Hankel Operators and Applications
(Lecture Notes)

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Abstract. A Hankel matrix is a (possibly infinite) matrix whose entries depend only on the sum of coordinates. These short notes can be considered as an introduction into a lively and beautiful area connecting the function theory in the unit disc (theory of Hardy spaces) with spectral theory of operators in Hilbert spaces. Applications to moment problems and the cubic Szegö equation are also discussed.

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Preface

These short notes grew out from an advanced course given in University of Vienna in 2015. The main goal is to give a quick introduction into a lively and beautiful area connecting theory of Hardy spaces in the unit disc with spectral theory of operators in Hilbert spaces. Another motivation is a recent work of P. Gerard and S. Gréllier [**7**]–[**10**] on the cubic Szegö equation, connecting non-dispersive PDEs with Hankel operators and hence motivating the study of inverse spectral problems for this class of operators.

The first part of the notes serves as an introduction to Hankel matrices and their basic properties. In Section 1.1, we introduce the class of Hankel operators as operators in $\ell_2 = \ell_2(\mathbb{Z}_+)$ with matrices of the form $(a_{i+j})_{i,j\in\mathbb{Z}_+}$ and then answer the question whether this matrix defines a bounded operator on ℓ_2 (Nehari's Theorem). The key role in solving this problem plays the most important realization of Hankel matrices as operators from the Hardy space H^2 to $H^2_- := L^2 \ominus H^2$. Section 1.2 deals with one of the earliest results on Hankel operators — Kronecker's Theorem, which describes all finite rank Hankel matrices. Next, we present Hartman's theorem describing compact Hankel operators as well the results of D. Sarason connecting compact Hankel operators with the space of functions of vanishing mean oscillation (Section 1.3). In Section 1.4 we briefly discuss vectorial Hankel operators.

The second part deals with applications of Hankel operators. We focus only on two such applications out of many. Perhaps, the most commonly known appearance of Hankel matrices is in the moment problem. In Section 2.1, we briefly introduce the classical moment problems (Hamburger and Stieltjes) and then proceed with the beautiful results of H. Widom [**39**], who characterized bounded/compact positive Hankel operators by means of the corresponding measures. Section 2.2 deals with the newly discovered cubic Szegö equation [7], [10]. It turns out that this Hamiltonian equation is completely integrable — it enjoys the Lax pair structure and the Lax operator in this case is nothing but a Hankel operator. The latter motivates the study of the so-called *inverse spectral problems* for Hankel operators and we review this material in Sections 2.3–2.4. Most of this material is given without proofs (e.g., a solution of the inverse spectral problem for self-adjoint Hankel operators by A. V. Megretskii, V. V. Peller and S. R. Treil [25, Chapter XII] is far beyond the scope of these notes).

The familiarity with basic functional analysis (e.g., bounded and compact operators in separable Hilbert spaces) and Fourier series is assumed. In fact, these notes can be considered as a good source of examples for students trying to learn the theory of linear operators in Hilbert spaces since the very basic questions on boundedness and compactness of a particular class of linear operators (or infinite matrices) lead to connections with deep results in a very rich and beautiful area of Hardy spaces. It is desirable to have some knowledge of the latter, however, in appendices basic facts on Fourier series (convergence and summability), Hardy spaces in the unit disc (boundary values and factorization), BMO and VMO spaces (including Fefferman's and Sarason's theorems) are provided.

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Chapter 1

Hankel, Kronecker and Nehari

1.1. Boundedness

Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. An infinite matrix $A = (a_{ij})_{i,j\ge 0}$ is called *a Hankel matrix* if $a_{ij} = \alpha_{i+j}$ for all i, $j \ge 0$. That is, the Hankel matrices are the matrices whose entries depend only on the sum of coordinates:

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \dots \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.1}$$

In the sequel we shall denote the Hankel matrix A by H_α (sometimes α is called a *symbol* of a Hankel matrix).

The classical framework for the theory of Hankel operators is the usual Hilbert space of square summable sequences $\ell_2 := \ell_2(\mathbb{Z}_+)$. By $\{e_n\}_{n=0}^{\infty}$ we denote the standard orthonormal basis in ℓ_2. It is well known that every bounded operator on a Hilbert space $\mathfrak{H}$ admits a matrix representation in an orthonormal basis. However, it is a notoriously difficult problem to decide whether an infinite matrix defines a bounded operator on a Hilbert space. We begin with the following simple fact.

Lemma 1.1.1. *If the Hankel matrix H_α generates a bounded operator on ℓ_2, then $\alpha \in \ell_2$.*

Proof. It suffices to note that $\alpha = H_\alpha e_0 \in \ell_2$. □

The problem whether H_α is bounded on ℓ_2 or not was solved by Z. Nehari in 1957. Before formulate his result, we need another representation of Hankel operators. It turns out that the theory of Hankel operators is closely connected with the theory of functions on the unit circle. More precisely, consider the Hilbert space of function on the unit circle $\mathfrak{H} = L^2(\mathbb{T})$ equipped with the inner product[1]

$$(f, g)_{L^2} = \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)g(\theta)^* d\theta.$$

Then the Hardy space H^2 is a subspace of $L^2(\mathbb{T})$ such that its negative Fourier coefficients vanish,[2]

$$H^2 = \{f \in L^2(\mathbb{T}) \mid \hat{f}_n = 0,\ n < 0\}. \tag{1.2}$$

Denote by P_+ and P_- the orthogonal projections in $L^2(\mathbb{T})$ onto H^2 and, respectively, its orthogonal complement $H^2_- := L^2(\mathbb{T}) \ominus H^2$. The operator P_+ is also known as *the Riesz projection.* The standard orthonormal basis of $L^2(\mathbb{T})$ is given by the family of exponents $\{\mathrm{e}_n\}_{n\in\mathbb{Z}}$, $\mathrm{e}_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}$, $\theta \in \mathbb{T}$. In particular, $\{\mathrm{e}_n\}_{n\geq 0}$ and $\{\mathrm{e}_n\}_{n<0}$ are the orthonormal bases of H^2 and H^2_-, respectively.

Take $\varphi \in L^2(\mathbb{T})$ and denote its Fourier coefficients by $\hat{\varphi}_n$, $n \in \mathbb{Z}$. Note that by Parceval's formula, $\varphi \in L^2(\mathbb{T})$ exactly when $\hat{\varphi} \in \ell_2(\mathbb{Z})$. Consider now the following operator $H_\varphi : H^2 \to H^2_-$ defined by

$$H_\varphi : f \mapsto P_-(\varphi f). \tag{1.3}$$

It is easy to see that

$$H_\varphi \mathrm{e}_k = P_-(\varphi \mathrm{e}_k) = P_-\left(\sum_{n\in\mathbb{Z}} \hat{\varphi}_{n-k}\mathrm{e}_n\right) = \sum_{n\in\mathbb{N}} \hat{\varphi}_{-(n+k)}\mathrm{e}_{-n} \tag{1.4}$$

for all $k \in \mathbb{Z}_+$. Therefore, the matrix representation of H_φ is given by the Hankel matrix with coefficients $\alpha_n = \hat{\varphi}_{-(n+1)}$, $n \geq 0$.

Now we are in position to formulate Nehari's theorem.

Theorem 1.1.2 (Nehari). *Let $\alpha \in \ell_2$. The Hankel matrix H_α generates a bounded operator on ℓ_2 if and only if there is a function $\varphi \in L^\infty(\mathbb{T})$ such that*

$$\alpha_n = \hat{\varphi}_n, \quad n \geq 0. \tag{1.5}$$

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[1]Throughout the text z^* denotes the complex conjugate of $z \in \mathbb{C}$.

[2]The Hardy space on the unit disk $H^2(\mathbb{D})$ can be identified with ℓ_2:

$$f \in H^2(\mathbb{D}) \quad \Longleftrightarrow \quad f(z) = \sum_{n\geq 0} \hat{f}_n z^n, \quad \{\hat{f}_n\}_{n=0}^\infty \in \ell_2.$$

Moreover, it turns out that $H^2 = H^2(\mathbb{D})$, that is, every function from $H^2(\mathbb{D})$ can be identified with its boundary values and vise versa (see Appendix A.2.2).

Moreover, in this case,

$$\|H_\alpha\| = \inf\{\|\psi\|_\infty | \, \hat{\psi}_n = \alpha_n, \; n \geq 0\}. \tag{1.6}$$

Proof. *Sufficiency.* Suppose there is $\varphi \in L^\infty(\mathbb{T})$ satisfying (1.5). For every $f,\, g \in \ell_2$ having finitely many nonzero entries, consider the bilinear form associated with the matrix H_α:

$$(H_\alpha f, g) = \sum_{k,n\geq 0} \alpha_{k+n} f_k g_n^*. \tag{1.7}$$

Let $f(\theta) = \sum_{k\geq 0} f_k \mathrm{e}_k(\theta)$ and $g(\theta) = \sum_{n\geq 0} g_n \mathrm{e}_n(\theta)$ be polynomials in the Hardy space H^2. Then we can rewrite (1.7) as follows

$$\begin{aligned}(H_\alpha f, g) = \sum_{k,n\geq 0} \hat{\varphi}_{k+n} f_k g_n^* &= \sum_{k\geq 0} \hat{\varphi}_k \sum_{n=0}^{k} f_n g_{k-n}^* \\ &= \sum_{k\geq 0} \hat{\varphi}_k \hat{q}_k^* = \frac{1}{2\pi}\int_{\mathbb{T}} \varphi(\theta) q(\theta)^* d\theta,\end{aligned} \tag{1.8}$$

where

$$q(\theta) = f(\theta)^* g(\theta) = \sum_{k\geq 0} q_k \mathrm{e}^{\mathrm{i}k\theta}, \quad q_k = \sum_{n=0}^{k} f_n^* g_{k-n}.$$

Therefore,

$$|(H_\alpha f, g)| \leq \|\varphi\|_\infty \|q\|_{L^1} \leq \|\varphi\|_\infty \|f\|_{L^2}\|g\|_{L^2} = \|\varphi\|_\infty \|f\|_{\ell_2}\|g\|_{\ell_2}, \tag{1.9}$$

which shows that this bilinear form is bounded on ℓ_2 and $\|H_\alpha\| \leq \|\varphi\|_\infty$.

Necessity. Assume now that H_α is a bounded operator on ℓ_2. Consider the linear functional defined in $H^1(\mathbb{D})$ on the set of analytic polynomials Pol_+ by

$$l_\alpha(q) := \sum_{n\geq 0} \alpha_n \hat{q}_n, \quad q \in \mathrm{Pol}_+. \tag{1.10}$$

By the Hahn–Banach theorem, l_α extends by continuity to a bounded functional on H^1 if and only if there is $\varphi \in L^\infty$ such that $l_\alpha(q) = (q, \varphi^*)_{L^2}$ and $\hat{\varphi}_n = \alpha_n$, $n \geq 0$. In particular, $\|l_\alpha\| = \inf\{\|\psi\|_\infty | \, \hat{\psi}_n = \alpha_n, \; n \geq 0\}$. Therefore, we need to show that $\|l_\alpha\| \leq \|H_\alpha\|$.

Assume additionally that $\alpha \in \ell_1$. Then l_α defines a bounded functional on $H^1(\mathbb{D})$. Assume that $q \in H^1$ with $\|q\|_{H^1} \leq 1$. Then q admits a representation $q = fg^*$ with $f,\, g \in H^2$ with $\|f\|_{H^2},\, \|g\|_{H^2} \leq 1$. Hence, similar to (1.8), we can show that

$$l_\alpha(q) = (H_\alpha f, g)_{\ell_2}, \quad f = \{\hat{f}_k\}_{k\geq 0}, \quad g = \{\hat{g}_k\}_{k\geq 0}. \tag{1.11}$$

Therefore, we get

$$\|l_\alpha\| = \sup_{q\in H^1, \|q\|_1 \leq 1} |l_\alpha(q)| \leq \|H_\alpha\|, \tag{1.12}$$

and hence the claim follows.

Assume now that α is an arbitrary sequence such that H_α is bounded. For $r \in (0,1)$ we set

$$\alpha^r = \{\alpha_n^r\}_{n\geq 0}, \qquad \alpha_n^r := \alpha_n r^n.$$

Since $\alpha_{k+n}^r = \alpha_{k+n} r^{k+n} = r^k \alpha_{k+n} r^n$, the matrix H_{α^r} admits the following factorization

$$H_{\alpha^r} = D_r H_\alpha D_r, \qquad D_r := \operatorname{diag}(1, r, r^2, \dots, r^n, \dots). \tag{1.13}$$

Clearly, $\|D_r\| = 1$ and we immediately conclude that

$$\|H_{\alpha^r}\| \leq \|D_r\|\|H_\alpha\|\|D_r\| = \|H_\alpha\| \tag{1.14}$$

for all $r \in (0,1)$. On the other hand, $\alpha^r \in \ell_1$ whenever $r \in (0,1)$ and hence, as we already proved, we get

$$\|l_{\alpha^r}\|_{H^1\to\mathbb{C}} \leq \|H_{\alpha^r}\| \leq \|H_\alpha\|.$$

It remains to note that l_{α^r} converges strongly to l_α as $r \to 1$ (that is, $l_{\alpha^r}(q) \to l_\alpha(q)$ for all $q \in H^1$) since this family of functionals is uniformly bounded on H^1. Therefore, l_α is bounded too. □

Remark 1.1.3. Nehari's theorem allows to reduce the problem of boundedness of a Hankel operator on ℓ_2 to the question of existence of an extension of the sequence α to the sequence of Fourier coefficients of a bounded function. This problem is non-trivial as the next example shows.

Example 1.1.4 (The Hilbert matrix)**.** The Hankel matrix with the coefficients $\alpha_n = \frac{1}{n+1}$, $n \geq 0$ is called the Hilbert matrix:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.15}$$

The Hilbert inequality [13, Chapter IX] states that the bilinear form generated by H (and hence the operator H) is bounded on ℓ_2:

$$|(Hf,g)_{\ell_2}| = \Big| \sum_{k,n\geq 0} \frac{f_k g_n^*}{k+n+1} \Big| \leq \pi \, \|f\|_2 \|g\|_2. \tag{1.16}$$

By Theorem 1.1.2, there is a function $\varphi \in L^\infty(\mathbb{T})$ such that $\hat{\varphi}_n = \frac{1}{n+1}$ for all $n \geq 0$. However, it is easy to see that the function

$$\varphi_+(\theta) = \sum_{n\geq 0} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{n+1}, \quad \theta \in \mathbb{T},$$

does not belong to $L^\infty(\mathbb{T})$. On the other hand, the function $\varphi(\theta) = \sum_{n\in\mathbb{Z}} \frac{e^{in\theta}}{|n|+1}$ belongs to L^∞.

Exercise 1.1.1. Compute φ and φ_+[3].

After the work of C. Fefferman on the space BMO (see Appendix A.4) it has become possible to judge whether H_α is bounded or not in terms of the sequence α itself.

Theorem 1.1.5. *The matrix H_α generates a bounded operator on ℓ_2 if and only if the function*

$$\alpha(z) = \sum_{n\geq 0} \alpha_n z^n, \qquad z \in \mathbb{D}, \tag{1.17}$$

belongs to the space $BMOA := BMO \cap H^1$.

Proof. Straightforward from Nehari's Theorem 1.1.2 and Fefferman's Theorem A.4.4 (see also Remark A.3.4). □

Of course we can immediately reformulate Theorems 1.1.2 and 1.1.5 for the operator H_φ defined by (1.3).

Theorem 1.1.6. *Let $\varphi \in L^2(\mathbb{T})$. The following statements are equivalent:*

(i) *H_φ is bounded,*

(ii) *there is a function $\psi \in L^\infty(\mathbb{T})$ such that*

$$\hat{\varphi}_{-n} = \hat{\psi}_{-n}, \qquad n \in \mathbb{N}, \tag{1.18}$$

(iii) *$P_-\varphi \in BMO(\mathbb{T})$.*

If one of the above conditions is satisfied, then

$$\|H_\varphi\| = \inf\{\|\psi\|_\infty \,|\, \psi \text{ satisfies } (1.18)\} \tag{1.19}$$

Remark 1.1.7. Let $\varphi \in L^\infty(\mathbb{T})$ and $\psi \in H^\infty$. Then

$$H_{\varphi+\psi} f = P_-\big((\varphi+\psi)f\big) = P_-(\varphi f + \psi f) = P_-(\varphi f) = H_\varphi.$$

This in particular implies that

$$\|H_\varphi\| \leq \|\varphi + \psi\|_\infty$$

for all $\psi \in H^\infty$. Moreover, Nehari's theorem states that

$$\|H_\varphi\| = \inf\{\|\varphi - \psi\|_\infty \,|\, \psi \in H^\infty\} = \operatorname{dist}(\varphi, H^\infty). \tag{1.20}$$

The problem of approximation of an L^∞ function by bounded analytic functions is called *Nehari's problem.*

[3]Note that $\varphi(\theta) = \sum_{n\geq 0} \frac{\cos(n\theta)}{n+1} \in L^\infty(\mathbb{T})$ and its harmonic conjugate $\tilde{\varphi}(\theta) = \sum_{n\in\mathbb{N}} \frac{\sin(n\theta)}{n+1}$ as well as $\varphi_+ = P_+\varphi$ do not belong to $L^\infty(\mathbb{T})$. Therefore, it might happen that $\varphi \in L^\infty$, however $P_+\varphi \notin L^\infty$. In other words, **the Riesz projection P_+ and the Hilbert transform are unbounded on L^∞!**

A standard compactness argument shows that the norm is always attained:

Lemma 1.1.8. *Let $\varphi \in L^\infty$. Then there is $f \in H^\infty$ such that*

$$\|\varphi - f\|_\infty = \operatorname{dist}(\varphi, H^\infty).$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of H^∞ functions such that

$$\lim_{n\to\infty} \|\varphi - f_n\|_\infty = \operatorname{dist}(\varphi, H^\infty).$$

Clearly, this sequence is bounded and hence there is a subsequence (also denoted by f_n) which converges in the weak-$*$ topology to some function $f \in H^\infty$. Hence

$$\operatorname{dist}(\varphi, H^\infty) \le \|\varphi - f\|_\infty \le \lim_{n\to\infty} \|\varphi - f_n\|_\infty = \operatorname{dist}(\varphi, H^\infty),$$

which proves the claim. □

Thus, we proved that for any bounded Hankel operator there exists a symbol of minimal norm. A natural question arises whether such a symbol is unique. The first results in this direction were obtained by S. Ya. Khavinson (1951) and W. Rogosinski and H. Shapiro (1953). For further details we refer to [**6**, Chapter IV], [**25**, Chapter I.1].

We complete this section with the description of bounded Hankel operators in terms of certain commutation relations.

Theorem 1.1.9. *Let A be a bounded operator on ℓ_2. Then A is a Hankel operator if and only if*

$$\mathcal{S}^* A = A\mathcal{S}, \tag{1.21}$$

where $\mathcal{S}$ is the shift operator on ℓ_2, $\mathcal{S}\colon (f_0, f_1, f_2, \dots) \mapsto (0, f_0, f_1, \dots)$, and S^* is its adjoint, $\mathcal{S}^*\colon (f_0, f_1, f_2, \dots) \mapsto (f_1, f_2, \dots)$.

Proof. Let $A = H_\alpha$ be a bounded Hankel matrix. Note that

$$H_\alpha f = \sum_{n\ge 0} (f, \mathcal{S}^{*n}\alpha)\mathrm{e}_n, \tag{1.22}$$

for every $f \in \ell_2$. Therefore, we get

$$\begin{aligned} \mathcal{S}^* H_\alpha f &= \mathcal{S}^*\Big(\sum_{n\ge 0} (f, \mathcal{S}^{*n}\alpha)\mathrm{e}_n\Big) = \sum_{n\ge 0} (f, \mathcal{S}^{*n}\alpha)\mathcal{S}^*\mathrm{e}_n \\ &= \sum_{n\ge 0} (f, \mathcal{S}^{*(n+1)}\alpha)\mathrm{e}_n = \sum_{n\ge 0} (\mathcal{S}f, \mathcal{S}^{*n}\alpha)\mathrm{e}_n = H_\alpha \mathcal{S} f. \end{aligned}$$

Assume now that A satisfies (1.21). Then

$$\begin{aligned} a_{k,n} = (A\mathrm{e}_k, \mathrm{e}_n)_{\ell_2} &= (A\mathrm{e}_k, \mathcal{S}\mathrm{e}_{n-1})_{\ell_2} = (\mathcal{S}^* A\mathrm{e}_k, \mathrm{e}_{n-1})_{\ell_2} \\ &= (A\mathcal{S}\mathrm{e}_k, \mathrm{e}_{n-1})_{\ell_2} = (A\mathrm{e}_{k+1}, \mathrm{e}_{n-1})_{\ell_2} = a_{k+1,n-1}, \end{aligned}$$

for all k, $n \in \mathbb{N}$. The latter immediately implies that $a_{k,n} = \alpha_{k+n}$ for all k, $n \in \mathbb{Z}_+$ and hence $A = H_\alpha$. □

Corollary 1.1.10. *Let A be a bounded operator from H^2 to H^2_-. Then A is a Hankel operator if and only if*

$$P_- \mathrm{S} A = A\mathcal{S}, \tag{1.23}$$

where $\mathcal{S}$ is the shift operator on H^2 and S *is the bilateral shift on L^2.*

Exercise 1.1.2. Prove Corollary 1.1.10.

Corollary 1.1.11. *Let H_φ be a bounded Hankel operator on H^2. Then* $\ker(H_\varphi)$ *is an invariant subspace of $\mathcal{S}$.*

If $\ker(H_\varphi) \neq \{0\}$, then by the Beurling Theorem A.2.12, there is an inner function $G \in H^2$ such that $\ker(H_\varphi) = GH^2$. This in particular implies that either $\dim\ker(H_\varphi) = 0$ or $\dim\ker(H_\varphi) = \infty$.

1.2. Finite Rank

One of the first results about Hankel matrices was a theorem of L. Kronecker (1881) that describes the Hankel matrices of finite rank. Corollary 1.1.11 allows to answer this question too, however, it requires the additional boundedness assumption and uses a heavy machinery (Beurling's description of invariant subspaces of the shift operator). Hence we would like to present the elementary proof.

We need to recall the following definition. A function f is called *a rational function* if it can be written in the following form

$$f(z) = \frac{P(z)}{Q(z)},$$

where P and Q are polynomials and $Q \not\equiv 0$. Without loss of generality we shall always assume that P and Q are *prime*, that is, there are no non-constant polynomial R and polynomials P_1 and Q_1 such that $P(z) = R(z)P_1(z)$, $Q(z) = R(z)Q_1(z)$. *The degree* of a rational function is the maximum of the degrees of its constituent polynomials P and Q. In other words, $\deg f$ is the multiplicity of poles of f (including a possible pole at infinity).

If $z = 0$ is not a pole of f, then f can be expanded into a Taylor series

$$f(z) = \sum_{n\geq 0} f_n z^n,$$

and its radius of convergence equals the distance from $z = 0$ to the set of zeros of $Q(z)$. For such a series let us define the shift operator $\mathcal{S}$

$$\mathcal{S} : f(z) \mapsto zf(z) = \sum_{n\geq 0} f_n z^{n+1}, \tag{1.24}$$

and the backward shift $\mathcal{S}^*$

$$\mathcal{S}^* : f(z) \mapsto \frac{f(z) - f_0}{z} = \sum_{n\ge 0} f_{n+1} z^n. \tag{1.25}$$

Note that

$$\mathcal{S}^k \mathcal{S}^{*j} f = \mathcal{S}^{k-j} f - \mathcal{S}^{k-j} \sum_{n=0}^{j-1} f_n z^n, \quad \mathcal{S}^{*k} \mathcal{S}^j f = \mathcal{S}^{*(k-j)} f, \tag{1.26}$$

for all $k,\ j \in \mathbb{Z}_+$ with $k \ge j$.

Theorem 1.2.1. *Let H_α be a Hankel matrix with symbol $\alpha = \{\alpha_n\}_{n\in\mathbb{Z}_+}$. Then H_α has a finite rank if and only if the formal power series $\alpha(z) = \sum_{n\ge 0} \alpha_n z^n$ determines a rational function. In this case,*

$$\operatorname{rank}(H_\alpha) = \deg z\alpha(z). \tag{1.27}$$

Proof. *Necessity.* Suppose $\operatorname{rank}(H_\alpha) = N < \infty$. Then the first $N+1$ rows are linearly dependent and hence there are complex numbers $c_0,\ c_1,\ \dots,\ c_N$ (not identically zero) such that

$$c_0\alpha + c_1\mathcal{S}^*\alpha + \cdots + c_N \mathcal{S}^{*N}\alpha = 0. \tag{1.28}$$

Applying $\mathcal{S}^N$ to both sides in (1.28) and using (1.26), we get

$$\begin{aligned} 0 = \mathcal{S}^N \Big(\sum_{k=0}^N c_k \mathcal{S}^{*k}\alpha \Big) &= \sum_{k=0}^N c_k \mathcal{S}^n \mathcal{S}^{*k}\alpha \\ &= \sum_{k=0}^N c_k \mathcal{S}^{N-k}\alpha - p = q\alpha - p, \end{aligned} \tag{1.29}$$

where $q(z) = \sum_{n=0}^N c_n z^{N-n}$ and p is a polynomial of degree $\deg p \le N-1$,

$$p(z) = \sum_{n=0}^{N-1} p_n z^n. \tag{1.30}$$

Therefore, (1.29) shows that $\alpha = p/q$ is a rational function and

$$\deg z\alpha(z) \le \max(\deg q, \deg zp) = N.$$

In particular, $c_N \ne 0$ since $\operatorname{rank}(H_\alpha) = N$ by the assumption.

Sufficiency. Suppose that $\alpha(z) = p(z)/q(z)$ is a rational function with $\deg p \le N-1$ and $\deg q \le N$. Hence we get

$$p = q\alpha = \sum_{n=0}^N c_n \mathcal{S}^{N-n}\alpha, \qquad q(z) = \sum_{n=0}^N c_n z^{N-n}.$$

Applying $\mathcal{S}^{*N}$ to both sides and using the second identity in (1.26), we get

$$0 = \mathcal{S}^{*N} p = \mathcal{S}^{*N} \sum_{n=0}^{N} c_n \mathcal{S}^{N-n} \alpha = \sum_{n=0}^{N} c_n \mathcal{S}^{*n} \alpha,$$

which means that the first $N+1$ rows of H_α are linearly dependent.

Set

$$m = \max\{n \,|\, n \le N,\ c_n \ne 0\}.$$

Clearly, $\mathcal{S}^{*m}\alpha$ is a linear combination of $\mathcal{S}^{*n}\alpha$ with $n \le m-1$, i.e.,

$$\mathcal{S}^{*m}\alpha = \sum_{n=0}^{m-1} d_n \mathcal{S}^{*n}\alpha,$$

with some constants d_0, d_1, ..., d_{m-1}. Then

$$\mathcal{S}^{*(m+j)}\alpha = \sum_{n=0}^{m-1} d_n \mathcal{S}^{*(n+j)}\alpha,$$

and $\mathcal{S}^{*(m+j)}\alpha$ is a linear combination of m rows $\mathcal{S}^{*(m+n)}\alpha$ with $0 \le n \le m-1$. Therefore, $\operatorname{rank}(H_\alpha) \le m$. □

Let us reformulate Kronecker's theorem for Hankel operators on H^2.

Corollary 1.2.2. *Let $\varphi \in L^\infty$. Then the Hankel operator H_φ has a finite rank if and only if $P_-\varphi$ is a rational function. In particular,*

$$\operatorname{rank}(H_\varphi) = \deg P_-\varphi. \tag{1.31}$$

Proof. One can deduce the proof from Theorem 1.2.1, however, one can get an alternative proof based on the Beurling theorem A.2.12.

Suppose that $\operatorname{rank}(H_\varphi) = N < \infty$. Hence $\ker(H_\varphi) \ne \{0\}$ and by Corollary 1.1.11 and Beurling's theorem A.2.12, $\ker(H_\varphi) = GH^2$ with some inner function $G \in H^\infty$. Clearly, $\operatorname{rank}(H_\varphi) = N < \infty$ only if $\dim H^2 \ominus GH^2 = N$. The latter holds if and only if $G = B$ is a finite Blaschke product and $\operatorname{rank}(H_\varphi) = \deg B$. Thus we proved that $\operatorname{rank}(H_\varphi) = N < \infty$ if and only $H_{B\varphi} = 0$, i.e., $B\varphi \in H^\infty$ with some finite Blaschke product B.

Conversely, if $P_-\varphi$ is a rational function of degree N, then there is a finite Blaschke product B of degree N such that $B\varphi \in H^\infty$ (the zeros of G are the poles of $P_-\varphi$). This observation completes the proof. □

Corollary 1.2.3. *The Hankel operator H_φ has finite rank if and only if there exists a finite Blaschke product B such that $B\varphi \in H^\infty$.*

Remark 1.2.4. In the finite rank case, $H_\varphi f$ can be computed explicitly. Namely, assume first that

$$P_-\varphi = \frac{1}{z-\lambda}, \qquad \lambda \in \mathbb{D}.$$

Then $\varphi = \frac{1}{z-\lambda} + \tilde{\varphi}$ with some $\tilde{\varphi} \in H^\infty$ and hence we can assume without generality that $\varphi = \frac{1}{z-\lambda}$. Therefore,

$$H_\varphi f = P_-\Big(\frac{1}{z-\lambda} f(z)\Big) = P_-\Big(\frac{f(z)-f(\lambda)}{z-\lambda} + \frac{f(\lambda)}{z-\lambda}\Big) = \frac{f(\lambda)}{z-\lambda}.$$

Similarly, if $P_-\varphi = (z-\lambda)^{-N}$ with some $\lambda \in \mathbb{D}$ and $N \in \mathbb{N}$, then

$$\begin{aligned} H_\varphi f = P_-\Big(\frac{1}{(z-\lambda)^N} f(z)\Big) &= \sum_{n=0}^{N-1} \frac{f^{(n)}(\lambda)}{n!(z-\lambda)^n} \\ &= \frac{1}{(N-1)!} \frac{\partial^{N-1}}{\partial \zeta^{(N-1)}} \left(\frac{f(\zeta)}{z-\zeta}\right)(z,\lambda). \end{aligned}$$

Hence in the general case

$$P_-\varphi = \sum_{n=1}^{N} \sum_{j=1}^{m_n} \frac{c_{nj}}{(z-\lambda_n)^j}$$

we get

$$H_\varphi f = \sum_{n=1}^{N} \sum_{j=1}^{m_n} \frac{c_{nj}}{(j-1)!} \frac{\partial^{j-1}}{\partial \zeta^{(j-1)}} \left(\frac{f(\zeta)}{z-\zeta}\right)(z,\lambda_n) \tag{1.32}$$

1.3. Compactness

Nehari's and Kronecker's theorems suggest that the class of compact Hankel operators is closely connected with the space of continuous functions. At least, it is not difficult to show by approximation that the Hankel operator H_φ with $\varphi \in H^\infty + C$ is compact. Here

$$H^\infty + C := \{f + g \mid f \in H^\infty(\mathbb{T}),\ g \in C(\mathbb{T})\}.$$

P. Hartman proved that the converse is also true. This naturally leads us to the study of the linear space $H^\infty + C$.

Theorem 1.3.1 (Sarason)**.** *$H^\infty + C$ is a closed subalgebra of L^∞.*

Proof. Let $A(\mathbb{D})$ be the disc algebra, $A(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\mathbb{D})$. Clearly, for any $\varphi \in C$,

$$\operatorname{dist}(\varphi, H^\infty) \le \operatorname{dist}(\varphi, A(D)).$$

On the other hand, take any $f \in L^\infty(\mathbb{T})$ and consider its harmonic extension into the disc $\mathbb{D}$. Note that $f_r(z) = f(rz)$, $z \in \mathbb{D}$ belongs to $A(D)$ for every $r \in (0,1)$. If $\varphi \in C(\mathbb{T})$ and $g \in H^\infty(\mathbb{T})$, then by the Young inequality (A.5) we get

$$\|\varphi - g\|_{L^\infty} \ge \lim_{r\to 1} \|(\varphi - g)_r\|_{L^\infty}.$$

Since $\varphi \in C(\mathbb{T})$, $\|\varphi - \varphi_r\|_{L^\infty} \to 0$ as $r \to 1$ (cf. Theorem A.1.11(iii)). Therefore,

$$\|\varphi - g\|_{L^\infty} \geq \lim_{r\to 1} \|\varphi - g_r\|_{L^\infty} \geq \operatorname{dist}(\varphi, A(\mathbb{D})),$$

and we end up with the equality

$$\operatorname{dist}(\varphi, H^\infty) = \operatorname{dist}(\varphi, A(D)). \tag{1.33}$$

Suppose that φ belongs to the closure of $H^\infty + C$. Then there are $\{f_n\} \subset H^\infty(\mathbb{T})$ and $\{g_n\} \subset C(\mathbb{T})$ such that

$$\|\varphi - \varphi_n\|_{L^\infty} \leq \frac{1}{2^n}, \quad \varphi_n := f_n + g_n,$$

for all $n \in \mathbb{N}$. Hence

$$\operatorname{dist}(g_n - g_{n+1}, H^\infty) < \frac{1}{2^{n-1}}$$

and by (1.33) there is $h_n \in A(\mathbb{D})$ such that

$$\|(g_n - g_{n+1}) - h_n\|_{L^\infty} < \frac{1}{2^{n-1}}, \qquad n \in \mathbb{N}.$$

Set $\mathbf{h}_1 = 0$ and $\mathbf{h}_n = h_1 + \cdots + h_{n-1}$ for all $n > 1$. Hence $\mathbf{g}_n := g_n + \mathbf{h}_n \in C(\mathbb{T})$ and $\|\mathbf{g}_n - \mathbf{g}_{n+1}\|_{L^\infty} \leq 1/2^{n-1}$ for all $n \in \mathbb{N}$. Therefore, there is $\mathbf{g} \in C(\mathbb{T})$ such that $\mathbf{g}_n \to \mathbf{g}$ as $n \to \infty$. However,

$$\mathbf{f}_n := f_n - \mathbf{h}_n = (f_n + g_n) - \mathbf{g}_n \in H^\infty(\mathbb{T})$$

for all $n \in \mathbb{N}$. It remains to note that $\mathbf{f}_n$ converges to $\varphi - \mathbf{g}$ in the uniform norm. Since $H^\infty(\mathbb{T})$ is closed, we conclude that $\mathbf{f} \in H^\infty(\mathbb{T})$ and hence $\varphi = \mathbf{f} + \mathbf{g} \in H^\infty + C$. □

Remark 1.3.2. A closed algebra $\mathcal{B}$ such that $H^\infty \subset \mathcal{B} \subset L^\infty$ is called *a Douglas algebra*. Sarason's Theorem 1.3.1 states that $H^\infty + C$ is the Douglas algebra. Note that $H^\infty + C = [H^\infty, z^*]$, that is, $H^\infty + C$ is the closed algebra generated by the set $H^\infty \cup z^*$. S. Chang and D. Marshall in 1977 proved a conjecture by R. Douglas that *every Douglas algebra $\mathcal{B}$ has the form $\mathcal{B} = [H^\infty, B^*]$, where B is a set of inner functions from H^∞*. For further details and results we refer to [**6**, Chapter IX].

Before formulate the next result we need the following definition.

Definition 1.3.3. Let A be a bounded operator on a Hilbert space $\mathcal{H}$. *The essential norm* of A is defined by

$$\|A\|_{ess} := \inf_{K \in \mathfrak{S}_\infty(\mathcal{H})} \|A - K\|_{\mathcal{H}}. \tag{1.34}$$

Here $\mathfrak{S}_\infty(\mathcal{H})$ is the ideal of compact operators in $\mathcal{H}$.

Exercise 1.3.1. Prove that $\|A\|_{ess} = 0$ if and only if $A \in \mathfrak{S}_\infty(\mathcal{H})$.

Theorem 1.3.4. *Let $\varphi \in L^\infty(\mathbb{T})$ and H_φ be the corresponding Hankel operator. Then*

$$\|H_\varphi\|_{ess} = \operatorname{dist}(\varphi, H^\infty + C). \tag{1.35}$$

Before proving Theorem 1.3.4 we need the following simple lemma.

Lemma 1.3.5. *Let K be a compact operator from H^2 to H^2_-. Then*

$$\lim_{n\to\infty} \|K\mathcal{S}^n\| = 0,$$

where $\mathcal{S}$ is the shift operator on H^2, $\mathcal{S}\colon f(z) \mapsto zf(z)$.

Proof. Since finite rank operators are dense in $\mathfrak{S}_\infty$, it suffices to prove the claim for rank 1 operators. Take $K\colon f \mapsto (f, \varphi)\psi$, where $\varphi \in H^2$ and $\psi \in H^2_-$. Hence $K\mathcal{S}^n f = (f, \mathcal{S}^n\varphi)\psi$ and therefore we get

$$\|K\mathcal{S}^n\| = \|\mathcal{S}^n\varphi\|_{H^2}\|\psi\|_{H^2_-} \to 0$$

as $n \to \infty$. □

Proof of Theorem 1.3.4. By Corollary 1.2.2, $H_\varphi \in \mathfrak{S}_\infty$ if φ is a trigonometric polynomial. Therefore, $H_\varphi \in \mathfrak{S}_\infty$ if $\varphi \in C(\mathbb{T})$ and hence

$$\operatorname{dist}(\varphi, H^\infty + C) = \inf_{\psi \in C(\mathbb{T})} \|H_\varphi - H_\psi\| \geq \|H_\varphi\|_{ess}.$$

On the other hand, for any $K \in \mathfrak{S}_\infty(H^2, H^2_-)$ we get

$$\begin{aligned} \|H_\varphi - K\| &\geq \|(H_\varphi - K)\mathcal{S}^n\| \geq \|H_\varphi\mathcal{S}^n\| - \|K\mathcal{S}^n\| \\ &= \operatorname{dist}(z^n\varphi, H^\infty) - \|K\mathcal{S}^n\| \\ &= \operatorname{dist}(\varphi, (z^*)^n H^\infty) - \|K\mathcal{S}^n\| \\ &\geq \operatorname{dist}(\varphi, H^\infty + C) - \|K\mathcal{S}^n\|. \end{aligned}$$

Applying Lemma 1.3.5, we end up with the following inequality

$$\|H_\varphi\|_{ess} \geq \operatorname{dist}(\varphi, H^\infty + C),$$

which completes the proof. □

Remark 1.3.6. Similar to Nehari's problem, the problem of approximation by $H^\infty + C$ functions was posed by V. M. Adamyan, D. Z. Arov and M. G. Krein in 1978. However, in contrast to Nehari's problem, it was shown by S. Axler, I. D. Berg, N. Jewell and A. Shields (1979) that for any $\varphi \in L^\infty$ there are infinitely many best approximants in $H^\infty + C$.

Combining Theorem 1.3.1 with Theorem 1.3.4 we arrive at the following description of compact Hankel operators.

Theorem 1.3.7 (Hartman)**.** *Let $\varphi \in L^\infty$. The following statements are equivalent:*

(i) $H_\varphi \in \mathfrak{S}_\infty(H^2, H^2_-)$,

(ii) $\varphi \in H^\infty + C$,

(iii) *there is* $\psi \in C(\mathbb{T})$ *such that* $H_\varphi = H_\psi$,

(iv) $P_-\varphi \in VMO(\mathbb{T})$.

Proof. Clearly, $(ii) \Leftrightarrow (iii)$. The equivalence $(i) \Leftrightarrow (ii)$ follows from Theorem 1.3.4 and Exercise 1.3.1. It remains to apply the description of the space $VMO(\mathbb{T})$ (see Theorem A.4.7) in order to see that $(ii) \Leftrightarrow (iv)$. □

Let us also mention the following important result.

Corollary 1.3.8. *Let* $\varphi \in L^\infty$. *Then*

$$\|H_\varphi\|_{ess} = \inf_{\psi \in C(\mathbb{T})} \|H_\varphi - H_\psi\| = \inf_{H_\psi \in \mathfrak{S}_\infty} \|H_\varphi - H_\psi\|. \tag{1.36}$$

Remark 1.3.9. For a compact operator $A \in \mathfrak{S}_\infty(\mathfrak{H})$ in a Hilbert space $\mathfrak{H}$, its singular numbers are given by

$$s_{n+1}(A) = \min_{\mathrm{rank}(K) \le n} \|A - K\|, \qquad n \ge 0.$$

Clearly, $(s_n)_{n\in\mathbb{N}}$ determines the rate of approximation of A by finite rank operators in $\mathfrak{H}$. In fact, it was observed by V. M. Adamyan, D. Z. Arov and M. G. Krein that in order to compute singular numbers of compact Hankel operators it suffices to look only at finite rank Hankel perturbations (see [**25**, Chapter IV.1]).

We finish this section with the following

Remark 1.3.10. It is a natural question to describe the Schatten–von Neumann ideals $\mathfrak{S}_p$ of Hankel operators for $p \in (0, \infty)$. The simplest case is the Hilbert–Schmidt ideal. Indeed, the Hankel matrix H_α (operator H_φ) belongs to $\mathfrak{S}_2$ if and only if its Hilbert–Schmidt norm (also known as the Frobenius norm for finite matrices) is finite. However, for Hankel matrices the Hilbert–Schmidt norm equals

$$\|H_\alpha\|^2_{\mathfrak{S}_2} = \sum_{n\ge 0} n|\alpha_n| < \infty \qquad (P_-\varphi \in W^{1,2}(\mathbb{T})).$$

For $p \ne 2$ the problem was solved by V. V. Peller in 1980. He proved that $H_\varphi \in \mathfrak{S}_p$ *for* $p \in (0, \infty)$ *if and only if* $P_-\varphi \in B_p^{1/p}$, where $B_p^{1/p}$ is the *Besov space*. The proof of this fact as well as further results can be found in [**25**, Chapter VI].

1.4. Vectorial Hankel operators

The form (1.3) of the Hankel operator suggests that every bounded Hankel operator can be lifted to a bounded operator on the Hilbert space $\ell_2(\mathbb{Z})$. Using Parrott's theorem (see below), we can obtain another proof of Nehari's Theorem 1.1.2. The advantage of this proof is that it applies to vectorial Hankel operators, i.e., Hankel operators on $\ell_2(\mathbb{Z}_+, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

Before we start, let us recall that every bounded operator T on a Hilbert space $\mathcal{H}$ admits a polar decomposition

$$T = U_T|T|,$$

where $|T| = (T^*T)^{1/2}$ is a nonnegative operator on $\mathcal{H}$ and U_T is a partial isometry with $\ker U_T = \ker T$ and $\operatorname{ran} U_T = \overline{\operatorname{ran} T}$. This decomposition is unique. Since $T^* = U_{T^*}|T^*| = |T|U_T^*$ and $U_{T^*}^* T^* = U_{T^*}^* U_{T^*}|T^*| = |T^*|$, we get

$$T = |T^*|U_{T^*}^* = U_{T^*}^*(U_{T^*}|T^*|U_{T^*}^*) = U_T|T|.$$

Since a polar decomposition is unique[4], we conclude that

$$U_{T^*}^* = U_T, \qquad U_T|T| = |T^*|U_T. \tag{1.37}$$

Exercise 1.4.1. Let f be a continuous function. Show that $U_T f(|T|) = f(|T^*|)U_T$. (Hint: prove it first when f is a polynomial).

Now we are in position to formulate the following

Theorem 1.4.1 (Parrott [**24**])**.** *Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ be a Hilbert space and let $T_X \in [\mathfrak{H}]$ be given by*

$$T_X = \begin{pmatrix} A & B \\ C & X \end{pmatrix}. \tag{1.38}$$

Define the operators $T_1 \colon \mathfrak{H} \to \mathfrak{H}_1$ and $T_2 \colon \mathfrak{H}_1 \to \mathfrak{H}$ by

$$T_1 = \begin{pmatrix} A & B \end{pmatrix}, \qquad T_2 = \begin{pmatrix} A \\ C \end{pmatrix}. \tag{1.39}$$

Then

$$\inf_{X \in [\mathfrak{H}_2]} \|T_X\| = \max\{\|T_1\|, \|T_2\|\}. \tag{1.40}$$

Proof. The inequality

$$\|T_X\| \geq \max\{\|T_1\|, \|T_2\|\}$$

is obvious since $T_1 = P_1 T_X$ and $T_2 = T_X P_1$, where P_1 is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}_1$. Hence we only need to show that

$$\inf_{X \in [\mathfrak{H}_2]} \|T_X\| \leq \max\{\|T_1\|, \|T_2\|\}. \tag{1.41}$$

[4] Verify that $\ker U_{T^*}^* = \ker T$!

Without loss of generality we can assume that $\max\{\|T_1\|, \|T_2\|\} = 1$. Suppose that $T_1T_1^* = AA^* + BB^* = I_{\mathfrak{H}_1}$ and $T_2^*T_2 = A^*A + C^*C = I_{\mathfrak{H}_1}$. Consider the polar decomposition of operators A, B and C: $A = U_A|A|$, $B = U_B|B|$ and $C = U_C|C|$. Set

$$X = -U_C A^* U_B \tag{1.42}$$

and consider the corresponding operator T_X. First observe that

$$T_X^* T_X = \begin{pmatrix} A^*A + C^*C & A^*B + C^*X \\ B^*A + X^*C & B^*B + X^*X \end{pmatrix}. \tag{1.43}$$

By the assumption, $(T_X^*T_X)_{11} = I_{\mathfrak{H}_1}$. Using (1.42), we get

$$\begin{aligned} C^*X &= -|C|U_C^*U_C A^*U_B = -|C|A^*U_B \\ &= -(I_{\mathfrak{H}_1} - A^*A)^{1/2}|A|U_A^*U_B = -|A|(I_{\mathfrak{H}_1} - A^*A)^{1/2}U_A^*U_B \\ &= -|A|U_A^*(I_{\mathfrak{H}_1} - AA^*)^{1/2}U_B = -|A|U_A^*|B^*|U_B = -A^*B. \end{aligned}$$

Therefore, $(T_X^*T_X)_{12} = 0$ and, moreover, $(T_X^*T_X)_{21} = 0$ since $T_X^*T_X$ is self-adjoint. Finally, note that

$$\begin{aligned} (T_X^*T_X)_{22} =& B^*B + X^*X \\ &= U_B^*BB^*U_B + U_B^*AU_C^*U_CA^*U_B \\ &\le U_B^*BB^*U_B + U_B^*AA^*U_B \\ &= U_B^*(AA^* + BB^*)U_B = U_B^*U_B. \end{aligned}$$

Therefore, $T_X^*T_X = I_{\mathfrak{H}_1} \oplus U_B^*U_B \le I_{\mathfrak{H}}$ and hence $\|T_X\| = 1$.

Now consider the general case, $F := AA^* + BB^* \le I_{\mathfrak{H}_1}$ and $G := A^*A + C^*C \le I_{\mathfrak{H}_1}$. Define $\mathfrak{H}_F = \overline{\operatorname{ran}(I_{\mathfrak{H}_1} - F)}$ and $\mathfrak{H}_G = \overline{\operatorname{ran}(I_{\mathfrak{H}_1} - G)}$ and consider the operator $\tilde{T}\colon \mathfrak{H} \oplus \mathfrak{H}_F \to \mathfrak{H} \oplus \mathfrak{H}_G$ defined by

$$\tilde{T} = \begin{pmatrix} A & B & (I_{\mathfrak{H}_1} - F)^{1/2} \\ C & 0 & 0 \\ (I_{\mathfrak{H}_1} - G)^{1/2} & 0 & 0 \end{pmatrix}$$

Clearly, with respect to the decompositions $\mathfrak{H} = \mathfrak{H}_1 \oplus (\mathfrak{H}_2 \oplus \mathfrak{H}_F)$ and $\mathfrak{H} = \mathfrak{H}_1 \oplus (\mathfrak{H}_2 \oplus \mathfrak{H}_G)$ the operator $\tilde{T}$ has the form

$$\tilde{T} = \begin{pmatrix} A & \tilde{B} \\ \tilde{C} & 0 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B & (I_{\mathfrak{H}_1} - F)^{1/2} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C \\ (I_{\mathfrak{H}_1} - G)^{1/2} \end{pmatrix}.$$

Moreover, by definition we get $A^*A + \tilde{C}^*\tilde{C} = I_{\mathfrak{H}_1}$ and $AA^* + \tilde{B}\tilde{B}^* = I_{\mathfrak{H}_1}$. Therefore, there is $\tilde{X}\colon \mathfrak{H}_2 \oplus \mathfrak{H}_F \to \mathfrak{H}_2 \oplus \mathfrak{H}_G$ such that

$$\tilde{T}_{\tilde{X}} = \begin{pmatrix} A & \tilde{B} \\ \tilde{C} & \tilde{X} \end{pmatrix}$$

satisfies $\|\tilde{T}_{\tilde{X}}\| = 1$. It remains to choose X by $X = P_{\mathfrak{H}_2}\tilde{X}P_{\mathfrak{H}_2}$ and hence the operator $T_X = P_{\mathfrak{H}}\tilde{T}_{\tilde{X}}P_{\mathfrak{H}}$ will be a contraction. This proves the claim. □

Let $\mathcal{H}$ be a Hilbert space and $\{\Omega_n\}_{n\in\mathbb{Z}_+}$ be a sequence of bounded operators on $\mathcal{H}$. In $\ell_2(\mathbb{Z}_+, \mathcal{H})$, consider the block matrix

$$H_\Omega = \begin{pmatrix} \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \dots \\ \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \dots \\ \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \dots \\ \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{1.44}$$

Such matrices are called *block Hankel matrices*. As in the scalar case, we can consider vectorial Hankel operators as operators on the Hardy space $H^2(\mathbb{T}, \mathcal{H})$, which is defined as follows:

$$H^2(\mathbb{T}, \mathcal{H}) = \{\Phi \in L^2(\mathbb{T}, \mathcal{H}) |\, \hat{\Phi}_n = 0,\ n < 0\}.$$

Here $L^2(\mathbb{T}, \mathcal{H})$ is the Hilbert space of weakly measurable $\mathcal{H}$-valued functions Φ such that

$$\|\Phi\|_{L^2(\mathbb{T},\mathcal{H})} := \frac{1}{2\pi}\int_{\mathbb{T}} \|\Phi(\theta)\|^2_{\mathcal{H}} d\theta < \infty.$$

By P_- we denote the orthogonal projection in $L^2(\mathbb{T}, \mathcal{H})$ onto $H^2_-(\mathbb{T}, \mathcal{H}) = L^2(\mathbb{T}, \mathcal{H}) \ominus H^2(\mathbb{T}, \mathcal{H})$.

Also, by $L^\infty(\mathbb{T}, [\mathcal{H}])$ we denote the space of bounded weakly measurable $[\mathcal{H}]$-valued functions. Fourier coefficients $\hat{\Phi}_n \in [\mathcal{H}]$ of $\Phi \in L^\infty(\mathbb{T}, [\mathcal{H}])$ are defined by

$$\hat{\Phi}_n f = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}n\theta}\Phi(\theta) f\, d\theta, \qquad n \in \mathbb{Z}, \quad f \in \mathcal{H}.$$

For a function $\Psi \in L^\infty(\mathbb{T}, [\mathcal{H}])$ we can also define the operator $H_\Psi \colon H^2(\mathbb{T}, \mathcal{H}) \to H^2_-(\mathbb{T}, \mathcal{H})$ by

$$H_\Psi \colon F \mapsto P_-(\Psi F), \qquad F \in H^2(\mathbb{T}, \mathcal{H}). \tag{1.45}$$

As in the scalar case, the operators H_Ω and H_Ψ are closely connected. Namely, the operator matrix representation of H_Ψ has the form (1.44) with

$$\Omega_n = \hat{\Psi}_{-n-1}, \qquad n \in \mathbb{Z}_+. \tag{1.46}$$

Theorem 1.4.2. *Let $\{\Omega_n\}_{n\in\mathbb{Z}_+}$ be a sequence of bounded operators on $\mathcal{H}$. The block Hankel matrix (1.44) determines a bounded linear operator on $\ell_2(\mathbb{Z}_+, \mathcal{H})$ if and only if there is a function $\Phi \in L^\infty(\mathbb{T}, [\mathcal{H}])$ such that*

$$\hat{\Phi}_n = \Omega_n \tag{1.47}$$

for all $n \in \mathbb{Z}_+$. Moreover, in this case

$$\|H_\Omega\| = \inf\{\|\Phi\|_{L^\infty(\mathbb{T},[\mathcal{H}])} |\, \hat{\Phi}_n = \Omega_n,\ n \in \mathbb{Z}_+\}. \tag{1.48}$$

Proof. The proof of sufficiency follows immediately from (1.45) and (1.46). Let us prove necessity. Assume that H_Ω is a bounded operator on $\ell_2(\mathbb{Z}_+, \mathcal{H})$ and consider the matrix

$$T_X := \begin{pmatrix} X & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \dots \\ \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \dots \\ \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \dots \\ \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 & \dots \\ \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 & \Omega_7 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1.49}$$

where $X \in [\mathcal{H}]$. Clearly, T_X has the form

$$T_X = \begin{pmatrix} X & C \\ B & A \end{pmatrix}$$

with

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \dots \\ \Omega_2 & \Omega_3 & \Omega_4 & \dots \\ \Omega_3 & \Omega_4 & \Omega_5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad C = \begin{pmatrix} \Omega_0 & \Omega_1 & \Omega_2 & \dots \end{pmatrix}, \quad B = \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \Omega_2 \\ \dots \end{pmatrix}.$$

Clearly,

$$\| \begin{pmatrix} B & A \end{pmatrix} \| = \|H_\Omega\|, \quad \left\| \begin{pmatrix} C \\ A \end{pmatrix} \right\| = \|H_\Omega\|.$$

By Parrott's Theorem 1.4.1, there exists $X = \Omega_{-1} \in [\mathcal{H}]$ such that $\|T_{\Omega_{-1}}\| = \|H_\Omega\|$. Continuing this process, we would end up with the sequence of bounded operators $\{\Omega_n\}_{n\in\mathbb{Z}_-}$ such that the matrix

$$\Lambda^m := (\Lambda^m_{jk})_{j\in\mathbb{Z},k\in\mathbb{Z}_+}, \qquad \Lambda^m_{jk} = \begin{cases} \Omega_{j+k}, & j \ge -m, \\ 0, & j < -m \end{cases},$$

defines a bounded operator and $\|\Lambda^m\| = \|H_\Omega\|$ for all $m \in N$. Clearly, Λ^m converges to $\Lambda = (\Omega_{j+k})_{j\in\mathbb{Z},k\in\mathbb{Z}_+}$ in the weak operator topology and hence $\|\Lambda\| = \|H_\Omega\|$.

Consider the matrix $Q = (\Omega_{j+k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$. It is not difficult to see that $\|(\Omega_{j+k})_{j\in\mathbb{Z},k\ge -m}\| = \|H_\Omega\|$ and hence $\|Q\| = \|H_\Omega\|$. Thus, Q defines a bounded operator on $\ell_2(\mathbb{Z}, \mathcal{H})$.

Let us identify $\ell_2(\mathbb{Z}, \mathcal{H})$ with $L^2(\mathbb{T}, \mathcal{H})$ in a standard way,

$$\{x_n\}_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\theta} x_n,$$

and consider Q as an operator on $L^2(\mathbb{T}, \mathcal{H})$. In particular, $Qx \in \ell_2(\mathbb{Z}, \mathcal{H})$ for every $x \in \mathcal{H}$ and hence

$$(Qx)(\theta) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\theta} \Omega_n x \in L^2(\mathbb{T}, \mathcal{H}).$$

Since $\mathcal{H}$ is a separable Hilbert space, we can define a $[\mathcal{H}]$-valued function for almost all $\theta \in \mathbb{T}$ by

$$\Phi(\theta)x := \sum_{n\in\mathbb{Z}} \mathrm{e}^{\imath n\theta}\Omega_n x.$$

It is easy to see that for all trigonometric polynomials F,

$$(QF)(\theta) = \Phi(\theta)F(-\theta).$$

This implies that the operator of multiplication $M_\Phi\colon F \to \Phi F$ extends to a bounded operator on $L^2(\mathbb{T},\mathcal{H})$ and its norm equals $\|Q\|$. To prove that $\Phi \in L^\infty(\mathbb{T},[\mathcal{H}])$ and $\|\Phi\|_{L^\infty(\mathbb{T},[\mathcal{H}])} \leq Q$, it suffices to show that

$$\sup_{\|f\|,\|g\|\leq 1} \|\langle\Phi(\cdot)f,g\rangle_{\mathcal{H}}\|_{L^\infty(\mathbb{T})} \leq \|Q\|.$$

However, setting $\varphi(\theta) := \langle\Phi(\cdot)f,g\rangle_{\mathcal{H}}$, this inequality immediately follows from the fact that a multiplication operator M_φ by a scalar function φ in $L^2(\mathbb{T})$ is bounded if and only if $\varphi \in L^\infty(\mathbb{T})$. Moreover, $\|M_\varphi\| = \|\varphi\|_{L^\infty(\mathbb{T})}$. □

Remark 1.4.3. For further results (compactness, finite rank etc.) we refer to [**25**, Chapter II].

Chapter 2

Applications

We start with H. Widom's characterization of bounded and compact positive Hankel matrices. Then we'll proceed with the newly discovered cubic Szegö equation, a non dispersive nonlinear evolution equation for which Hankel operators serve as isospectral operators. This discovery leads to the study of direct and inverse spectral problems for Hankel matrices.

2.1. Hankel matrices and moment sequences

A connection between Hankel matrices and moment problems goes back at least to the works of H. L. Hamburger and T. J. Stieltjes. More precisely, we will be concerned with two basic moment problems. Let $\{s_k\}_{k\in\mathbb{Z}_+}$ be a sequence of real numbers.

The classical Hamburger moment problem is to find a positive measure μ on $\mathbb{R}$ such that s_k are its moments of order k, i.e.,

$$s_k = \int_{\mathbb{R}} \lambda^k d\mu(\lambda), \qquad k \in \mathbb{Z}_+. \tag{2.1}$$

If $\mu \in \mathcal{M}^+(\mathbb{R})$ satisfies (2.1), then μ is called a solution of the power moment problem with data $\{s_k\}_{k\in\mathbb{Z}_+}$.

The Stieltjes moment problem is to find a positive measure μ on $\mathbb{R}_+ = [0,\infty)$ such that s_k, $k \in \mathbb{Z}_+$ are its moments of order k, i.e.,

$$s_k = \int_{\mathbb{R}_+} \lambda^k d\mu(\lambda), \qquad k \in \mathbb{Z}_+. \tag{2.2}$$

There are two main questions: (i) For which sequences $\{s_k\}_{k\in\mathbb{Z}_+}$ the Hamburger/Stieltjes moment problem is solvable? (ii) If such a measure μ exists, is it unique? If no, how to describe the set of solutions?

We are not going to give neither a comprehensive historical details nor a complete solution to both problems. In this respect we refer to the excellent book by N. I. Akhiezer [**1**] (see also [**33**]).

Let us start with the following simple observation. Suppose $\mu \in \mathcal{M}^+(\mathbb{R})$ is a solution to the Hamburger moment problem. Let $\mathrm{p} = \{p_k\}_{k\in\mathbb{Z}_+} \in \ell_{2,c}$ and $p(\lambda) = \sum_k p_k\lambda^k \in \mathrm{Pol}_+(\mathbb{R})$. Consider the integral

$$0 \le \int_{\mathbb{R}} |p(\lambda)|^2 d\mu(\lambda) = \int_{\mathbb{R}} \Big| \sum_k p_k\lambda^k \Big|^2 d\mu(\lambda) = \int_{\mathbb{R}} \sum_k p_k p_j^* \lambda^{k+j} d\mu(\lambda)$$
$$= \sum_{k,j} p_k p_j^* \int_{\mathbb{R}} \sum_{k,j=0}^{N} p_k p_j^* \lambda^{k+j} d\mu(\lambda) = \sum_{k,j} s_{k+j} p_k p_j^* = (H_s\mathrm{p}, \mathrm{p})_{\ell_2},$$

where $H_s = (s_{k+j})_{k,j\in\mathbb{Z}_+}$ is the Hankel matrix with the symbol $s = \{s_k\}_{k\in\mathbb{Z}_+}$. The Hankel matrix H_s satisfying

$$(H_s\mathrm{p}, \mathrm{p})_{\ell_2} \ge 0 \tag{2.3}$$

for all $\mathrm{p} \in \ell_{2,c}$ is called *nonnegative*.

Exercise 2.1.1. Let $\Delta_N^0 := \det H_s(N)$ be the leading minor of the Hankel matrix H_s, where $H_s(N) = (s_{k+j})_{k,j=0}^N$. Show that H_s is nonnegative if and only if $\Delta_N^0 \ge 0$ for all $N \in \mathbb{Z}_+$.

Exercise 2.1.2. Let H_s be a nonnegative Hankel matrix. Show that if $\Delta_{N_0}^0 = 0$ for some $N_0 \in \mathbb{Z}_+$, then $\Delta_N^0 = 0$ for all $N > N_0$. Show also that $\Delta_N^0 = 0$ for some $N \in \mathbb{Z}_+$ if and only if the solution $\mu \in \mathcal{M}^+(\mathbb{R})$ to the moment problem (2.1) is supported on a finite set of points of $\mathbb{R}$.

Thus the positivity of the Hankel matrix H_s is necessary for the existence of solutions to the Hamburger moment problem. It turns out that the converse is also true!

Theorem 2.1.1. *Let $\{s_k\}_{k\in\mathbb{Z}_+}$ be a sequence of reals and H_s be the corresponding Hankel matrix with symbol s. Then:*

(i) *There is a solution $\mu \in \mathcal{M}^+(\mathbb{R})$ to the Hamburger moment problem (2.1) if and only if the Hankel matrix H_s is nonnegative.*

(ii) *There is a solution $\mu \in \mathcal{M}^+(\mathbb{R}_+)$ to the Stieltjes moment problem (2.2) if and only if the Hankel matrices H_s and $H_{\mathcal{S}^*s}$ are nonnegative.*

Remark 2.1.2. Theorem 2.1.1 (i) is due to Hamburger. The second part of Theorem 2.1.1 was proved by Stieltjes. Further details as well as the proof of Theorem 2.1.1 can be found in [**1, 33**].

Nehari's and Hartman's theorems provide criteria for a Hankel matrix H_α with symbol $\alpha \in \ell_2$ to be bounded/compact. It turns out that in the case of positive Hankel matrices the answer is much more transparent. First we need the following simple fact.

Lemma 2.1.3. *Let $s \in \ell_2$ be such that the Hankel matrix H_s is positive (the Hankel matrices H_s and $H_{\mathcal{S}^*s}$ are positive). Then there is a unique positive measure μ on $[-1,1]$ (on $[0,1]$) such that*

$$s_k = \int_{(-1,1)} \lambda^k d\mu(\lambda), \quad \left(s_k = \int_{[0,1)} \lambda^k d\mu(\lambda) \right) \qquad k \in \mathbb{Z}_+. \tag{2.4}$$

Note that the moment problem (2.4) is called *the Hausdorff moment problem.*

Proof. Since H_s is positive, there is $\mu \in \mathcal{M}^+(\mathbb{R})$ such that (2.1) holds. Moreover, such a measure μ is unique since $s \in \ell_2$ (see, e.g., the Carleman test, [**1**, Problem II.11] or [**33**, Proposition 1.5]). Hence it remains to show that $\mathrm{supp}(\mu) \subseteq [-1,1]$ and $\mu(\{-1\}) = \mu(\{1\}) = 0$. Notice that

$$s_{2k} = \int_{\mathbb{R}} \lambda^{2k} d\mu(\lambda) \geq \mu(\mathbb{R} \setminus (-1,1)).$$

However, $s_k \to 0$ since $s \in \ell_2$ and hence $\mu(\mathbb{R} \setminus (-1,1)) = 0$. □

Consider the Hilbert matrix $H = ((k+j+a)^{-1})_{k,j\in\mathbb{Z}_+}$ with $a \notin \mathbb{Z}_-$. Clearly, $H = H_s$ with

$$s_k = \frac{1}{k+a+1} = \int_0^1 \lambda^k d\rho_a(\lambda), \qquad k \in \mathbb{Z}_+,$$

where $\rho_a(\lambda) = \frac{\lambda^{a+1}}{a+1} \mathbb{1}_{[0,1]}(\lambda)$. The Hilbert matrix generates a bounded operator on ℓ_2. The next result shows that the estimate $s_k = \mathcal{O}(k^{-1})$ as $k \to \infty$ is necessary and sufficient for a positive Hankel matrix to be bounded.

Theorem 2.1.4 (Widom)**.** *Let $\alpha \in \ell_2$ be such that the Hankel matrix H_α is positive. Then the following statements are equivalent:*

(i) *The Hankel matrix H_α is bounded on ℓ_2,*

(ii) *$\alpha_k = \mathcal{O}(k^{-1})$ as $k \to \infty$,*

(iii) *There exists a positive measure μ on $(-1,1)$ such that*

$$\alpha_k = \int_{(-1,1)} \lambda^k d\mu(\lambda) \tag{2.5}$$

holds for all $k \in \mathbb{Z}_+$ and μ is a Carleson measure, i.e.,

$$\mu((-1,-t) \cup (t,1)) = \mathcal{O}(1-t) \tag{2.6}$$

as $t \uparrow 1$.

(iv) $H^2(\mathbb{D})$ *is continuously embedded into* $L^2((-1,1);d\mu)$*, i.e., there is* $C>0$ *such that* $\|f\|_{L^2(d\mu)} \le C\|f\|_{H^2}$ *for all* $f \in H^2(\mathbb{D})$.

Proof. $(iii) \Rightarrow (ii)$ Note that

$$|\alpha_k| = \Big|\int_{(-1,1)} \lambda^k d\mu(\lambda)\Big| \le \int_{[0,1)} |\lambda|^k d\mu(\lambda) + \int_{(-1,0]} |\lambda|^k d\mu(\lambda).$$

Integrating by parts (see, for example, [**5**, Exercise 5.8.112]) and using (2.6), we get

$$\begin{aligned}\int_{[0,1)} |\lambda|^k d\mu(\lambda) =& \mu([0,1)) - \int_{[0,1)} k\lambda^{k-1}\mu([0,\lambda))d\lambda \\ =& \int_{[0,1)} k\lambda^{k-1}\mu([\lambda,1)))d\lambda \\ \le& C\int_{[0,1)} k\lambda^{k-1}(1-\lambda)d\lambda = \frac{C}{k+1}.\end{aligned}$$

Similarly,

$$\int_{(-1,0]} |\lambda|^k d\mu(\lambda) \le \frac{C}{k+1}$$

for all $k \in \mathbb{Z}_+$ and hence we are done.

$(ii) \Rightarrow (i)$ This implication follows from Hilbert's inequality (1.16). Indeed, for any f, $g \in \ell_{2,0}$ we get

$$|(H_\alpha f, g)|_{\ell_2} = \Big|\sum_{k,j} \alpha_{k+j} f_k g_j^*\Big| \le C\sum_{k,j} \frac{|f_k||g_j^*|}{k+j+1} \le C\pi\|f\|_{\ell_2}\|g\|_{\ell_2},$$

which immediately implies that $\|H_\alpha\| \le C\pi$.

$(i) \Rightarrow (iv)$ If $f \in \ell_{2,c}$ and $f(\lambda) \in \mathrm{Pol}_+(\mathbb{R})$ is the corresponding polynomial, then

$$\|f\|^2_{L^2(d\mu)} = (H_\alpha f, f)_{\ell_2} \le \|H_\alpha\|\|f\|^2_{\ell_2} = \|H_\alpha\|\|f\|^2_{H^2}.$$

Since H_α is bounded on ℓ_2, the latter extends by continuity for all $f \in H^2(\mathbb{D})$.

It remains to show that $(iv) \Rightarrow (iii)$. Let

$$f_r(\lambda) = \frac{\sqrt{1-r^2}}{1-r\lambda} = \sqrt{1-r^2}\sum_{k\in\mathbb{Z}_+} (r\lambda)^k \in L^2(-1,1)$$

for all $r \in (0,1)$. Moreover, $\hat{f}_r = \{\sqrt{1-r^2}r^k\}_{k\in\mathbb{Z}_+}$ and $\|\hat{f}_r\|_{\ell_2} = 1$. Clearly,

$$\|f_r\|^2_{L^2(d\mu)} = \int_{(-1,1)} \frac{1-r^2}{(1-r\lambda)^2} d\mu(\lambda) = (H_\alpha \hat{f}_r, \hat{f}_r) \le \|H_\alpha\| < \infty$$

for all $r \in (0,1)$. On the other hand,

$$\frac{\mu((r,1))}{1-r^2} = \int_{(r,1)} \frac{d\mu(\lambda)}{1-r^2} \le \int_{(r,1)} \frac{1-r^2}{(1-r\lambda)^2} d\mu(\lambda) \le \|H_\alpha\|.$$

Similarly, one gets $(1-r^2)^{-1}\mu(-1,r) \le \|H_\alpha\|$ and we are done. □

Compactness of positive Hankel operators can be characterized in a similar way.

Theorem 2.1.5 (Widom)**.** *Let $\alpha \in \ell_2$ be such that the Hankel matrix H_α is positive. Then the following statements are equivalent:*

(i) *The Hankel matrix H_α is compact on ℓ_2,*

(ii) *$\alpha_k = o(k^{-1})$ as $k \to \infty$,*

(iii) *There exists a positive measure μ on $(-1,1)$ such that (2.5) holds for all $k \in \mathbb{Z}_+$ and μ is a vanishing Carleson measure, i.e.,*

$$\mu((-1,-t) \cup (t,1)) = o(1-t) \tag{2.7}$$

as $t \uparrow 1$.

(iv) *$H^2(\mathbb{D})$ is compactly embedded into $L^2((-1,1); d\mu)$.*

Proof. The equivalence $(i) \Leftrightarrow (iv)$ follows from the equality $H_\alpha = \mathcal{F}J^*J\mathcal{F}^{-1}$, where $\mathcal{F}\colon H^2 \to \ell_2$ is the Fourier transform, $\mathcal{F}\colon f \mapsto \hat{f}$, and $J\colon H^2(\mathbb{D}) \to L^2((-1,1); d\mu)$ is the embedding, $J\colon f(z) \mapsto f(\lambda)$. Indeed, for all $f \in \mathrm{Pol}_+$

$$\|Jf\|^2_{L^2(d\mu)} = (Jf, Jf)_{L^2(d\mu)} = (J^*Jf, f)_{L^2(d\mu)} = (H_\alpha \hat{f}, \hat{f})_{\ell_2}.$$

$(iii) \Rightarrow (ii)$ For any $\varepsilon > 0$ there exists $\lambda_\varepsilon \in (0,1)$ such that

$$\mu((-1,-\lambda) \cup (\lambda,1)) = o(1-\lambda)$$

for all $\lambda \in (\lambda_0, 1)$. Arguing as in the the proof of Theorem 2.1.4, it suffices to show the following estimate

$$\int_{[0,1)} |\lambda|^k d\mu(\lambda) = \int_{[0,1)} k\lambda^{k-1}\mu([\lambda,1)))d\lambda = \int_{[0,\lambda_\varepsilon]} + \int_{(\lambda_\varepsilon,1)} k\lambda^{k-1}\mu([\lambda,1)))d\lambda$$
$$\le k\mu([0,1))\lambda_\varepsilon^k + \varepsilon \int_{(\lambda_\varepsilon,1)} k\lambda^{k-1}(1-\lambda)d\lambda \le \frac{2\varepsilon}{k+1},$$

which holds for all k large enough.

$(ii) \Rightarrow (i)$ Again, choose $N_\varepsilon \in \mathbb{N}$ such that $\alpha_k < \varepsilon k^{-1}$ for all $k > N_\varepsilon$. Let us represent H_α as a sum of two Hankel matrices $H_\alpha = H_{\alpha,1} + H_{\alpha,2}$, where H^1_α and H^2_α are defined by

$$\alpha_{k,1} = \begin{cases} \alpha_k, & k \le N_\varepsilon, \\ 0, & k > N_\varepsilon \end{cases}, \qquad \alpha_{k,2} = \begin{cases} 0, & k \le N_\varepsilon, \\ \alpha_k, & k > N_\varepsilon \end{cases}.$$

Clearly, H_α^1 is a finite rank Hankel matrix. By the proof of the implication $(ii) \Rightarrow (i)$ of Theorem 2.1.4, we conclude that $\|H_\alpha^2\| \leq \varepsilon\pi$. Thus H_α can be approximated (in the uniform operator topology) by finite rank operators and hence it is compact.

$(iv) \Rightarrow (iii)$ By Hartman's theorem, there is $\varphi \in C(\mathbb{T})$ such that $\hat{\varphi}_k = \alpha_k$ for all $k \in \mathbb{Z}_+$. Consider $\varphi_r = P_r * \varphi$. Since $\varphi \in C(\mathbb{T})$, $\|\varphi_r - \varphi\|_\infty \to 0$ as $r \to 1$. Set $\alpha_{k,r} := \hat{\varphi}_{r\,k} = r^k \alpha_k$, $k \in \mathbb{Z}_+$ and consider the corresponding Hankel matrix H_{α_r}. By Nehari's theorem, for every $\varepsilon > 0$ there is $r_\varepsilon \in (0,1)$ such that $\|H_{\alpha_r} - H_\alpha\| \to 0$ for all $r \in (r_\varepsilon, 1)$. On another hand,

$$\alpha_{k,r} = r^k \alpha_k = r^k \int_{(-1,1)} \lambda^k d\mu(\lambda) = \int_{(-1,1)} \lambda^k d\mu_r(\lambda),$$

where $\mu_r((a,b)) = \mu((ra, rb))$ for all $(a,b) \subset (-1,1)$. Thus we get

$$|((H_\alpha - H_{\alpha_r})\hat{f}_s, \hat{f}_s)| = \Big| \int_{(-1,1)} \frac{1-s^2}{(1-s\lambda)^2} d(\mu - \mu_r) \Big| \leq \varepsilon$$

for all $r \in (r_\varepsilon, 1)$. Since $\operatorname{supp}(\mu_r) \subseteq [-r, r]$, one can choose s_ε such that

$$\int_{(-1,1)} \frac{1-s^2}{(1-s\lambda)^2} d\mu_r(\lambda) \leq \varepsilon$$

and hence

$$\int_{(-1,1)} \frac{1-s^2}{(1-s\lambda)^2} d\mu \leq 2\varepsilon$$

for all $s \in (s_\varepsilon, 1)$. The last inequality immediately implies that

$$\frac{\mu((r,1)) + \mu((-1,-r))}{1-r^2} \leq \varepsilon$$

for all $r \in (r_\varepsilon, 1)$, which completes the proof. □

Remark 2.1.6. The family of functions

$$f_\zeta(z) = \frac{\sqrt{1-|\zeta|^2}}{1-\zeta z}, \qquad \zeta \in \mathbb{D},$$

is called *the normalized reproducing kernel*. It is possible to show that the Hankel operator H_φ is bounded (compact) if and only if the set $\{H_\varphi f_\zeta \,|\, \zeta \in \mathbb{D}\}$ is uniformly bounded (uniformly tends to 0 as $|\zeta| \to 0$) in H_-^2 (see [**25**, Chapter I.6]). This result provides another way of proving the implication $(iv) \Rightarrow (iii)$.

Let μ be a finite complex Borel measure on $\overline{\mathbb{D}}$. Set

$$\alpha_k = \int_{\overline{\mathbb{D}}} z^k d\mu(z), \qquad k \in \mathbb{Z}_+. \tag{2.8}$$

and consider the Hankel matrix H_α with the symbol $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}_+}$. Then for any f, $g \in \mathrm{Pol}_+$ one easily gets

$$(H_\alpha f, g)_{\ell_2} = \int_{\overline{\mathbb{D}}} f(z) g(z^*)^* d\mu(z).$$

Hence similar to the proof of Theorem 2.1.4, it is not difficult to show that the Hankel matrix H_α is bounded if the embedding $J\colon H^2(\mathbb{D}) \to L^2(\mathbb{D}, d\mu)$ is bounded. *The Carleson embedding theorem* states that the embedding J is bounded if and only if μ is *a Carleson measure* on $\mathbb{D}$. The latter means that

$$\sup_I \frac{|\mu|(R_I)}{|I|} < \infty, \tag{2.9}$$

where sup is taken over all subarcs of $\mathbb{T}$, $|I| = \frac{1}{2\pi}\int_I d\theta$ is the arc length, and R_I is the so-called *Carleson window* $R_I = \{r\mathrm{e}^{\mathrm{i}\theta} \,|\, \mathrm{e}^{\mathrm{i}\theta} \in I,\ |I| < r < 1\}$.

If

$$\lim_{|I|\to 0} \frac{|\mu|(R_I)}{|I|} = 0, \tag{2.10}$$

then μ is called *a vanishing Carleson measure*. Note that in the case when μ is a positive measure and $\mathrm{supp}(\mu) \subseteq [-1, 1]$, conditions (2.9) and (2.10) are equivalent to (2.6) and (2.7), respectively.

It turns out that every function $f \in BMO(\mathbb{T})$ ($f \in VMO(\mathbb{T})$) is a convolution of the Poisson kernel with the Carleson measure (vanishing Carleson measure). Thus applying the Nehari and Hartman theorems one can prove the following result.

Theorem 2.1.7. *The Hankel matrix H_α is bounded (compact) on ℓ_2 if and only if there exists a Carleson measure (a vanishing Carleson measure) μ on $\mathbb{D}$ such that (2.8) is satisfied.*

Remark 2.1.8. Further details about Carleson measures and the Carleson embedding theorem can be found in [6]. Connection between BMO, VMO and Carleson measures was noticed by P. Jones [15] and E. Amar and A. Bonami [3].

Remark 2.1.9. If H_α is a positive Hankel matrix, then it belongs to the trace class if and only if its trace is finite, that is,

$$\sum_{k\in\mathbb{Z}_+} s_{2k} = \sum_{k\in\mathbb{Z}_+} \int_{(-1,1)} \lambda^{2k} d\mu = \int_{(-1,1)} \frac{d\mu}{1-\lambda^2} < \infty.$$

2.2. The cubic Szegö equation

Let P_+ be the Szegö projection in $L^2(\mathbb{T})$ onto H^2. Consider the following nonlinear equation

$$\mathrm{i}\partial_t u = P_+(|u|^2 u), \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{2.11}$$

This equation first appeared in [7, 8] as a toy model for totally non dispersive evolution equations. The study of (2.11) is motivated by the study of the nonlinear Schrödinger equation

$$\mathrm{i}\partial_t u + \Delta u = |u|^2 u, \qquad (t,x) \in \mathbb{R}_+ \times \mathcal{M}, \tag{2.12}$$

where $\mathcal{M}$ is a Riemannian manifold. Note that the boundary value problem for another non dispersive equation (which is similar to (2.11))

$$\mathrm{i}\partial_t u = |u|^2 u, \qquad u(0,x) = u_0(x), \tag{2.13}$$

admits an explicit solution $u(t,x) = \mathrm{e}^{-\mathrm{i}t|u_0(x)|^2} u_0(x)$. Clearly, it generates a nonsmooth map in L^2. On the other hand, replacing Δ in (2.12) by the Grushin operator $G = \partial_x^2 + x^2\partial_y^2$ acting on $L^2(\mathbb{R}^2)$ and making a separation of variables leads to a system of coupled transport equations

$$\mathrm{i}(\partial_t \pm (2m+1)\partial_y)u_m = P_m^{\pm}(|u|^2 u), \tag{2.14}$$

That is why the study of (2.11) is important in understanding the interaction between the Szegö projection P_+ and the nonlinearity $|u|^2 u$.

Denote by $W^{s,2}(\mathbb{T})$, $s \geq 0$ the standard Sobolev spaces ($u \in W^{s,2}(\mathbb{T})$ if $\sum_{k\in\mathbb{Z}}(1+|k|^2)^s|\hat{u}_k| < \infty$) and set $W_+^{s,2}(\mathbb{T}) = W^{s,2}(\mathbb{T}) \cap H^2(\mathbb{T})$. Note that $W^{0,2}(\mathbb{T}) = L^2(\mathbb{T})$ and $W_+^{0,2}(\mathbb{T}) = H^2(\mathbb{T})$. We begin with following well-posedness result.

Theorem 2.2.1. *Let $s \geq 1/2$. Then for any $u_0 \in W_+^{s,2}(\mathbb{T})$ there exists a unique solution $u \in C(\mathbb{R}, W_+^{s,2}(\mathbb{T}))$ of (2.11) such that $u(0,x) = u_0(x)$. Moreover, for every $T > 0$, the mapping $u \in W_+^{\frac{1}{2},2}(\mathbb{T}) \mapsto C([-T,T], W_+^{\frac{1}{2},2}(\mathbb{T}))$ is continuous.*

Remark 2.2.2. The proof of this result can be found in [7].

Consider the following symplectic form on $H^2(\mathbb{T})$

$$\omega(f,g) := 4\mathrm{Im}(f,g)_{L^2(\mathbb{T})} = \frac{2}{\pi}\mathrm{Im}\int_{\mathbb{T}} f g^* d\theta. \tag{2.15}$$

Let $\mathcal{D}$ be a dense subspace of $H^2(\mathbb{T})$. Let also F be a real valued functional defined on $\mathcal{D}$. F is called *Gâteaux differentiable* if the following limit

$$dF(u;h) = \lim_{t\to 0} \frac{F(u+th) - F(u)}{t} \tag{2.16}$$

exists for all $h \in \mathcal{D}$. We shall say that F *admits a Hamiltonian vector field* if there exists a mapping

$$X_F \colon \mathcal{D} \to H^2(\mathbb{D})$$

such that the Gâteaux derivative of F satisfies

$$dF(u;h) = \omega(h, X_F(u)). \tag{2.17}$$

A Hamiltonian curve associated to F is a solution $u = u(t)$ of

$$\dot{u} = X_F(u).$$

Finally, for two functionals F and G admitting Hamiltonian vector fields, the Poisson bracket of F, G is defined on $\mathcal{D}$ by

$$\{F, G\}(u) = \omega(X_F(u), X_G(u)).$$

Define the energy functional

$$E(u) := \|u^2\|^2_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_{\mathbb{T}} |u|^4 d\theta, \qquad u \in \mathcal{D}_E := L^4(\mathbb{T}) \cap H^2(\mathbb{T}). \tag{2.18}$$

Lemma 2.2.3. *Equation* (2.11) *is the equation of Hamiltonian curves for* E*, that is,*

$$\dot{u} = X_E(u) = -\mathrm{i}P_+(|u|^2 u). \tag{2.19}$$

Proof. First observe that for any u, $h \in L^4(\mathbb{T}) \cap H^2(\mathbb{T})$

$$\begin{aligned}\lim_{t\to 0} \frac{E(u+th) - E(u)}{t} =& \lim_{t\to 0} \frac{1}{2\pi t}\int_{\mathbb{T}} |u+th|^4 - |u|^4 d\theta \\ &= \frac{1}{\pi}\int_{\mathbb{T}} (uh^* + u^*h)|u|^2 d\theta = 4\mathrm{Re}(h, |u|^2 u)_{L^2(\mathbb{T})}.\end{aligned}$$

Since $h \in H^2(\mathbb{T})$, we get

$$\begin{aligned}4\mathrm{Re}(h, |u|^2 u)_{L^2(\mathbb{T})} =& 4\mathrm{Re}(h, P_+(|u|^2 u))_{L^2(\mathbb{T})} \\ &= 4\mathrm{Im}(h, -\mathrm{i}P_+(|u|^2 u))_{L^2(\mathbb{T})} = \omega(h, X_E(u)).\end{aligned}$$

Therefore, E admits a Hamiltonian vector field and Hamiltonian curves associated to E are solutions of (2.19). □

Remark 2.2.4. This lemma implies that the cubic Szegö equation (2.11) is formally Hamiltonian. In particular, the Hamiltonian E generates a conservation law: *the energy is conserved* $E(u(t)) = E(u(0))$ *for all* $t \in \mathbb{R}$.

Exercise 2.2.1. Show that the functionals

$$Q(u) := \|u\|^2_{L^2(\mathbb{T})}, \qquad M(u) := -\mathrm{i}(u', u), \tag{2.20}$$

defined on $H^2(\mathbb{T})$ and $W^{1,2}_+(\mathbb{T})$, respectively, admit Hamiltonian vector fields

$$X_Q(u) = -\frac{\mathrm{i}}{2}u, \qquad X_M(u) = -\frac{1}{2}u'. \tag{2.21}$$

Exercise 2.2.2. Show that Q and M are *integrals of motion* for E, i.e.,

$$\{Q, E\} = 0, \qquad \{M, E\} = 0.$$

Moreover, Q and M are *in involution*, $\{Q, M\} = 0$.

Our main aim is to show that the cubic Szegö equation is formally integrable (admits a Lax pair representation). The latter will enable us to construct infinitely many conserved quantities for (2.11) and to describe finite dimensional isospectral tori.

Let $u \in W_+^{\frac{1}{2},2}(\mathbb{T})$ and consider the anti-linear Hankel operator with symbol u:

$$H_u\colon \quad \begin{array}{ccc} H^2(\mathbb{T}) & \to & H^2(\mathbb{T}) \\ h & \mapsto & P_+(uh^*) \end{array}. \tag{2.22}$$

Note that the matrix representation of H_u is given by

$$(H_u h)_k = \sum_{n\geq 0} \hat{u}_{n+k} h_n^*.$$

Clearly, H_u is Hilbert–Schmidt. Indeed, according to Remark 1.3.10,

$$\|H_u\|^2_{\mathfrak{S}_2} = \sum_{k,n} |\hat{u}_{k+n}|^2 = \sum_{k\in\mathbb{Z}_+} (1+k)|\hat{u}_k|^2 = \|u\|^2_{W^{\frac{1}{2},2}}.$$

Moreover, using the functionals Q and M, one gets

$$\|H_u\|^2_{\mathfrak{S}_2} = Q(u) + M(u), \tag{2.23}$$

and hence the Hilbert–Schmidt norm of H_u is a conserved quantity under the cubic Szegö flow (2.11).

Finally, since $(H_u h, f) = (H_u f, h)$, the operator $H_u^2 = H_u H_u$ is a positive self-adjoint (linear! since the product of two nonlinear operators is a linear operator) operator and its matrix representation is given by

$$(H_u^2)_{kn} = \sum_{j\in\mathbb{Z}_+} \hat{u}_{k+j}\hat{u}^*_{j+n}.$$

The main result of this section is the following

Theorem 2.2.5. *Let $u \in C(\mathbb{R}, W_+^{s,2}(\mathbb{T}))$ with some $s > \frac{1}{2}$. Then u is a solution to the cubic Szegö equation (2.11) if and only if the Hankel operator H_u satisfies the following evolution equation*

$$\frac{d}{dt}H_u = [B_u, H_u] = B_u H_u - H_u B_u, \tag{2.24}$$

where B_u is a skew-self-adjoint operator given by

$$B_u := \frac{\mathrm{i}}{2} H_u r - \mathrm{i}\, T_{|u|^2}, \tag{2.25}$$

and $T_{|u|^2}\colon h \mapsto P_+(|u|^2 h)$ is a Toeplitz operator.

Proof. If u solves (2.11), then we get

$$\frac{d}{dt}H_u h = \frac{d}{dt}P_+(uh^*) = P_+(\dot{u}h^*) = -\mathrm{i}P_+\big(P_+(|u|^2u)h^*\big) = -\mathrm{i}H_{P_+(|u|^2u)}h,$$

and hence we only need to show that

$$H_{P_+(|u|^2u)} = T_{|u|^2}H_u + H_uT_{|u|^2} - H_u^3.$$

Since $P_+(I - P_+) = 0$,

$$P_+\big(P_+(|u|^2u)h^*\big) = P_+(|u|^2uh^*)$$

for all $h \in H^2(\mathbb{T})$. Further, note that

$$P_+(|u|^2uh^*) = P_+(|u|^2P_+(uh^*)) + P_+(|u|^2(I - P_+)(uh^*)).$$

The first summand gives

$$P_+(|u|^2P_+(uh^*)) = (T_{|u|^2}H_u)(h).$$

Since $|u|^2 = uu^*$ and $u((I - P_+)(uh^*))^* \in H^2(\mathbb{T})$, we get

$$\begin{aligned} P_+(|u|^2(I - P_+)(uh^*)) &= H_u(u((I - P_+)(uh^*))^*) \\ &= H_u(P_+u((I - P_+)(uh^*))^*) \\ &= H_u(P_+(|u|^2h - u(P_+uh^*)^*) = (H_uT_{|u|^2})(h) - (H_u^3)(h). \qquad \square \end{aligned}$$

The representation (2.24) is called *the Lax pair*. If $U(t)$ is a family of unitary operators solving the equation

$$\frac{d}{dt}U(t) = B_{u(t)}U(t), \qquad U(0) = I,$$

then the family of Hankel operators $H_{u(t)}$ are unitary equivalent and satisfy

$$H_{u(t)} = U(t)^{-1}H_{u(0)}U(t), \quad t \in \mathbb{R}. \tag{2.26}$$

In particular, this immediately implies the following result.

Corollary 2.2.6. *The family $H_{u(t)}$ is isospectral under the cubic Szegö flow. In particular, every eigenvalue of $H_{u(t)}$ is a constant of motion.*

Let us state some immediate consequences. First of all, $\|H_{u(t)}\| = \|H_{u(0)}\|$ and by Nehari's theorem 1.1.5, these norms are equivalent to the BMO norm of u. Moreover, $H_u \in \mathfrak{S}_p$ for some p if and only if $H_{u(t)} \in \mathfrak{S}_p$ for all $t \in \mathbb{R}$; their von Neumann–Schatten norms are equal and, by Peller's theorem, are further equivalent to the norm of u in the Besov space $B_p^{1/p}$. The most interesting case is the trace class. In this case, $\|H_u\|_{\mathfrak{S}_1}$ is equivalent to $\|u''\|_{L^1(\mathbb{D})}$ and this allows to improve some long time estimates for $u(t)$ (see [**7**, Corollary 2]).

Finally, for $N \in \mathbb{N}$ denote by $M(N)$ the set of rational functions u of the form

$$u(z) = \frac{P(z)}{Q(z)},$$

where P, Q are polynomials with complex coefficients having no common zeros and such that $\deg P = N - 1$, $\deg Q = N$, $Q(0) = 1$ and $Q(z) \neq 0$ if

$|z| \leq 1$. By Kronecker's theorem 1.2.1, $\mathrm{rank} H_u = N$. Thus, Corollary 2.2.6 implies the following

Theorem 2.2.7. *Let $u_0 \in M(N)$ and $u(t)$ be a solution to (2.11) with $u(0) = u_0$. Then $u(t) \in M(N)$ for all $t \in \mathbb{R}$, that is, the submanifolds $M(N)$ are invariant under the cubic Szegö flow (2.11).*

Exercise 2.2.3. Since $\dim M(N) = 2N$, equation (2.11) is reduced to a finite dimensional Hamiltonian system on $M(N)$. Take

$$u = \sum_{k=1}^{N} \frac{a_k}{1 - b_k z}, \qquad b_k \in \mathbb{D}, \quad k \in \{1, \dots, N\},$$

and find the system of equations for a_k and b_k if u solves (2.11).

2.3. Inverse spectral problems for Hankel operators

The results in the previous section motivate the study of the *inverse spectral problem for Hankel matrices*. The general inverse spectral problem can be stated as follows. Given a spectral data S (e.g., spectrum), describe Hankel matrices with this spectral data. If H_α is a bounded Hankel operator, then there are some necessary restrictions on its spectrum. First of all, as we already mentioned in Corollary 1.1.11, the kernel of H_α is either empty or infinite dimensional. On the other hand, by Theorem 1.1.9,

$$\mathcal{S}^{*k} H_\alpha = H_\alpha \mathcal{S}^k$$

for all $k \in \mathbb{N}$. Therefore,

$$H_\alpha \mathrm{e}_k = H_\alpha \mathcal{S}^k \mathrm{e}_0 = \mathcal{S}^{*k} H_\alpha \mathrm{e}_0 = \mathcal{S}^{*k} \alpha \to 0 \tag{2.27}$$

as $k \to \infty$. Thus, by Weyl's criterion, we arrive at the following

Lemma 2.3.1. *Let H_α be a bounded Hankel operator. Assume also that $\mathrm{rank} H_\alpha = \infty$. Then*

$$0 \in \sigma_{\mathrm{ess}}(H_\alpha). \tag{2.28}$$

Thus, Hankel operators satisfy the following two conditions:

(i) $0 \in \sigma(H)$ and $0 \in \sigma_{\mathrm{ess}}(H)$,

(ii) if $\ker(H) \neq \{0\}$, then $\dim \ker(H) = \infty$.

So, the following question naturally arises in this context: Let $\sigma \subset \mathbb{C}$ be a closed bounded subset such that $0 \in \sigma$. Does there exists a bounded Hankel operator H such that $\sigma(H) = \sigma$? In a particular case $\sigma = \{0\}$ this problem was posed by S. C. Power. The next result shows that there is no nilpotent Hankel operators.

Lemma 2.3.2 (Power). *If $\varphi \in L^\infty(\mathbb{T})$ is such that H_φ is nilpotent, i.e., $H_\varphi^N = \mathbb{O}$ for some $N \in \mathbb{N}$, then $H_\alpha = \mathbb{O}$.*

Proof. Let us consider Hankel operators in the following form $H_\varphi f = P_+(J(\varphi f))$, where $J(f)(z) = f(z^*)$. It is straightforward to check that the matrix of the operator H_φ in the basis $\{\mathrm{e}_k\}_{k\in\mathbb{Z}_+}$ coincides with the Hankel matrix with coefficients $\alpha_k = \hat{\varphi}_{-k}$, $k \in \mathbb{Z}_+$.

Since H_φ is nilpotent, its kernel $\ker H_\varphi$ is nontrivial and hence by Beurling's theorem A.2.12, $\ker H_\varphi = \Theta H^2$, where Θ is an inner function. Noting that $H_{\varphi\Theta} = \mathbb{O}$, we conclude

$$\varphi\Theta = zh, \qquad h \in H^\infty.$$

Without loss of generality we can assume that Θ and h have no common inner divisors. Consider the operator $H_{z\Theta^*}$. It is easy to see that $H_{z\Theta^*}$ is a partial isometry with the initial space $K_\Theta := H^2 \ominus \Theta H^2$ and the final space $K_{\Theta^\#} := H^2 \ominus \Theta^\# H^2$, where $\Theta^\#(z) := \Theta(z^*)^*$. Indeed,

$$H_{z\Theta^*}(\Theta f) = P_+(J(z|\Theta|^2 f)) = P_+(J(zf)) = 0$$

for all $f \in H^2$. On another hand, for any $g \in K_\Theta$ we get

$$H_{z\Theta^*} g = P_+(J(z\Theta^* g)) = P_+(\Theta^\# J(zg)).$$

Since $J(zK_\Theta) = H^2_- \ominus (J\Theta)H^2_-$, we get $\Theta^\# J(zK_\Theta) = K_{\Theta^\#}$.

Noting that $H_\varphi f = H_{z\Theta^*}(hf)$ and using the nilpotence of H_φ we conclude that there exists $f \in K_{\Theta^\#}$ such that $hf \in \Theta H^2$. Therefore, $f_1 := f/\Theta \in H^2$. However, $f \in K_{\Theta^\#}$ and hence

$$P_+((\Theta^\#)^* f) = P_+((\Theta^\#)^* \Theta f_1) = 0.$$

Therefore, the Toeplitz operator $T_{(\Theta^\#)^*\Theta}$ has a nontrivial kernel. Noting that

$$\ker T^*_{(\Theta^\#)^*\Theta} = \ker T_{\Theta^\#\Theta^*} = \ker T_{(\Theta(\Theta^\#)^*)^\#} = (\ker T_{(\Theta^\#)^*\Theta})^\#.$$

However, Coburn's alternative (see [**25**, Theorem III.1.4]) states that either a kernel or a co-kernel of a non-zero Toeplitz operator is trivial. This contradiction completes the proof. □

Remark 2.3.3. An explicit example of a quasi-nilpotent[1] Hankel operator was constructed by A. V. Megretskii. For instance, the matrix

$$H = \begin{pmatrix} \mathrm{i} & 1/2 & 0 & 1/4 & 0 & \dots \\ 1/2 & 0 & 1/4 & 0 & 0 & \dots \\ 0 & 1/4 & 0 & 0 & 0 & \dots \\ 1/4 & 0 & 0 & 0 & 1/8 & \dots \\ 0 & 0 & 0 & 1/8 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

[1] An operator T is called quasi-nilpotent if $\|T^n\|^{1/n} \to 0$ as $n \to \infty$. Clearly, the latter is equivalent to the condition $\sigma(T) = \{0\}$.

is compact and $\sigma(H) = \{0\}$ (for further details we refer to [**25**, Chapter X.3]).

Motivated by numerous applications, S. V. Khruschev and V. V. Peller conjectured in 1984 that *every positive bounded operator T satisfying conditions (i) and (ii) is unitarily equivalent to a modulus of a Hankel operator.* This problem was solved in affirmative by V. V. Vasyunin and S. R. Treil (see [**36**]). Moreover, it turns out that the condition $0 \in \sigma$ is the only restriction on the spectrum of Hankel operators.

Theorem 2.3.4 ([**20**])**.** *Let σ be any compact subset of the complex plane containing* 0*. Then there exists a Hankel operator H such that $\sigma(H) = \sigma$.*

Note that it is easy to construct a linear operator T such that $\sigma(T) = \sigma$, where $\sigma \subset \mathbb{C}$ is any given compact set. Indeed, take a disjoint sequence of points $\{\lambda_k\}_{k=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$ which is dense in σ, $\overline{\{\lambda_k\}} = \sigma$, and then set $T = \mathrm{diag}(\lambda_k)$. Clearly, every λ_k is an eigenvalue of T and $\sigma(T) = \overline{\{\lambda_k\}}$. Unfortunately, there are no simple "building blocks" for Hankel operators. Detailed proof of Theorem 2.3.4 can be found in [**20**].

Theorem 2.3.4 describes all possible spectra of bounded Hankel operators. The next question is about the spectral structure of Hankel operators. We start with the following result.

Theorem 2.3.5 (Abakumov)**.** *Let $\{\lambda_j\}_{j=1}^N$ be a finite set of non-zero points and let $\{k_j\}_{j=1}^N \subset \mathbb{N}$. Then there is a finite rank Hankel operator such that its non-zero eigenvalues are precisely λ_j and their algebraic multiplicities*[2] *are k_j, $j = 1, \ldots, N$.*

According to Theorem 2.3.5, Hankel operator might have eigenvalues of an arbitrary algebraic multiplicity. However, the situation with the geometric multiplicity is a bit different.

Theorem 2.3.6 (Peller)**.** *Let H be a Hankel operator. Then*

$$|\dim\ker(H - z) - \dim\ker(H + z)| \leq 1 \tag{2.29}$$

for all $z \in \mathbb{C}$.

Proof. Let $H = H_\alpha$ be a Hankel matrix in ℓ_2. Set $\mathcal{N}_z := \ker(H - z)$, $z \in \mathbb{C}$. Clearly, it suffices to show that

$$\dim \mathcal{N}_{-z} \geq \dim \mathcal{N}_z - 1 \tag{2.30}$$

[2] The algebraic multiplicity of an eigenvalue is the dimension of the corresponding root subspace $\cup_{n\in\mathbb{N}} \ker(T - z)^n$. The number $\dim\ker(T - z)$ is called the geometric multiplicity of an eigenvalue.

for all $z \in \mathbb{C}$. Using (1.21), we get for all $f \in \ell_2$

$$\mathcal{S}^* H_\alpha f - H_\alpha \mathcal{S} f = 0, \tag{2.31}$$

and, moreover,

$$\mathcal{S} H_\alpha f - H_\alpha \mathcal{S}^* f = (f, \mathrm{e}_0)_{\ell_2} \mathcal{S}\alpha - (\mathcal{S}\alpha, f^*)_{\ell_2}. \tag{2.32}$$

Assume first that there is $f \in \mathcal{N}_z$ such that $f_0 = (f, \mathrm{e}_0)_{\ell_2} \neq 0$. Take any $g \in \mathcal{N}_z$ such that $g_0 = (g, \mathrm{e}_0)_{\ell_2} = 0$. By (2.31) and (2.32) we get

$$H_\alpha(\mathcal{S} + \mathcal{S}^*)g - (\mathcal{S} + \mathcal{S}^*)H_\alpha g = -(\mathcal{S}\alpha, g^*)\mathrm{e}_0.$$

Furthermore,

$$\big((H_\alpha(\mathcal{S} + \mathcal{S}^*) - (\mathcal{S} + \mathcal{S}^*)H_\alpha)g, f^*\big)_{\ell_2} = -(\mathcal{S}\alpha, g^*) f_0,$$

where $f_0 \neq 0$ by the assumption. On the other hand,

$$\begin{aligned}
&\big((H_\alpha(\mathcal{S} + \mathcal{S}^*) - (\mathcal{S} + \mathcal{S}^*)H_\alpha)g, f^*\big)_{\ell_2} \\
&\quad = \big((\mathcal{S} + \mathcal{S}^*)g, H_\alpha^* f^*\big)_{\ell_2} - \big(H_\alpha g, (\mathcal{S} + \mathcal{S}^*) f^*\big)_{\ell_2} \\
&\quad = z\big((\mathcal{S} + \mathcal{S}^*)g, f^*\big)_{\ell_2} - z\big(g, (\mathcal{S} + \mathcal{S}^*) f^*\big)_{\ell_2} = 0,
\end{aligned}$$

which implies that $(\mathcal{S}\alpha, g^*) = 0$ and hence

$$H_\alpha(\mathcal{S} + \mathcal{S}^*)g = (\mathcal{S} + \mathcal{S}^*)H_\alpha g, \qquad H_\alpha \mathcal{S}^* g = \mathcal{S} H_\alpha g. \tag{2.33}$$

These equalities imply that

$$H_\alpha(\mathcal{S} - \mathcal{S}^*)g = -(\mathcal{S} - \mathcal{S}^*)H_\alpha g = -z(\mathcal{S} - \mathcal{S}^*)g,$$

that is, $(\mathcal{S} - \mathcal{S}^*)g \in \mathcal{N}_{-z}$ whenever $g \in \mathcal{N}_z$ and $g_0 = 0$. Since $\ker(\mathcal{S} - \mathcal{S}^*) = \{0\}$, we conclude that $\mathcal{S} - \mathcal{S}^*$ is a 1-to-1 map of $\{g \in \mathcal{N}_z \,|\, g_0 = 0\}$ into $\mathcal{N}_{-z}$. This proves (2.30).

Finally, if $(f, \mathrm{e}_0) = 0$ for all $f \in \mathcal{N}_z$, then (2.31) and (2.32) imply that (2.33) holds for all f such that $(\mathcal{S}\alpha, f^*)_{\ell_2} = 0$. As before, this implies that $\mathcal{S} - \mathcal{S}^*$ is a 1-to-1 map of $\{f \in \mathcal{N}_z \,|\, (\mathcal{S}\alpha, f^*)_{\ell_2} = 0\}$ into $\mathcal{N}_{-z}$, which proves (2.30) □

A complete description of a spectral structure of self-adjoint Hankel operators was obtained by A. V. Megretskii, V. V. Peller and S. R. Treil. Before formulate their result, let us recall that by von Neumann's theorem, every self-adjoint operator T on separable Hilbert space $\mathfrak{H}$ is unitarily equivalent to multiplication by the independent variable on a direct integral $\int_\oplus \mathfrak{H}(t) d\mu(t)$:

$$(\mathcal{M} f)(t) = t f(t), \qquad f \in \int_\oplus \mathfrak{H}(t) d\mu(t). \tag{2.34}$$

Without loss of generality we can assume that $\mathfrak{H}(t) \neq \{0\}$ μ-almost everywhere. In this case, μ is called *a scalar spectral measure* and

$$\nu_A \colon t \mapsto \dim \mathfrak{H}(t) \tag{2.35}$$

is *the spectral multiplicity* of A. Two self-adjoint operators are unitarily equivalent if and only if their scalar spectral measures are mutually absolutely continuous and spectral multiplicities are equal almost everywhere.

Theorem 2.3.7 (Megretskii–Peller–Treil)**.** *Let $T \in [\mathfrak{H}]$ be self-adjoint. Let also μ be its scalar spectral measure and ν_T its spectral multiplicity function. Then T is unitarily equivalent to a Hankel operator if and only if the following conditions are satisfied:*

(i) *either* $\ker T = \{0\}$ *or* $\dim \ker T = \infty$,

(ii) $0 \in \sigma(T)$,

(iii) $|\nu_T(t) - \nu_T(-t)| \le 2$ μ_{ac}*-almost everywhere and* $|\nu_T(t) - \nu_T(-t)| \le 1$ μ_s*-almost everywhere.*

The proof of this result can be found in [**25**, Chapter XII]. We complete this section with the following corollary.

Corollary 2.3.8. *Let $T \in [\mathfrak{H}]$ be self-adjoint and positive. Then T is unitarily equivalent to a Hankel operator if and only if the following conditions are satisfied:*

(i) *either* $\ker T = \{0\}$ *or* $\dim \ker T = \infty$,

(ii) $0 \in \sigma(T)$,

(iii) $\nu_T(t) \le 2$ μ_{ac}*-almost everywhere and* $\nu_T(t) \le 1$ μ_s*-almost everywhere.*

In particular, if T is compact and $\ker T = \{0\}$, then it is similar to a positive Hankel operator if and only if the operator T is simple.

2.4. Inverse spectral problem for self-adjoint compact Hankel operators

We start with the following identity.

Lemma 2.4.1. *Let $H_\alpha = (\alpha_{j+k})_{j,k\in\mathbb{Z}_+}$ be a bounded Hankel matrix and $H_{\mathcal{S}^*\alpha} = (\alpha_{j+k+1})_{j,k\in\mathbb{Z}_+}$. Then*

$$H_\alpha H_\alpha^* = H_{\mathcal{S}^*\alpha} H_{\mathcal{S}^*\alpha}^* + (\cdot, \alpha)\alpha. \tag{2.36}$$

Proof. Indeed, using (1.21) we get

$$H_{\mathcal{S}^*\alpha} = \mathcal{S}^* H_\alpha = H_\alpha \mathcal{S}, \quad H_{\mathcal{S}^*\alpha}^* = \mathcal{S}^* H_\alpha^* = H_\alpha^* \mathcal{S}.$$

Noting that $\mathcal{S}\mathcal{S}^* = I - (\cdot, \mathrm{e}_0)\mathrm{e}_0$ and $H_\alpha \mathrm{e}_0 = \alpha$, we end up with (2.37). □

Remark 2.4.2. Since $H_\alpha^* = H_{\alpha^*}$, we get

$$H_\alpha^* H_\alpha = H_{\mathcal{S}^*\alpha}^* H_{\mathcal{S}^*\alpha} + (\cdot, \alpha^*)\alpha^*. \tag{2.37}$$

Assume now that $\alpha = \alpha^*$ and $\alpha(z) = \sum_{k\in\mathbb{Z}_+} \alpha_k z^k \in VMOA$, that is, H_α is a compact self-adjoint Hankel operator. Let $\{\lambda_j\}$ be the sequence of non-zero eigenvalues of H_α ordered in decreasing order

$$0 < \dots\dots \le |\lambda_n| \le \cdots \le |\lambda_2| \le |\lambda_1|.$$

Theorem 2.3.6 immediately implies the following restrictions on multiplicities of non-zero eigenvalues of H_α.

Corollary 2.4.3. *If $\lambda \in \sigma(H_\alpha)$ and $\lambda \neq 0$, then*

$$|\dim\ker(H_\alpha - \lambda) - \dim\ker(H_\alpha + \lambda)| \le 1. \tag{2.38}$$

If in addition H_α is a nonnegative Hankel operator, then all its non-zero eigenvalues are simple.

Remark 2.4.4. In general, one can not say much on the spectrum of $H_{\mathcal{S}^*\alpha}$. In particular, the positivity of H_α does not imply the positivity of $H_{\mathcal{S}^*\alpha}$ (cf. Lemma 2.1.3). On the other hand, there are certain restriction on their absolute values. Namely, by (2.37), $H^2_{\mathcal{S}^*\alpha}$ is a rank 1 perturbation of H^2_α, which suggests certain interlacing properties (see Lemma 2.4.7 below).

We need the following simple result.

Lemma 2.4.5. *Let A_1 and A_0 be bounded self-adjoint operators on a Hilbert space $\mathfrak{H}$ such that*

$$A_1 = A_0 + \langle \cdot, \phi\rangle\phi. \tag{2.39}$$

Set

$$\mathcal{M}_j := \operatorname{span}\{A_j^n\phi\}_{n\in\mathbb{Z}_+}, \qquad j \in \{0, 1\}. \tag{2.40}$$

Then:

(i) $\mathcal{M}_0 = \mathcal{M}_1$,

(ii) $\mathcal{M}_j$ *is a reducing subspace for* A_j,

(iii) $A_0|_{\mathcal{M}_0^\perp} = A_1|_{\mathcal{M}_1^\perp}$.

Exercise 2.4.1. Prove Lemma 2.4.5 (Hint: Show that every $f = A_1^n\phi \in \mathcal{M}_0$. To prove (iii), use the implication $f \perp \mathcal{M}_j \Rightarrow f \perp \phi$.)

Remark 2.4.6. Note that the operators $A_0|_{\mathcal{M}_0}$ and $A_1|_{\mathcal{M}_1}$ are simple, that is, they are unitarily equivalent to a multiplication operator in $L^2(d\mu_j)$, where the measures $d\mu_j$ are defined by the Stieltjes inversion formula applied to Herglotz–Nevanlinna functions

$$m_j(z) = \langle (A_j - z)^{-1}\phi, \phi\rangle = \int_{\mathbb{R}} \frac{d\mu_j(s)}{s - z} = \int_{\mathbb{R}} \frac{1}{s - z} d\langle E_j(s)\phi, \phi\rangle, \tag{2.41}$$

where E_j is the distribution of identity for A_j, $j \in \{0,1\}$. Further information about simple operators can be found in [2].

Since $\mathcal{M}_0 = \mathcal{M}_1$, let us denote these spaces by $\mathcal{M}$. By Lemma 2.4.5, the operators A_0 and A_1 admit the following representation with respect to the decomposition $\mathfrak{H} = \mathcal{M} \oplus \mathcal{M}^\perp$:

$$A_j = A_j|_{\mathcal{M}} \oplus A_j|_{\mathcal{M}^\perp}, \qquad j \in \{0,1\}. \tag{2.42}$$

In particular, $\sigma(A_j) = \sigma(A_j|_{\mathcal{M}}) \cup \sigma(A_j|_{\mathcal{M}^\perp})$ and, by Lemma 2.4.5(iii), $\sigma(A_0|_{\mathcal{M}^\perp}) = \sigma(A_1|_{\mathcal{M}^\perp})$.

Lemma 2.4.7. *Let A_1 and A_0 be compact self-adjoint operators on a Hilbert space $\mathfrak{H}$ such that (2.39) holds. Suppose also that $\mathcal{M} = \mathfrak{H}$, i.e., A_0 and A_1 are simple. Then the eigenvalues of A_0 and A_1 interlace.*

Proof. Let $\{\lambda_k\}$ and $\{\mu_k\}$ be the eigenvalues of A_0 and A_1, respectively. Define the functions m_0 and m_1 by (2.41). Then

$$m_1(z) = \frac{m_0(z)}{1+m_0(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.43}$$

Indeed, by (2.39),

$$(A_0 - z)^{-1}\phi - (A_1 - z)^{-1}\phi = \langle (A_1 - z)^{-1}\phi, \phi\rangle (A_0 - z)^{-1}\phi,$$

which immediately gives (2.43).

Now observe that m_0 and m_1 are analytic away of $\sigma(A_0) = \{0\} \cup \{\lambda_k\}$ and $\sigma(A_1) = \{0\} \cup \{\mu_k\}$. Moreover, each eigenvalue of A_j is a pole of m_j. By (2.43), m_1 is analytic at λ_k for all k and hence $\{\lambda_k\} \notin \sigma(A_1)$.

Assume now that λ_k and λ_{k+1} are two consecutive (positive) eigenvalues of A_0. By (2.43), we only need to show that $1 + m_0(z) = 0$ has precisely one solution in the interval $(\lambda_{k+1}, \lambda_k)$. But this follows from the Herglotz properties of m_0. Indeed, $m_0(\lambda_{k+1}+) = -\infty$ and $m_0(\lambda_k-) = +\infty$. On the other hand, m_0 is strictly increasing on $(\lambda_{k+1}, \lambda_k)$ since

$$m_0'(x) = \int_{\mathbb{R}} \frac{1}{|s-x|^2} d\mu_0(s) > 0, \qquad x \in \mathbb{R} \setminus \sigma(A_0).$$

Finally, since

$$m_0(z) = \frac{m_1(z)}{1-m_1(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

an analogues arguments show that between any two consecutive eigenvalues μ_k and μ_{k+1} of A_1 there is precisely one eigenvalue of A_0. □

The following result was obtained in [11].

Lemma 2.4.8. *If H_α and $H_{\mathcal{S}^*\alpha}$ are positive bounded Hankel operators, then $\overline{\mathrm{ran} H_\alpha} = \mathcal{M}$, that is, the operator $H_\alpha|_{\ker(H_\alpha)^\perp}$ (as well as $H_{\mathcal{S}^*\alpha}$) is simple*

Proof. Consider the subspace

$$\tilde{\mathcal{M}} = \operatorname{span}\{H_\alpha^{2n}\alpha\}_{n\in\mathbb{Z}_+}.$$

Actually, $\tilde{\mathcal{M}} = \mathcal{M}$, however, it would more convenient for us to deal with $\tilde{\mathcal{M}}$. By (2.37) and Lemma 2.4.5, $\tilde{\mathcal{M}} = \operatorname{span}\{H_{\mathcal{S}^*\alpha}^{2n}\alpha\}_{n\in\mathbb{Z}_+}$. Clearly, $\tilde{\mathcal{M}} \subseteq \mathcal{M}$. By Lemma 2.4.5, $H_\alpha^2|_{\tilde{\mathcal{M}}^\perp} = H_{\mathcal{S}^*\alpha}^2|_{\tilde{\mathcal{M}}^\perp}$. Since both operators are positive, the decomposition (2.42), implies that $H_\alpha|_{\tilde{\mathcal{M}}^\perp} = H_{\mathcal{S}^*\alpha}|_{\tilde{\mathcal{M}}^\perp}$. Hence for each $f \in \tilde{\mathcal{M}}^\perp$ we get

$$H_\alpha f = H_{\mathcal{S}^*\alpha} f = \mathcal{S}^* H_\alpha f.$$

Since $\ker(\mathcal{S}^* - I) = \{0\}$, we obtain $H_\alpha f = 0$ and $H_{\mathcal{S}^*\alpha} f = 0$. Therefore, $\tilde{\mathcal{M}}^\perp \subset \ker H_\alpha$ and $\tilde{\mathcal{M}}^\perp \subset \ker H_\alpha$, which implies that

$$\operatorname{ran} H_\alpha \subset \tilde{\mathcal{M}}, \qquad \operatorname{ran} H_{\mathcal{S}^*\alpha} \subset \tilde{\mathcal{M}}.$$

It remains to note that $\alpha = H_\alpha \mathrm{e}_0$ and hence $P_{\mathcal{M}}\mathrm{e}_0 \in \mathcal{M}$, which implies that $\mathcal{M} \subset \overline{\operatorname{ran} H_\alpha}$. □

Remark 2.4.9. It might happen that the inclusion $\overline{\operatorname{ran} H_{\mathcal{S}^*\alpha}} \subset \mathcal{M}$ is strict. Indeed, take $\alpha = \mathrm{e}_0$. Then $H_\alpha = (\cdot, \mathrm{e}_0)\mathrm{e}_0$ and $\operatorname{ran} H_\alpha = \operatorname{span}\{\mathrm{e}_0\}$, however, $H_{\mathcal{S}^*\alpha} = \mathbb{O}$ and $\operatorname{ran} H_{\mathcal{S}^*\alpha} = \{0\}$.

As an immediate consequence of Lemma 2.4.8 and Lemma 2.4.7 we obtain the following result.

Corollary 2.4.10. *Let H_α and $H_{\mathcal{S}^*\alpha}$ be positive compact Hankel operators. Then their eigenvalues arranged in the decreasing order satisfy*

$$0 < \cdots < \lambda_{k+1} < \mu_k < \lambda_k < \mu_{k-1} < \cdots < \mu_1 < \lambda_1. \tag{2.44}$$

Remark 2.4.11. It is absolutely unclear what happens without the double positive condition (H_α and $H_{\mathcal{S}^*\alpha}$ are positive). First of all, one needs to clarify the relationship between $\mathcal{M}$ and $\operatorname{ran} H_\alpha$. Another problem is the relationship between the spectral of $H_\alpha|_{\mathcal{M}}$ and $H_\alpha|_{\mathcal{M}^\perp}$.

We finish this section with the following result due to P. Gérard and S. Grellier.

Theorem 2.4.12 ([9])**.** *Let $\{\lambda_k\}$ and $\{\mu_k\}$ be two real sequences such that $\lambda_k \to 0$ and $\mu_k \to 0$ as $k \to \infty$, and their absolute values satisfy*

$$0 < \cdots < |\lambda_{k+1}| < |\mu_k| < |\lambda_k| < |\mu_{k-1}| < \cdots < |\mu_1| < |\lambda_1|. \tag{2.45}$$

Then there is a unique α such that the corresponding Hankel operator H_α is compact and $\{\lambda_k\}$ are the non-zero eigenvalues of H_α and $\{\mu_k\}$ are the non-zero eigenvalues of $H_{\mathcal{S}^*\alpha}$.

The proof of this result can be found [9].

Corollary 2.4.13. *Let* $\{\lambda_k\}$ *and* $\{\mu_k\}$ *be two positive sequences such that* $\lambda_k \to 0$*,* $\mu_k \to 0$ *as* $k \to \infty$*, and* (2.44) *holds. Then there is a unique* α *such that the Hankel operators* H_α *and* $H_{\mathcal{S}^*\alpha}$ *are positive and compact and their non-zero eigenvalues are* $\{\lambda_k\}$ *and* $\{\mu_k\}$*, respectively.*

A few remarks are in order.

Remark 2.4.14.

(i) An explicit formula for α given in terms of $\{\lambda_k\}$ and $\{\mu_k\}$ can be found in [**9**, Theorem 3].

(ii) An extension of Corollary 2.4.13 to the case of bounded Hankel operators can be found in [**11**].

(iii) It turns out that $\ker H_\alpha$ is trivial (and hence so is $\ker H_{\mathcal{S}^*\alpha}$) if and only if

$$\sum_{j\in\mathbb{Z}_+} \left(1 - \frac{\mu_j^2}{\lambda_j^2}\right) = \infty, \qquad \sup_N \frac{1}{\lambda_{N+1}^2} \prod_{j=1}^{N} \frac{\mu_j^2}{\lambda_j^2} = \infty. \tag{2.46}$$

Appendix A

Function Theory on the Unit Circle

A.1. Fourier Series: Convergence and Summability

For any Borel measure μ on the unit circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ one can associate a Fourier series

$$\mu \sim \sum_{n\in\mathbb{Z}} \hat{\mu}_n \mathrm{e}^{\mathrm{i}n\theta}, \qquad \hat{\mu}_n := \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}n\theta} d\mu(\theta). \tag{A.1}$$

If $\mu = f d\theta$ with some $f \in L^1(\mathbb{T})$, then we shall write $\hat{f}_n$ instead of $\hat{\mu}_n$.

It is natural to ask *to what extent μ is determined by its Fourier coefficients and how one can recover μ from $\{\hat{\mu}_n\}_{n\in\mathbb{Z}}$?*

A.1.1. The Dirichlet Kernel. Consider the partial sums of the Fourier series of f

$$\begin{aligned}(S_N f)(\theta) := \sum_{n=-N}^{N} \hat{f}_n \mathrm{e}_n(\theta) &= \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n\theta} \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\mathrm{e}^{-\mathrm{i}nt} dt \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(t) \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}n(\theta-t)} dt \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(t) D_N(\theta - t) dt \\ &= (f * D_N)(\theta),\end{aligned}$$

where D_N is the Dirichlet kernel,

$$D_N(\theta) = \sum_{n=-N}^{N} \mathrm{e}^{in\theta} = \frac{\sin((N+\frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}, \quad \theta \in \mathbb{T}. \tag{A.2}$$

Exercise A.1.1. Verify (A.2).

The next lemma summarizes the basic properties of *the convolution*.

Lemma A.1.1. *The operation of convolution satisfies:*

(i) *If f, $g \in L^1(\mathbb{T})$, then $f(\cdot)g(t-\cdot) \in L^1(\mathbb{T})$ for almost all $t \in \mathbb{T}$ and*

$$\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}. \tag{A.3}$$

Moreover,

$$\widehat{(f * g)}_n = \hat{f}_n \hat{g}_n, \quad n \in \mathbb{Z}. \tag{A.4}$$

(ii) *(Young's inequality) If $1 \le p, q, r \le \infty$ satisfy $1 + 1/r = 1/p + 1/q$, then*

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}, \tag{A.5}$$

(iii) *If $f \in C(\mathbb{T})$ and $\mu \in \mathcal{M}(\mathbb{T})$, then $f * \mu$ is well defined and*

$$\|f * \mu\|_{L^p} \le \|f\|_{L^p}\|\mu\|, \tag{A.6}$$

where $\|\mu\|$ is the total variation of μ, $\|\mu\| = |\mu|(\mathbb{T})$.

Remark A.1.2. Lemma A.1.1 (i) states that $L^1(\mathbb{T})$ is a commutative Banach algebra (with convolution as a multiplication). Moreover, Young's inequality states that $L^p(\mathbb{T})$ is an ideal of $L^1(\mathbb{T})$ for every $p \in (1, \infty]$.

Exercise A.1.2. Let $\mu \in \mathcal{M}(\mathbb{T})$ satisfy $\hat{\mu} \in \ell_1(\mathbb{Z})$, i.e., $\sum_{n\in\mathbb{Z}} |\hat{\mu}_n| < \infty$. Show that $\mu = f d\theta$ with $f \in C(\mathbb{T})$.

Definition A.1.3 (Wiener algebra)**.** The subspace of measures with the above summability property is called *the Wiener algebra* and is denoted by $\mathcal{A}(\mathbb{T})$. Show that $\mathcal{A}(\mathbb{T})$ is an algebra under multiplication and

$$\widehat{(fg)}_n = \sum_{m\in\mathbb{Z}} \hat{f}_n \hat{g}_{n-m}.$$

Moreover, $\|fg\|_{\mathcal{A}} \le \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$, where $\|f\|_{\mathcal{A}} := \|\hat{f}\|_{\ell_1}$.

Note that the constant function $\mathbb{1}$ is a unit in $\mathcal{A}(\mathbb{T})$ and hence $1/f \in \mathcal{A}(\mathbb{T})$ whenever $f \neq 0$ on $\mathbb{T}$ (this is the content of *Wiener's Lemma*).

The family of exponents $\mathrm{e}_n(\theta) = \mathrm{e}^{in\theta}$, $n \in \mathbb{Z}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{T})$ and hence

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}_n \mathrm{e}^{in\theta}, \qquad \hat{f}_n = (f, \mathrm{e}_n)_{L^2}, \quad n \in \mathbb{Z}, \tag{A.7}$$

where the convergence is understood in the L^2 sense. In particular, $f \in L^2(\mathbb{T})$ if and only if $\hat{f} \in \ell_2(\mathbb{Z})$. It is possible to show that $S_N f$ approximates $f \in L^p$ in the L^p norm whenever $p \in (1, \infty)$. However, there is no nice characterization (as in the L^2 space) of Fourier coefficients in this case. Moreover, this sort of approximation fails if $p = 1$ or $p = \infty$.

Theorem A.1.4. *There is $f \in C(\mathbb{T})$ ($\in L^1(\mathbb{T})$) such that*

$$\|S_N f - f\|_\infty \not\to 0 \quad (\|S_N f - f\|_{L^1} \not\to 0) \tag{A.8}$$

as $N \to \infty$.

However, as for the space $C(\mathbb{T})$ it is possible to show that $S_N f$ approximates f under certain additional regularity assumptions on f.

Theorem A.1.5. *If $f \in Lip_\alpha(\mathbb{T})$*[1] *with some $\alpha \in (0, 1]$, then $\|S_N f - f\|_\infty \to 0$ as $n \to \infty$.*

A.1.2. Cesáro Means and Fejer's Kernel. According to the Weierstrass approximation theorem, polynomials are dense in $C(\mathbb{T})$. Hence this result suggests that there exists another way to approximate f by trigonometric polynomials. *An approximate identity* is the key concept in our further considerations.

We begin with *the Cesáro means* defined by

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_N f = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}_n e^{in\theta} = \sum_{n \in \mathbb{Z}} \left(1 - \frac{|n|}{N}\right)^+ \hat{f}_n e^{in\theta} \tag{A.9}$$

for all $N \in \mathbb{N}$. Clearly,

$$\sigma_N f = K_N * f, \qquad K_N := \frac{1}{N} \sum_{n=0}^{N-1} D_n = \frac{1}{N} \left(\frac{\sin(\frac{N}{2}x)}{\sin(\frac{1}{2}x)} \right)^2 . \tag{A.10}$$

The function K_N is called *the Fejer kernel.*

Exercise A.1.3. Verify (A.10).

The properties of Fejer's kernels are summarized in the following lemma.

Lemma A.1.6. *The family $\{K_N\}_{N \in \mathbb{N}}$ is* an approximate identity*, that is:*

(i) $K_N(\theta) \geq 0$ *for all $\theta \in \mathbb{T}$ and $N \in \mathbb{N}$,*

(ii)
$$\frac{1}{2\pi} \int_{\mathbb{T}} K_N(\theta) d\theta = 1, \qquad N \in \mathbb{N}, \tag{A.11}$$

---

[1] $Lip_\alpha(\mathbb{T})$ is the class of Hölder continuous functions, i.e., $f \in Lip_\alpha(\mathbb{T})$ if there exists a constant $c > 0$ such that $|f(x) - f(y)| < c|x - y|^\alpha$ for all $x, y \in \mathbb{T}$.

(iii) *for all* $\delta \in (0,\pi)$,

$$\lim_{N\to\infty} \int_{\delta}^{2\pi-\delta} K_N(\theta)d\theta = 0. \tag{A.12}$$

The basic convergence properties of families that form an approximate identity are collected in the following theorem.

Theorem A.1.7. *Let* $\{\Phi_n\}_{n\in\mathbb{N}}$ *be an approximate identity. Then:*

(i) *If* $f \in C(\mathbb{T})$, *then* $\|\Phi_N * f - f\|_\infty \to 0$ *as* $N \to \infty$,

(ii) *If* $f \in L^p(\mathbb{T})$ *with* $p \in [1,\infty)$, *then* $\|\Phi_N * f - f\|_{L^p} \to 0$ *as* $N \to \infty$,

(iii) *If* $\mu \in \mathcal{M}(\mathbb{T})$, *then* $\Phi_N * \mu$ *converges to* μ *in the weak-*$*$ *topology.*

As an immediate corollary of Lemma A.1.6 and Theorem A.1.7 we obtain the following result.

Corollary A.1.8. (i) *If* $f \in C(\mathbb{T})$, *then* $\|\sigma_N f - f\|_\infty \to 0$ *as* $N \to \infty$,

(ii) *If* $f \in L^p(\mathbb{T})$ *with* $p \in [1,\infty)$, *then* $\|\sigma_N f - f\|_{L^p} \to 0$ *as* $N \to \infty$,

(iii) *If* $\mu \in \mathcal{M}(\mathbb{T})$, *then* $\sigma_N \mu$ *converges to* μ *in the weak-*$*$ *topology.*

If $\hat{\mu} = 0$, then $\sigma_N \mu = K_N * \mu = 0$, $n \in \mathbb{Z}_+$. Hence Corollary A.1.8(iii) implies uniqueness:

Corollary A.1.9. *If* $\mu \in \mathcal{M}(\mathbb{T})$ *and* $\hat{\mu}_n = 0$ *for all* $n \in \mathbb{Z}$, *then* $\mu \equiv 0$.

If $f \in L^1(\mathbb{T})$ and $|n| > N$, then we get

$$|\hat{f}_n| = |\widehat{(\sigma_N f)}_n - \hat{f}_n| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} (\sigma_N f - f)\mathrm{e}^{-\mathrm{i}n\theta} d\theta\right| \le \|\sigma_N f - f\|_{L^1}.$$

Applying Corollary A.1.8(ii), we arrive at *the Riemann–Lebesgue lemma.*

Corollary A.1.10. *If* $f \in L^1(\mathbb{T})$, *then* $\hat{f}_n = o(1)$ *as* $N \to \infty$, *i.e.*, $\widehat{L^1(\mathbb{T})} \subseteq c_0$.

Exercise A.1.4. Show by examples that the Riemann–Lebesgue lemma is not valid for measures.

Moreover, Fourier coefficients of L^1 functions can go to 0 arbitrarily slowly and the inclusion in Corollary A.1.10 is strict[2]. However, one can characterize the properties of f in terms of Cesáro means of its formal Fourier series.

Theorem A.1.11. *Let* $\mu \sim \sum_{n\in\mathbb{Z}} \hat{\mu}_n \mathrm{e}_n$ *be a formal Fourier series and let* σ_N, $N \in \mathbb{N}$ *be the corresponding Cesáro means. Then:*

[2]For example, the series $\sum_{n\ge 2} \frac{\sin(n\theta)}{\log(n)}$ is not a Fourier series of an L^1 function

(i) *$\mu = f d\theta$ with $f \in L^p(\mathbb{T})$ for some $p \in (1,\infty]$ if and only if $\sup_N \|\sigma_N\|_{L^p} < \infty$,*

(ii) *$\mu = f d\theta$ with $f \in L^1(\mathbb{T})$ if and only if σ_N converges in the L^1 norm,*

(iii) *$\mu = f d\theta$ with $f \in C(\mathbb{T})$ if and only if σ_N converges uniformly,*

(iv) *$\mu \in \mathcal{M}(\mathbb{T})$ if and only if $\sup_N \|\sigma_N\|_{L^1} < \infty$,*

(v) *$\mu \in \mathcal{M}^+(\mathbb{T})$ if and only if $\sigma_N \geq 0$ for all $N \in \mathbb{Z}_+$.*

A.2. Hardy Spaces

A.2.1. Harmonic Functions: The Poisson Kernel. Let $f \in L^1(\mathbb{T})$. Consider *the Abel means* of its formal Fourier series

$$f_r(\theta) := \sum_{n\in\mathbb{Z}} \hat{f}_n r^{|n|} \mathrm{e}^{\mathrm{i}n\theta}, \qquad r \in (0,1). \tag{A.13}$$

Clearly, $f_r \in \mathcal{A}(\mathbb{T})$ for all $r \in (0,1)$. Moreover, by Lemma A.1.1(i)

$$f_r = P_r * f, \tag{A.14}$$

where

$$P_r(\theta) := \sum_{n\in\mathbb{Z}} r^{|n|} \mathrm{e}^{\mathrm{i}n\theta} = \mathrm{Re}\Big(\frac{1+r\mathrm{e}^{\mathrm{i}\theta}}{1-r\mathrm{e}^{\mathrm{i}\theta}}\Big) = \frac{1-r^2}{1-2r\cos(\theta)+r^2} \tag{A.15}$$

is *the Poisson kernel*.

Lemma A.2.1. *The family P_r, $r \in (0,1)$ is an approximate identity.*

Therefore, one can immediately formulate the analogs of Theorem A.1.7 and Theorem A.1.11 for the Abel means of a Fourier series. We left that as an exercise.

Theorem A.2.2 (Fatou)**.** *Let $\mu \in \mathcal{M}(\mathbb{T})$. If $\mu = \frac{1}{2\pi} f d\theta + \mu_s$, where μ_s is a singular measure, then the following limit $\lim_{r\to 1}(P_r * \mu)(\theta)$ exists and equals $f(\theta)$ for almost all $\theta \in \mathbb{T}$.*

In particular, if μ is absolutely continuous, $\mu = \frac{1}{2\pi} f d\theta$, and $\theta_0 \in \mathbb{T}$ is a Lebesgue point of f, then $\lim_{r\to 1}(P_r * f)(\theta) = \lim_{r\to 1} f_r(\theta) = f(\theta)$.

Now let us consider the function $f(r\mathrm{e}^{\mathrm{i}\theta}) := f_r(\theta)$ as a function of a complex variable $z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$. Since $r^{|n|}\mathrm{e}^{\mathrm{i}n\theta}$ is harmonic in $\mathbb{D}$ and the series in (A.13) converges on compact subsets of $\mathbb{D}$, the function $f(r\mathrm{e}^{\mathrm{i}\theta})$ is harmonic in $\mathbb{D}$. Hence *the Poisson integral* $P_r * f$ provides an extension of $f(\theta) =: f(\mathrm{e}^{\mathrm{i}\theta})$ from the circle $\mathbb{T} = \partial\mathbb{D}$ to a harmonic function $f(r\mathrm{e}^{\mathrm{i}\theta})$ in the disc $\mathbb{D}$. On the other hand, we infer from Fatou's Theorem that $f(r\mathrm{e}^{\mathrm{i}\theta})$ is a Poisson integral of its boundary values:

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi}\int_{\mathbb{T}} P_r(\theta - t) f(\mathrm{e}^{\mathrm{i}t})dt. \tag{A.16}$$

Let us state the analog of Theorem A.1.11 for functions harmonic in $\mathbb{D}$.

Theorem A.2.3. *Let u be a function harmonic in the unit disc $\mathbb{D}$. Set $u(r,\theta) := u(r\mathrm{e}^{\mathrm{i}\theta})$*

(i) *if $u \in C(\overline{\mathbb{D}})$, then $u(r\mathrm{e}^{\mathrm{i}\theta}) = (P_r * f)(\theta)$ for all $r \in (0,1)$, where $f(\mathrm{e}^{\mathrm{i}\theta}) := u(\mathrm{e}^{\mathrm{i}\theta})$,*

(ii) *$u(r\mathrm{e}^{\mathrm{i}\theta}) = (P_r * f)(\theta)$ with some $f \in L^p(\mathbb{T})$, $p \in (1,\infty]$ if and only if $\sup_{r\in(0,1)} \|u(r,\cdot)\|_{L^p} < \infty$,*

(iii) *$u(r\mathrm{e}^{\mathrm{i}\theta}) = (P_r * \mu)(\theta)$ with some $\mu \in \mathcal{M}(\mathbb{T})$ if and only if*

$$\sup_{r\in(0,1)} \|u(r,\cdot)\|_{L^1} < \infty,$$

(iv) *u is positive in $\mathbb{D}$ if and only if $u(r\mathrm{e}^{\mathrm{i}\theta}) = (P_r * \mu)(\theta)$, where μ is a positive Borel measure on $\mathbb{T}$, $\mu \in \mathcal{M}^+(\mathbb{T})$.*

A.2.2. H^p Spaces. Let f be a function analytic in the unit disc $\mathbb{D}$, i.e., it is the sum of a convergent power series

$$f(z) = \sum_{n\geq 0} f_n z^n, \qquad z \in \mathbb{D}. \tag{A.17}$$

In polar coordinates, $z = r\mathrm{e}^{\mathrm{i}\theta}$, we can rewrite the above series as follows

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\geq 0} f_n r^n \mathrm{e}^{\mathrm{i}n\theta}. \tag{A.18}$$

The results from the previous subsection on harmonic functions clearly apply to analytic functions.

Definition A.2.4. If $p \in (0,\infty]$, we denote by $H^p(\mathbb{D})$ the *Hardy space* of analytic functions in the disc $\mathbb{D}$ such that the functions $f_r(\theta) = f(r\mathrm{e}^{\mathrm{i}\theta})$, $r \in (0,1)$ are uniformly bounded in the L^p norm, i.e.,

$$\|f\|_{H^p} := \sup_{r\in(0,1)} \|f_r\|_{L^p} = \sup_{r\in(0,1)} \left(\frac{1}{2\pi}\int_{\mathbb{T}} \left|f(r\mathrm{e}^{\mathrm{i}\theta})\right|^p d\theta\right)^{1/p} < \infty. \tag{A.19}$$

In fact, it turns out that $\|f_r\|_{L^p}$ is increasing as $r \to 1$ and hence $\|f\|_{H^p} = \lim_{r\to 1}\|f_r\|_{L^p}$. By Theorem A.2.3, if $p \in (1,\infty]$, then the space $H^p(\mathbb{D})$ can be identified with the subspace H^p of $L^p(\mathbb{T})$, which consists of functions $f \in L^p(\mathbb{T})$ such that their Poisson integral is analytic in $\mathbb{D}$, i.e., all $f \in L^p(\mathbb{T})$ such that $\hat{f}_{-n} = 0$ for all $n \in \mathbb{N}$. When $p = 1$ we obtain an identification of $H^1(\mathbb{D})$ with the subspace H^1 of analytic measures, i.e., $\mu \in H^1$ if $\hat{\mu}_{-n} = 0$

for all $n \in \mathbb{N}$. However, in contrast to harmonic functions, according to the theorem of F. and M. Riesz the space H^1 is a subspace of L^1!

Theorem A.2.5 (F. and M. Riesz)**.** *If $\mu \in \mathcal{M}(\mathbb{T})$ is analytic, $\hat{\mu}_{-n} = 0$ for all $n \in \mathbb{N}$, then $d\mu = \frac{1}{2\pi} f d\theta$ and $|\{f(\mathrm{e}^{\mathrm{i}\theta}) = 0 \,|\, \theta \in \mathbb{T}\}| = 0$, i.e., μ and the Lebesgue measure are equivalent on $\mathbb{T}$.*

In particular, the theorem of Riesz brothers implies that $f \equiv 0$ whenever $f \in H^1$ and $f = 0$ on a subset of a positive Lebesgue measure. One can say even more.

Theorem A.2.6 (Szegö–F. Riesz)**.** *If $f \in H^1(\mathbb{D})$, then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log\left(|f(\mathrm{e}^{\mathrm{i}\theta})|\right) d\theta > -\infty. \tag{A.20}$$

A.2.3. Factorization for H^p Functions. Let $f \in H^1(\mathbb{D})$ be a non-zero function. Then f has non-tangential limits at almost every point of the unit circle,

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \lim_{z \to \mathrm{e}^{\mathrm{i}\theta}} f(z),$$

and f is a Poisson integral its of boundary values,

$$f(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_{\mathbb{T}} P_r(\theta - t) f(\mathrm{e}^{\mathrm{i}t}) dt.$$

Moreover, f satisfies (A.20) and hence we can define the following function

$$F(z) = \exp\left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \log\left(|f(\mathrm{e}^{\mathrm{i}\theta})|\right) d\theta\right) \tag{A.21}$$

for all $z \in \mathbb{D}$. Clearly, F is analytic in $\mathbb{D}$ and $|F(z)| = \mathrm{e}^{u(z)}$, where u is a Poisson integral of $\log |f(\mathrm{e}^{\mathrm{i}\theta})|$. The later implies that $F \in H^1(\mathbb{D})$ and $|F| = |f|$ almost everywhere on $\mathbb{T}$. Moreover, F has no zeros in $\mathbb{D}$ and hence $\log(|F|)$ is a harmonic function in $\mathbb{D}$,

$$\log\left(|F(r\mathrm{e}^{\mathrm{i}\theta})|\right) = (\log(|F|) * P_r)(\theta) = (\log(|f|) * P_r)(\theta). \tag{A.22}$$

Thus, applying Jensen's inequality[3], we get $|F(z)| \geq |f(z)|$ for all $z \in \mathbb{D}$. Therefore, the function

$$G(z) := \frac{f(z)}{F(z)}, \qquad z \in \mathbb{D}, \tag{A.23}$$

is *an inner function*, that is, G is analytic in $\mathbb{D}$ and satisfies the following conditions

$$|G(z)| \leq 1, \quad z \in \mathbb{D}; \quad |G(\mathrm{e}^{\mathrm{i}\theta})| = 1 \ \text{ a. e. on } \mathbb{T}. \tag{A.24}$$

[3]Let $\mu \in \mathcal{M}^+(\mathbb{T})$ be a probability measure on $\mathbb{T}$, $\mu(\mathbb{T}) = 1$, and let φ be a convex function on $\mathbb{T}$. Then $\varphi\Big(\int_{\mathbb{T}} f d\mu\Big) \leq \int_{\mathbb{T}} \varphi(f) d\mu$ for any real-valued function $f \in L^1(\mathbb{T})$.

The function $F \in H^1(\mathbb{D})$ is called *an outer function* if there is a positive function $k \in L^1(\mathbb{T})$ and a number $\lambda \in \mathbb{T}$ such that

$$F(z) = \lambda \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} k(\theta) d\theta\right), \quad z \in \mathbb{D}. \tag{A.25}$$

Lemma A.2.7. *Let $F \in H^1(\mathbb{D})$, $F \not\equiv 0$. Then the following are equivalent:*

(i) *F is an outer function,*

(ii)
$$\log\big(|F(0)|\big) = \frac{1}{2\pi}\int_{\mathbb{T}} \log\big(|F(e^{i\theta})|\big) d\theta, \tag{A.26}$$

(iii) *if $f \in H^1(\mathbb{D})$ satisfies $|f| = |F|$ a.e. on $\mathbb{T}$, then $|f(z)| \le |F|(z)$ for all $z \in \mathbb{D}$.*

Summarizing, we see that every non-zero function $f \in H^1(\mathbb{D})$ admits the factorization $f = FG$, where F is outer and G is an inner function. Moreover, this factorization is unique up to a multiplicative constant $\lambda \in \mathbb{T}$. Note also that $f \in H^1 \cap H^p$ with some $p > 1$ if and only if $F \in H^p$. Thus we get the following useful result.

Corollary A.2.8. *Let $f \in H^1(\mathbb{D})$. Then $f = f_1 f_2$, where $f_j \in H^2(\mathbb{D})$ and $\|f_j\|_{H^2} \le \|f\|_{H^1}$, $j = 1, 2$.*

Proof. Since $f = FG$, we can set $f_1 = F^{1/2}$ and $f_2 = GF^{1/2}$. □

Our next aim is to show that every inner function can be factored into a product of two more specialized inner functions.

Lemma A.2.9. *Let $f \in H^\infty(\mathbb{D})$. Then the the sequence of its zeros $\{z_k\} \subset \mathbb{D}$ (counting multiplicities and arranged such that $|z_1| \le |z_2| \le \dots$) satisfies* the Blaschke condition

$$\sum_k (1 - |z_k|) < \infty. \tag{A.27}$$

Let us form *the Blaschke product*[4]

$$B(z) = z^p \prod_{z_k \neq 0} \frac{z_k^*}{|z_k|} \frac{z_k - z}{1 - z_k^* z}, \quad z \in \mathbb{D}. \tag{A.28}$$

Clearly, if $\{z_k\}$ is a finite sequence, then B is a rational inner function since so is each of its multiple.

Lemma A.2.10. *Let $\{z_k\} \subset \mathbb{D}$ be a sequence of nonzero numbers. Then the Blaschke product (A.28) converges locally uniformly in $\mathbb{D}$ if and only if $\{z_k\}$ satisfies the Blaschke condition (A.27). In this case B is an inner function whose zeros are $\{z_k\}$.*

[4] Here $p = \#\{k \,|\, z_k = 0\}$ is the multiplicity of $z = 0$ as the zero of B.

Lemma A.2.10 implies that every $f \in H^\infty(\mathbb{D})$ admits a unique factorization

$$f(z) = B(z)S(z), \qquad z \in \mathbb{D}, \tag{A.29}$$

where B is a Blaschke product (A.28) and $S \in H^\infty(\mathbb{D})$ has no zeros in $\mathbb{D}$. In particular, if f is an inner function, then S is an inner function without zeros. An example of an inner function without zeros is the function

$$S(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}+z} d\mu(\theta)\right), \tag{A.30}$$

where $\mu \in \mathcal{M}^+(\mathbb{T})$ is a singular measure. The function (A.30) is called *a singular inner function*. It turns out that all inner functions that have no zeros in $\mathbb{D}$ has the form (A.30). Thus we arrive at the following factorization of H^1 functions.

Theorem A.2.11. *If $f \in H^1(\mathbb{D})$ is a non-zero function, then f admits a unique (up to a constant multiple $\lambda \in \mathbb{T}$) representation*

$$f(z) = \lambda B(z)S(z)F(z), \tag{A.31}$$

where B is a Blaschke product, S is a singular inner function, and F is an outer function.

We complete this subsection with a description of invariant subspaces of the shift operator

$$\begin{array}{rccc} \mathcal{S}\colon & \ell_2 & \to & \ell_2 \\ & e_n & \mapsto & e_{n+1}. \end{array}$$

Note that $\mathcal{S}$ can be identified with the multiplication operator on $H^2(\mathbb{D})$,

$$(Sf)(z) := zf(z). \tag{A.32}$$

A (closed) subspace $\mathcal{H} \subseteq H^2(\mathbb{D})$ is called an invariant subspace of $\mathcal{S}$, $\mathcal{H} \in Lat(\mathcal{S})$, if $\mathcal{S}f \in \mathcal{H}$ for every $f \in \mathcal{H}$.

Theorem A.2.12 (Beurling)**.** *If $\mathcal{H} \in Lat(S)$ and $\mathcal{H} \neq \{0\}$, then there is a unique (up to a constant multiple $\lambda \in \mathbb{T}$) inner function G such that*

$$\mathcal{H} = GH^2 := \{Gf \mid f \in H^2(\mathbb{D})\}. \tag{A.33}$$

Corollary A.2.13. *A function $F \in H^2(\mathbb{D})$ is outer if and only if the set $\{Ff \mid f \in \mathrm{Pol}_+\}$ is dense in $H^2(\mathbb{D})$.*

A.3. A Conjugate Function and the Hilbert Transform

Let f be a function analytic in the disc $\mathbb{D}$. Consider its real and imaginary parts

$$u(z) := f(z) + f(z)^*, \qquad v(z) := -i(f(z) - f(z)^*).$$

Then the Taylor series expansion (A.17) implies

$$u_r(\theta) = u(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} c_n r^{|n|}\mathrm{e}^{in\theta}, \quad c_n = \begin{cases} f_n, & n > 0, \\ 2\mathrm{Re} f_0, & n = 0, \\ f_{-n}^*, & n < 0, \end{cases} \tag{A.34}$$

and

$$v_r(\theta) = v(r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} d_n r^{|n|}\mathrm{e}^{in\theta}, \quad d_n = -\mathrm{i}\begin{cases} f_n, & n > 0, \\ 2\mathrm{Im} f_0, & n = 0, \\ -f_{-n}^*, & n < 0. \end{cases} \tag{A.35}$$

Both functions are harmonic in $\mathbb{D}$ and their sum is an analytic function. This suggests the following definition.

Definition A.3.1. (i) Let $f \in L^1(\mathbb{T})$ and let $\hat{f}_n$, $n \in \mathbb{Z}$ be its Fourier coefficients. The Fourier series

$$\tilde{f} \sim -\mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\hat{f}_n \mathrm{e}^{in\theta} \tag{A.36}$$

is called *a conjugate Fourier series* and the corresponding function $\tilde{f}$ is called *a (formal) conjugate function*.

(ii) If $f(r\mathrm{e}^{\mathrm{i}\theta}) = (P_r * f)(\theta)$ is a Poisson transform of $f \in L^1(\mathbb{T})$, then the function

$$\tilde{f}(r\mathrm{e}^{\mathrm{i}\theta}) := -\mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\hat{f}_n r^{|n|}\mathrm{e}^{in\theta} \tag{A.37}$$

is called *a harmonic conjugate of f*.

First of it is not at all clear whether a conjugate Fourier series converges and in what sense. Concerning a harmonic conjugate, take a look at the kernel

$$Q_r(\theta) := -\mathrm{i}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) r^{|n|}\mathrm{e}^{in\theta} = \frac{2r\sin(\theta)}{1 - 2r\cos(\theta) + r^2} = \mathrm{Im}\Big(\frac{1 + r\mathrm{e}^{\mathrm{i}\theta}}{1 - r\mathrm{e}^{\mathrm{i}\theta}}\Big), \tag{A.38}$$

which is called *the conjugate Poisson kernel* (normalized by the condition $Q(0,\theta) = \mathrm{sgn}(0) = 0$). It is not difficult to check that the family Q_r is not an approximate identity ($Q_r(\theta) = -Q_r(-\theta)$ and $\|Q_r\|_1 \sim -\log(1-r)$) and hence we cannot apply the previous results. However, it turns out that a harmonic conjugate has non-tangential limits almost everywhere on $\mathbb{T}$.

Lemma A.3.2. *Let $f \in L^1(\mathbb{T})$ and let $\tilde{f}$ be its harmonic conjugate defined by (A.35). Then $\tilde{f}(\mathrm{e}^{\mathrm{i}\theta}) := \lim_{r\to 1} \tilde{f}(r\mathrm{e}^{\mathrm{i}\theta})$ exists for a.a. $\theta \in \mathbb{T}$.*

This lemma enables us to define *a harmonic conjugate* of $f \in L^1(\mathbb{T})$ as the boundary values $\tilde{f}(e^{i\theta})$ of its harmonic conjugate (A.35). In particular, if (A.34) is a Fourier series of some function $g \in L^1(\mathbb{T})$, then the corresponding harmonic conjugate (A.34) is a Poisson integral of g and hence $\tilde{f}(e^{i\theta}) = g(\theta)$ almost everywhere on $\mathbb{T}$. Thus this new definition of a conjugate function extends the definition of a formal conjugate function.

For $1 < p < \infty$, $S_N f$ approximate $f \in L^p(\mathbb{T})$ in the L^p norm as $N \to \infty$, and hence it is straightforward to show that a conjugate function $\tilde{f}$ of $f \in L^p(\mathbb{T})$ belongs to $L^p(\mathbb{T})$ if $1 < p < \infty$. However, in view of Lemma A.1.4, we cannot define a harmonic conjugate of $f \in L^1(\mathbb{T})$ in this way[5].

Theorem A.3.3 (M. Riesz)**.** *For $p \in (1,\infty)$, the mapping $f \mapsto \tilde{f}$ is a bounded map on $L^p(\mathbb{T})$. If $f \in L^1(\mathbb{T})$, then the conjugate function $\tilde{f}$ belongs to the weak L^1 space.*

Remark A.3.4. (i) It is immediate from the Parseval identity that for $f \in L^2(\mathbb{T})$,

$$\|\tilde{f}\|_{L^2}^2 = \|f\|_{L^2}^2 - |\hat{f}_0|^2. \tag{A.39}$$

(ii) Consider the Riesz projection P_+, the operator which transforms a Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}_n e_n$ into $\sum_{n\geq 0} \hat{f}_n e_n$, i.e., P_+ discards the $\hat{f}_n$ for $n < 0$. Clearly, P_+ considered on $L^2(\mathbb{T})$ is an orthogonal projection onto $H^2(\mathbb{T})$. Notice that

$$P_+ f = \frac{1}{2}(f + i\tilde{f}) + \frac{1}{2}\hat{f}_0. \tag{A.40}$$

Therefore, in any norm under which the linear functional $f \mapsto \hat{f}_0$ is continuous, the Riesz projection is bounded if and only if the conjugation operator is bounded. In particular, P_+ is bounded on $L^p(\mathbb{T})$ whenever $p \in (1,\infty)$. However, it is unbounded when either $p = 1$ or $p = \infty$!

Now our strategy would be to investigate the non-tangential limits of harmonic conjugates and then to establish a connection between boundary values of a harmonic conjugate (A.35) and a conjugate Fourier series (A.34). Since the limit

$$Q_1(\theta) := \lim_{r\to 1} Q_r(\theta) = \frac{\sin(\theta)}{1-\cos(\theta)} = \cot(\theta/2), \quad \theta \in \mathbb{T}, \tag{A.41}$$

is so explicit, it would be natural to try to change the order of operations in Lemma A.3.2 and to write $\tilde{f} = Q_1 * f$. The difficulty is that $Q_1 \notin L^1$ and

[5] For instance, $\sum_{n\geq 2} \frac{\cos(\theta)}{\log(n)}$ is a Fourier series of an L^1 function, however, its conjugate $\sum_{n\geq 2} \frac{\sin(\theta)}{\log(n)}$ is not!

hence one cannot define the convolution in a straightforward way. However, the following result is true.

Theorem A.3.5. *Let $f \in L^1(\mathbb{T})$. Then for almost all $\theta \in \mathbb{T}$ the limit*

$$\begin{aligned}(Hf)(\theta) &= P.V. \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta - t)\cot(t/2)dt \\ &:= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon}^{2\pi-\varepsilon} f(\theta - t)\cot(t/2)dt,\end{aligned} \tag{A.42}$$

exists and, moreover, $(Hf)(\theta) = \tilde{f}(\mathrm{e}^{\mathrm{i}\theta})$.

The mapping H is called *the Hilbert transform*.

A.4. BMO and VMO

Let $f \in L^1(\mathbb{T})$. We shall say that $f \in BMO(\mathbb{T})$, the space of functions of *bounded mean oscillation*, if

$$\sup_I \frac{1}{|I|}\int_I |f - f_I|d\theta =: \|f\|_* < \infty. \tag{A.43}$$

Here I is any arc on $\mathbb{T}$, $|I| = \int_I d\theta$ and

$$f_I := \frac{1}{2\pi}\int_I f d\theta \tag{A.44}$$

is the average of f over I.

Exercise A.4.1.

(i) $\|f\|_* = 0$ if and only if $f \equiv const$ on $\mathbb{T}$.

(ii) Show that $\|\cdot\|_*$ is a semi-norm.

Hence we can make BMO into a normed space by defining on it the norm

$$\|f\|_{BMO} = \|f\|_* + |f_{\mathbb{T}}|. \tag{A.45}$$

If $f \in L^\infty(\mathbb{T})$, then by the Cauchy–Schwarz inequality

$$\|f\|_* \le \sup_I \left(\frac{1}{|I|}\int_I |f - f_I|^2 d\theta\right)^{1/2} \le \sup_I \left(\frac{1}{|I|}\int_I |f|^2 d\theta\right)^{1/2} \le \|f\|_\infty,$$

and hence $f \in BMO(\mathbb{T})$. In particular,

$$\|f\|_* \le \inf_{c\in\mathbb{C}} \|f - c\|_\infty.$$

However, the inclusion $L^\infty/\mathbb{C} \subset BMO$ is proper, that is, BMO contains unbounded functions.

Exercise A.4.2.

(i) Show that $\log|\theta - \pi|$ is in BMO.

(ii) Show that $\log|\theta-\pi|\cdot \mathbb{1}_{(0,\pi)}(\theta)\notin BMO(\mathbb{T})$.

The space BMO was introduced by F. John and L. Nirenberg in 1961. The *John–Nirenberg inequality* provides a characterization of functions in BMO in terms of their distribution functions.

Theorem A.4.1 (John–Nirenberg)**.** *If $f\in BMO(\mathbb{T})$, then for every arc $I\subset\mathbb{T}$ and $\lambda>0$*

$$|\{\theta\in\mathbb{T}\,|\,|f(\theta)-f_I|>\lambda\}\cap I|\le K|I|\mathrm{e}^{-\frac{\lambda k}{\|f\|_*}}, \tag{A.46}$$

where K, $k>0$ are absolute constants.

Conversely, if $f\in L^1(\mathbb{T})$ and for every arc I there is $c_I\in\mathbb{C}$ such that

$$|\{\theta\in\mathbb{T}\,|\,|f(\theta)-c_I|>\lambda\}\cap I|\le K|I|\mathrm{e}^{-\lambda k} \tag{A.47}$$

with some constants K, $k>0$ independent of I, then $f\in BMO(\mathbb{T})$ and $\|f\|_*\le 2K/k$.

Remark A.4.2. The John–Nirenberg inequality shows that the distribution function of $|f-f_I|$ is not worse than the distribution of the logarithm. The papers [**34, 37**] discuss sharp constants in the John–Nirenberg inequality.

One of the first results establishing a deep connection between BMO and the Hilbert transform is due to S. Spanne and E. Stein.

Theorem A.4.3 (Spanne–Stein)**.** *If $f\in L^\infty(\mathbb{T})$, then $\tilde{f}\in BMO$.*

The following remarkable result of C. Fefferman shows that BMO can be characterized in terms of the Hilbert transform.

Theorem A.4.4 (Fefferman)**.** *Let $f\in L^\infty(\mathbb{T})$. Then the following conditions are equivalent:*

(i) *$f\in BMO(\mathbb{T})$,*
(ii) *$f=u+\tilde{v}$, where u, $v\in L^\infty(\mathbb{T})$,*
(iii) *the measure $|\nabla f(z)|^2(1-|z|)dxdy$ is a Carleson measure on $\mathbb{D}$*[6]*.*

Remark A.4.5. Theorem A.4.4 is called *the Fefferman duality theorem* and sometimes it is informally stated as follows: *BMO is the dual of H^1*. In fact, $BMO_{\mathbb{R}}$ is the dual of $H^1_{\mathbb{R}}$.

We also need the following space of functions, which was introduced by D. Sarason in the 1970s.

[6] Let μ be a positive finite measure on $\mathbb{D}$. If $\sup_{r\in(0,1),\,\theta\in\mathbb{T}} r^{-1}\mu(B_r(\mathrm{e}^{\mathrm{i}\theta})\cap\mathbb{D})<\infty$, then μ is called *a Carleson measure*. Carleson measures play an important role in analysis because they allow to answer the question whether the embedding of $L^2(\mathbb{T})$ into $L^2(\mathbb{D},\mu)$ is bounded or not (*the Carleson embedding theorem*)

Definition A.4.6. The BMO function f is said to have *vanishing mean oscillation*, $f \in VMO(\mathbb{T})$ if

$$\lim_{\varepsilon \to 0} \sup_{|I|<\varepsilon} \frac{1}{|I|} \int_I |f - f_I| d\theta = 0. \tag{A.48}$$

Clearly, $C(\mathbb{T}) \subset VMO(\mathbb{T})$. On another hand, any $f \in BMO(\mathbb{T})$ having a jump discontinuity on $\mathbb{T}$ does not belong to VMO. However, the are unbounded and discontinuous function on $\mathbb{T}$ that belong to VMO. It is immediate to verify that VMO is a closed subspace of BMO and hence it contains the closure of $C(\mathbb{T})$ with respect to the BMO norm. It turns out that this closure coincides with VMO and the relation between BMO and VMO is similar to that of $L^\infty(\mathbb{T})$ and $C(\mathbb{T})$.

Theorem A.4.7 (Sarason)**.** *Let $f \in BMO(\mathbb{T})$. Then the following are equivalent:*

(i) $f \in VMO(\mathbb{T})$,

(ii) *there is a sequence* $\{f_n\} \in C(\mathbb{T})$ *such that* $\|f - f_n\|_* \to 0$ *as* $n \to \infty$,

(iii) $f = u + \tilde{v}$, *where* u, $v \in C(\mathbb{T})$.

Notes and comments on the literature:

Here I would like to document sources from which I have learned the material and which I have used during the preparation of this text. General references for Appendix are the monographs [**6**, **14**, **18**, **21**, **30**]. As a reference for general background on Fourier series I can recommend Katznelson's classical book [**18**] and the recent book by Muscalu and Schlag [**21**]. Concerning Hardy spaces, I would recommend the classics by Hoffman [**14**] and Garnett [**6**]. The books of Hoffman [**14**] and Nikolski [**22**] give a comprehensive discussion of invariant subspaces of the shift operator. The material in Appendix A.4 is taken from [**6**] and [**30**].

Bibliography

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd Ltd, Edinburgh, London, 1965.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Spaces*, Moskow: Nauka, 1978.

[3] E. Amar and A. Bonami, *Mesures de Carleson d'order α et solutions au bord de l'equation $\overline{\partial}$*, Bull. Soc. Math. France **107**, 23–48 (1979).

[4] M. Sh. Birman and M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Spaces*, D. Reidel Publ., Kluwer, 1987.

[5] V. I. Bogachev, *Measure theory*, Vol. I, II, Springer-Verlag, Berlin, 2007.

[6] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[7] P. Gérard and S. Grellier, *The cubic Szegö equation*, Ann. Scient. Éc. Norm. Sup. **43**, 761–810 (2010).

[8] P. Gérard and S. Grellier, *Invariant tori for the cubic Szegö equation*, Invent. Math. **187**, 707–754 (2012).

[9] P. Gérard and S. Grellier, *Inverse spectral problems for compact Hankel operators*, J. Inst. Math. Jussieu **13**, 273–301 (2014).

[10] P. Gérard and S. Grellier, *The Cubic Szegö Equation and Hankel Operators*, Astérisque **389**, Soc. Math. France, 2017.

[11] P. Gérard and A. Pushnitski, *An inverse problems for self-adjoint positive Hankel operators*, Intern. Math. Res. Notices **2015**, 4505–4535 (2015).

[12] I. C. Gokhberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Am. Math. Soc., Providence, 1969.

[13] G. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge Univ. Press, 1934.

[14] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, 1962.

[15] P. Jones, *Extension theorems for BMO*, Indiana Univ. Math. J. **29**, 41–66 (1980).

[16] I. S. Kac and M. G. Kreĭn, *R–functions — analitic functions mapping the upper halfplane into itself* , Amer. Math. Soc. Transl. Ser. 2, **103** (1974), 1–18.

[17] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer-Verlag, Berlin-Heidelberg, New York, 1966.

[18] I. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publ., New York, 1976.

[19] E. Lieb and M. Loss, *Analysis*, Amer. Math. Soc., Providence, 1997.

[20] R. A. Martínez-Avendaño and S. R. Treil, *An inverse spectral problem for Hankel operators*, J. Oper. Theory **48**, 83–93 (2002).

[21] C. Muscalu and W. Schlag, *Classical and Multilinear Harmonic Analysis, Vol. I*, Cambridge Univ. Press, 2013.

[22] N. K. Nikolski, *Operators, Functions and Systems: An Easy Reading, Vol. I*, Amer. Math. Soc., Providence, 2002.

[23] F. W. J. Olver et al., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[24] S. Parrott, *On a quotient norm and the Sz.-Nagy–Foias lifting theorem*, J. Funct. Anal. **30**, 311–328 (1978).

[25] V. V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003.

[26] S. R. Power, *Hankel Operators on Hilbert Space*, Pitnam, Boston, 1982.

[27] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, rev. and enl. edition, Academic Press, San Diego, 1980.

[28] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[29] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, New York, 1987.

[30] D. Sarason, *Function Theory on the Unit Circle*, Notes for lectures at Virginia Polytechnic Ints., 1978.

[31] K. Schmüdgen, *Unbounded Self–Adjoint operators on Hilbert Space*, Springer, 2012.

[32] B. Simon, *Trace Ideals and Their Applications*, 2nd edn., Amer. Math. Soc., Providence, RI, 2005.

[33] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137**, 82–203 (1998).

[34] L. Slavin and V. Vasyunin, *Sharp results in the integral-form John–Nirenberg inequality*, Trans. Amer. Math. Soc. **363**, 4135–4169 (2011).

[35] T.-J. Stieltjes, *Recherches sur les Fractions Continues*, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **8**, no. 4, 1–122 (1894).

[36] S. R. Treil, *An inverse spectral problem for the modulus of the Hankel operator, and balanced realizations*, Leningrad Math. J. **2**, 353–375 (1991).

[37] V. Vasyunin and A. Volberg, *Sharp constants in the classical weak form of the John–Nirenberg inequality*, Proc. London Math. Soc. **108**, 1417–1434 (2014).

[38] J. Weidmann, *Lineare Operatoren in Hilberträumen I: Grundlagen*, B.G.Teubner, Stuttgart, 2000.

[39] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121**, no. 1, 1–35 (1966).

[40] D. R. Yafaev, *Criteria for Hankel operators to be sign-definite*, arXiv: 1303.4040.

[41] D. R. Yafaev, *Quasi-diagonalization of Hankel operators*, J. d'Anal. Math. (2014), to appear; arXiv: 1403.3941.

[42] D. R. Yafaev, *Quasi–Carleman operators and their spectral properties*, Integr. Equ. Oper. Theory **81**, 499–534 (2015); arXiv: 1404.6742.