Copulas arising in shock models

Colloquium talk at the Technical University Eindhoven, Eindhoven, The Netherlands

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[1] A report on joint work with Damjana Kokol Bukovšek, Matjaž Omladič, and Blaž Mojškerc.

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Sklar's Theorem, 1959

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- The beginning of the subject is the well-known theorem by A. Sklar that provided an answer to a question by M. Fréchet.

Theorem (A. Sklar, 1959)

Let $H(x, y)$ be a joint distribution function with marginal distribution functions F and G. Then there exists a copula C such that

$$H(x, y) = C\left(F(x), G(y)\right) \quad \text{for all } x, y \in \mathbb{R}.$$

If F and G are continuous then C is uniquely determined.

Examples of bivariate copulas

- The three most important copulas are:

  $$M(u,v) = \min\{u,v\},\ \Pi(u,v) = uv,\ W(u,v) = \max\{0, u+v-1\}.$$

  The Fréchet-Hoeffding upper bound M, the independence copula Π and the Fréchet-Hoeffding lower bound W.

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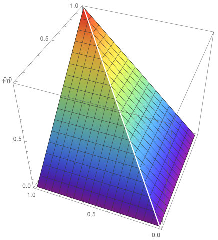

Figure: $M(u,v)$

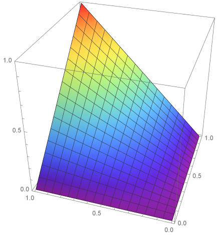

Figure: $\Pi(u,v)$

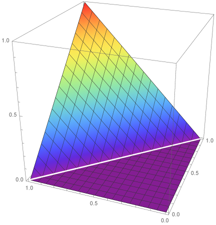

Figure: $W(u,v)$

Properties of bivariate copulas

- For any copula $C(u, v)$ we have

$$W(u, v) \leq C(u, v) \leq M(u, v) \text{ for all } u, v \in [0, 1].$$

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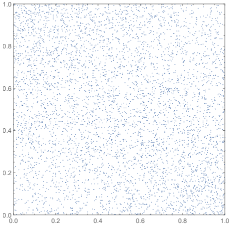
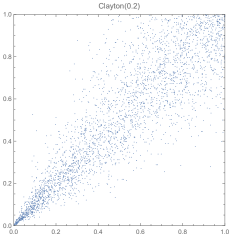
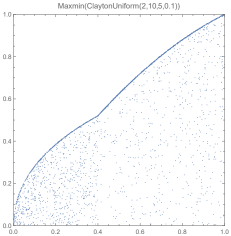
- We often present copulas with scatterplots:







More basics on copulas-2

- Suppose $C(u,v)$ is a copula corresponding to a random vector (X,Y). Then:
  - $C^{\sigma}(u,v) = u - C(u, 1-v)$ is a copula corresponding to the random vector $(X,-Y)$. It is called a *reflected copula* of C.

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  - $\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ is a copula corresponding to the random vector $(-X, -Y)$. $\widehat{C}$ is called the *survival copula* of C.

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- A copula C is called *positively quadrant dependent*, or *PQD* for short, if $C(u,v) \geq \Pi(u,v)$ for all $u,v \in [0,1]$.

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- A copula C is called *positively quadrant dependent*, or *PQD* for short, if $C(u, v) \geq \Pi(u, v)$ for all $u, v \in [0, 1]$.
- A copula C is called *negatively quadrant dependent*, or *NQD* for short, if $C(u, v) \leq \Pi(u, v)$ for all $u, v \in [0, 1]$.

More basics on copulas-3

- A copula C is *nonsingular (or absolutely continuous)* if it has density
$$c(u,v) = \frac{\partial^2 C}{\partial u\, \partial v}(u,v)$$
such that
$$C(u,v) = \int_0^u \int_0^v c(s,t)\, ds\, dt.$$

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such that
$$C(u, v) = \int_0^u \int_0^v c(s, t) \, ds \, dt.$$
- A copula has no isolated singular points (no mass in a single point). Singularities may be on arcs or more complicated (on Cantor-like sets, etc.).

Ljubljana, last Saturday

Marshall-Olkin Model, 1967

- Consider lifetimes U and V of two components of a system. Three different independent shocks act on it:
  - Shock X represented by a Poisson process with intensity $\lambda > 0$ acts on the first component only.
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- Then $U = \min\{X, Z\}$ and $V = \min\{Y, Z\}$.

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- The model is motivated by applications: in medicine (e.g. cancer treatment), hydrology (floods modelling), finance and economics, engineering, etc.

Marshall-Olkin Copulas-2

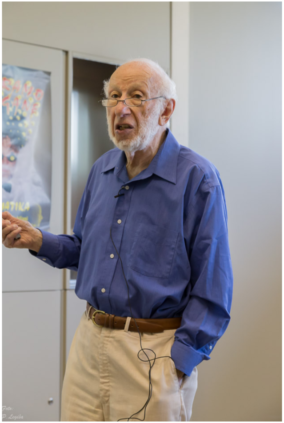

Figure: Ingram Olkin

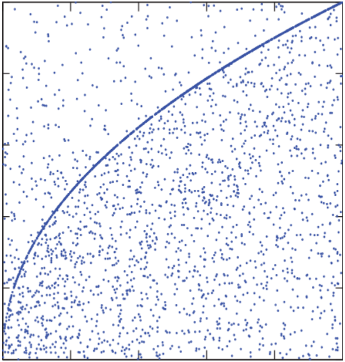

Figure: Scatterplot of a Marshall-Olkin copula

Marshall's Observation

- Observe that function $f(u) = u^a$ for $0 < a < 1$ is such that f maps $[0, 1]$ to itself, $f(0) = 0$, $f(1) = 1$, and function $f^*(u) = \frac{f(u)}{u} = u^{a-1}$ is decreasing on $(0, 1]$.

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- Note that $f^*(1) = 1$ and $f^*(u) \geq 1$ for $u \in (0, 1)$.
- We write $\mathcal{F}$ for the set of all such functions f.

Marshall's Theorem

Theorem (A. W. Marshall, 1996)

Suppose that f, g belong to $\mathcal{F}$. Then the function

$$C(u, v) = \min\{u\, g(v), f(u)\, v\} = u\, v \min\{f^*(u), g^*(v)\}$$

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- A Marshall copula is PQD.

Marshall's Shock Models, 1996

Theorem (A. W. Marshall, 1996)

Suppose C is a Marshall copula with generators f and g, and $H(x,y) = C(F(x), G(y))$ for some distribution functions F and G. Then the following are equivalent:

1. *Random variables U and V with joint distribution function H have a representation of the form $U = \max\{X, Z\}$ and $V = \max\{Y, Z\}$, where X, Y and Z are independent random variables (representing shocks).*
2. *H has the form $H(x,y) = F_X(x)F_y(y)F_Z(\min\{x,y\})$, where F_X, F_Y and F_Z are distribution functions.*
3. *$f^*(F(x)) = g^*(G(x))$, i.e., $G(x) = g^{*-}(f^*(F(x))$, where g^{*-} is left inverse of g^*.*

23rd International Workshop on Matrices and Statistics - Ljubljana 2014

Definition of a maxmin copula

- Suppose that ψ is an increasing function (not necessarily strictly) defined on $[0,1]$, such that $\psi(0)=0$, $\psi(1)=1$, and function $\psi_*(u)=\frac{1-\psi(u)}{u-\psi(u)}$ is decreasing on $(0,1)$.

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Suppose that $\varphi \in \mathcal{F}$ and $\psi \in \mathcal{G}$. Then the function

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Maxmin Shock Models

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Suppose C is a maxmin copula with generators φ and ψ, and $H(x,y) = C(F(x), G(y))$ for some distribution functions F and G. Then the following are equivalent:

1. *Random variables U and V with joint distribution function H have a representation of the form $U = \max\{X, Z\}$ and $V = \min\{Y, Z\}$, where X, Y and Z are independent random variables.*
2. *$\varphi(F(x))\,(G(x) - \psi(G(x)) = F(x)\,(1 - \psi(G(x))$, i.e., $G(x) = \psi_*^-(\varphi^*(F(x))$, where ψ_*^- is left inverse of ψ_*.*

23rd IWMS, Ljubljana 2014

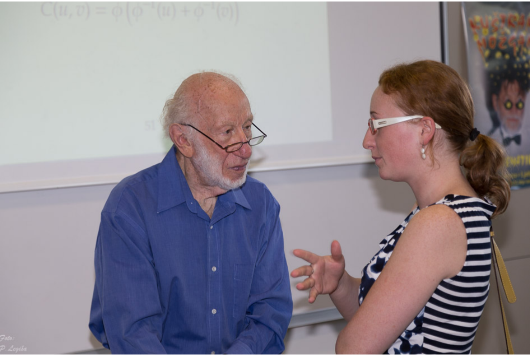

Figure: Ingram Olkin and Nina Ružić

Figure: Matjaž Omladič

Reflected maxmin copulas

Theorem (TK, M. Omladič, 2020)

Suppose that φ and ψ are the generators of a maxmin copula C. Then

$$C^\sigma(u,v) = \max\{0, uv - f(u)g(v)\},$$

where $f(u) = \varphi(u) - u$ and $g(v) = 1 - v - \psi(1-v)$, is its reflected copula. Generators f and g of C^σ belong to $\mathcal{F}'$.

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- We call C^σ a *reflected maxmin copula*, or an *RMM copula* for short. It is NQD.
- RMM copulas are nicer to deal with. They are symmetric if $f = g$, which is not the case for maxmin copulas. With reflection operation it is easy to transfer properties from RMM to maxmin copulas.

Some scatterplots

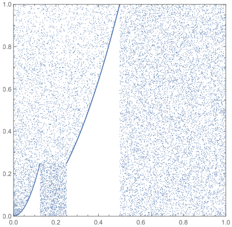


Figure: a Marshall copula

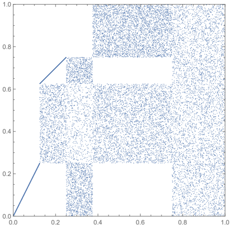


Figure: a maxmin copula

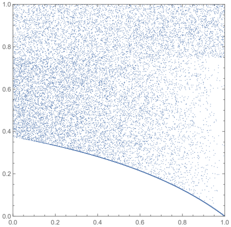
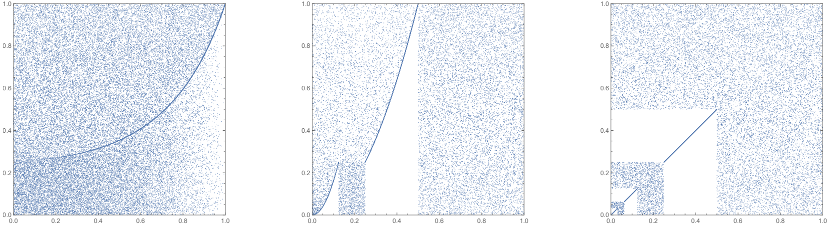


Figure: an RMM copula

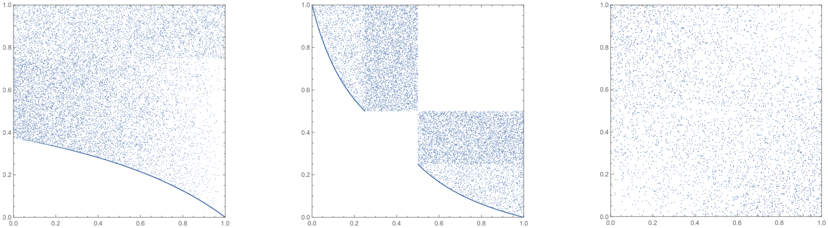
Singularities, TK., M. Omladič, 2022

- The only nonsingular Marshall copula is Π. A Marshall copula is singular on certain arcs of the form $(u, \chi(u))$, $u \in (a, b)$. The 'density' on an arc is given by $h(u) = \chi(u)\,[f^*(u) - f'(u)]$.



Singularities-2, TK., M. Omladič, 2022

- An RMM copula is nonsingular if and only if $uv \geq f(u)g(v)$ for all $u, v \in [0.1]$. If it is singular then it is so on certain arcs of the form $(u, \chi(u))$, $u \in (a, b)$. The 'density' on an arc is given by $h(u) = \chi(u)\left[1 - \frac{f'(u)}{f^*(u)}\right]$.



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- The maximal value of μ_∞ on the set of all copulas is $\frac{1}{3}$, on the set of all PQD copulas it is $3 - 2\sqrt{2} \approx 0.172$ and on the set of all NQD copulas it is $\sqrt{5} - 2 \approx 0.236$.

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- The maximal value of $\mu_\infty(u,v)$ on the set of Marshall copulas is $\frac{4}{27} \approx 0.148$. The same is its maximal value on the set of all maxmin copulas, while on the set of all RMM copulas it is $3 - 2\sqrt{2}$.

Nonexchangeability-2

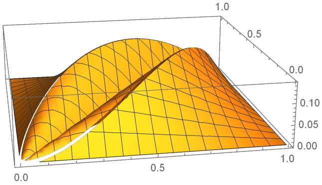
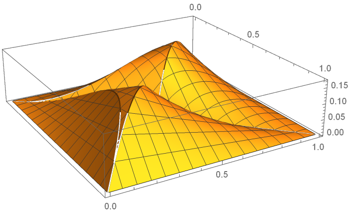


Figure: Function μ_∞ for the set of all Marshall copulas.



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Other Topics on Shock model topics

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- Copulas for Dependent Shocks.[Durante, Gerard, Mazo, 2016],[Durante, Omladič, Oražem, Ružić, 2017],[Kokol Bukovšek, TK, Omladič, Mojškerc, 2020]

Other Topics on Shock model topics

- Concordances and Tail Dependences.[Durante, Omladič, Oražem, Ružić, 2017]
- Extreme Generators.[Kokol Bukovšek, TK, Omladič, Mojškerc, 2022]
- Copulas for Dependent Shocks.[Durante, Gerard, Mazo, 2016],[Durante, Omladič, Oražem, Ružić, 2017],[Kokol Bukovšek, TK, Omladič, Mojškerc, 2020]
- All of these has been generalized to d-dimensional copulas ($d \geq 3$).

Ljubljana, last Saturday-2

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12. Scatterplots by Blaž Mojškerc, photos from the 23rd IWMS by Peter Legiša.