Singular components in shock models

Talk at the 11th International Conference on Soft Methods in Probability and Statistics Salzburg, Austria, September 2024

Tomaž Košir[1]

Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia

September 4, 2024

[1] A report on joint work with Matjaž Omladič.

Contents

1 Copulas for shock models

Contents

1. Copulas for shock models
2. Singularities of shock model copulas
    - Marshall Copulas
    - RMM Copulas
    - Maxmin Copulas

1. Copulas for shock models

2. Singularities of shock model copulas
   - Marshall Copulas
   - RMM Copulas
   - Maxmin Copulas

Definitions

- A **bivariate copula** $C(u,v)$ is a function $C : [0,1]^2 \to [0,1]$ such that:
  1. $C(u,0) = 0 = C(0,v)$, $C(u,1) = u$, $C(1,v) = v$ for all $u, v \in [0,1]$,
  2. $C(u_2,v_2) + C(u_1,v_1) - C(u_2,v_1) - C(u_1,v_2) \geq 0$ for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$.

Definitions

- A **bivariate copula** $C(u,v)$ is a function $C:[0,1]^2 \to [0,1]$ such that:
  1. $C(u,0)=0=C(0,v)$, $C(u,1)=u$, $C(1,v)=v$ for all $u,v\in[0,1]$,
  2. $C(u_2,v_2)+C(u_1,v_1)-C(u_2,v_1)-C(u_1,v_2)\geq 0$ for all $0\leq u_1\leq u_2\leq 1$ and $0\leq v_1\leq v_2\leq 1$.
- Let $\mathcal{F}$ be the set of all functions $f:[0,1]\to[0,1]$ such that:
  1. $f(0)=0$, $f(1)=1$,
  2. f is nondecreasing,
  3. function $f^*(u)=\frac{f(u)}{u}:(0,1]\to\mathbb{R}$ is nonincreasing.

Marshall Copulas

Theorem (A. W. Marshall, 1996)

Suppose that f, g belong to $\mathcal{F}$. Then the function

$$C(u, v) = \min\{u\, g(v), f(u)\, v\} = u\, v \min\{f^*(u), g^*(v)\}$$

is a copula.

Marshall Copulas

Theorem (A. W. Marshall, 1996)

Suppose that f, g belong to $\mathcal{F}$. Then the function

$$C(u,v) = \min\{u\,g(v), f(u)\,v\} = u\,v\min\{f^*(u), g^*(v)\}$$

is a copula.

- A copula C described in the theorem is called a *Marshall copula*, functions f and g are its *generators*.

Marshall Copulas

Theorem (A. W. Marshall, 1996)

Suppose that f, g belong to $\mathcal{F}$. Then the function

$$C(u, v) = \min\{u\, g(v), f(u)\, v\} = u\, v \min\{f^*(u), g^*(v)\}$$

is a copula.

- A copula C described in the theorem is called a *Marshall copula*, functions f and g are its *generators*.
- Given three independent shocks X, Y and Z suppose that $U = \max\{X, Z\}$ and $V = \max\{Y, Z\}$. Then the copula connecting U and V is a Marshall copula:

Marshall's Shock Model, 1996

Theorem (A. W. Marshall, 1996)

Suppose C is a Marshall copula with generators f and g, and $H(x,y) = C(F(x), G(y))$ for some distribution functions F and G. Then the following are equivalent:

1. *Random variables U and V with joint distribution function H have a representation of the form $U = \max\{X, Z\}$ and $V = \max\{Y, Z\}$, where X, Y and Z are independent random variables (representing shocks).*
2. *H has the form $H(x,y) = F_X(x)F_y(y)F_Z(\min\{x,y\})$, where F_X, F_Y and F_Z are distribution functions.*
3. *$f^*(F(x)) = g^*(G(x))$, i.e., $G(x) = g^{*-}(f^*(F(x))$, where g^{*-} is left inverse of g^*.*

Marshall-Olkin Copulas, 1967

- When X, Y and Z have Poisson distribution then the copula corresponding to (U, V) is given by

$$C(u, v) = \min\{uv^{\alpha}, u^{\beta}v\}$$

for some $\alpha, \beta \in [0, 1]$.

Marshall-Olkin Copulas, 1967

- When X, Y and Z have Poisson distribution then the copula corresponding to (U, V) is given by

$$C(u, v) = \min\{uv^{\alpha}, u^{\beta}v\}$$

  for some $\alpha, \beta \in [0, 1]$.
- The model is motivated by several applications: in medicine (e.g. cancer treatment), hydrology (floods modelling), finance and economics, engineering, etc.

Maxmin Copulas

- Suppose that ψ is a nondecreasing function on $[0, 1]$, such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi_*(u) = \frac{1-\psi(u)}{u-\psi(u)}$ is nonincreasing on $(0, 1)$.
- Write $\mathcal{G}$ for the set of all such functions ψ.

Maxmin Copulas

- Suppose that ψ is a nondecreasing function on $[0, 1]$, such that $\psi(0) = 0$, $\psi(1) = 1$, and $\psi_*(u) = \frac{1-\psi(u)}{u-\psi(u)}$ is nonincreasing on $(0, 1)$.
- Write $\mathcal{G}$ for the set of all such functions ψ.

Theorem (M. Omladič, N. Ružić, 2016)

Suppose that $\varphi \in \mathcal{F}$ and $\psi \in \mathcal{G}$. Then the function

$$C(u, v) = uv + \min\{u(1 - v),\ (\varphi(u) - u)(v - \psi(v))\}$$

is a copula.

- A copula C described in the theorem is called a *maxmin copula*, functions φ and ψ are its *generators*.

Maxmin Shock Model

Theorem (M. Omladič, N. Ružić, 2016)

Suppose C is a maxmin copula with generators φ and ψ, and $H(x,y) = C(F(x), G(y))$ for some distribution functions F and G. Then the following are equivalent:

1. *Random variables U and V with joint distribution function H have a representation of the form $U = \max\{X, Z\}$ and $V = \min\{Y, Z\}$, where X, Y and Z are independent random variables.*
2. $\varphi(F(x))\,(G(x) - \psi(G(x)) = F(x)\,(1 - \psi(G(x))$, *i.e.,* $G(x) = \psi_*^-(\varphi^*(F(x))$, *where ψ_*^- is left inverse of ψ_*.*

23rd IWMS, Ljubljana 2014 (PHOTOS BY PETER LEGIŠA)

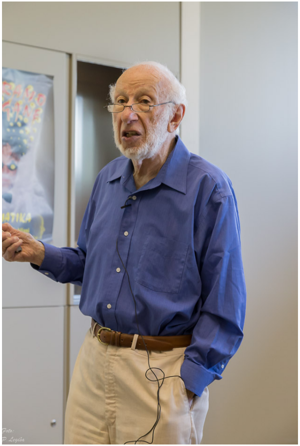


Figure: Ingram Olkin

Figure: Ingram Olkin and Nina Ružić

Reflected maxmin copulas

- We denote by $\mathcal{F}'$ the set of all functions $f : [0,1] \to [0,1]$ such that $x + f(x)$ belong to $\mathcal{F}$.

Theorem (TK, M. Omladič, 2020)

Suppose that φ and ψ are the generators of a maxmin copula C. Then

$$C^{\sigma_2}(u,v) = \max\{0, uv - f(u)g(v)\},$$

where $f(u) = \varphi(u) - u$ and $g(v) = 1 - v - \psi(1-v)$, is its reflected copula. Generators f and g of C^{σ_2} belong to $\mathcal{F}'$.

Reflected maxmin copulas

- We denote by $\mathcal{F}'$ the set of all functions $f : [0,1] \to [0,1]$ such that $x + f(x)$ belong to $\mathcal{F}$.

Theorem (TK, M. Omladič, 2020)

Suppose that φ and ψ are the generators of a maxmin copula C. Then

$$C^{\sigma_2}(u,v) = \max\{0, uv - f(u)g(v)\},$$

where $f(u) = \varphi(u) - u$ and $g(v) = 1 - v - \psi(1-v)$, is its reflected copula. Generators f and g of C^{σ_2} belong to $\mathcal{F}'$.

- C^{σ_2} is a *reflected maxmin copula* (or an *RMM copula*).

Reflected maxmin copulas

- We denote by $\mathcal{F}'$ the set of all functions $f : [0, 1] \to [0, 1]$ such that $x + f(x)$ belong to $\mathcal{F}$.

Theorem (TK, M. Omladič, 2020)

Suppose that φ and ψ are the generators of a maxmin copula C. Then

$$C^{\sigma_2}(u, v) = \max\{0, uv - f(u)g(v)\},$$

where $f(u) = \varphi(u) - u$ and $g(v) = 1 - v - \psi(1 - v)$, is its reflected copula. Generators f and g of C^{σ_2} belong to $\mathcal{F}'$.

- C^{σ_2} is a *reflected maxmin copula* (or an *RMM copula*).
- RMM copulas are nicer to deal with. They are symmetric if $f = g$, which is not the case for maxmin copulas. With reflection operation it is easy to transfer properties from RMM to maxmin copulas.

1. Copulas for shock models

2. Singularities of shock model copulas
   - Marshall Copulas
   - RMM Copulas
   - Maxmin Copulas

Constancy segments

- Consider a Marshall copula $C(u, v) = \max\{ug(v), f(u)v\}$. Then C is absolutely continuous everywhere except possibly on the set

$$L_0(f, g) = \{(u, v) | \; f(u)v = ug(v)\} = \{(u, v) | \; f^*(u) = g^*(v)\}.$$

Constancy segments

- Consider a Marshall copula $C(u,v) = \max\{ug(v), f(u)v\}$. Then C is absolutely continuous everywhere except possibly on the set

$$L_0(f,g) = \{(u,v) |\ f(u)v = ug(v)\} = \{(u,v) |\ f^*(u) = g^*(v)\}.$$

- Call a segment $S = [a,b] \subseteq [0,1], a < b$, a *constancy segment* (for f^*) if for all $u, u' \in [a,b]$ we have $f^*(u) = f^*(u')$. We assume that every constancy segment is maximal in the sense that for all $u \in (0,1] \setminus S$ we have $f^*(u) \neq f^*(S)$.
- If $f^*(S) = \lambda$ for some $\lambda \in [0,\infty)$ we denote this segment by $S^f_\lambda = [a^f_\lambda, b^f_\lambda]$. The set of all constancy segments of the function f^* will be denoted by $\mathcal{I}_f$ and the set of all possible λ's such that $\lambda = f^*(S)$, for some $S \in \mathcal{I}_f$, by $\mathcal{D}_f$.

Monotonicity segments

- An interval (a, b) will be called a *monotonicity interval* for f^* if for all $u, u' \in (a, b), u < u'$, we have $f^*(u) > f^*(u')$. Again, we assume that every monotonicity interval is maximal.
- The set of all monotonicity intervals of the function f^* will be denoted by $\mathcal{M}_f$.

Monotonicity segments

- An interval (a, b) will be called a *monotonicity interval* for f^* if for all $u, u' \in (a, b), u < u'$, we have $f^*(u) > f^*(u')$. Again, we assume that every monotonicity interval is maximal.
- The set of all monotonicity intervals of the function f^* will be denoted by $\mathcal{M}_f$.

Lemma

If for either f or g the corresponding set $\mathcal{M}_f$ or $\mathcal{M}_g$ is empty, then $C_{f,g} = \Pi$. If $C_{f,g} \neq \Pi$, then both sets $\mathcal{M}_f$ and $\mathcal{M}_g$ are nonempty.

Singularities of Marshall copulas

Lemma

1. *If for some $\lambda \in \mathcal{D}_f$ we have $\lambda \in \operatorname{Im} g^* \setminus \mathcal{D}_g$, then*

$$\{(u,v); u \in S_\lambda, v = {g^*}^{-1}(\lambda)\} \subset L_0(f,g).$$

2. *If for some $\mu \in \mathcal{D}_g$ we have $\mu \in \operatorname{Im} f^* \setminus \mathcal{D}_f$, then*

$$\{(u,v); u = {f^*}^{-1}(\mu), v \in S_\mu\} \subset L_0(f,g).$$

3. *If $\lambda \in \mathcal{D}_f \cap \mathcal{D}_g$, then*

$$\{(u, b^g_\mu); u \in S_\lambda\} \cup \{(a^f_\lambda, v); v \in S_\lambda\} \subset L_0(f,g).$$

4. *The copula $C_{f,g}$ is not singular along either of the parts of (1), (2), or (3).*

Singularities of Marshall Copulas - 2

Theorem

For every interval $M \in \mathcal{M}_g$ there exists an interval $\widehat{M} \in \mathcal{M}_f$, and strictly increasing bijective functions $\chi : \widehat{M} \to M, \omega : M \to \widehat{M}$, such that

1. *χ and ω are inverses of each other,*
2. *The arc $A_M = \{(u, \chi(u)); u \in \widehat{M}\} = \{(\omega(v), v); v \in M\}$ belongs to $L_0(f, g)$.*

The union

$$\bigcup_{M \in \mathcal{M}_g} A_M$$

is the entire singularity locus of $C_{f,g}$.

Singularities of Marshall Copulas - 3

Theorem

For any $\nu = (\widehat{M}, M)$*, we have the density* h *of the arc* A_ν*, where*

1. *under parametrization* $(u, \chi(u)), u \in \widehat{M}$,

$$h(u) = \chi(u)\left[f^*(u) - f'(u)\right];$$

2. *under parametrization* $(\omega(v), v), v \in M$,

$$h(v) = \omega(v)\left[g^*(v) - g'(v)\right]$$

3. *under parametrization* $({f^*}^{-1}(\tau^{-1}), {g^*}^{-1}(\tau)), \tau \in g^*(M)$,

$$h(\tau) = \frac{{g^*}^{-1}(\tau){f^*}^{-1}(\tau^{-1})}{\tau}.$$

Singularities of Marshall Copulas - 4

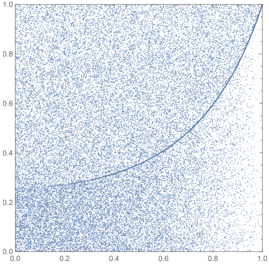


Figure: Connected singular component

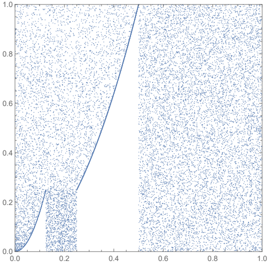


Figure: Disconnected singular component

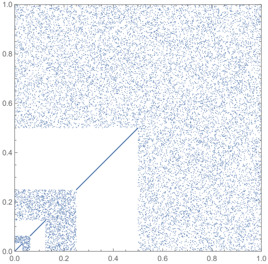


Figure: Disconnected symmetric case

Nonsingular RMM copulas

- Now we denote by $C_{f,g}$ RMM copula

$$C_{f,g}(u,v) = \max\{0, uv - f(u)g(v)\}.$$

Theorem (Kamnitui, Trutschnig, 2020)

An RMM copula $C_{f,g}$ is nonsingular if and only if $uv \geq f(u)g(v)$ for all $u, v \in [0.1]$.

Nonsingular RMM copulas

- Now we denote by $C_{f,g}$ RMM copula

$$C_{f,g}(u,v) = \max\{0, uv - f(u)g(v)\}.$$

Theorem (Kamnitui, Trutschnig, 2020)

An RMM copula $C_{f,g}$ is nonsingular if and only if $uv \geq f(u)g(v)$ for all $u, v \in [0.1]$.

- RMM copula $C_{f,g}$ is absolutely continuous everywhere except possibly on the set

$$\widetilde{L}_0(f,g) = \{(u,v) |\ uv = f(u)g(v)\} = \{(u,v) |\ f^*(u) = g^*(v)^{-1}\}.$$

Nonsingular RMM copulas

- Now we denote by $C_{f,g}$ RMM copula

$$C_{f,g}(u,v) = \max\{0, uv - f(u)g(v)\}.$$

Theorem (Kamnitui, Trutschnig, 2020)

An RMM copula $C_{f,g}$ is nonsingular if and only if $uv \geq f(u)g(v)$ for all $u, v \in [0.1]$.

- RMM copula $C_{f,g}$ is absolutely continuous everywhere except possibly on the set

$$\widetilde{L}_0(f,g) = \{(u,v)|\ uv = f(u)g(v)\} = \{(u,v)|\ f^*(u) = g^*(v)^{-1}\}.$$

- We consider the constancy and monotonicity intervals for f^* and g^* as we did for the Marshall copulas and prove analoquous results:

Singularities of RMM copulas

Lemma

1. *If for some* $\lambda \in \mathcal{D}_f$ *we have* $\mu = \frac{1}{\lambda} \in \operatorname{Im} g^* \setminus \mathcal{D}_g$*, then*
$$\{(u,v);\, u \in S_\lambda, v = f^*(\mu)^{-1}\} \subset \widetilde{L}_0(f,g).$$
2. *If for some* $\mu \in \mathcal{D}_g$ *we have* $\lambda = \frac{1}{\mu} \in \operatorname{Im} f^* \setminus \mathcal{D}_f$*, then*
$$\{(u,v);\, u = g^*(\lambda)^{-1}, v \in S_\mu\} \subset \widetilde{L}_0(f,g).$$
3. *If* $\lambda\mu = 1$ *for some* $\lambda \in \mathcal{D}_f$ *and* $\mu \in \mathcal{D}_g$*, then*
$$\{(u, b^g_\mu);\, u \in S_\lambda\} \cup \{(a^f_\lambda, v);\, v \in S_\mu\} \subset \widetilde{L}_0(f,g).$$
4. *The copula* $C_{f,g}$ *is not singular along either of the parts of (1), (2), or (3).*

Singularities of RMM Copulas - 2

Theorem

For every interval $M \in \mathcal{M}_g$ *there exists an interval* $\widehat{M} \in \mathcal{M}_f$*, and strictly decreasing bijective functions* $\chi : \widehat{M} \to M, \omega : M \to \widehat{M}$*, such that*

1. χ *and* ω *are inverses of each other,*
2. *The arc* $A_M = \{(u, \chi(u)); u \in \widehat{M}\} = \{(\omega(v), v); v \in M\}$ *belongs to* $\widetilde{L}_0(f, g)$*.*

The union

$$\bigcup_{M \in \mathcal{M}_g} A_M$$

is the entire singularity locus of $C_{f,g}$*.*

Singularities of RMM Copulas - 3

Theorem

For any $\nu = (\widehat{M}, M)$, we have the density h of the arc A_ν, where

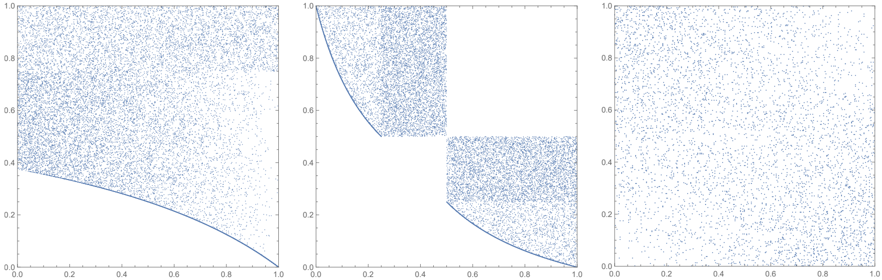
1. *under parametrization* $(u, \chi(u)), u \in \widehat{M}$,
$$h(u) = \chi(u)\left[1 - \frac{f'(u)}{f^*(u)}\right];$$
2. *under parametrization* $(\omega(v), v), v \in M$,
$$h(v) = \omega(v)\left[1 - \frac{g'(v)}{g^*(v)}\right];$$
3. *under parametrization* $(f^{*-1}(\tau^{-1}), g^{*-1}(\tau)), \tau \in g^*(M)$,
$$h(\tau) = \tau^{-1} g^{*-1}(\tau) f^{*-1}(\tau^{-1}).$$

Singularities of RMM copulas - 4

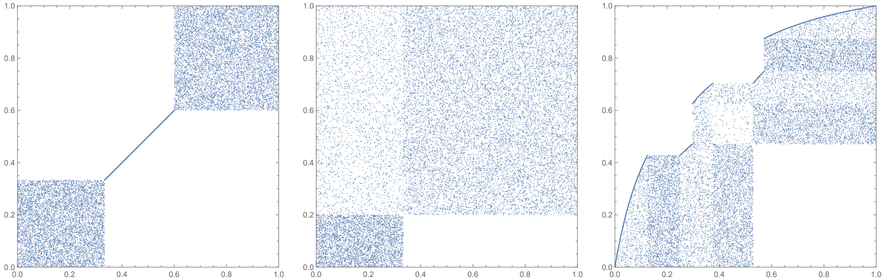
Theorem (Kamnitui, Trutschnig, 2020)

The family of all RMM copulas with non-degenerated singular component is dense in the set of all RMM copulas with respect to d_∞. The family of all absolutely continuous RMM copulas is a compact, nowhere dense subset.

Singularities of RMM copulas - 5



Singularities of maxmin copulas



References

1. A.W. Marshall, I. Olkin. A generalized bivariate exponential distribution, J. Appl. Prob. 4 (1967) 291-302.
2. A.W. Marshall. Copulas, Marginals, and Joint Distributions. In LMS Lecture Notes Series, vol., 28, 1996, pp. 213-222.
3. M. Omladič, N. Ružić. Shock models with recovery option via the maxmin copulas. Fuzzy Sets Syst. 284 (2016) 113-128.
4. TK, M. Omladič. Reflected maxmin copulas and modeling quadrant subindependence. Fuzzy Sets Syst. 378 (2020) 125-143.
5. N. Kamnitui, W. Trutschnig. On some properties of reflected maxmin copulas. Fuzzy Sets Syst. 393 (2020) 53-74.
6. TK, M. Omladič. Singular components of shock model copulas. J. Comp. Appl. Math. 400 (2022), 14, 113749.