A proof of the Box Conjecture for commuting pairs of matrices

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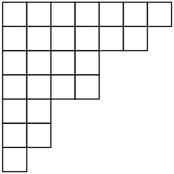
Contents

Formal setup

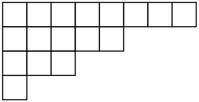
- F an infinite field, $B \in M_n(F)$ a **nilpotent matrix**,
- $\mathcal{N}_B = \{A \in M_n(F);\ A^n = 0,\ AB = BA\}$ the *nilpotent commutator*.
- The *Jordan type* of B is the partition $P \in \mathcal{P}$ that determines the Jordan canonical structure of B.
- $\mathcal{P}$ the set of all partitions of all natural numbers $P = (p_1, p_2, \ldots, p_k) \in \mathcal{P}$, $k \in \mathbb{N}$, $p_i \geq p_{i+1}$ for all i and $p_k > 0$.
- $Q = (q_1, q_2, \ldots, q_m)$ is a *Rogers-Ramanujan* (R-R) (or *super-distinct*) partition $\iff q_i - q_{i+1} \geq 2$ for all i.
  $\mathcal{Q}$ the set of all R-R partitions.
- $P = (p_1, p_2, \ldots, p_k) \in \mathcal{P}$ is *almost-rectangular* $\iff p_1 - p_k \leq 1$.

Ferrers diagrams of different types of partitions

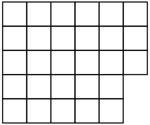
Partition: $(7, 6, 4, 4, 2, 2, 1)$ - a general partition



Partition: $(8, 5, 3, 1)$ - an R-R partition



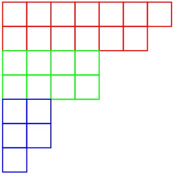
Partition: $(6, 6, 6, 5, 5)$ - an AR partition



The story of the map $\mathfrak{D} : \mathcal{P} \to \mathcal{P}$

- $\mathcal{N}_B$ is an irreducible variety (Basili 2003, Baranovsky 2002).
- Thus, there is a nilpotent orbit with respect to $\mathrm{GL}_n(F)$-action on $M_n(F)$ such that its intersection with $\mathcal{N}_B$ is Zariski dense in $\mathcal{N}_B$.
- Panyushev (2009) defined a map $\mathfrak{D} : \mathcal{P} \to \mathcal{P}$ so that $\mathfrak{D}(P)$ is the partition of the *dense orbit* in $\mathcal{N}_B$ for P being the Jordan type of B.
- The number of parts of $\mathfrak{D}(P)$ is equal to the smallest number of AR subpartitions needed to cover P (Basili 2000).
- $\mathfrak{D}(P)$ is an R-R partition for each $P \in \mathcal{P}$ (Basili, Iarrobino 2008).
- The longest part of $\mathfrak{D}(P)$ (Oblak, 2008)
- The smallest part of $\mathfrak{D}(P)$ (Khatami, 2014)
- $\mathfrak{D}$ is an idempotent map (K., Oblak, 2009).
- The Box conjecture on the form od $\mathfrak{D}^{-1}(Q)$ for a given $Q \in \mathcal{Q}$ (Iarrobino, Khatami, Van Steirteghem, Zhao, 2014).
- Proof of the Table Theorem for $Q \in \mathcal{Q}$ with two parts (Iarrobino, Khatami, Van Steirteghem, Zhao, 2014).

$\mathfrak{D}(P)$ for partition $P = (7, 6, 4, 4, 2, 2, 1)$ of 26



$|\mathfrak{D}(P)| = 3$
We will see how to show that $\mathfrak{D}(P) = (13, 9, 4)$.

The statement of the Box Conjecture

- $Q = (q_1, q_2, \dots, q_k)$ an R-R partition. The key of Q is the sequence $(s_1, s_2, \dots, s_k)$ where

$$s_i = q_i - q_{i+1} - 1 \text{ for } 1 \le i \le k-1, \text{ and } s_k = q_k.$$

Example

Q	the key
$(11, 7, 3)$	$(3, 3, 3)$
$(9, 6)$	$(2, 6)$
$(7, 5, 3, 1)$	$(1, 1, 1, 1)$
$(23, 18, 8, 3)$	$(4, 9, 4, 3)$

Conjecture (Iarrobino et al, 2014)

Q an R-R partition. The elements of $\mathfrak{D}^{-1}(Q)$ can be arranged in an array (box) of sizes $s_1 \times s_2 \times \cdots \times s_k$ such that the partition in the $(i_1, i_2, \dots, i_k)$-th position has exactly $\sum_{j=1}^{k} i_j$ parts.

The setup

- $P = (p_1, \ldots, p_k) \in \mathcal{P}$ a partition, p_i the *parts* of P.
- The *size* of P is $|P| = \sum_{i=1}^{k} p_i$.
- The *length* of P is the number of parts, $\ell(P) = k$.
- The *2-measure* of P, denoted $\mu_2(P)$, is the maximum length of a super-distinct subpartition of P, or equivalently, the minimal number of AR subpartitions to cover P.
- The *empty partition* is the unique element $\varepsilon \in \mathcal{P}$ of size and length 0.

Example

Partition $P = (8, 7, 4, 4, 3, 2, 2, 1)$ has $|P| = 31$, $\ell(P) = 8$.
It contains the subpartition $(7, 4, 1) \in \mathcal{Q}$ of length 3 and $\mu_2(P) = 3$.
Alternatively, P can be covered by AR subpartitions $(8, 7)$, $(4, 4, 3)$ and $(2, 2, 1)$.

The Burge correspondence

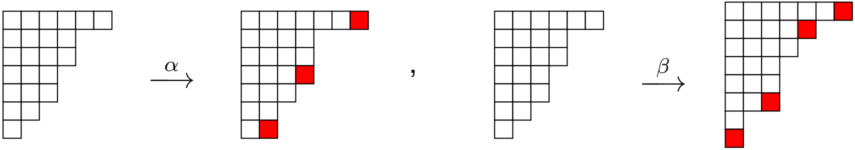
- $\{\alpha, \beta\}^*$ the free monoid on two symbols α and β
- $\mathcal{W}$ the subset of of all words that end with a singleton α.
- α and β are represented by injective maps $\alpha, \beta : \mathcal{P} \to \mathcal{P}$.
- α takes $P \in \mathcal{P}$ and adds one square consecutively to each AR block of P starting from the smallest part, while β adds 1 part of size 1, and then acts as α on the parts od size 2 or more.
- Denote by $\mathcal{A}$ the image of α and by $\mathcal{B}$ the image of β. Then:

Lemma (Burge, 1981)

Maps α and β are bijections from $\mathcal{P}$ to $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, $\mathcal{A} \cup \mathcal{B} = \mathcal{P}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Burge maps on partitions

Given a partition $P = (6, 4, 4, 3, 3, 2, 1)$ of $n = 23$ we show how the maps α and β work:



So $\alpha(P) = (7, 4, 4, 4, 3, 2, 2)$ and $\beta(P) = (7, 5, 4, 3, 3, 3, 1, 1)$.

We have $|\alpha(P)| = 26$ and $|\beta(P)| = 27$.

The inverse of the Burge maps

- α and β have surjective inverses $\alpha^{-1} : \mathcal{A} \to \mathcal{P}$ and $\beta^{-1} : \mathcal{B} \to \mathcal{P}$.
- Their 'union' is well defined map $\partial : \mathcal{P} \to \mathcal{P}$:

$$\partial(P) = \begin{cases} \alpha^{-1}(P) & \text{if } P \in \mathcal{A}, \\ \beta^{-1}(P) & \text{if } P \in \mathcal{B}. \end{cases}$$

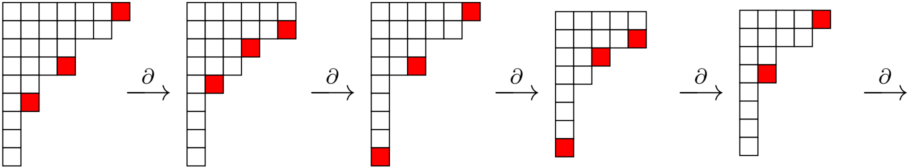
- Map ∂ is a 2-to-1 map.
- Obviously $|\partial(P)| < |P|$ for $P \neq \varepsilon$.
- Applying ∂ repeatedly to any $P \in \mathcal{P}$ will result in $\partial^k P = \varepsilon$ for some least $k \geq 1$.
- Recording along the way which map was ∂ inverting gives a unique word $\Omega(P)$ in $\mathcal{W}$.
- Inversely, taking a word ω in $\mathcal{W}$ and applying its letters iteratively starting with empty partition ε will lead to a unique partition $P(\omega)$.

Theorem (Burge, 1981)

Maps $\Omega : \mathcal{P} \to \mathcal{W}$ and $P : \mathcal{W} \to \mathcal{P}$ give a bijective correspondence between the set of all partitions $\mathcal{P}$ and the set $\mathcal{W}$ of all binary words that end with singleton α.

Burge code for a partition

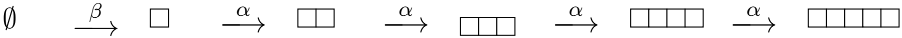
For partition $P = (7, 6, 4, 4, 2, 2, 1)$ of $n = 26$ one has:



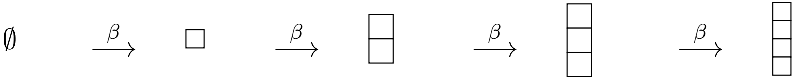
So the Burge code $\Omega(P)$ starts with $\alpha\alpha\beta\beta\alpha\ldots$

Partitions from Burge codes

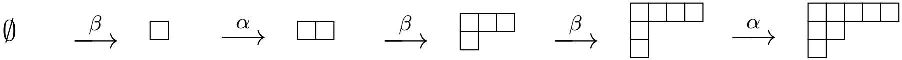
Word $\alpha\alpha\alpha\alpha\beta\alpha$ corresponds to partition (5):



Word $\beta\beta\beta\beta\alpha$ corresponds to partition $(1,1,1,1)$:



Given word $\omega = \alpha\beta\beta\alpha\beta\alpha \in \mathcal{W}$ we draw Ferrers diagrams for emerging partitions and obtain $P(\omega) = (5,2,1)$:



Descent sets, descent numbers and major indices

Example

Consider

$$\omega = \beta|\alpha\beta|\alpha\alpha\alpha\beta\beta\beta|\alpha.$$

Red lines | denote positions of descents, that is, places where β preceeds α. Natural order is $\alpha < \beta$.

- For $\omega = \omega_1\omega_2\cdots\omega_n \in \mathcal{W}$ its *descent set* $\mathrm{Des}(\omega)$ is the set of all indices i for which $\omega_i\omega_{i+1} = \beta\alpha$.
- These indeces differ pairwise by at least 2.

Example

The descent set of $\omega = \beta|\alpha\beta|\alpha\alpha\alpha\beta\beta\beta|\alpha$ is

$$\mathrm{Des}(\omega) = \{1, 3, 9\}.$$

Descent sets, descent numbers and major indices – 2

- The *descent number* and *major index* of ω are

$$\mathrm{des}(\omega) = |\mathrm{Des}(\omega)| \text{ and } \mathrm{maj}(\omega) := \sum_{i \in \mathrm{Des}(\omega)} i.$$

Proposition

Let $P \in \mathcal{P}$ and let $\omega = \Omega(P)$. Then:

1. $\ell(P) = \#$*occurrences of* β *in* ω
2. $|P| = \mathrm{maj}(\omega)$
3. $\mu_2(P) = \mathrm{des}(\omega) = \#$*occurrences of* $\beta\alpha$ *in* ω

Example

For $\omega = \beta|\alpha\beta|\alpha\alpha\alpha\beta\beta\beta|\alpha$ we have $P = P(\omega) = (5, 3, 2^2, 1)$.
ω contains 5 copies of β, its descent set is $\mathrm{Des}(\omega) = \{1, 3, 9\}$, hence

$$\ell(P) = 5, \ |P| = 13 = \mathrm{maj}(\omega), \ \mu_2(P) = 3 = \mathrm{des}(\omega).$$

Characterisation of the descent map

Definition

For a partition P, let $P-1$ be the *reduced partition* obtained by subtracting 1 from each part of P and eliminating any resulting zeros.

For example: $P = (6, 4^3, 2^4, 1^3)$ and $P - 1 = (5, 3^3, 1^4)$.

Theorem

The descent map $P \mapsto \mathrm{Des}(P)$ is a size-preserving function from $\mathcal{P}$ to $\mathcal{Q}$ satisfying $\mathrm{Des}(\partial P) = \mathrm{Des}(P) - 1$. It is the unique such function.

The descent map and the dense orbit map coincide

Based on Shayman's description of the variety of invariant subspace of a nilpotent matrix from 1982 we get the following results:

Theorem

$P = P(B)$ the Jordan type of a nilpotent matrix B. Suppose that A is a generic nilpotent matrix commuting with B and that $W = \operatorname{Im} A$ is its image. Then the Jordan type of the restriction $B|_W$ is given by ∂P.

Corollary

$$\mathfrak{D}(P) = \mathrm{Des}(P)$$

for every $P \in \mathcal{P}$

The Box Theorem

Theorem

Q an R-R partition. The elements of $\mathfrak{D}^{-1}(Q)$ are arranged in an array of sizes $s_1 \times s_2 \times \cdots \times s_k$ such that the partition in the $(i_1, i_2, \ldots, i_k)$-th position has Burge code of the form

$$\alpha^{s_k - i_k}\beta^{i_k}|\alpha^{s_{k-1}+1-i_{k-1}}\beta^{i_{k-1}}|\alpha^{s_{k-2}+1-i_{k-2}}\beta^{i_{k-2}}|\cdots|\alpha^{s_1+1-i_1}\beta^{i_1}|\alpha. \tag{1}$$

The partition determined by (1) has exactly $\sum_j i_j$ (= the number of βs) parts.

$\mathfrak{D}^{-1}(9,6,2)$

(i_1, i_2, i_3)	code ω	partition $\Omega^{-1}(\omega)$	# parts
$(1,1,1)$	$\alpha\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$(9,6,2)$	3
$(2,1,1)$	$\beta\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$(9,6,1^2)$	4
$(1,2,1)$	$\alpha\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$(9,4,2^2)$	4
$(2,2,1)$	$\beta\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$(9,3^2,1^2)$	5
$(1,3,1)$	$\alpha\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$(9,4,2,1^2)$	5
$(2,3,1)$	$\beta\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$(9,4,1^4)$	6
$(1,1,2)$	$\alpha\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$(8,4,3,2)$	4
$(2,1,2)$	$\beta\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$(8,4,3,1^2)$	5
$(1,2,2)$	$\alpha\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$(8,4,2^2,1)$	5
$(2,2,2)$	$\beta\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$(8,3^2,1^3)$	6
$(1,3,2)$	$\alpha\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$(8,4,2,1^3)$	6
$(2,3,2)$	$\beta\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$(8,4,1^5)$	7

Some references

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Propetries of ∂

Lemma

For any $P \in \mathcal{P}$ we have:

1. $|\partial P| = |P| - \mu_2(P)$
2. $\ell(\partial P) = \begin{cases} \ell(P) - 1 & \text{if } P \in \mathcal{B} \\ \ell(P) & \text{if } P \in \mathcal{A}. \end{cases}$
3. $\mu_2(\partial P) = \begin{cases} \mu_2(P) - 1 & \text{if } P \in \mathcal{B} \text{ and } \partial P \in \mathcal{A} \\ \mu_2(P) & \text{otherwise.} \end{cases}$