)\right\}, \\ D_O^Q(r,s) &= \min\left\{D_{\swarrow}^Q(r,s), D_{\nearrow}^Q(r,s)\right\}. \end{aligned}$$

- We replace the subscript Q by A when we consider an ABM matrix A instead of the corresponding quasi-copula $Q(A)$. We write $D_{\searrow}^A$, $D_{\swarrow}^A$, $D_{\nwarrow}^A$, etc.

Example

Consider $F_3^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and the discrete quasi-copula $Q = Q(F_3^2)$. The corresponding directional defect matrices are

$$D_{\searrow}^{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_{\swarrow}^{Q} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_{\nwarrow}^{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, D_{\nearrow}^{Q} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The main and the opposite defect matrices are

$$D_M^Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } D_O^Q = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example

Another matrix and two of its defect matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad D^A_{\searrow} = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^A_M = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Using the defect matrices we obtain six transformation on quasi-copulas. They are:

$$\begin{aligned} Q_{\searrow} &= Q - D^Q_{\searrow}, \\ Q_{\nwarrow} &= Q - D^Q_{\nwarrow}, \\ Q_{M} &= Q - D^Q_{M}, \end{aligned} \qquad \begin{aligned} Q_{\swarrow} &= Q + D^Q_{\swarrow}, \\ Q_{\nearrow} &= Q + D^Q_{\nearrow}, \\ Q_{O} &= Q + D^Q_{O}. \end{aligned}$$

Theorem (Dibala et al, 2016)

All six transformations applied to a (discrete) quasi-copula yield (discrete) quasi-copulas.

Theorem (K., Peronne, Stopar, 2025)

All six transformations applied to a quasi-copula of minimal range yield quasi-copulas of minimal range.

Example

The six transformations applied to Q give discrete quasi-copulas that are in fact discrete copulas. Their ASM matrices are the six permutation matrices.

$$Q_{\searrow} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \; Q_{\swarrow} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \; Q_{\nwarrow} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$Q_{\nearrow} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \; Q_M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \; \text{and } Q_O = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Definition

A pair (P, Q) of functions $P, Q : L_n^2 \to [0, n]$ is called a **discrete imprecise copula** if P and Q satisfy the following conditions:

(IC1) Both P and Q satisfy condition (Q1').

(IC2) For each rectangle $R = [i, j] \times [k, l] \in \mathcal{R}$ we have:

$$\begin{aligned}
Q(i,k) + P(j,l) - P(i,l) - P(j,k) &\geq 0, \\
P(i,k) + Q(j,l) - P(i,l) - P(j,k) &\geq 0, \\
Q(i,k) + Q(j,l) - Q(i,l) - P(j,k) &\geq 0, \\
Q(i,k) + Q(j,l) - P(i,l) - Q(j,k) &\geq 0.
\end{aligned}$$

Example

Take

$$A = F_4^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = F_4^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then

$$P = Q(A) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, Q = Q(B) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Pair (P, Q) is an imprecise copula.

- Montes, Miranda, Pelessoni, and Vicig in 2015 proved that that given an imprecise copula then both P and Q are quasi-copulas and $P(r,s) \leq Q(r,s)$ for all (r,s), i.e., $P \leq Q$ in the point-wise order.
- We also assume for any imprecise copula (P,Q) that both P and Q are proper quasi-copulas.
- Dibala, Saminger-Platz, Mesiar, and Sempi in 2016 showed that a pair (P,Q) of quasi-copulas is an imprecise copula if and only if $P_M \leq Q$ and $P \leq Q_O$.
- Hence, (P, P_M) and (Q_O, Q) are imprecise copulas for any proper quasi-copulas P and Q.
- The main defect $-D_M^Q$ and the opposite defect $-D_O^P$ bound the difference $Q-P$ from below: $Q-P \geq -D_M^Q$, and $Q-P \geq -D_O^P$.
- Repeating the operations on an imprecise copula (P,Q) one obtains $P \leq (P_M)_O \leq P_M \leq Q$, so that $((P_M)_O, P_M)$ is an imprecise copula inside the original imprecise copula.

- Iterating the process further, we get a sequence of embedded quasi-copulas (P_k, Q_k) given by $P_0 = P, Q_0 = Q$ and $P_k = (Q_k)_O, Q_k = (P_{k-1})_M$.
- The limiting pair $(\overline{P}, \overline{Q})$ is also an imprecise copula which satisfies the equalities $(\overline{P})_M = \overline{Q}$ and $(\overline{Q})_O = \overline{P}$.

Definition

An imprecise copula (P, Q) is **self-dual** if $P_M = Q$ and $Q_O = P$. Also, a pair of quasi-copulas (P, Q) is **a dual pair** if (P, Q) is a self-dual imprecise copula.
A pair of ASM (A, B) is **a dual pair** if $(Q(A), Q(B))$ is a self-dual imprecise copula.

- It follows from definition that an imprecise copula (P, Q) is self-dual if and only if $D_M^P = D_O^Q$.

- An example of a self-dual discrete imprecise copula (P, Q), where both P and Q are quasi-copulas of minimal range of size $n = 7$, was given by Omladič and Stopar in 2020. We generalized this example to finer grids of size $n \geq 8$.

Example

The corresponding ASM are:

$$A_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Definition

An ASM is called **dense** if it has no zero entries between any two nonzero entries in either a column or in a row.

Example

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Example

Brualdi and Schroeder, 2017, denoted by F_n^k, $k = 1, 2, \ldots, n$, $n \geq 1$, the (irreducible) dense matrix that has the $(1, k)$ entry equal to 1. By these conditions it is uniquely determined for $2 \leq k \leq n-1$.

For instance, when $n = 4$ or $n = 5$ we have

$$F_4^1 = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, F_4^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, F_4^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$F_5^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, F_5^3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Theorem (K., Perrone, 2024)

Suppose $n \geq 3$. Then

$$Q(F_n^k)_M = Q(F_n^{k-1}) \text{ and } Q(F_n^k)_O = Q(F_n^{k+1})$$

for $2 \leq k \leq n-1$.

Corollary

Suppose that $n \geq 4$. Then (F_n^k, F_n^{k-1}) is a dual pair of ASM for $3 \leq k \leq n-1$.

Imprecise copulas $(Q(F_n^{n-1}), Q(F_n^{n-2}))$, $(Q(F_n^{n-2}), Q(F_n^{n-3}))$, $\ldots$, $(Q(F_n^3), Q(F_n^2))$ form a maximal chain of self-dual imprecise copulas.

The chain cannot be extended further since $Q(F_n^{n-1})_O = W$ and $Q(F_n^2)_M = M$ are copulas.

Definition

An imprecise copula (P, Q) **avoids sure loss** if there exists a discrete copula C such that $P \leq C \leq Q$.

Definition

Suppose that an imprecise copula (P, Q) *avoids sure loss*. Then (P, Q) is **coherent** if $P(r, s) = \inf_{C \in \mathcal{C}(P,Q)}\{C(r, s)\}$ and $Q(r, s) = \sup_{C \in \mathcal{C}(P,Q)}\{C(r, s)\}$ for all $r, s \in L_n$. Here $\mathcal{C}(P, Q)$ is the set of all copulas such that $P(r, s) \leq C(r, s) \leq Q(r, s)$ for all r and s.

- Omladič and Stopar in 2020 gave an example of imprecise copula that **does not avoid sure loss** (nor it is coherent).
- All imprecise copulas in the maximal chain above are coherent, so are those of the example generalizing Omladič and Stopar example (A_7, B_7).

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