Eigenvalue problems for two and more parameters

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Contents

Setup

- Given matrices

$$A_{10}, A_{11}, A_{12} \in M_{n_1}(\mathbb{C})$$

and

$$A_{20}, A_{21}, A_{22} \in M_{n_2}(\mathbb{C}),$$

- we consider twoparameter eigenvalue problem

$$\begin{aligned}(A_{10} - \lambda_1 A_{11} - \lambda_2 A_{12})v_1 &= 0,\\ (A_{20} - \lambda_1 A_{21} - \lambda_2 A_{22})v_2 &= 0,\end{aligned}$$

for $\lambda_1, \lambda_2 \in \mathbb{C}$, $v_1 \in \mathbb{C}^{n_1} \backslash 0$ and $v_2 \in \mathbb{C}^{n_2} \backslash 0$.

Tensor Product

- The tensor product $V := \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ is isomorphic to $\mathbb{C}^{n_1 \cdot n_2}$.
- Vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_1} \end{bmatrix} \in \mathbb{C}^{n_1}$$

and $v \in \mathbb{C}^{n_2}$ induce $u \otimes v$ in V represented by 'stacked' vector

$$\begin{bmatrix} u_1 v \\ u_2 v \\ \vdots \\ u_{n_1} v \end{bmatrix}.$$

Tensor Product 2

- Matrices $A \in M_{n_1}(\mathbb{C})$ and $B \in M_{n_2}(\mathbb{C})$ induce $A \otimes B$ acting on V represented by block-matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \vdots & \vdots & & \\ \vdots & & & \\ a_{n_1 1}B & a_{n_1 2}B & \cdots & a_{n_1 n_1}B \end{bmatrix} \in M_{n_1 n_2}(\mathbb{C}).$$

- For $A \in M_{n_1}(\mathbb{C})$ we write $A^\dagger = A \otimes I$ and for $B \in M_{n_2}(\mathbb{C})$ we write $B^\dagger = I \otimes B$.

Associated Tensor Determinants

- Given twoparameter eigenvalue problem we want to study it via a pair of standard eigenvalue problems.
- To do that Atkinson in 1960's introduced a 'Cramer Rule' type approach.
- Define first the determinant of the system:

$$\Delta_0 = A_{11} \otimes A_{22} - A_{12} \otimes A_{21}.$$

- Treat A_{10}, A_{20} as the right hand side and define two more determinants by:

$$\begin{aligned} & \Delta_1 = A_{10} \otimes A_{22} - A_{12} \otimes A_{20} \\ \text{and} \quad & \Delta_2 = A_{11} \otimes A_{20} - A_{10} \otimes A_{21}. \end{aligned}$$

Regularity Condition

Definition

A twoparameter system is called *regular* if the associated Δ_0 is invertible.

- Consider from now on only regular twoparameter eigenvalue problems.
- Define the Cramer type maps

$$\begin{aligned} &\Gamma_1 = \Delta_0^{-1}\Delta_1 \\ \text{and} \quad &\Gamma_2 = \Delta_0^{-1}\Delta_2. \end{aligned}$$

Cramer's Rule

Theorem (Atkinson, 1968)

For a given regular twoparameter eigenvalue problem we have:

- *Cramer's Rule:*

$$A_{11}^{\dagger}\Gamma_1 + A_{12}^{\dagger}\Gamma_2 = A_{10}^{\dagger},$$
$$A_{21}^{\dagger}\Gamma_1 + A_{22}^{\dagger}\Gamma_2 = A_{20}^{\dagger},$$

- *Commutativity:*

$$\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1.$$

Spectral Decomposition for Commuting Matrices

Theorem

Given a pair of commuting matrices A_1 and A_2 in $M_n(\mathbb{C})$ then their eigenvalues form pairs $(\alpha_{j1}, \alpha_{j2})$, $j = 1, 2, \ldots, k$, such that: Spectral decomposition:

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k,$$

where

$$V_j = \ker(A_1 - \alpha_{j1} I)^n \cap \ker(A_2 - \alpha_{j2} I)^n$$

is the joint spectral subspace for a joint eigenvalue $(\alpha_{j1}, \alpha_{j2})$.

- The dimension $\dim V_j$ is the multiplicity of $(\alpha_{j1}, \alpha_{j2})$. The sum of all multiplicities is equal to n.

Spectral Theorems for Two Parameters

Theorem (Atkinson, 1965)

The eigenvalues of a twoparameter eigenvalue problem are exactly the joint eigenvalues $(\alpha_{j1}, \alpha_{j2})$, $j = 1, 2, \ldots, k$ *of the associated commuting matrices* Γ_1 *and* Γ_2.
Furthermore, we have

$$ker(\Gamma_1 - \alpha_{j1} I) \cap \ker(\Gamma_2 - \alpha_{j2} I) = \ker(W_1(\alpha_j)) \otimes \ker(W_2(\alpha_j)),$$

where $W_i(\alpha_j) = A_{i0} - \alpha_{j1} A_{i1} - \alpha_{j2} A_{i2}$, $i = 1, 2$.
There are $n_1 n_2$ *eigenvalues counting multiplicities.*

Cayley-Hamilton Theorem

Theorem (TK, 2003)

For a given twoparameter eigenvalue problem write

$$w_i(\lambda_1, \lambda_2) = \det(A_{i0} - \lambda_1 A_{1i} - \lambda_2 A_{i2})$$

for $i = 1, 2$. *Then*

$$w_i(\Gamma_1, \Gamma_2) = 0, \quad i = 1, 2.$$

- All the above results were originally proved in general multiparameter setup.
- A number of numerical methods for standard eigenvalue problems were generalized to two and more parameters. Plestenjak maintains an extensive Matlab toolbox for multiparameter eigenvalue problems at www.mathworks.com.

Setup

- Given $k \geq 2$, an integer, and rectangular matrices

$$A, B_1, B_2 \ldots, B_k \in M_{(n+k-1)\times n}(\mathbb{C}),$$

- we consider multiparameter eigenvalue problem

$$(A - \lambda_1 B_1 - \lambda_2 B_2 - \ldots - \lambda_k B_k)v = 0.$$

  for $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$, and $v \in \mathbb{C}^n \backslash 0$.
- Write $M(\lambda) = A - \lambda_1 B_1 - \lambda_2 B_2 - \ldots - \lambda_k B_k$ and $\mathcal{L}$ for the linear subspace spanned by $B_1, B_2, \ldots, B_k$.

Regularity Assumption

Definition

The *normal rank* of $M(\lambda)$ is defined as

$$\operatorname{nrank}(M) := \max_{\lambda \in \mathbb{C}^k} \{\operatorname{rank}(M(\lambda))\}.$$

Definition

A rectangular eigenvalue problem is *regular* if $\operatorname{nrank}(M) = n$, $\dim \mathcal{L} = k$ (i.e. $B_1, B_2, \ldots, B_k$ are linearly independent), and $\mathcal{L}$ is in generic position with respect to $\mathcal{M}$, the variety of matrices of rank at most $n - 1$.

- From now we consider only regular rectangular eigenvalue problems.

Eigenvalues of a Rectangular Eigenvalue Problem

Theorem (Shapiro, Shapiro, 2009)

A regular rectangular eigenvalue problem has exactly $\binom{n+k-1}{k}$ eigenvalues counting multiplicities. In particular, it has finitely many eigenvalues.

- Consider $k = 1$ and $B_1 = I$ first.
- Twoparameter rectangular eigenvalue problem has $\binom{n+1}{2} = \frac{n(n+1)}{2}$ eigenvalues counting multiplicities.
- Geometry behind the result: The variety $\mathcal{M}$ in $M_{(n+k-1)\times n}$ has codimension equal to k and degree equal to $\binom{n+k-1}{k}$.

Twoparameter Rectangular Eigenvalue Problem via Atkinson's Approach

- Given a rectangular twoparameter eigenvalue problem

$$M(\lambda)v = (A - \lambda_1 B_1 - \lambda_2 B_2)v = 0$$

  choose two matrices $P_1, P_2 \in M_{n\times(n+1)}(\mathbb{C})$.
- Multiply $M(\lambda)v = 0$ by P_1 and P_2 to obtain

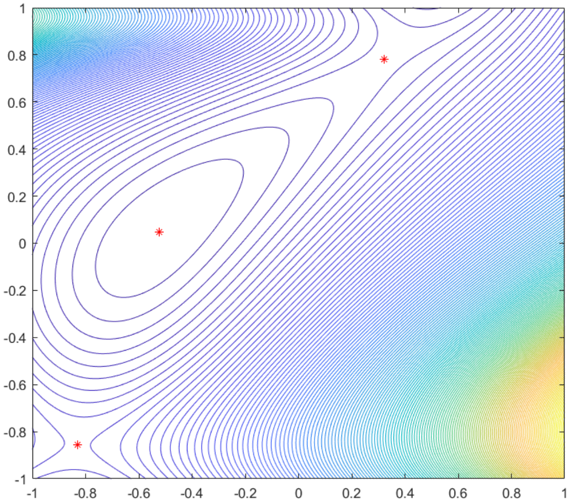
$$(P_1 A - \lambda_1 P_1 B_1 - \lambda_2 P_1 B_2)v = 0,$$
$$(P_2 A - \lambda_1 P_2 B_1 - \lambda_2 P_2 B_2)v = 0.$$

- For a generic choice of P_1 and P_2 the new twoparameter eigenvalue problem with square matrices is regular.
- One may use any numerical method for such problems to find eigenvalues of the original rectangular eigenvalue problem.

Caveat

- The sum of multiplicities of the new eigenvalue problem is n^2, while the sum of multiplicities of the rectangular eigenvalue problem is $\frac{n(n+1)}{2}$. So we get an access of solutions (eigenvalue–eigenvector pairs). We detect the access ones by substituting them back to the original eigenvalue problem.
- The numerical methods for rectangular eigenvalue problems are presented in a joint paper with Michiel Hochstenbach and Bor Plestenjak. We generalized the theory also to polynomial multiparameter eigenvalue problems that occur in applications.

Graphics



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