A proof of the Box Conjecture for commuting nilpotent matrices

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Contents

The setup

- F an infinite field, $B \in M_n(F)$ a nilpotent matrix,
- $\mathcal{N}_B = \{A \in M_n(F);\ A^n = 0,\ AB = BA\}$ the *nilpotent commutator of* B.
- $\mathcal{P}$ the set of all partitions of all natural numbers (including the empty partition): $P = (p_1, p_2, \ldots, p_k) \in \mathcal{P}$ where $k \in \mathbb{N}$, $p_i \geq p_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $p_k > 0$.
- $\mathcal{Q}$ the subset of all *Rogers-Ramanujan* (or *super-distinct*) partitions, i.e., $Q = (q_1, q_2, \ldots, q_m) \in \mathcal{Q}$ if and only if $\sum_{j=1}^{m} q_j = n$ and $q_i - q_{i+1} \geq 2$ for $i = 1, 2, \ldots, m-1$,
- $P = (p_1, p_2, \ldots, p_k) \in \mathcal{P}$ is *almost-rectangular* if $p_1 - p_k \leq 1$.
- The *Jordan type* of B is the partition $P \in \mathcal{P}$ that determines the Jordan canonical structure of B.

Introducing the map $\mathfrak{D} : \mathcal{P} \to \mathcal{P}$

- $\mathcal{N}_B$ is an irreducible variety (Basili 2003),
- Thus, there is a nilpotent orbit (with respect to $\mathrm{GL}_n(F)$-action on $M_n(F)$) such that its intersection with $\mathcal{N}_B$ is Zariski dense in $\mathcal{N}_B$.
- So, we have a map $\mathfrak{D} : \mathcal{P} \to \mathcal{P}$ such that $\mathfrak{D}(P)$ is the partition corresponding to the *dense orbit* in the nilpotent commutator $\mathcal{N}_B$, where B is of Jordan type P.
- Moreover, $\mathfrak{D}(P)$ is the *dominant partition* of $\mathcal{N}_B$, i.e., it is the maximal partition of any element of $\mathcal{N}_B$ in the dominance order on $\mathcal{P}$.

The history of results on $\mathfrak{D}$

- The number of parts of $\mathfrak{D}(P)$ is equal to the smallest number of almost rectangular subpartitions needed to cover P (Basili 2000),
- $\mathfrak{D}(P) \in \mathcal{Q}$ for each $P \in \mathcal{P}$ (Basili, Iarrobino 2008),
- Description of the largest part of $\mathfrak{D}(P)$ (Oblak 2008).
- The extension of $\mathfrak{D}$ to the Lie algebra setup and description of its image for simple Lie algebras (Panyushev 2008).
- A conjecture on recursive process to construct $\mathfrak{D}(P)$ (Oblak 2008).
- $\mathfrak{D}$ is idempotent map, i.e., $\mathfrak{D}^2 = \mathfrak{D}$ (K., Oblak 2009). So, $\mathfrak{D} : \mathcal{P} \to \mathcal{Q}$ and $\mathfrak{D}(Q) = Q$ for $Q \in \mathcal{Q}$. Partitions in $\mathcal{Q}$ are called *stable* partitions.
- The Oblak process produces lower bound for $\mathfrak{D}(P)$ in the dominance order for partitions (Iarrobino, Khatami 2013).
- Description of the smallest part of $\mathfrak{D}(P)$ (Khatami 2014).

The history of results on $\mathfrak{D}$

- The Box conjecture on the form od $\mathfrak{D}^{-1}(Q)$ for a given $Q \in \mathcal{Q}$ (Iarrobino, Khatami, Van Steirtenghem, Zhao, 2014).
- Proof of the Table Theorem for $Q \in \mathcal{Q}$ with two parts $Q = (q_1, q_2)$ (Iarrobino, Khatami, Van Steirtenghem, Zhao, 2014).
- Proof that the Oblak process gives $\mathfrak{D}(P)$ (Basili 2022).

The statement of the Box Conjecture

Conjecture (Iarrobino et al, 2014)

Given a stable partition $Q = (q_1, q_2, \ldots, q_k) \in \mathcal{Q}$, the elements of $\mathfrak{D}^{-1}(Q)$ can be arranged in a box (i.e. an array) of sizes $q_k \times (q_{k-1} - q_k - 1) \times (q_{k-2} - q_{k-1} - 1) \times \cdots \times (q_1 - q_2 - 1)$ such that the partition in the $(i_1, i_2, \ldots, i_k)$-th position has exactly $\sum_{j=1}^{k} i_j$ parts.

Setup

- $P = (p_1, \ldots, p_k) \in \mathcal{P}$ a partition, p_i the *parts* of P.
- The *size* of P is $|P| = \sum_{i=1}^{k} p_i$, the *length* of P is the number of parts, $\ell(P) = k$.
- The *empty partition* is the unique element $\varepsilon \in \mathcal{P}$ of size (and length) 0.
- The *2-measure* of P, denoted $\mu_2(P)$, is the maximum length of a super-distinct subpartition of P, or equivalently, the minimal number of almost rectangular partitions to cover P.

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Example

Partition $P = (8, 7, 4, 4, 3, 2, 2, 1)$ has $|P| = 31$, $\ell(P) = 8$. It contains the subpartition $(7, 4, 1) \in \mathcal{Q}$ of length 3 and none longer, so $\mu_2(P) = 3$.

Frequency representation of partitions

- A partition can equivalently be regarded as unordered multiset of positive integers. By $[1^{f_1}, 2^{f_2} \cdots]$ we denote the partition whose parts consist of f_1 copies of 1, f_2 copies of 2, etc.

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- Let $\mathcal{F}$ be the set of all finitely supported sequences of nonnegative integers. We work with partitions via their "frequency" representations in $\mathcal{F}$ given by the trivial correspondence $[1^{f_1} 2^{f_2} \cdots] \leftrightarrow (f_1, f_2, \ldots)$.
- This identification of $\mathcal{F}$ and $\mathcal{P}$ is used throughout. We write $f(P)$ for the frequency sequence of $P \in \mathcal{P}$ and $P(f)$ for the partition corresponding to $f \in \mathcal{F}$.

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Example

Partition $P = (8, 7, 4, 4, 3, 2, 2, 1)$ has multiset representation $[1, 2^2, 3, 4^2, 7, 8]$ and frequency representation $(1, 2, 1, 2, 0, 0, 1, 1)$.

We omit the trailing zeros and use $f_0 = 0$ if needed.

Notation for frequencies

- Given $f \in \mathcal{F}$. The *support* of f is denoted $\mathcal{S}(f) = \{i \geq 1 \,:\, f_i \neq 0\}$. The size and length on $\mathcal{P}$ are extended to $\mathcal{F}$ by
$$|f| = |P(f)| = \sum_i i\, f_i,$$
and
$$\ell(f) = \ell(P(f)) = \sum_i f_i.$$
Also, we let $\mu_2(f) = \mu_2(P(f))$. This is the maximum size of a subset of $\mathcal{S}(f)$ that contains no consecutive pairs $\{i, i+1\}$.

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Example

Suppose $f = (0, 2, 3, 1, 0, 0, 1, 0, 1)$. Then its support is $\mathcal{S}(f) = \{2, 3, 4, 7, 9\}$, the size is $|f| =$, and the length is $\ell(f) = 7$ and the 2-measure is $\mu_2(f) = 4$.

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Here the corresponding multiset is $P(f) = [2^2, 3^3, 4, 7, 9]$.

Spreads, left and right pairings

- A *spread* of f is a maximal interval $[i,j] \subseteq \mathcal{S}(f)$. A spread $[i,i] = \{i\}$ of size 1 is said to be *trivial*.
- We define

$$\mathrm{L}(f) := \bigcup \{i, i+2, \ldots, i+2\lfloor \tfrac{j-i}{2} \rfloor\}$$

$$\mathrm{R}(f) := \bigcup \{j, j-2, \ldots, j-2\lfloor \tfrac{j-i}{2} \rfloor\},$$

  where the unions run over all spreads $[i,j]$ of f.
- Observe that they are of equal size, namely

$$|\mathrm{L}(f)| = |\mathrm{R}(f)| = \mu_2(f) = \sum \lceil \tfrac{j-i+1}{2} \rceil.$$

Example

Let $f = (2,1,0,3,2,2,0,0,1)$. Then $P = P(f) = [9, 6^2, 5^2, 4^3, 2, 1^2]$, and $|f| = |P| = 47$ and $\ell(f) = \ell(P) = 11$. The spreads of f are $\{1,2\}$, $\{4,5,6\}$ and $\{9\}$, so $\mathrm{L}(f) = \{1,4,6,9\}$, $\mathrm{R}(f) = \{2,4,6,9\}$ and $\mu_2(f) = \mu_2(P) = 4$.

Left and right pairings

- The elements $i_1 < i_2 < \ldots < i_m$ of $\mathrm{L}(f)$ specify the pairs (f_{i_1}, f_{i_1+1}), $\ldots$, (f_{i_m}, f_{i_m+1}) that result from parsing f from left-to-right and grouping consecutive entries f_i, f_{i+1} with $f_i > 0$. We call these the *forward pairs* of f.
- The elements $j_1 > j_2 > \ldots > j_m$ of $\mathrm{R}(f)$ determine the *backward* pairs (f_{j_1-1}, f_{j_1}), $\ldots$, (f_{j_m-1}, f_{j_m}) obtained by parsing f from right-to-left and grouping consecutive entries f_j, f_{j-1} with $f_j > 0$.

Left and right pairings – example

Example

For $f = (2, 2, 1, 3, 1, 0, 4, 0, 0, 2, 1)$ we have $\mathrm{L}(f) = \{1, 3, 5, 7, 10\}$ and $\mathrm{R}(f) = \{1, 3, 5, 7, 11\}$. The forward/backward pairs of f are indicated with arrows pointed in the direction of parsing.

$$\color{red}{0}\ (2, 2, 1, 3, 1, 0, 4, 0, 0, 2, 1)$$

forward

$$\color{red}{0}\ (2, 2, 1, 3, 1, 0, 4, 0, 0, 2, 1)$$

backward

The fictional entry $f_0 = 0$ has been prepended in red. Observe that every nonzero entry of f appears in one forward and one backward pair, while all unpaired entries of f are 0.

Observe that $\mathrm{L}(f)$ and $\mathrm{R}(f)$ are different only in parts corresponding to spreads of even lengths, while forward/backward pairs are equal for spreads of even lengths and distinct for spreads of odd lenghts.

Burge correspondence - the setup

- Let $\mathcal{A} = \{f \in \mathcal{F} \,:\, 1 \notin \mathrm{R}(f)\}$ and $\mathcal{B} = \{f \in \mathcal{F} \,:\, 1 \in \mathrm{R}(f)\}$.
- Observe that these sets partition $\mathcal{F}$: $\mathcal{A} \cup \mathcal{B} = \mathcal{F}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$.
- Also, $f \in \mathcal{B} \iff (f_0, f_1)$ is a backward pair $\iff$ 1 is contained in a spread of f of odd size.

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- Also, $f \in \mathcal{B} \iff (f_0, f_1)$ is a backward pair $\iff$ 1 is contained in a spread of f of odd size.
- Now, introduce **two central transformations** $\alpha : \mathcal{F} \longrightarrow \mathcal{A}$ and $\beta : \mathcal{F} \longrightarrow \mathcal{B}$, along with a **mapping** $\partial : \mathcal{F} \longrightarrow \mathcal{F}$ that undoes them.

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- Each of these acts as a sequence of raising/lowering operators on the forward or backward pairs of f.
  - $\alpha(f)$ is obtained from f by replacing (f_i, f_{i+1}) with $(f_i - 1, f_{i+1} + 1)$ for each $i \in \mathrm{L}(f)$.
  - $\beta(f) := (f_1 + 1, \alpha(f_2, f_3, \ldots))$
  - $\partial(f)$ is obtained from f by replacing (f_{j-1}, f_j) with $(f_{j-1} + 1, f_j - 1)$ for each $j \in \mathrm{R}(f)$, where in the case $j = 1$ only f_1 is reduced by 1 (i.e., the fictional f_0 is ignored).

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Burge transformations – comments

- In effect, α scans f from left-to-right and “promotes” each forward pair (f_i, f_{i+1}) to $(f_i - 1, f_{i+1} + 1)$.
- α transforms the forward pairs of f into the backward pairs of $\alpha(f)$.
- That is, $i \in \mathrm{L}(f) \iff i + 1 \in \mathrm{R}(\alpha(f))$.
- In particular, for any $f \in \mathcal{F}$ we have $1 \notin \mathrm{R}(\alpha(f))$ and hence $1 \in \mathrm{R}(\beta(f))$. Thus, the claim that α and β map $\mathcal{F}$ into $\mathcal{A}$ and $\mathcal{B}$, respectively, follows.
- Moreover, f can be recovered from either $\alpha(f)$ or $\beta(f)$ by scanning from right-to-left and “demoting” backward pairs. This is exactly the action of ∂.
- α and β are bijections from $\mathcal{F}$ to $\mathcal{A}$ and $\mathcal{B}$, respectively, while $\partial : \mathcal{F} \longrightarrow \mathcal{F}$ is 2-to-1 and restricts to α^{-1} on $\mathcal{A}$ and β^{-1} on $\mathcal{B}$.

Burge transformations – example

Example

Let $f = (2,2,1,3,1,0,4,0,0,2,1)$ as in the previous example. Then

$$\alpha(f) = \alpha(\overrightarrow{2,2},\overrightarrow{1,3},\overrightarrow{1,0},\overrightarrow{4,0},0,\overrightarrow{2,1}) = (1,3,0,4,0,1,3,1,0,1,2)$$

$$\beta(f) = (2+1, \alpha(\overrightarrow{2,1},\overrightarrow{3,1},0,\overrightarrow{4,0},0,\overrightarrow{2,1})) = (3,1,2,2,2,0,3,1,0,1,2)$$

$$\partial(f) = \partial(\overleftarrow{2,2},\overleftarrow{1,3},\overleftarrow{1,0},\overleftarrow{4,0},0,\overleftarrow{2,1}) = (1,3,0,4,0,1,3,0,0,3,0).$$

We have $\partial(\alpha(f)) = \partial(\beta(f)) = \beta(\partial(f)) = f$. Note, however, that $\alpha(\partial(f)) = (0,4,0,3,1,0,4,0,0,2,1) \neq f$. This inequality is due to the fact that $f \notin \mathcal{A}$ (since $1 \in \mathrm{R}(f)$).

The Burge chain and the Burge code

- Clearly we have $0 \leq |\partial(f)| < |f|$ for all $f \neq \varepsilon$.
- Therefore, applying ∂ repeatedly to any $f \in \mathcal{F}$ results in $\partial^k f = \varepsilon$ for a positive integer $k \geq 1$.

Definition

We define the *Burge chain* of $f \in \mathcal{F}$ to be the sequence $\partial^0 f, \partial^1 f, \partial^2 f, \ldots, \partial^k f$, where k is the smallest positive integer such that $\partial^k f = \varepsilon$. The length of this sequence (namely $k+1$) is the *Burge length* of f. The *Burge code* of f is the binary word $\Omega(f) = \omega_1 \omega_2 \cdots \omega_{k+1} \in \{\alpha, \beta\}^*$ defined by

$$\omega_i = \begin{cases} \alpha & \text{if } \partial^{i-1} f \in \mathcal{A}, \\ \beta & \text{if } \partial^{i-1} f \in \mathcal{B}. \end{cases}.$$

Here $\{\alpha, \beta\}^*$ is the free monoid on two symbols α and β.

Burge bijection Ω

- $f = \varepsilon$ has trivial Burge chain ε and Burge code $\Omega(\varepsilon) = \alpha$,
- Burge codes of all other sequences f are of length at least 2.
- If $\partial f = \varepsilon$ then $f = \varepsilon$ or $f = (1)$.
- The chain of every $f \neq \varepsilon$ ends with $\partial^{k-1} f = (1) \in \mathcal{B}$ and $\partial^k f = \varepsilon \in \mathcal{A}$. Thus the Burge code $\Omega(f)$ for $f \neq \varepsilon$ ends with $\ldots \beta\alpha$.
- Since $\partial|_{\mathcal{A}} = \alpha^{-1}$ and $\partial|_{\mathcal{B}} = \beta^{-1}$, the definition of $\Omega(f) = \omega_1 \cdots \omega_n$ ensures that $\partial^{i-1} f = (\omega_i \circ \partial)(\partial^{i-1} f) = \omega_i(\partial^i f)$ for all i.
- Therefore, we reconstruct f from $\Omega(f)$ by applying it to ε

$$f = \omega_1(\partial^1 f) = \omega_1\omega_2(\partial^2 f) = \omega_1\omega_2\omega_3(\partial^3 f) = \cdots = (\omega_1\omega_2 \cdots \omega_n)(\varepsilon).$$

Burge bijection Ω

- So,
$$f = \Omega(f)(\varepsilon).$$
- The products of ω_i are to be interpreted as functional composition in the usual right-to-left order.
- That is, the right-to-left reading of $\Omega(f)$ specifies the unique manner by which f can be "built" from ε via iterative applications of α and β, beginning with a single 'non-effective' application of α: $\alpha\varepsilon = \varepsilon$.

Proposition

The Burge encoding $f \mapsto \Omega(f)$ is a one-one correspondence between $\mathcal{F}$ and the set $\mathcal{W} = (\alpha^*\beta)^*\alpha$ of all finite words on $\{\alpha, \beta\}$ that end with a singleton α.

Example of Burge correspondence

Example

Let $f = (1,2,1,0,1)$. The Burge chain of f is displayed below, along with the values of ω_i. The Burge code is $\Omega(f) = \beta\alpha\beta\alpha\alpha\alpha\beta\beta\beta\alpha$.

i	$\partial^i f$	ω_{i+1}
0	$(1,2,1,0,1)$	β
1	$(0,3,0,1)$	α
2	$(1,2,1)$	β
3	$(0,3)$	α
4	$(1,2)$	α
5	$(2,1)$	α
6	(3)	β
7	(2)	β
8	(1)	β
9	ε	α

Example of Burge correspondence – 2

Example

Given the word $\omega = \alpha\beta\beta\alpha\beta\alpha \in \mathcal{W}$ we build its corresponding partition by iterating α and β as follows:

$$\varepsilon \overset{\alpha}{\mapsto} \varepsilon \overset{\beta}{\mapsto} (1) \overset{\alpha}{\mapsto} (0,1) \overset{\beta}{\mapsto} (1,0,1) \overset{\beta}{\mapsto} (2,0,0,1) \overset{\alpha}{\mapsto} (1,1,0,0,1).$$

Thus ω is the Burge code of $f = (1,1,0,0,1)$. Here $P(f) = [1,2,5]$ is a partition with $|P(f)| = 8, \ell(P(f)) = 3$ and $\mu_2(P(f)) = 2$.

Descent sets, descent numbers and major indices

Lemma

For any $f \in \mathcal{F}$ we have:

1. $|\partial f| = |f| - \mu_2(f)$
2. $\ell(\partial f) = \begin{cases} \ell(f) - 1 & \text{if } f \in \mathcal{B} \\ \ell(f) & \text{if } f \in \mathcal{A}. \end{cases}$
3. $\mu_2(\partial f) = \begin{cases} \mu_2(f) - 1 & \text{if } f \in \mathcal{B} \text{ and } \partial f \in \mathcal{A} \\ \mu_2(f) & \text{otherwise.} \end{cases}$

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- For $\omega = \omega_1\omega_2 \cdots \omega_n \in \{\alpha, \beta\}^*$ we define the *descent set* of ω, denoted $\mathrm{Des}(\omega)$, as the set of all indices i for which $\omega_i\omega_{i+1} = \beta\alpha$.
- So, $\mathrm{Des}(\omega)$ records the positions of descents in ω relative to the ordering $\beta > \alpha$. These positions differ pairwise by at least 2.
- The *descent number* and *major index* of ω are defined by $\mathrm{des}(\omega) = |\mathrm{Des}(\omega)|$ and $\mathrm{maj}(\omega) := \sum_{i \in \mathrm{Des}(\omega)} i$, respectively.

Descent sets, descent numbers and major indices – 2

Proposition

Let $f \in \mathcal{F}$ and let $\omega = \Omega(f)$. Then:

1. $\ell(f) = \#$ *occurrences of* β *in* ω
2. $|f| = \mathrm{maj}(\omega)$
3. $\mu_2(f) = \mathrm{des}(\omega) = \#$ *occurrences of* $\beta\alpha$ *in* ω

Example

In the example before we saw that $f = (1, 2, 1, 0, 1)$ has Burge code $\omega = \beta\alpha\beta\alpha\alpha\alpha\beta\beta\beta\alpha$. Note that ω contains $5 = \ell(f)$ copies of β, and we have $\mathrm{Des}(\omega) = \{1, 3, 9\}$, $\mathrm{maj}(\omega) = 13 = |f|$ and $\mathrm{des}(\omega) = 3 = \mu_2(f)$.

Burge correspondence

- We have a three-way bijective correspondence between partitions, frequency vectors, and binary words, namely

$$\mathcal{P} \xrightarrow{\text{frequency } f} \mathcal{F} \xrightarrow{\text{Burge code } \Omega} \mathcal{W}.$$

- All notation defined previously for elements of $\mathcal{F}$ and $\mathcal{W}$ is extended to $\mathcal{P}$ through $f(\cdot)$ and $\Omega(\cdot)$.
- We define $\partial P = P(\partial f(P))$, $\partial^i P = P(\partial^i f(P))$, $\Omega(P) = \Omega(f(P))$ and $\mathrm{Des}(P) = \mathrm{Des}(\Omega(P))$.
- If $\mathrm{Des}(P) = \{i_1, i_2, \ldots, i_m\}$ where $i_1 < i_2 < \cdots < i_m$ then $i_j - i_{j-1} \geq 2$ for all j, and $\sum_j i_j = \mathrm{maj}(\Omega(P)) = |P|$.
- Thus $\mathrm{Des}(P)$ can be regarded as a super-distinct partition of size $|P|$.
- If $\Omega(P) = \omega_1\omega_2\cdots\omega_n$ then $\Omega(\partial P) = \omega_2\cdots\omega_n$, so we have $\mathrm{Des}(\partial P) = \{i_1 - 1, i_2 - 1, \ldots, i_m - 1\} \setminus \{0\}$.

Example

Example

i	$\partial^i f$	$\Omega(\partial^i f)$	$\partial^i P$	$\mathrm{Des}(\partial^i P)$
0	$(1,1,0,1,0,0,1)$	$\alpha\beta\alpha\beta\beta\alpha\beta\alpha$	$[7,4,2,1]$	$[7,5,2]$
1	$(2,0,1,0,0,1)$	$\beta\alpha\beta\beta\alpha\beta\alpha$	$[6,3,1^2]$	$[6,4,1]$
2	$(1,1,0,0,1)$	$\alpha\beta\beta\alpha\beta\alpha$	$[5,2,1]$	$[5,3]$
3	$(2,0,0,1)$	$\beta\beta\alpha\beta\alpha$	$[4,1^2]$	$[4,2]$
4	$(1,0,1)$	$\beta\alpha\beta\alpha$	$[3,1]$	$[3,1]$
5	$(0,1)$	$\alpha\beta\alpha$	$[2]$	$[2]$
6	(1)	$\beta\alpha$	$[1]$	$[1]$
7	ε	α	ε	ε

Characterisation of the descent map

Definition

For a partition P, let $P-1$ be the *reduced partition* obtained by subtracting 1 from each part of P and eliminating any resulting zeros.

For example: $P = [6, 4^3, 2^4, 1^3] = [5, 3^3, 1^4]$.

Theorem

The descent map $P \mapsto \mathrm{Des}(P)$ *is a size-preserving function from* $\mathcal{P}$ *to* $\mathcal{Q}$ *satisfying* $\mathrm{Des}(\partial P) = \mathrm{Des}(P) - 1$. *It is the unique such function.*

Invariant subspaces and the descent map

Based on Shayman's description of the variety of invariant subspace of a nilpotent matrix from 1982 and 1986 we prove the following results:

Proposition

Each invariant subspace of B is equal to the image of an element of the commutator of $\mathcal{C}_B$ of B.

Theorem

Suppose that B is a nilpotent matrix and that $P = P(B)$ is its Jordan type. Suppose that A is a generic nilpotent matrix commuting with B and that $W = \operatorname{Im} A$ is its image. Then the Jordan type of the restriction $B|_W$ is given by ∂P.

$\mathfrak{D} = \mathrm{Des}$

Corollary

$$\mathfrak{D}(P) = \mathrm{Des}(P).$$

Proof.

Theorem gives $\mathfrak{D}(\partial P) = \mathfrak{D}(P) - 1$ since the Jordan type of the restriction of a nilpotent matrix of Jordan type T to its own image is equal to $T - 1$. Obviously $|\mathfrak{D}(P)| = |P|$, so the uniqueness result identifies $\mathfrak{D}$ as the descent map. □

The Box Theorem

Corollary

Suppose $P \in \mathcal{P}$ is the Jordan type of B and $\Omega(P) = \omega_1 \cdots \omega_n$ its Burge code. Then $\mathfrak{D}(P) = (q_1, q_2, \ldots, q_k)$, where $q_1 > q_2 > \cdots > q_k$ is the complete list of indices q for which $\omega_q = \beta$ and $\omega_{q+1} = \alpha$.

Theorem

Let $Q = (q_1, q_2, \ldots, q_k) \in \mathcal{Q}$ and set $\delta_1 = q_k$ and $\delta_i = q_{k-i+1} - q_{k-i+2} - 1$ for $2 \leq i \leq k$. Then $\mathfrak{D}^{-1}(Q)$ is of size $\delta_1 \delta_2 \cdots \delta_k$ and consists of precisely those partitions whose Burge code is of the form

$$\alpha^{\delta_1 - i_1} \beta^{i_1} \alpha^{\delta_2 - i_2 + 1} \beta^{i_2} \alpha^{\delta_3 - i_3 + 1} \beta^{i_3} \cdots \alpha^{\delta_k - i_k + 1} \beta^{i_k} \alpha, \tag{1}$$

for $(i_1, \ldots, i_k) \in [1, \delta_1] \times [1, \delta_2] \times \cdots \times [1, \delta_k]$. The partition determined by (1) has exactly $\sum_j i_j$ (= the number of βs) parts.

(i_1, i_2, i_3)	code ω	partition $\Omega^{-1}(\omega)$	# parts
$(1,1,1)$	$\alpha\alpha\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,3]$	3
$(2,1,1)$	$\alpha\beta\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,2,1]$	4
$(3,1,1)$	$\beta\beta\beta\alpha\alpha\alpha\beta\alpha\alpha\beta\alpha$	$[10,7,1^3]$	5
$(1,2,1)$	$\alpha\alpha\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,5,3,2]$	4
$(2,2,1)$	$\alpha\beta\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,4,3,2,1]$	5
$(3,2,1)$	$\beta\beta\beta\alpha\alpha\beta\beta\alpha\alpha\beta\alpha$	$[10,4,3,1^3]$	6
$(1,3,1)$	$\alpha\alpha\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,2^2,1]$	5
$(2,3,1)$	$\alpha\beta\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,2,1^3]$	6
$(3,3,1)$	$\beta\beta\beta\alpha\beta\beta\beta\alpha\alpha\beta\alpha$	$[10,5,1^5]$	7
$(1,1,2)$	$\alpha\alpha\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,5,3^2]$	4
$(2,1,2)$	$\alpha\beta\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,4^2,2,1]$	5
$(3,1,2)$	$\beta\beta\beta\alpha\alpha\alpha\beta\alpha\beta\beta\alpha$	$[9,4^2,1^3]$	6
$(1,2,2)$	$\alpha\alpha\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,5,2^3]$	5
$(2,2,2)$	$\alpha\beta\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,4,3,2,1^2]$	6
$(3,2,2)$	$\beta\beta\beta\alpha\alpha\beta\beta\alpha\beta\beta\alpha$	$[9,4,3,1^4]$	7
$(1,3,2)$	$\alpha\alpha\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,2^2,1^2]$	6
$(2,3,2)$	$\alpha\beta\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,2,1^4]$	7
$(3,3,2)$	$\beta\beta\beta\alpha\beta\beta\beta\alpha\beta\beta\alpha$	$[9,5,1^6]$	8

Further comments

- The Box Theorem holds over all infinite fields.
- If we define $\mathfrak{D}(P)$ to be the maximal partition of $\mathcal{N}_B$ in the dominance order then the Box Theorem holds over any field.
- The new definition of $\mathfrak{D}(P)$ is not field dependent as is the case for general pairs of partitions for commuting pairs of nilpotent matrices (a result by Britnell and Wildon, 2011).
- Using the Burge correspondence we provide also another proof of Oblak process and obtain another process to build $\mathfrak{D}(P)$.
- The Burge process is using $P \mapsto \partial P$ map. At each step each almost-rectangular part of P is shortened by 1.