The String Density Problem and the Camassa–Holm Equation

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(**joint work with Jonathan Eckhardt**)

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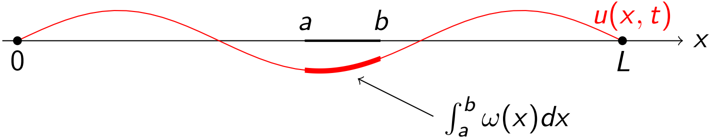
Vibrating inhomogeneous strings

Vibrating string with mass density given by ω:



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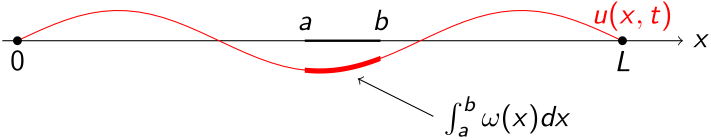

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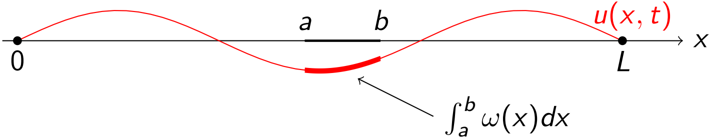

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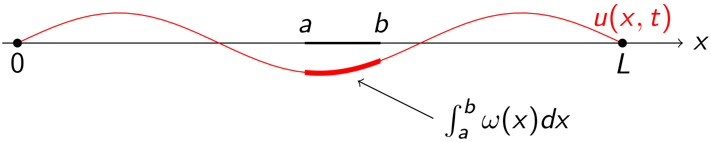

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$$-f'' = \lambda^2\,\omega(x)\,f \quad \text{on } [0,L); \quad f(0) = f(L) = 0$$

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- The spectrum σ (the set of zeros of $s(\cdot, L)$) consists of simple and positive eigenvalues, $\sigma = \{\lambda_n\}_{n\in\mathbb{N}}$

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots; \qquad \lim_{n\to\infty} \frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi}\int_0^L \sqrt{\omega(x)}dx.$$

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- The family of eigenfunctions $\{s(\lambda_n, \cdot)\}_{n\in\mathbb{N}}$ forms an orthogonal basis of the Hilbert space $L^2([0, L); \omega)$.

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The Inverse Spectral Problem:

Can we recover L and ω from $\sigma = \{\lambda_n\}_{n\in\mathbb{N}}$?

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G. Borg, V. A. Marchenko, I. M. Gelfand and B. M. Levitan,. . . :
Either a knowledge of two spectra is required or 1 spectrum and norming constants,

$$\{\lambda_n\}_{n\in\mathbb{N}} \quad \text{and} \quad \{\gamma_n\}_{n\in\mathbb{N}}, \quad \gamma_n^{-1} = \|s(\lambda_n, \cdot)\|^2.$$

I. M. Gelfand & B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izvestiya AN SSSR **15**, (1951)

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- Fundamental system of solutions $c(z, x)$ and $s(z, x)$:

$$c(z, 0) = s'(z, -0) = 1, \quad c'(z, -0) = s(z, 0) = 0$$

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$$m(z) = \lim_{x \uparrow L} -\frac{c(z,x)}{z\, s(z,x)}, \quad z \in \mathbb{C} \backslash [0, \infty)$$

... also called *a coefficient of a dynamical compliance of a string*

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- ρ is a spectral measure, which satisfies $\int_{(0,\infty)} \frac{d\rho(\lambda)}{1+\lambda} < \infty$.

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- For regular strings ($L < \infty$ and $\omega([0, L)) < \infty$):

$$m(z) = \omega(\{0\}) - \frac{1}{Lz} + \sum_{\lambda\in\sigma} \frac{\gamma_\lambda}{\lambda - z}$$

where σ is the Dirichlet spectrum and $\{\gamma_\lambda\}_{\lambda\in\sigma}$ are the norming constans.

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The map Φ is one-to-one.

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The map Φ is one-to-one. Moreover, Φ is a homeomorphism.

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Relevant for particular nonlinear wave equations:

Camassa–Holm: $\omega_t + 2u_x\omega + u_x\omega = 0, \quad \omega = u - u_{xx},$
Hunter–Saxton: $(u_t + uu_x)_x = \frac{1}{2}u_x^2,$
Dym: $u_t = u^3 u_{xxx}$

The Camassa–Holm Equation

$$u_t - u_{xxt} + 2\kappa u_x = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \quad u|_{t=t_0} = u_0(x) \qquad \text{(CH)}$$

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- R. Bhatt & A. V. Mikhailov, *On the inconsistency of the Camassa–Holm model with the shallow water theory*, ArXiv:1010.1932

Camassa–Holm equation: Wave breaking

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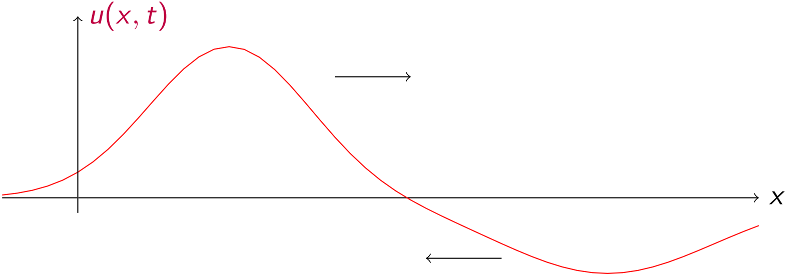

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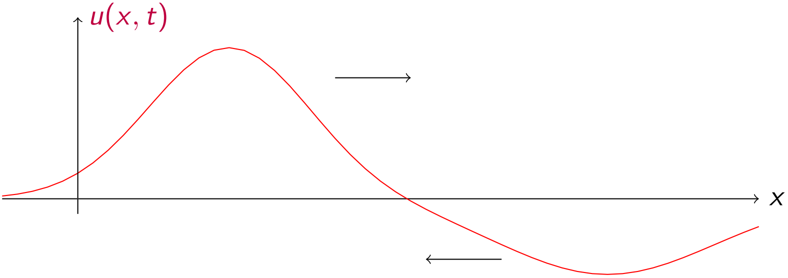

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- Wave breaking only happens when $\omega = u - u_{xx}$ changes sign

Camassa–Holm equation: Peakon solutions ($\kappa = 0$)

The Camassa–Holm equation (weak formulation)

$$u_t + uu_x + p_x = 0, \qquad p = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-s|}\Big(u^2 + \frac{1}{2}u_x^2\Big)ds,$$

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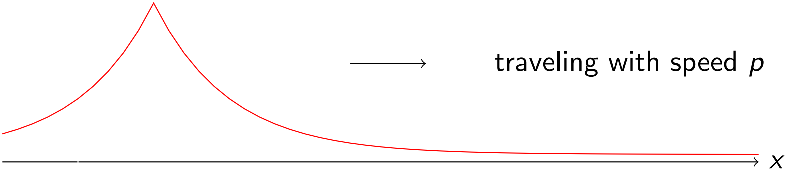

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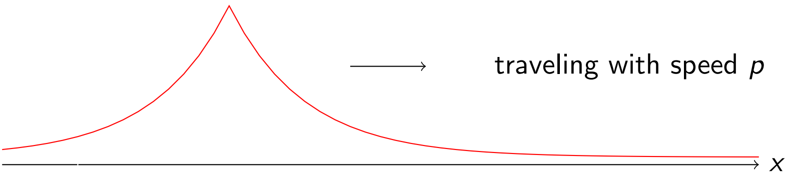

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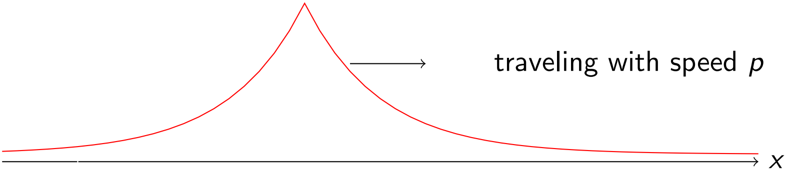

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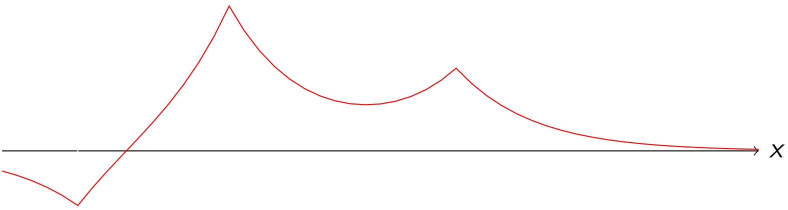

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This system is explicitly solvable,

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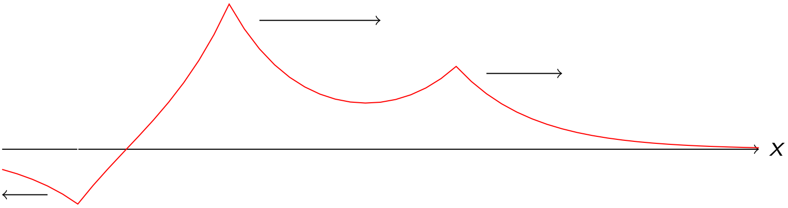
$$H(p,q) = \frac{1}{2}\sum_{n,k=1}^{N} p_n p_k \mathrm{e}^{-|q_n-q_k|} = \frac{1}{4}\|u\|^2_{H^1(\mathbb{R})}$$

This system is explicitly solvable, however,
the Liouville–Arnold theorem does not apply! ($H \notin C^1$)

Camassa–Holm equation: Multi-peakon solutions

More generally a multi-peakon is given by

$$u(x,t) = \sum_{n=1}^{N} p_n(t)\mathrm{e}^{-|x-q_n(t)|}, \quad x,\ t \in \mathbb{R},$$



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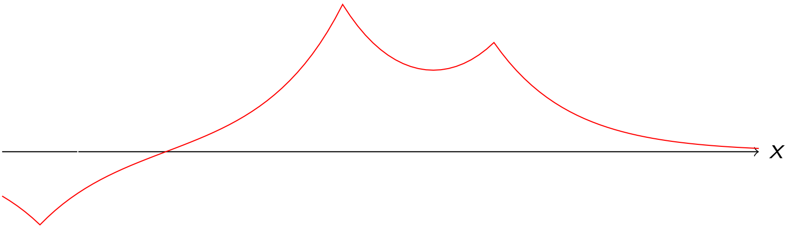

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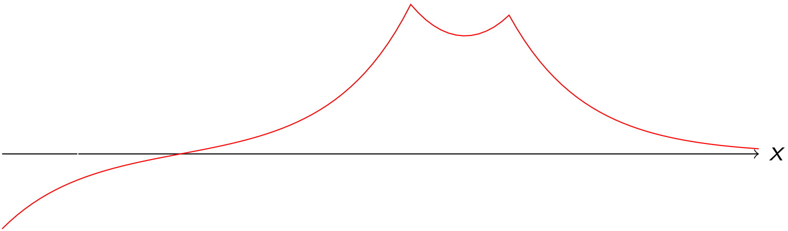

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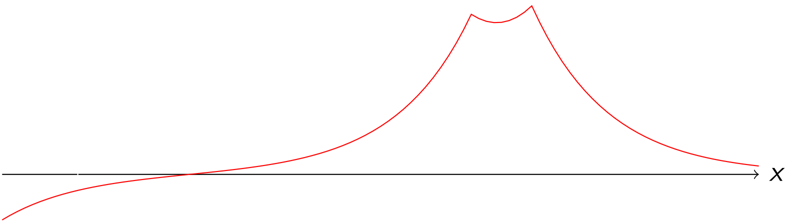

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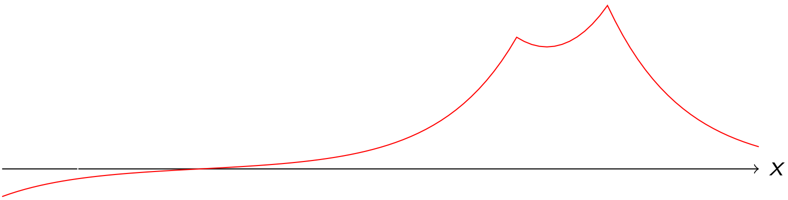

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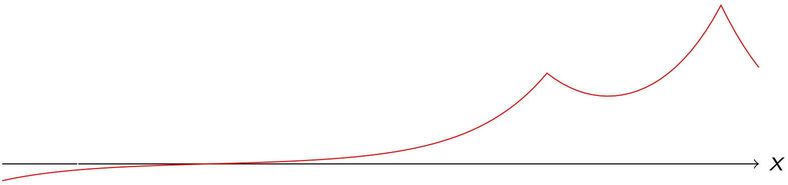

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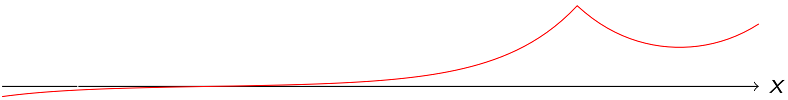

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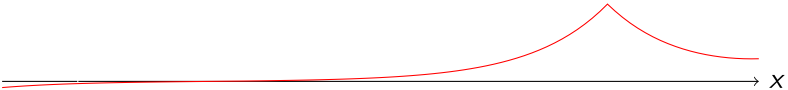

Inverse Spectral/Scattering Transform

Setting

$$\omega(x,t) = u(x,t) - u_{xx}(x,t)$$

consider the family of Sturm–Liouville problems ("Lax operators" for (CH))

$$-f'' + \frac{1}{4}f = z\,\omega(\,\cdot\,,t)f \quad \text{on} \quad \mathbb{R} \tag{Iso}$$

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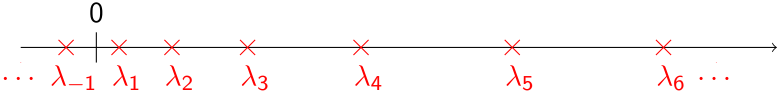

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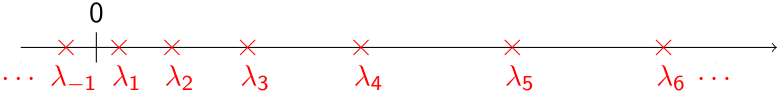

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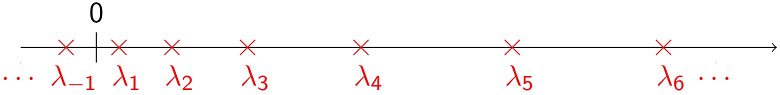

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$$\sigma(t) = \sigma(0) \quad \text{and} \quad \gamma_\lambda(t) = \mathrm{e}^{-\frac{t}{2\lambda}}\gamma_\lambda(0), \quad t \in \mathbb{R}$$

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C. S. Gardner, J. M. Greene, M. D. Kruskal & R. M. Miura, *Method for solving the Korteweg–de Vries equation*, Phys. Rev. Lett. **19** (1967)

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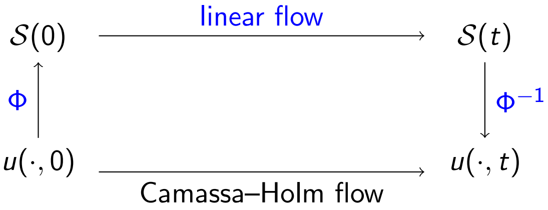

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- Existence ↔ $\mathrm{ran}(\Phi)$?
- Stability ↔ Is Φ^{-1} continuous?

The isospectral problem

Spectral theory for the weighted Sturm–Liouville problem

$$-f''(x) + \frac{1}{4}f(x) = z\omega(x)f(x), \quad x \in \mathbb{R},\ z \in \mathbb{C}$$

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  $\rightarrow$ For $u \in H^1_{\mathrm{loc}}$, ω is a H^{-1}_{loc} distribution

The Camassa–Holm equation: multi-peakons

N-Peakon Solutions

$$u(x,t) = \sum_k p_k(t) \mathrm{e}^{-|x-q_k(t)|} \quad \Leftrightarrow \quad \omega(x,t) = 2\sum_k p_k(t)\, \delta_{q_k(t)}(x).$$

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- The Weyl (Jost) solutions: $\phi_\pm(z,x) = \mathrm{e}^{\mp\frac{x}{2}}$ as $x \to \pm\infty$,
- The Weyl function: Set for $z \in \mathbb{C}$

$$M(z) = -\lim_{x\to+\infty} \frac{W(\phi_-(z,x), \mathrm{e}^{x/2})}{W(\phi_-(z,x), \mathrm{e}^{-x/2})} = z\sum_{j=1}^{N} \frac{\gamma_j}{\lambda_j - z}$$

where $\lambda_j \in \mathbb{R}\setminus\{0\}$ and $\gamma_j^{-1} = \lambda_j \int_{\mathbb{R}} |\phi_+|^2 d\omega > 0$.

Stieltjes Moment Problem

- M is a rational function and as $|z| \to \infty$

$$M(z) = 1 - \sum_{n \in \mathbb{Z}_+} \frac{s_n}{z^n}, \qquad s_n = \int_{\mathbb{R}} \lambda^n d\rho(\lambda) = \sum_{j=1}^{N} \lambda_j^n \gamma_j.$$

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- Continued fraction expansion:

$$M(z) = 1 + \cfrac{1}{-\ell_N + \cfrac{1}{m_N(z) + \cfrac{1}{\dots + \cfrac{1}{-\ell_1 + \cfrac{1}{m_1(z) + \cfrac{1}{-\ell_0}}}}}},$$

$$m_n(z) = 8\cosh^2(q_n/2)\, p_n z, \qquad \ell_n = \frac{\tanh(q_{n+1}/2) - \tanh(q_n/2)}{2}.$$

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T. Stieltjes, *Recherches sur les Fractions Continues* (1894)

If all $\lambda_j > 0$

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- Stieltjes formulas work if and only if $\Delta_{1,1}, \ldots, \Delta_{1,N} \neq 0$
- Allows to discuss the behavior of multi-peakons in detail
- $\Delta_{1,j} = 0$ for some j precisely at the times of blow-ups!

Peakon–Antipeakon Interaction

$$u(x,t) = p(t)\mathrm{e}^{-|x-q(t)|} - p(t)\mathrm{e}^{-|x+q(t)|}; \qquad p(0) > 0, \quad q(0) < 0.$$

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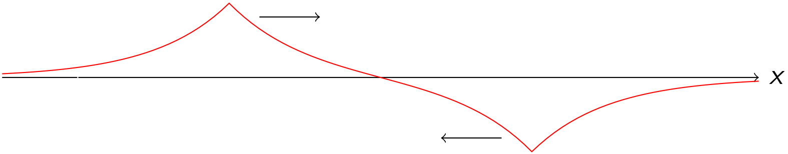

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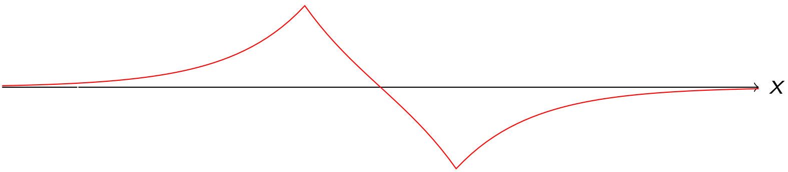

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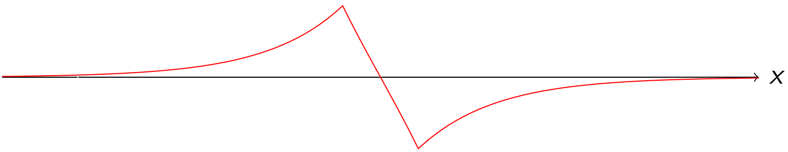
A. Bressan & A. Constantin, *Global conservative solutions of the Camassa–Holm equation*, Arch. Ration. Mech. Anal. **183** (2007)



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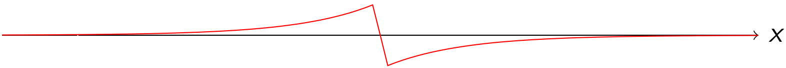

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$\longrightarrow u(x, t^{\times}) \equiv 0$

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For all $t \in (0, t^{\times})$,

$$\|u(\,\cdot\,,t)\|_{H^1(\mathbb{R})} = 4p(t)^2(1-\mathrm{e}^{2q(t)}) = 4H_0^2.$$

However, $u(x,t) \to 0$ as $t \uparrow t^{\times}$ for all $x \in \mathbb{R}$ and

$$u_x(t,0) = -2p(t)\mathrm{e}^{q(t)} \downarrow -\infty, \quad q(t) \uparrow 0, \ \text{ as } \ t \uparrow t^{\times}.$$

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Conservative solutions (u,μ): additional quantity μ measuring the loss of energy at the times of blow-ups . . . (also in *H. Holden & X. Raynaud* (2007))

Peakon–Antipeakon Interaction: The Weyl function

$$M_t(z) = 1 + \cfrac{1}{-\ell_2(t) + \cfrac{1}{m_2(z,t) + \cfrac{1}{-\ell_1(t) + \cfrac{1}{m_1(z,t) + \cfrac{1}{-\ell_0(t)}}}}},$$

where

$$m_1(z,t) = -m_2(z,t) = 8\cosh^2(q(t)/2)p(t)\,z,$$

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Take the limit as $t \to t^\times$: $\ell_0(t^\times) = \ell_2(t^\times) = \frac{1}{2}$ and $\ell_1(t^\times) = 0$.
However, $m_1(z,t^\times)/z = +\infty$ and $m_2(z,t^\times)/z = -\infty$!

Peakon–Antipeakon Interaction: The Weyl function

However, it turns out that for every z

$$\lim_{t\to t^\times} M_t(z) = M_{t^\times}(z) := 1 + \cfrac{1}{-1/2 + \cfrac{1}{16H_0^2\, z^2 + \cfrac{1}{-1/2}}} = \frac{4H_0^2 z^2}{1 - 4H_0^2 z^2}.$$

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First of all, $M_{t^\times}$ is Herglotz. Moreover, $M_{t^\times}$ is the Weyl function for the quadratic spectral problem

$$-y'' + \frac{1}{4}y = z^2\, \upsilon(x) y, \quad x \in \mathbb{R},$$

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A New Isospectral Problem for Multi-Peakons

$$-y'' + \frac{1}{4}y = z\omega(x)\,y + z^2\upsilon(x)\,y, \quad x \in \mathbb{R}, \tag{Iso}$$

where $\omega = 2\sum_k p_k\delta_{x_k}$ and $\upsilon = \sum_k \upsilon_k\delta_{x_k}$ with $p_k \in \mathbb{R}$, $\upsilon_k \geq 0$ and $|\omega_k| + \upsilon_k > 0$ for all $k \in \{1, .., N\}$ and $-\infty < x_1 < ... < x_N < \infty$.

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Lemma (Trace Formulas)

$$\sum_{\lambda\in\sigma} \frac{1}{\lambda} = 2\sum_k p_k, \qquad \sum_{\lambda\in\sigma} \frac{1}{\lambda^2} = 4\sum_{k,n} p_k p_n \mathrm{e}^{-|x_k - x_n|} + 2\sum_k \upsilon_k \qquad (1)$$

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Let $\omega(x,t) = 2\sum_{k=1}^{N(t)} p_k(t)\delta_{x_k(t)}(x)$ and $\upsilon(x,t) = \sum_{k=1}^{N(t)} \upsilon_k(t)\delta_{x_k(t)}(x)$.

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Theorem (Eckhardt & Kostenko (2014))

The pair (u,μ) is a global conservative multi-peakon solution of the Camassa–Holm equation if and only if the problems (Iso) are isospectral with

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...with $L \in (0, \infty]$ and ω and υ . . . on $[0, L)$.

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- Fundamental system of solutions $c(z, x)$ and $s(z, x)$:

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Define the map $\Phi : \mathcal{S} \to \mathcal{N}$ by $\Phi : (L, \omega, \upsilon) \mapsto m$.

Generalized Indefinite Strings

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The map Φ is one-to-one. Moreover, Φ is a homeomorphism.

Applications to the CH equation (in progress...)

$$-f'' + \frac{1}{4}f = z\,\omega f + z^2 \upsilon f \quad \text{on} \quad \mathbb{R} \tag{S2}$$

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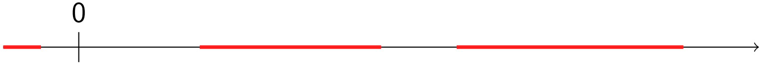

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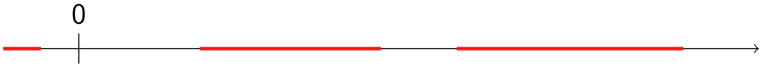

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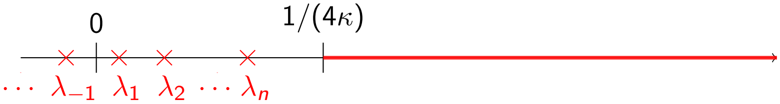

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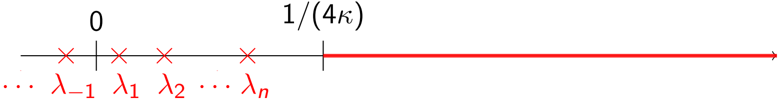
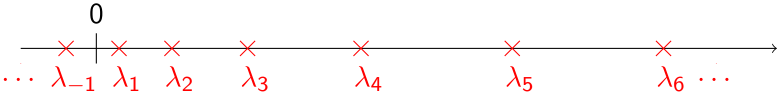


- The decaying case:



J. Eckhardt & A. Kostenko, *An isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation*, Comm. Math. Phys. **329**, 893–918 (2014).

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Thank you for your attention!

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How to understand this equation?

A solution of (S2) is a function $f \in H^1_{\mathrm{loc}}([0, L))$ such that

$$bh(0) + \int_0^L f'(x)h'(x)dx = z\omega(f\ h) + z^2\upsilon(f\ h)$$

for some constant $b =: f'(0-)$ and all $f \in H^1([0, L))$.

A. M. Savchuk & A. A. Shkalikov, *Sturm–Liouville operators with distribution potentials*, Trans. Moscow Math. Soc., 143–190 (2003).

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A Hilbert space setting

In a Hilbert space $\mathcal{H} = \dot{H}^1([0, L)) \times L^2([0, L); \upsilon)$ equipped with the norm

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{H}} = \int_{[0,L)} f_1'(x) g_1'(x)^* dx + \int_{[0,L)} f_2(x) g_2(x)^* d\upsilon(x),$$

we define the maximal linear relation $T_{\max}$ associated with (S2) by saying that $(\mathbf{f}, \mathbf{g}) \in T_{\max}$ iff

$$-f_1'' = z\omega g_1 + z^2 \upsilon g_2, \quad \upsilon f_2 = \upsilon g_1.$$

Conservative Solutions

Bressan, Constantin (2007) and Holden, Raynaud (2007):
If $u(x,t)$ is a solution and $y(t,\xi)$ denotes the corresponding characteristics, $y_t(t,\xi) = u(t, y(t,\xi))$, then the system

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = U^3 - 2PU, \end{cases} \tag{2}$$

where

$$U(t,\xi) = u(t, y(t,\xi)), \quad H(t,\xi) = \int_{-\infty}^{y(t,\xi)} u^2 + u_x^2 \, dx,$$

$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \mathrm{sgn}\,(\xi-\eta) \mathrm{e}^{-\mathrm{sgn}\,(\xi-\eta)(y(\xi)-y(\eta))}(U^2 y_\xi + H_\xi)(\eta) d\eta,$$

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{sgn}\,(\xi-\eta)(y(\xi)-y(\eta))}(U^2 y_\xi + H_\xi)(\eta) d\eta,$$

is equivalent to (**??**).

Conservative Solutions

Using the contraction principle, one can prove global existence of solutions to (2). The uniqueness issue is resolved by considering the set $\mathcal{D}$ of pairs (u, μ), where $u \in H^1(\mathbb{R})$ and μ is a positive Borel measure with

$$\mu_{ac} = (u^2 + u_x^2)dx \quad \text{and} \quad \mu \in BV(R).$$

Holden and Raynaud (2007) proved that there is a metric $d_{\mathcal{D}}$ on $\mathcal{D}$ such that $(\mathcal{D}, d_{\mathcal{D}})$ is a complete metric space and the transformation from Lagrangian to Eulerian coordinates generates a continuous semigroup on $(\mathcal{D}, d_{\mathcal{D}})$. In particular,

$$\mu(t)(\mathbb{R}) = \mu(0)(\mathbb{R})$$

for all $t \in \mathbb{R}$, and

$$\mu(t)(\mathbb{R}) = \mu_{ac}(t)(\mathbb{R}) = \|u\|^2_{H^1(\mathbb{R})}$$

for almost all $t \in \mathbb{R}$.