Spectral Asymptotics for 2×2 Canonical Systems

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(joint work with J. Eckhardt and G. Teschl)

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The Marchenko Formula (1952)

Consider the 1-D Schrödinger operator in $L^2(\mathbb{R}_+)$

$$H_q = -\frac{d^2}{dx^2} + q(x), \quad q \in L^1_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+). \tag{1}$$

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$$m(z) = \frac{1}{\sqrt{-z}}(1 + o(1)), \qquad z \to \infty \tag{2}$$

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Consider the general Strum–Liouville operator in $L^2(\mathbb{R}_+; wdx)$

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If for some $\alpha > -1$

$$\lim_{x\to 0} \frac{1}{x^{1+\alpha}} \int_0^x w(t)dt = 1, \tag{4}$$

then as $z \to \infty$

$$m(z) = C_{\frac{1}{2+\alpha}}(-z)^{-\frac{1}{2+\alpha}}(1 + o(1)), \qquad C_\nu = \frac{\nu^{1-\nu}\Gamma(\nu)}{(1-\nu)^\nu\Gamma(1-\nu)}. \tag{5}$$

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- $\alpha = 0$ gives (2).
- Necessity: Y. Kasahara (1975) and C. Bennewitz (1989).

The Marchenko Formula for Dirac Operators

Consider the 1-D Dirac equation on $\mathbb{R}_+ = [0, \infty)$

$$Jy' + Q(x)y = z\,\mathcal{H}(x)y, \qquad z \in \mathbb{C}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

where $Q = Q^*$, $\mathcal{H} = \mathcal{H}^* \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{2\times 2})$, $\mathcal{H} \geq 0$ a.e. on $\mathbb{R}_+$.

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The analog of Marchenko's formula (2): if $\mathcal{H} \equiv I_2$ on $\mathbb{R}_+$, then

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W. N. Everitt, D. B. Hinton and J. K. Shaw (1983) proved (7) if

$$\mathcal{H}(x) = \begin{pmatrix} a(x) & 0 \\ 0 & c(x) \end{pmatrix}, \quad \lim_{x\to 0} \frac{1}{x}\int_0^x |\mathrm{tr}(\mathcal{H}(t) - a_0 I_2)|dt = 0$$

with some $a_0 > 0$.

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PROBLEM (Everitt, Hinton, and Shaw' 1983):

Characterize those Q and $\mathcal{H}$ such that (7) holds.

Motivation

- Inverse spectral problem

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- The HELP inequality: *W. N. Everitt* (1971)

$$\left(\int_{\mathbb{R}_+} |f'|^2 dx\right)^2 \le K^2 \int_{\mathbb{R}_+} |f|^2 w\, dx \int_{\mathbb{R}_+} |w^{-1} f''|^2 w\, dx \qquad (8)$$

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for all $f \in \mathcal{D}_{\max}$ iff the m-function of $-w^{-1} f''$ satisfies

$$-\mathrm{Im}(z^2 m(z)) \ge 0, \qquad z \in \Gamma_\theta = \{z \in \mathbb{C}_+ : |\arg(z) - \frac{\pi}{2}| \le \theta\}$$

with $\theta = \arccos(1/K)$.

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- The similarity problem: *Kostenko* (2013)
Let $w\in L^1_{\mathrm{loc}}(\mathbb{R})$ be even and positive on $\mathbb{R}$.
The operator $L=\frac{\mathrm{sgn}(x)}{w(x)}\frac{d^2}{dx^2}$ acting in $L^2(\mathbb{R})$ is similar to a s.-a. iff

$$\sup_{y>0}\frac{\mathrm{Im}\,m_+(\mathrm{i}y)}{\mathrm{Re}\,m_+(\mathrm{i}y)}<\infty.$$

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$$\left(\int_{\mathbb{R}_+} |f'|^2 dx\right)^2 \le K^2 \int_{\mathbb{R}_+} |f|^2 w\, dx \int_{\mathbb{R}_+} |w^{-1} f''|^2 w\, dx \qquad (9)$$

  for all $f \in \mathcal{D}_{\max}$ and some $K > 0$ iff the m-function of $-w^{-1} f''$ satisfies

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$$\sup_{y>0} \frac{\mathrm{Im}\, m_+(\mathrm{i}y)}{\mathrm{Re}\, m_+(\mathrm{i}y)} < \infty.$$

2×2 Canonical Systems

$$Jy' = z\,\mathcal{H}(x)y, \qquad x \in \mathcal{I}, \quad z \in \mathbb{C}. \tag{10}$$

We assume that $\mathcal{I} = [0, \ell)$ is finite or infinite interval, i.e., $\ell \in (0, +\infty]$,

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{H}(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix}, \tag{11}$$

where a, b and $c \in L^1_{\mathbb{R},\mathrm{loc}}[0, \ell)$. Moreover,

$$\mathcal{H}(x) \geq 0 \quad \text{for a.a. } \ x \in \mathcal{I}, \qquad \int_{\mathcal{I}} \mathrm{tr}\,\mathcal{H}(x)dx = +\infty. \tag{12}$$

We shall also assume that

$$\mathcal{H}(x) \neq \mathcal{H}_a(x) := \begin{pmatrix} a(x) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \ [0, \ell). \tag{13}$$

The Weyl–Titchmarsh m-function

Consider the matrizant

$$\mathbb{U}(z,x) = \begin{pmatrix} \theta_1(z,x) & \phi_1(z,x) \\ \theta_2(z,x) & \phi_2(z,x) \end{pmatrix}, \qquad \mathbb{U}(z,0) = I_2. \tag{14}$$

Conditions (12) enables us to define *the m-function* $m : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by

$$\Psi(z,x) := \mathbb{U}(z,x) \begin{pmatrix} 1 \\ m(z) \end{pmatrix} \in L^2(\mathcal{I}; \mathcal{H}(x)) \tag{15}$$

The function m is Herglotz.

Theorem (de Branges)

Each Herglotz functiont is an m-function for a canonical system (10). If $\ell = \infty$ and $\operatorname{tr} \mathcal{H}(x) \equiv 1$, then this correspondence is one-to-one. Moreover, in this case $m_n \to m$ on compact subsets of $\mathbb{C}_+$ iff $\int_0^x \mathcal{H}_n dt \to \int_0^x \mathcal{H} dt$ locally uniformly on $\mathbb{R}_+$.

The Main Result. I

Consider the canonical system (10). Let also m be the corresponding m-function (15).

Theorem 1 (Eckhardt, Kostenko, Teschl).

Let $a_0 \in [0,1)$, $b_0 \in (-1,1)$ be such that $h_0^2 := a_0(1-a_0) - b_0^2 \geq 0$. Then the following conditions are equivalent:

(i)

$$m(z) = \frac{\mathrm{i}h_0 - b_0}{1 - a_0} + o(1), \qquad z \to \infty,$$

uniformly in any nonreal sector in $\mathbb{C}_+$,

(ii)

$$\lim_{x\to 0} \frac{1}{\eta(x)} \int_0^x \mathcal{H}(t)dt = \mathcal{H}_0 := \begin{pmatrix} a_0 & b_0 \\ b_0 & 1-a_0 \end{pmatrix}.$$

Here $\eta(x) = \int_0^x \mathrm{tr}\, \mathcal{H}(t)dt$ and $h_0 = \sqrt{\det \mathcal{H}_0}$.

The Main Result. I (continued)

Let $Q = Q^* \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{2\times 2})$. Consider the 1-D Dirac equation

$$Jy' + Q(x)y = z\,\mathcal{H}(x)y, \quad x \in \mathbb{R}_+. \tag{16}$$

Using the standard gauge transformation, (16) is unitarily equivalent to (10) with $U(0,x)^*\mathcal{H}(x)U(0,x)$ in place of $\mathcal{H}$. Thus, applying Theorem 1 and noting that $U(0,0) = I_2$, we end up with

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Corollary 1 (Eckhardt, Kostenko, Teschl).

Let m be the m-function of (16). The following conditions are equivalent:

(i)
$$m(z) = \mathrm{i} + o(1), \qquad z \to \infty,$$

(ii)
$$\lim_{x\to 0} \frac{1}{\eta(x)} \int_0^x \mathcal{H}(t)dt = \frac{1}{2} I_2.$$

Here $\eta(x) = \int_0^x \mathrm{tr}\,\mathcal{H}(t)dt$.

Sketch of the proof:

- If $\mathcal{H}(x) = \mathcal{H}_0 = \begin{pmatrix} a_0 & b_0 \\ b_0 & 1-a_0 \end{pmatrix}$ on $\mathbb{R}_+$ with $a_0 \in [0,1)$ and $h_0^2 = \det \mathcal{H}_0 \geq 0$, then the corresponding m-function is

$$m_0(z) = \frac{\mathrm{i}h_0 - b_0}{1-a_0} =: \zeta_0, \quad z \in \mathbb{C}_+.$$

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- If $\mathcal{H}_r(x) := \mathcal{H}(r^{-1}x)$, $r > 0$, then the corresponding m-function m_r is given by $m_r(z) = m(rz)$.

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- $m(z) = \zeta_0 + o(1)$ as $z \to \infty$ iff $m_r \to \zeta_0$ as $r \to \infty$ on compact subsets of $\mathbb{C}_+$. It remains to apply the de Branges Theorem.

The Main Result. II

Let $\alpha > 0$. Set

$$m_\alpha(z) = C_{\frac{1}{2+\alpha}} e^{i\pi\frac{1+\alpha}{2+\alpha}} z^{-\frac{\alpha}{2+\alpha}}, \qquad C_\nu := \frac{(1-\nu)^\nu \Gamma(1-\nu)}{\nu^{1-\nu}\Gamma(\nu)} \tag{17}$$

for $z \in \mathbb{C}_+$. m_α is the m-function of (10) with

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Theorem 2 (Eckhardt, Kostenko, Teschl).

Suppose $\operatorname{tr}\mathcal{H}(x) \equiv 1$ on $\mathbb{R}_+$. Then the following are equivalent:

(i) for some $\alpha > 0$

$$\lim_{x\to 0} \frac{1}{x^{1+\alpha}} \int_0^x a(t)dt = 1, \quad \lim_{x\to 0} \frac{1}{x^{1+\alpha}} \int_0^x b(t)dt = 0,$$

(ii)

$$m(z) = m_\alpha(z)(1+o(1)), \qquad z \to \infty.$$

Radial Dirac operators

Consider (16) with $R \equiv I_2$ and

$$Q(x) = \begin{pmatrix} q_{sc}(x) & \frac{\kappa}{x} + q_{am}(x) \\ \frac{\kappa}{x} + q_{am}(x) & -q_{sc}(x) \end{pmatrix}, \quad x \in \mathbb{R}_+,$$

where $q_{am}, q_{sc} \in L^1_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$ and $\kappa \geq 0$.

Theorem 3 (Eckhardt, Kostenko, Teschl).

Let ρ be the corresponding (Dirichlet) spectral function. Then

$$\rho(\pm\lambda) = \pm\frac{1}{\pi(2\kappa+1)}\lambda^{2\kappa+1}(1+o(1)), \quad \lambda \to +\infty.$$

Remark. Spectral measure is defined by $U : L^2(\mathbb{R}_+;\mathbb{C}^2) \to L^2(\mathbb{R}_+; d\rho)$

$$U : f \mapsto \hat{f}, \quad \hat{f}(\lambda) := \lim_{c \to +\infty} \int_0^c \langle \Phi(\lambda, x), f(x)^* \rangle_{\mathbb{C}^2} dx$$

where the solution $\Phi(\cdot, x) \sim (o(x^\kappa), x^\kappa)$ as $x \to 0$.

Further applications...

- Radial Schrödinger operators:

$$H = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + q(x), \quad x > 0,$$

where $q \in W^{-1,2}_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$ and $\ell \geq -\frac{1}{2}$.

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- Krein strings:

$$-f'' = z\,\omega f, \qquad x \in [0,\ell),$$

where ω is a positive Borel measure on $[0,\ell)$
(one recovers the results of *I. S. Kac, Y. Kasahara and C. Bennewitz*).

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- Generalized indefinite strings:

$$-f'' = z\,\omega f + z^2\,\upsilon f, \qquad x \in [0,\ell),$$

where $\omega \in W^{-1,2}_{\mathbb{R},\mathrm{loc}}[0,\ell)$ and υ is a positive Borel measure on $[0,\ell)$.

- J. Eckhardt, A. Kostenko and G. Teschl, *Spectral asymptotics for canonical systems*, J. Reine Angew. Math., to appear (arXiv:1412.0277)

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- A. Kostenko, *The similarity problem for indefinite Sturm–Liouville operators and the HELP inequality*, Adv. Math. **246**, 368–413 (2013).

THANK YOU FOR YOUR ATTENTION!

Radial Schrödinger operators

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$$H = -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + q(x), \quad x > 0,$$

where $q \in W^{-1,2}_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$ and $\ell \geq -\frac{1}{2}$. Setting $q_{sc} \equiv 0$, we get

Corollary 3.

Let ρ be the corresponding (*Dirichlet*) spectral function. Then

$$\rho(\lambda) \sim \frac{1}{\pi(\ell + \frac{3}{2})} \lambda^{\ell + \frac{3}{2}}, \quad \lambda \to +\infty,$$

- $\ell = 0$, $q \in L^1_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$: V. A. Marchenko (1952),
- $\ell \in \mathbb{N}$, $q \in L^1_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$: M. G. Krein (1957),
- $\ell \geq -\frac{1}{2}$, $xq \in L^1_{\mathbb{R},\mathrm{loc}}(\mathbb{R}_+)$: Kostenko, Teschl (2013).

The isospectral problem for the Camassa-Holm equation

$$-f'' + \frac{1}{4}f = z\,\omega(x)f + z^2\,\upsilon(x)f, \quad x \in [0, l) \tag{18}$$

Here $\omega \in W^{-1,2}_{\mathbb{R},\mathrm{loc}}[0, l)$ and $\upsilon \in \mathcal{M}^+_{\mathrm{loc}}[0, l)$. The m-function is defined by

$$\psi(z, x) = \theta(z, x) + zm(z)\phi(z, x) \in W^{1,2}(0, l).$$

Theorem 4.

Let $m(z)$ be the Dirichlet m-function for (18). Then

$$m(z) \sim \mathrm{i}, \qquad z \to \infty$$

in any nonreal sector of $\mathbb{C}_+$ if and only if (with $\mathrm{w} := \int_0^x \omega$)

$$\int_0^x \mathrm{w}(t)dt = o(x), \qquad \int_0^x \mathrm{w}^2(t)dt + \int_0^x d\upsilon = x(1 + o(1)), \quad x \to 0.$$