Trace Ideals with Applications
(Lecture Notes)

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2010 *Mathematics Subject Classification.* Primary 47B10

Key words and phrases. Hilbert space; Linear operator; Compact operator; Schatten–von Neumann ideal

Abstract. These notes grew out from advanced courses given in University of Ljubljana in 2018 and Graz University of Technology in 2019. The main goal is to give a quick introduction into a rich theory of compact operators in Hilbert spaces, which has numerous applications in other branches of mathematics and mathematical physics.

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Preface

These notes grew out from advanced courses given in University of Ljubljana in 2018 and Graz University of Technology in 2019. The main goal is to give a quick introduction into a rich theory of compact operators in Hilbert spaces, which has numerous applications in other branches of mathematics and mathematical physics.

We begin with reminding the basic facts on linear operators in finite dimensional vector spaces and also quickly outlining the fundamental problems appearing when one tries to extend the results from finite to infinite-dimensional setting.

The second chapter deals with the general theory of compact operators in Banach and Hilbert spaces. First, we give the definition of a compact operator in an abstract Banach space and also provide some indications why the general theory of compact operators in Banach spaces is out of reach (e.g., Grothendieck's approximation property). Thus, for the rest of the notes we are focused on separable Hilbert spaces only. We then prove two main results: Fredholm's alternative, which might be familiar from the PDEs when solving the Dirichlet problem for elliptic equations in a bounded domain, and the canonical form of compact operators in Hilbert spaces, which can be thought of as a far reaching extension of the canonical Jordan form for matrices.

Chapter 3 is the main part of the notes, where we answer the question on how to define the trace and the determinant of a linear operator and, the most important, what is a reasonable class of linear operators on which we can define these notions. The fundamental role here is played by *singular numbers* and we begin the discussion by explaining their role in the the class of compact operators. Next, we introduce the trace class $\mathfrak{S}_1$ as the set of compact operators such that their singular numbers belong to ℓ^1. It turns out that the class $\mathfrak{S}_1(\mathfrak{H})$ of operators such that their singular values belong to ℓ^1 is the only reasonable class on which we can define a trace. Moreover, we show that as in the finite dimensional case, the trace can be defined as a matrix trace. The fact that the matrix trace coincides with the spectral trace is the content of the deep Lidskii theorem, which we prove in Section 3.4. The class $\mathfrak{S}_2$ of Hilbert–Schmidt operators is discussed in Section 3.3, where we also touch the problem of characterization of trace class integral operators. In contrast to the Hilbert–Schmidt class, which can be characterized as the set of integral operators with square summable kernels, trace class operators do not admit such a transparent characterization. There are manifold reasons for this (for instance, one may recall that L^2 functions admit a very transparent characterization in terms of their Fourier coefficients, which is very different for L^1 functions). Section 3.5 contains explicit formulas for the resolvent $(I + zA)^{-1}$. We finish this chapter with a brief discussion of regularized determinants.

In the final chapter we present several important applications of compact operators and trace ideals. The main source of such applications is quantum mechanics, where spectral theory of linear operators in Hilbert spaces plays the fundamental role. Our main focus is on the Schrödinger equation in $\mathbb{R}^n$. Section 4.1 deals with the bound state problem, that is, estimates on the number of negative eigenvalues of

$$-\Delta + V(x)$$

for sufficiently "small" potentials V. We prove the Birman–Schwinger principle and apply it to establish the Birman–Schwinger estimate in $\mathbb{R}^3$ as well as the Bargmann bound in $\mathbb{R}^1$. In Section 4.2, we briefly touch the scattering in 1D and show that the transmission coefficient is nothing but the Fredholm determinant of the corresponding Birman–Schwinger operator. The latter enables us to discuss the KdV equation

$$u_t = -u_{xxx} + 6uu_x,$$

probably the most studied nonlinear PDE. As it was observed in 1965 by Greene, Gardner, Kruskal and Miura, the direct and inverse scattering in 1D has deep connections with the KdV equation. We briefly outline some of those connections (e.g., Lax formalism, the inverse scattering transform and Dyson's formula) and finish the chapter with the recent beautiful derivation of asymptotic conserved quantities by R. Killip, M. Visan and X. Zhang [**27**], which is heavily based on the tools developed in Chapter 3.

In our notes we mostly follow the classical text [**50**] (see also the recent [**51**]). The familiarity with basic functional analysis is assumed.

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Chapter 1

Introduction

1.1. Linear operators in finite dimensional spaces

For a moment, let $\mathcal{V}$ be a finite dimensional linear *vector space* over a field $\mathcal{K}$ (for simplicity, one can always assume that either $\mathcal{V} = \mathbb{R}^n$ or $\mathcal{V} = \mathbb{C}^n$).

Definition 1.1.1. A *linear operator* $\mathcal{A}$ in a vector space $\mathcal{V}$ is a linear map of $\mathcal{V}$ into itself.

Recall that a map A is called *linear* if

(i) $\mathcal{A}(f+g) = \mathcal{A}f + \mathcal{A}g$ for all $f, g \in \mathcal{V}$, and

(ii) $\mathcal{A}(zf) = z\,\mathcal{A}f$ for all $f \in \mathcal{V}$ and $z \in \mathcal{K}$.

Remark 1.1.1. More generally, a linear map $\mathcal{A}$ between vector spaces $\mathcal{V}$ and $\mathcal{U}$ is a linear map of $\mathcal{V}$ to $\mathcal{U}$. However, in this section we restrict to the case $\mathcal{V} = \mathcal{U}$.

If $\{e_1, \dots, e_n\}$ is a basis in $\mathcal{V}$, then $\mathcal{A}$ can be defined as a matrix.

Definition 1.1.2. A *matrix of a linear operator* $\mathcal{A}$ in a basis $\{e_1, \dots, e_n\}$ is called a matrix $A = (a_{ij})$ defined by the following equations

$$\mathcal{A}e_j = \sum_i a_{ij} e_i. \tag{1.1.1}$$

Notice that (1.1.1) can be written as

$$\big(\mathcal{A}e_1, \dots, \mathcal{A}e_n\big) = \big(e_1, \dots, e_n\big)A$$

One then can easily conclude that every linear operator is uniquely determined by its matrix. The converse is also true: every matrix defines a unique linear operator. However, a change in the basis may result in a change of the matrix representation, which is given by

$$\widetilde{A} = C^{-1}AC,$$

where C is the transformation matrix determined by

$$(\widetilde{e}_1, \dots, \widetilde{e}_n) = (e_1, \dots, e_n)C.$$

The next definition plays crucial role.

Definition 1.1.3. A subspace $\widetilde{\mathcal{V}} \subset \mathcal{V}$ is called an *invariant subspace* of an operator $\mathcal{A}$ if $\mathcal{A}f \in \widetilde{\mathcal{V}}$ for all $f \in \widetilde{\mathcal{V}}$,

$$\mathcal{A}\widetilde{\mathcal{V}} \subseteq \widetilde{\mathcal{V}}. \tag{1.1.2}$$

The set of all invariant subspaces of $\mathcal{A}$ is usually denoted by $\mathrm{Lat}(\mathcal{A})$.

The importance of invariant subspaces stems from the following observation. If in the basis $\{e_j\}_{j=1}^n$ the first k vectors are such that $\widetilde{\mathcal{V}} = \mathrm{span}\{e_1, \dots, e_k\}$ is invariant for $\mathcal{A}$, then the corresponding matrix has the form

$$A = \begin{pmatrix} B & C \\ \mathbb{O} & D \end{pmatrix}. \tag{1.1.3}$$

If in addition $\mathrm{span}\{e_{k+1}, \dots, e_n\}$ is also invariant, then $C = \mathbb{O}$ as well in the above representation. This gives the key to the main problem in the theory of linear operators, which consists in finding the simplest form of a given linear operator. Clearly, we would be able to do this once invariant subspaces are known. The most important role is played by one dimensional invariant subspaces:

Definition 1.1.4. A non-zero vector $f \in \mathcal{V}$ is called an *eigenvector* of an operator $\mathcal{A}$ if $\mathcal{A}f = \lambda f$ for some $\lambda \in \mathcal{K}$. In this case, λ is called an *eigenvalue* corresponding to the eigenvector f.

Clearly, f is an eigenvector if and only if $\mathrm{span}\{f\} \in \mathrm{Lat}(\mathcal{A})$. If the basis consists only of eigenvectors, then the corresponding matrix A is diagonal and the elements on its main diagonal are eigenvalues. This is the simplest of all possible forms. Unfortunately, it often happens that it is not possible to find such a basis (take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $\mathcal{V} = \mathbb{R}^2$).

On the other hand, λ is an eigenvalue if and only if equation $(A - \lambda)f = 0$ has a nontrivial solution, which happens exactly when

$$p_A(\lambda) := \det(\lambda - A) = 0.$$

p_A is called a *characteristic polynomial* of A. Notice that it does not depend on a choice of a basis (why?). Thus eigenvalues of A coincide with zeros of the characteristic polynomial p_A. In particular, every linear operator in a complex vector space has at least one eigenvector. If $\mathcal{V}$ is a real vector space, then A admits a nontrivial one or two dimensional invariant subspace. Now for simplicity we restrict ourselves to complex vector spaces. The above considerations lead to the *canonical (or Jordan) form* of linear operators in vector spaces. In the following, denote by $\mathcal{V}_\lambda(\mathcal{A}) := \ker(\mathcal{A}-\lambda)$ the eigenspace corresponding to the e.v. λ. It is easy to show that $\mathcal{V}_\lambda(\mathcal{A}) \cap \mathcal{V}_\mu(\mathcal{A}) = \{0\}$ if $\lambda \neq \mu$. If p_A has n distinct zeros, then all eigenvalues of A are distinct and hence there is a basis consisting of eigenvectors of A. However, this condition is only sufficient (take $A = I_n$; what is a criterion?). The crucial observation is the following: if $\widetilde{\mathcal{V}} \in \mathrm{Lat}(\mathcal{A})$ and $p_{\mathcal{A}|\widetilde{\mathcal{V}}}$ is the characteristic polynomial of the restriction of $\mathcal{A}$ onto $\widetilde{\mathcal{V}}$, then $p_{\mathcal{A}|\widetilde{\mathcal{V}}}$ divides $p_{\mathcal{A}}$. Therefore, (1.1.3) implies that $p_{\mathcal{A}} = p_B p_D$. Moreover, $\dim \mathcal{V}_\lambda$ is not greater than the multiplicity of the root of $p_{\mathcal{A}}$ (how is this related with teh diagonal representation). However, they are not necessarily equal: $p_A = \lambda^2$ for $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\dim\ker(A) = 1$.

Clearly, we have the chain of inclusions

$$\ker(\mathcal{A}-\lambda) \subseteq \ker(\mathcal{A}-\lambda)^2 \subseteq \ker(\mathcal{A}-\lambda)^3 \subseteq \dots$$

Since $\dim \mathcal{V} = n < \infty$, there is a minimal $p \geq 1$ such that $\ker(\mathcal{A}-\lambda)^p = \ker(\mathcal{A}-\lambda)^j$ for all $j \geq p$.

Definition 1.1.5. The subspace $\ker(\mathcal{A}-\lambda)^p$ is called the root subspace and is denoted by $\mathfrak{R}_\lambda(\mathcal{A})$. A non-zero vector $f \in \mathcal{V}$ is called a *root vector* of an operator $\mathcal{A}$ if $(\mathcal{A}-\lambda)^n f = 0$ for some $\lambda \in \mathcal{K}$ and $n \in \mathbb{N}$. In particular,

$$\mathfrak{R}_\lambda(\mathcal{A}) = \{f \in \mathcal{V} |\, (\mathcal{A}-\lambda)^n f = 0\}. \tag{1.1.4}$$

Now observe that $\mathfrak{R}_\lambda \in \mathrm{Lat}(\mathcal{A})$ and $p_{\mathcal{A}|\mathfrak{R}_\lambda}(z) = (z-\lambda)^p$. Noting that $\mathfrak{R}_\lambda \cap \mathfrak{R}_\mu = \{0\}$ if $\lambda \neq \mu$, we end up with the canonical form of the operator $\mathcal{A}$. Indeed, if λ_k, $k = 1, \dots, m$ are zeros of $p_{\mathcal{A}}$, then

$$\mathcal{V} = \oplus_{j=1}^m \mathfrak{R}_{\lambda_j}.$$

And the final step in this puzzle is the form of $\mathcal{A}$ in each subspace $\mathfrak{R}_\lambda$. The key notion is a *cyclic subspace*: $\widetilde{\mathcal{V}} \in \mathrm{Lat}(\mathcal{A})$ is called cyclic if $\widetilde{\mathcal{V}} = \mathrm{span}\{f, \mathcal{A}f, \mathcal{A}^2 f, \dots\}$ for some f. It turns out that every $\mathfrak{R}_\lambda$ can be decomposed as a direct sum of cyclic subspaces and in each cyclic subspace $\mathcal{A}$ acts as a Jordan block:

$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}.$$

1.2. Linear operators in (separable) Hilbert spaces

So far we considered finite dimensional vector spaces. **And now the question is what happens if** $\dim \mathcal{V} = \infty$**?** In orther words, *what are the properties of linear functions of infinitely many variables?*

First of all, in order to build up a reasonable theory, it is desirable to equip $\mathcal{V}$ with some additional structure. For example, let $\mathcal{V} = C([0,1])$. Consider $f_n := x^n$, $n \in \mathbb{Z}_{\geq 0}$. Clearly, this family of functions is linearly independent. Moreover, linear combinations (polynomials) are dense in $C([0,1])$ (by the Weierstrass theorem). But $\{f_n\}_{n\geq 0}$ is not a basis! (*Exercise: Explain why*). However, what is a basis actually? (Hamel basis, Schauder basis, ...)

Suppose, we have a linear operator in $\mathcal{V}$. When does it make sense to speak about matrix representation? This leads us to bounded linear operators.

We shall always assume that $\mathfrak{H}$ is a complex separable Hilbert space. All linear operators if not stated explicitly are assumed to be bounded.

Definition 1.2.1. A *linear operator* $\mathcal{A}$ in $\mathfrak{H}$ is called bounded if

$$\sup_{f\in \mathrm{dom}(\mathcal{A})\setminus\{0\}} \frac{\|\mathcal{A}f\|_{\mathfrak{H}}}{\|f\|_{\mathfrak{H}}} =: \|\mathcal{A}\|_{\mathfrak{H}} < \infty.$$

It is well known that $\mathcal{A}$ is bounded if and only if it is continuous. We shall always assume that $\mathcal{A}$ is closed, that is, $\mathrm{dom}(\mathcal{A}) = \mathfrak{H}$.

Q#1: What can we say about matrix representation?

Q#2: Does every bounded linear operator in $\mathfrak{H}$ have an eigenvalue? Invariant subspace?

Q#3: Is there the analog of the following alternative: *either* $\mathcal{A}x = y$ *has a unique solution for every* $y \in \mathcal{V}$ *or* $\mathcal{A}x = 0$ *has a nontrivial solution*?

Q#4: For (finite dimensional) matrices we have characteristic polynomials

$$p_{\mathcal{A}}(\lambda) = \lambda^n - \mathrm{tr}(\mathcal{A})\lambda^{n-1} + \cdots + (-1)^n \det(\mathcal{A}).$$

Are there the analogs of tr *and* det *for linear operators in* $\mathfrak{H}$*?*

Chapter 2

Compact operators in Hilbert spaces

2.1. Compact operators in Banach spaces

Let us start with an example. Many problems of analysis and classical mathematical physics can be handled by reformulating them in terms of integral equations (the most famous example is the Dirichlet problem for elliptic equations).

Example 2.1.1. Consider the operator K defined in $C([0,1])$ by

$$(Kf)(x) = \int_0^1 K(x,s)f(s)ds, \tag{2.1.1}$$

where the function (kernel) K is continuous on the square $0 \le x, y \le 1$. Since

$$\|Kf\|_C = \sup_{x\in[0,1]} |(Kf)(x)| \le \sup_{x,y} |K(x,y)| \sup_x |f| = \sup_{x,y} |K(x,y)| \cdot \|f\|_C,$$

the operator K is bounded on $C([0,1])$. However, K has another property which is very important. Let B_R be the set of continuous functions such that $\|f\|_C \le R$ (a ball of radius $R > 0$). Since $[0,1] \times [0,1]$ is compact, $K(\cdot,\cdot)$ is uniformly continuous, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|K(x,y) - K(x',y)| < \varepsilon$ for all $|x - x'| < \delta$. Thus, if $f \in B_R$, then

$$|(Kf)(x) - (Kf)(x')| \le \|f\|_C \sup_{y\in[0,1]} |K(x,y) - K(x',y)| \le \varepsilon\|f\|_C \le R\varepsilon,$$

and hence the functions from $K(B_R)$ are equicontinuous. Since this set is also bounded, we can use the Ascoli–Arzela theorem to conclude that for every sequence $\{f_n\} \subset B_R$, the sequence $\{Kf_n\}$ has a convergent subsequence, that is, the set $K(B_R)$ is pre-compact (its closure is compact in $C([0,1])$).

Clearly, $R > 0$ is not important in the above consideration and, in fact, we have shown that K *maps bounded sets into pre-compact sets*. It is this property which makes the so called "Fredholm alternative" hold for nice integral equations like (2.1.1). This section is devoted to studying such operators.

Definition 2.1.1. Let X and Y be Banach spaces. A linear operator $T\colon X \to Y$ is called *compact* if T takes bounded sets in X into precompact sets in Y.

Equivalently, T is compact if and only if for every bounded sequence $\{x_n\} \in X$, $\{Tx_n\}$ has a subsequence convergent in Y.

NOTE: *Sometimes (especially in the old texts) the word "completely continuous operators" was used for compact operators. However, now* completely continuous *has a different meaning! On the other hand, these notions coincide for operators in Hilbert spaces. For more details see* [**51**, Chapter 3.1].

The integral operator (2.1.1) is one example of a compact operator. Another class of examples is:

Example 2.1.2 (Finite rank operators)**.** Suppose that the range of T is finite dimensional. That is, every vector in the range of T can be written

$$Tf = \sum_{k=1}^{N} \alpha_k f_k$$

for some fixed family $\{f_k\}_{k=1}^N \in Y$. If $\{g_n\}$, is any bounded sequence in X, the corresponding $\alpha_{k,n}$ are bounded since T is bounded. The usual subsequence trick allows one to extract a convergent subsequence from $\{Tg_n\}$ which proves that T is compact. (Exercise: Provide the details!)

Exercise 2.1.1. Prove that $T\colon X \to Y$ is a finite rank operator if and only if there are vectors $\{\ell_n\}_{n=1}^N \in X^*$ and $\{\varphi_n\}_{n=1}^N \in Y$ such that

$$Tf = \sum_{k=1}^{N} \ell_n(f)\varphi_n \tag{2.1.2}$$

for all $f \in X$. Find the rank of F.

An important property of compact operators is given by next result:

Theorem 2.1.1. *A compact operator maps* weakly convergent *sequences into* norm convergent *sequences.*

Proof. Suppose f_n converges weakly to f in X. By the uniform boundedness theorem, $\|f_n\|$ are bounded. Let $g_n := Tf_n \in Y$ and $g := Tf \in Y$. Then for every $\ell \in Y^*$, $\ell(g_n) - \ell(g) = (T^\times \ell)(f_n - f)$, where $T^\times$ is the adjoint operator (see Definition 2.1.2 below). Thus g_n converges weakly to g in Y. Suppose that g_n does not converge to g in $\|\cdot\|_Y$. Then there is $\varepsilon > 0$ and a subsequence $\{g_{n_k}\}$ such that $\|g_{n_k} - g\|_Y \geq \varepsilon$ for all n_k. However, T is compact and $\{f_n\}$ is bounded, thus $\{Tf_{n_k}\}$ has a convergent subsequence, $Tf_{n_k} \to \widetilde{g} \neq g$. Then this subsequence converges also weakly, but this contradicts to $\widetilde{g} \neq g$. Thus $g_n \to g$ in Y. □

In the above proof we have used the notion of the *adjoint operator.*

Definition 2.1.2. Let X and Y be Banach spaces, T a bounded linear operator from X to Y. The *Banach space adjoint* of T, denoted by $T^\times$, is the bounded linear operator from Y^* to X^* defined by

$$(T^\times \ell)(f) := \ell(Tf) \tag{2.1.3}$$

for all $\ell \in Y^*$ and $f \in X$.

Notice that the map $T \to T^\times$ defines a Banach space isomorphism of $B(X,Y)$ to $B(Y^*, X^*)$.

Exercise 2.1.2. Find $T^\times$ of (2.1.2).

Remark 2.1.1. We note that if X is reflexive, then the converse of Theorem 2.1.1 holds (Prove this!). In particular, this holds true for operators in Hilbert spaces.

The following theorem is important since one can use it to prove that an operator is compact by exhibiting it as a norm limit of compact operators or as an adjoint of a compact operator.

Theorem 2.1.2. *Let $T\colon X \to Y$ be a bounded operator. If $\{T_n\}$ are compact and $T_n \to T$ in the norm topology, then T is compact.*

Proof. Let $\{f_m\}$ be a sequence in the unit ball of X. Since T_n is compact for each n, we can use the diagonalization trick to find a subsequence $\{f_{m_k}\}$ such that $T_n f_{m_k} \to g_n$ in Y for all n as $k \to \infty$. Since $\|f_n\| \leq 1$ and $\|T_n - T\| \to 0$ as $n \to \infty$, an $\varepsilon/3$-argument shows that $\{g_n\}$ is Cauchy, so $g_n \to g$. Indeed, choosing k large enough, we get

$$\|g_n - g_m\| = \|g_n - T_n f_{m_k} + T_n f_{m_k} - T_m f_{m_k} + T_m f_{m_k} - g_m\| \leq \varepsilon.$$

Applying an $\varepsilon/3$-argument once again, we get

$$\|Tf_{m_k} - g\| = \|Tf_{m_k} - T_n f_{m_k} + T_n f_{m_k} - T_m f_{m_k} + T_m f_{m_k} - g\| \le \varepsilon,$$

which shows that $Tf_{m_k} \to g$. Thus T is compact. □

As an immediate corollary we get

Corollary 2.1.1. *Let $T\colon X \to Y$ be a bounded operator. If $\{T_n\}$ are finite rank operators and $T_n \to T$ in the norm topology, then T is compact.*

Next we need the following important result.

Theorem 2.1.3. *Let $\mathfrak{H}$ be a separable Hilbert space. Then every compact operator in $\mathfrak{H}$ is the norm limit of a sequence of finite rank operators.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathfrak{H}$. Set

$$\lambda_n := \sup_{f\in \mathrm{span}\{e_1,\dots,e_n\}^\perp\colon \|f\|=1} \|Tf\|, \qquad n \ge 1.$$

Clearly, the sequence $\{\lambda_n\}$ is monotone decreasing and hence it converges to a non-negative limit $\lambda_0 \ge 0$. Let us show that $\lambda_0 = 0$. Choose a sequence $\psi_n \in \mathrm{span}\{e_1,\dots,e_n\}^\perp$ with $\|\psi_n\| = 1$ and $\|T\psi_n\| \ge \frac{1}{2}\lambda_0$. Observe that ψ_n converges weakly to 0. Indeed, each ψ_n admits the representation

$$\psi_n = \sum_{k>n} \alpha_{k,n} e_k,$$

where $\alpha_{k,n}$ are the Fourier coefficients, $\alpha_{k,n} = (f, e_n)_{\mathfrak{H}}$. Then we get by applying the Cauchy–Schwarz inequality

$$\begin{aligned} \big|(f,\psi_n)_{\mathfrak{H}}\big|^2 &= \Big|\sum_{k>n} \alpha_{k,n}(f,e_k)_{\mathfrak{H}}\Big|^2 \\ &\le \sum_{k>n} |\alpha_{k,n}|^2 \sum_{k>n} |(f,e_k)_{\mathfrak{H}}|^2 \\ &= \sum_{k>n} |(f,e_k)_{\mathfrak{H}}|^2 \end{aligned}$$

for each $f \in \mathfrak{H}$. However, the latter tends to 0 as $n \to \infty$, which proves the claim.

Next, by compactness (Theorem 2.1.1), $T\psi_n \to 0$ in $\mathfrak{H}$. Thus $\lambda_0 = 0$. It remains to note that

$$\|T - \sum_{k=1}^{n} (\cdot, e_k)Te_k\|_{\mathfrak{H}} = \lambda_n, \tag{2.1.4}$$

which implies that T is the norm limit of finite rank operators. □

Remark 2.1.2. Notice that Theorem 2.1.3 provides the converse to Theorem 2.1.2 in the case of Hilbert spaces. It turns out that Theorem 2.1.3 is not true for Banach spaces! A counterexample was constructed by P. Enflo [**13**]. More precisely, A. Grothendieck introduced in [**19**] the so-called *approximation property*: a Banach space is said to have the approximation property, if every compact operator is a limit of finite-rank operators. The converse is always true by Theorem 2.1.2(i). Also Grothendieck proved that *if every Banach space had the approximation property, then every Banach space would have a Schauder basis*. The latter is known as the basis problem of S. Banach and closely related to the "live goose" problem of S. Mazur[1]. In 1972, Enflo constructed a **separable** Banach space that lacks the approximation property and a Schauder basis thus solving three longstanding fundamental problems in functional analysis.

Exercise 2.1.3. Using the Weierstrass theorem, show that every integral operator $\mathcal{K}$ from Example 2.1.1 with a continuous kernel can be approximated by finite-rank operators.

2.2. Fredholm's alternative

Taking into account the above remark, we are mostly interested in the case where T is a compact operator from a separable Hilbert space to itself, so we will not pursue the general case any further. We denote the Banach space of compact operators on a separable Hilbert space $\mathfrak{H}$ by $\mathfrak{S}_\infty(\mathfrak{H})$ or simply $\mathfrak{S}_\infty$ if the corresponding Hilbert space is clear from the context (this notation will become clear later on). Notice that by Theorem 2.1.2(i) and Theorem 2.1.3, compact operators in a Hilbert space coincide with the closure of finite rank operators in the operator norm topology.

We have discussed a wide variety of properties of compact operators but we have not yet described any property which explains our special interest in them. The basic principle which makes compact operators important is the *Fredholm alternative* (perhaps, you might remember it from the PDEs course):

If A is compact, then either $Af = f$ has a solution or $(I - A)^{-1}$ exists.

This is not a property shared by all bounded linear transformations. For example, if A is the operator $(Af)(x) = xf(x)$ on $L^2([0,1])$, then $Af = f$ has no solutions in $L^2([0,1])$ (indeed, $1/x \notin L^2([0,1])$), but $(I - A)^{-1}$ does not exist (as a bounded operator). In terms of "solving equations" the Fredholm alternative is especially nice: *It tells us that if for any f there is at most one*

[1]On 6 November 1936, Stanislaw Mazur posed a problem on representing continuous functions. Formally writing down problem 153 in the Scottish Book, Mazur promised as the reward a "live goose", an especially rich prize during the Great Depression and on the eve of World War II.

g with $g = f + Ag$, then there is always exactly one. That is, compactness and uniqueness together imply existence!

Compactness combines very nicely with analyticity so we first prove an elegant result which is of great use in itself.

Theorem 2.2.1 (Analytic Fredholm theorem)**.** *Let $\mathcal{D}$ be an open connected subset of $\mathbb{C}$. Let $f\colon \mathcal{D} \to [\mathfrak{H}]$ be an analytic operator-valued function*[2] *such that $f(z) \in \mathfrak{S}_\infty(\mathfrak{H})$ for each $z \in \mathcal{D}$. Then, either*

(i) *$(I - f(z))^{-1}$ exists for no $z \in \mathcal{D}$,*
or

(ii) *$(I - f(z))^{-1}$ exists for all $z \in \mathcal{D} \setminus S$, where S is a discrete subset of $\mathcal{D}$ (i.e., no accumulation points in $\mathcal{D}$). In this case, $(I - f(z))^{-1}$ is meromorphic in $\mathcal{D}$, analytic in $\mathcal{D} \setminus S$, the residues at the poles are finite rank operators, and if $z \in S$, then $f(z)\psi = \psi$ has a nontrivial solution in $\mathfrak{H}$.*

Before proving this theorem let us first underline its importance. We are going to apply Theorem 2.2.1 to the following function: $f(z) = zT$, where T is a compact operator. Notice that

$$(I - f(z))^{-1} = (I - zT)^{-1} = z^{-1}(z^{-1} - T)^{-1},$$

which is a scalar multiple of the resolvent of T. Thus all important structure theorems for compact operators are immediate corollaries of the analytic Fredholm Theorem.

Proof of Theorem 2.2.1. Since the Fredholm alternative holds for finite-dimensional matrices (see Section 1.1), it is possible to prove the Fredholm alternative for compact operators in the Hilbert space case by using the fact that, by Theorem 2.1.3, any compact operator A can be written as $A = \widetilde{A} + F$ where F has finite rank and $\widetilde{A}$ has small norm.

First, we we will prove that near any $z_0 \in \mathcal{D}$ either (i) or (ii) holds. Then a simple connectedness argument would enable us to convert this into a statement about all of $\mathcal{D}$. Given $z_0 \in \mathcal{D}$, choose $r > 0$ so that $|z - z_0| < r$ implies $\|f(z) - f(z_0)\| < \frac{1}{2}$. We can also find a finite rank operator F such that $\|f(z_0) - F\| \leq \frac{1}{2}$ (see Theorem 2.1.3). Hence for all $z \in B_r(z_0)$

$$\|f(z) - F\| < 1.$$

Thus expanding in a geometric series, we see that $(I - f(z) + F)^{-1}$ exists and is analytic in $B_r(z_0)$. Since F is finite rank, by the Riesz representation

[2] For definitions and properties see Appendix B.

theorem, there are $\{\varphi_n\}_{n=1}^N$ and $\{\psi_n\}_{n=1}^N$ such that

$$Fg = \sum_{n=1}^{N} (g, \psi_n)\varphi_n, \qquad g \in \mathfrak{H}.$$

Set

$$\phi_n(z) := \big((I - f(z) + F)^{-1}\big)^* \psi_n. \tag{2.2.1}$$

Then

$$G(z) := F(I - f(z) + F)^{-1} = \sum_{n=1}^{N} (\,\cdot\,, \phi_n(z))\varphi_n.$$

By writing

$$\begin{aligned} I - f(z) = (I - f(z) + F) - F &= (I - f(z) + F) - G(z)(I - f(z) + F) \\ &= (I - G(z))(I - f(z) + F), \end{aligned}$$

we see that $(I - f(z))^{-1}$ exists if and only if $(I - G(z))^{-1}$ exists. Moreover, $f(z)\psi = \psi$ has a nontrivial solution only if $G(z)\psi = \psi$ has a solution.

Now observe that if ψ solves $G(z)\psi = \psi$, then $\psi = \sum_{n=1}^N \beta_n \varphi_n$ for some constants $\beta_n \in \mathbb{C}$. Moreover, taking into account

$$\sum_{n=1}^{N} \beta_n \varphi_n = \sum_{n=1}^{N} \big(\sum_{k=1}^{N} \beta_k \varphi_k, \phi_n(z)\big)\varphi_n$$

we get

$$\beta_n = \sum_{k=1}^{N} \beta_k (\varphi_k, \phi_n(z)). \tag{2.2.2}$$

Conversely, if $\{\beta_n\}$ solves (2.2.2), then the corresponding $\psi = \sum_{n=1}^N \beta_n \varphi_n$ solves $G(z)\psi = \psi$. Thus the latter equation has a nontrivial solution only if the determinant

$$d(z) := \det\big(\delta_{kn} - (\varphi_k, \phi_n(z))\big) \tag{2.2.3}$$

vanishes. Since all $\phi_n(\cdot)$ are analytic, so is $d(\cdot)$. This means that either the set of zeros S of d is a discrete subset or $d \equiv 0$.

Now suppose $d(z) \neq 0$. Then, given ψ, we can solve $(I - G(z))\phi = \psi$ by setting $\phi = \psi + \sum_{n=1}^N \beta_n \varphi_n$ if we can find betas satisfying

$$\beta_n = (\psi, \phi_n(z)) + \sum_{k=1}^{N} \beta_k (\varphi_k, \phi_n(z)). \tag{2.2.4}$$

Since $d(z) \neq 0$, this equation has a solution. Thus $(I - G(z))^{-1}$ exists if and only if $z \notin S$.

The meromorphic nature of $(I - f(z))^{-1}$ and the finite rank residues follow from the fact that there is an explicit formula for the β_n, in in terms of cofactor matrices. $\square$

This theorem has four important consequences.

Theorem 2.2.2 (Fredholm's alternative)**.** *If A is a compact operator on $\mathfrak{H}$, then either $(I-A)^{-1}$ exists or $A\psi = \psi$ has a nontrivial solution.*

Proof. Take $f = zA$, $\mathcal{D} = \mathbb{C}$ and apply Theorem 2.2.1 near $z = 1$. Notice also that the spectrum of A is not an empty set and hence (i) of Theorem 2.2.1 is not possible (or simply notice that $f(0) = I$). □

2.3. Canonical form of compact operators in Hilbert spaces

We continue with the corollaries of the analytic Fredholm alternative in this section.

Theorem 2.3.1 (Riesz–Schauder)**.** *If A is a compact operator on $\mathfrak{H}$, then $\sigma(A)$ is a discrete set of points having no limit accumulation points except possibly $z = 0$. Moreover, every non-zero $z \in \sigma(A)$ is an eigenvalue of finite algebraic multiplicity.*

Proof. Again, take $f(z) = zA$, which is entire (in fact, it is just a linear function in z). Thus the set $S := \{z \in \mathbb{C} |\, zA\psi = \psi \text{ has a nontrivial solution}\}$ is a discrete set by Theorem 2.2.1. If $z \notin S$, then for $\lambda = 1/z$,

$$(\lambda - A)^{-1} = \frac{1}{\lambda}\Big(I - \frac{1}{\lambda}A\Big)^{-1} = z(I - zA)^{-1}$$

exists. Thus $\sigma(A) \subseteq \{z |\, 1/z \in S\}$. Moreover, each $\lambda = 1/z$ with $z \in S$ is an eigenvalue of A. The fact that the nonzero eigenvalues have finite multiplicity follows immediately from the compactness of A. □

Theorem 2.3.2 (Hilbert–Schmidt)**.** *Let A be a self-adjoint compact operator on $\mathfrak{H}$. Then there is a complete orthonormal basis $\{\varphi_n\}$ of $\mathfrak{H}$ such that $A\varphi_n = \lambda_n\varphi_n$ and $\lambda_n \to 0$ as $n \to \infty$.*

Proof. For each eigenvalue $\lambda \in \sigma(A)$ choose an orthonormal basis of $\ker(A - \lambda)$. The collection of all these vectors $\{\varphi_n\}$ forms an orthonormal system in $\mathfrak{H}$ since eigenvectors corresponding to different eigenvalues are orthogonal (for self-adjoint operators!). Let $\mathcal{M} := \operatorname{span}\{\varphi_n\}$. Clearly, $\mathcal{M} \in \operatorname{Lat}(A)$. Moreover, $\mathcal{M}^\perp \in \operatorname{Lat}(A)$ as well since A is self-adjoint. Consider

$$\widetilde{A} = A \restriction \mathcal{M}^\perp.$$

Every eigenvalue of $\widetilde{A}$ is an eigenvalue of A as well and hence by the Riesz–Schauder theorem, $\sigma(\widetilde{A}) = \{0\}$. However, $\widetilde{A}$ is a self-adjoint operator and hence $\widetilde{A}$ is a zero operator on $\mathcal{M}^\perp$. Thus $\mathcal{M}^\perp = \{0\}$ and we are done.

The claim $\lambda_n \to 0$ as $n \to \infty$ follows from the Riesz–Schauder theorem since non-zero eigenvalues have finite multiplicity and can accumulate only at 0. □

Theorem 2.3.3 (Canonical form of compact operators)**.** *Let $A \neq \mathbb{O}_{\mathfrak{H}}$ be a compact operator on $\mathfrak{H}$. Then there are orthonormal sets (not necessary complete in $\mathfrak{H}$, that is, not necessary basis) $\{\varphi_n\}_{n=1}^N$ and $\{\psi_n\}_{n=1}^N$ in $\mathfrak{H}$ and positive numbers $\{s_n\}_{n=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$ such that*

$$A = \sum_{n=1}^{N} s_n(\cdot, \psi_n)\varphi_n. \tag{2.3.1}$$

The sum converges in the operator norm if $N = \infty$.

Definition 2.3.1. The numbers $s_n = s_n(A)$, $n \in \{1, \dots, N\}$ are called the *singular values* (or singular numbers) of the operator A.

Later we'll see that the singular values have a number of important properties and play a crucial role in the analysis of compact operators.

Before going to the proof of Theorem 2.3.3, we need some prerequisites.

2.3.1. Polar decomposition. We want to prove the existence of a special decomposition for operators on a Hilbert space which is analogous to the decomposition $z = |z|\mathrm{e}^{\mathrm{i}\arg(z)}$ for complex numbers.

2.3.1.1. *Non-negative operators and the square root lemma.* First we must describe a suitable analogue of the positive numbers.

Definition 2.3.2. A bounded operator A on a (complex) Hilbert space $\mathfrak{H}$ is called non-negative if its quadratic form is non-negative, that is,

$$\mathfrak{t}_A[f] := (Af, f) \geq 0$$

for every $f \in \mathfrak{H}$. We shall write $A \geq 0$ if A is non-negative and $A \geq B$ if $A - B \geq 0$.

Remark 2.3.1. Every bounded non-negative operator is self-adjoint. Indeed, by the *polarization identity*,

$$\begin{aligned} 4(Af, g) &= (A(f+g), f+g) - (A(f-g), f-g) \\ &\quad + \mathrm{i}(A(f+\mathrm{i}g), f+\mathrm{i}g) - \mathrm{i}(A(f-\mathrm{i}g), f-\mathrm{i}g), \end{aligned}$$

we immediately obtain that $(Af, g) = (f, Ag)$ if the quadratic from $\mathfrak{t}_A$ is real.

Next observe that for every bounded operator A, the operator A^*A is non-negative:

$$\mathfrak{t}_{A^*A}[f] = (A^*Af, f) = (Af, Af) = \|Af\|^2 \geq 0.$$

Hence just as $|z| = \sqrt{z^*z}$, we would like to find $|A|$ as $\sqrt{A^*A}$. And for this we need to show that we can take squares of non-negative operators.

Remark 2.3.2. For compact non-negative operators we can employ the Hilbert–Schmidt theorem, which allows to build up a functional calculus for self-adjoint compact operators. However, this can be done in a unified way avoiding the use of the spectral theorem (the Hilbert–Schmidt theorem is a compact operator version of the spectral theorem).

Theorem 2.3.4 (Square root lemma)**.** *Let A be a non-negative bounded operator on $\mathfrak{H}$. Then there is a unique $B \in [\mathfrak{H}]$ such that $B \geq 0$ and $A = B^2$. Furthermore, B commutes with every bounded operator commuting with A.*

Proof. It suffices to prove the claim for operators satisfying $\|A\| \leq 1$ (use a simple scaling argument). First observe that the power series near zero for $\sqrt{1-z}$ converges absolutely for all $|z| < 1$ since it is analytic inside the unit disc $\mathbb{D}$. In fact, it is given by

$$\sqrt{1-z} = 1 + \sum_{n\geq 1} \frac{(-1/2)_n}{n!} z^n, \tag{2.3.2}$$

where

$$(x)_n := x(x+1)\dots(x+n-1) = \prod_{j=0}^{n-1}(x+j), \quad x \in \mathbb{C}, \ \ n \in \mathbb{N}; \qquad (x)_0 := 1,$$

is the *Pochhammer symbol* (or shifted factorial). In fact,

$$\frac{(-1/2)_n}{n!} = \frac{\Gamma(n-1/2)}{\Gamma(-1/2)\Gamma(n+1)} = \frac{-1}{\sqrt{\pi n^3}}(1 + o(n^{-1})),$$

as $n \to \infty$ (see, e.g., [**39**, (5.2.5)], [**39**, (5.4.6)], [**39**, (5.5.1)], and [**39**, (5.11.12)]), and hence the series (2.3.2) converges absolutely for all $|z| \leq 1$.

Exercise 2.3.1. Show that for a bounded self-adjoint operator A,

$$\|A\| = \sup_{\|f\|\leq 1} |(Af, f)|.$$

Hint: By the polarization identity

$$\mathrm{Re}(Af, g) = \frac{1}{4}(A(f+g), f+g) - (A(f-g), f-g).$$

Then using the inequality

$$|(f, Af)| \leq \|f\|^2 \sup_{\|g\|=1} |(Ag, g)|$$

and the parallelogram law ($\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$), prove that

$$|(Af, g)| \leq \sup_{\|\phi\|=1} |(A\phi, \phi)|$$

Exercise 2.3.2. Show that $\|I - A\| \leq 1$ if A is non-negative and $\|A\| \leq 1$.

Thus we set (by replacing z by $I-A$ in (2.3.2))

$$B := I + \sum_{n\geq 1} \frac{(-1/2)_n}{n!}(I-A)^n.$$

The series converges in the operator norm topology. Moreover, it is straightforward to show that $B^2 = A$ (by squaring and rearranging terms in the series). Since $I-A$ is non-negative, it is easy to see that $(I-A)^n$ is non-negative for all $n \geq 0$ and, moreover,

$$0 \leq (f, (I-A)^n f) \leq 1$$

for all $f \in \mathfrak{H}$ with $\|f\| = 1$. Hence taking into account the negativity of coefficients in (2.3.2), we get

$$(Bf, f) = \|f\|^2 + \sum_{n\geq 1} \frac{(-1/2)_n}{n!}((I-A)^n f, f) \geq 1 + \sum_{n\geq 1} \frac{(-1/2)_n}{n!} = 0.$$

Hence B is non-negative. Also, B commutes with every operator commuting with A since the series converges in the operator norm.

So, it remains to show that B is unique. Let $\widetilde{B}$ be another square root of A with $\widetilde{B} \geq 0$ and $\widetilde{B}^2 = A$. Clearly,

$$\widetilde{B}A = \widetilde{B}\widetilde{B}^2 = \widetilde{B}^2\widetilde{B} = A\widetilde{B},$$

and hence B and $\widetilde{B}$ commute. Therefore,

$$(B-\widetilde{B})B(B-\widetilde{B}) + (B-\widetilde{B})\widetilde{B}(B-\widetilde{B}) = (\underbrace{B^2 - \widetilde{B}^2}_{A-A=0})(B-\widetilde{B}) = 0$$

Two summands on the LHS are non-negative (since so are both B and $\widetilde{B}$). Therefore, both are zero and hence also so is their difference, which is $(B-\widetilde{B})^3$. Since $B-\widetilde{B}$ is self-adjoint, we finally get $\|B-\widetilde{B}\|^4 = \|(B-\widetilde{B})^4\| = 0$, which proves the claim. $\square$

Now we are ready to define $|A|$.

Definition 2.3.3. For a bounded operator A on $\mathfrak{H}$, $|A| := \sqrt{A^*A}$.

Remark 2.3.3. One has to be very careful with $|\cdot|$. Of course, $|zA| = |z||A|$ for all $z \in \mathbb{C}$, however, it is not true in general that $|AB| = |A||B|$. Even more, it is in general false that $|A| = |A^*|$ (take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$) and $|A+B| \leq |A| + |B|$ (take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$).

2.3.1.2. *Partial isometries.* The analogue of the complex numbers of modulus one is a little bit more complicated. At first one might expect that the unitary operators would be sufficient (*U is unitary if $\|Uf\| = \|f\|$ for all $f \in \mathfrak{H}$ and U is onto*), but the following example shows that this is not the case.

Example 2.3.1. Let V be the shift operator on $\ell^2(\mathbb{Z}_{\geq 0})$:

$$V\colon (f_0, f_1, \dots, f_n, \dots) \mapsto (0, f_0, \dots, f_{n-1}, \dots).$$

Clearly, V^* is the left shift operator,

$$V^*\colon (f_0, f_1, \dots, f_n, \dots) \mapsto (f_1, f_2, \dots, f_{n+1}, \dots).$$

And hence $V^*V = I$. Thus $|V| = \sqrt{V^*V} = \sqrt{I} = I$. If $V = U|V|$ for some unitary U, then $V = U$, however, V is not unitary since $(1, 0, \dots, 0, \dots)$ is not in the range of V.

Definition 2.3.4. The operator V is called an isometry if $\|Vf\| = \|f\|$ for all $f \in \mathfrak{H}$. V is called a *partial isometry* if it is an isometry when restricted to the closed subspace $\ker(V)^\perp$. The subspaces $\ker(V)^\perp$ and $\operatorname{ran}(V)$ are called the initial and final subspaces, respectively.

Exercise 2.3.3. Show that V^* is a partial isometry if so is V. Find its initial and final subspaces. Show that V^*V and VV^* are the projections onto the initial and final subspaces of V, respectively.

Theorem 2.3.5 (Polar decomposition)**.** *Let $A \in [\mathfrak{H}]$. Then there is a partial isometry U such that $A = U|A|$. U is uniquely determined by the condition* $\ker(A) = \ker(U)$. *Moreover,* $\operatorname{ran}(U) = \overline{\operatorname{ran}(A)}$.

Proof. First define $\widetilde{U}\colon \operatorname{ran}(|A|) \to \operatorname{ran}(A)$ by $U(|A|f) := Af$. Since

$$\||A|f\|^2 = (|A|^2 f, f) = (A^*Af, f) = \|Af\|^2,$$

we conclude that $|A|f = |A|g$ implies that $Af = Ag$. Hence U is well defined. Moreover, it is isometric and hence admits an isometric extension $U\colon \overline{\operatorname{ran}(|A|)} \to \overline{\operatorname{ran}(A)}$. Since $|A|$ is self-adjoint, $\ker(|A|) = \operatorname{ran}(|A|)^\perp$ and hence we can extend U on $\mathfrak{H}$ by setting to be zero on $\ker(|A|)$. Since $\ker(A) \subseteq \ker(|A|)$ (just by definition of $|A|$), we in fact have $\ker(A) = \ker(|A|)$ and hence $\ker(A) = \ker(U)$. Uniqueness is left to the reader. $\square$

Corollary 2.3.1. *Let T be a bounded linear linear operator in $\mathfrak{H}$. Then T is compact if and only if T^* is compact.*

Proof. By Theorem 2.3.5, $T = U|T|$. Moreover, it is easy to see that this implies $|T| = U^*T$ and hence $|T|$ is compact if so is T (see Exercise 2.3.4 below). However, $T^* = |T|U^*$, which is compact as well by (i). $\square$

Exercise 2.3.4. Show that if S is bounded linear operator in $\mathfrak{H}$, then ST is compact if either T or S is compact.

2.3.2. Proof of Theorem 2.3.3. Finally we are in position to complete the proof of Theorem 2.3.3.

Proof of Theorem 2.3.3. Since A is compact, so is A^* and hence A^*A, which is also self-adjoint and non-negative (Exercise 2.3.4). By the Hilbert–Schmidt theorem, there is an orthonormal set $\{\psi_n\}$ such that $A^*A\psi_n = \lambda_n\psi_n$ for all n and $A^*A = 0$ on the subspace orthogonal to $\{\psi_n\}$. Moreover, $\lambda_n > 0$ for all n due to positivity of A^*A. Setting

$$s_n := \sqrt{\lambda_n}, \qquad \varphi_n := \frac{1}{s_n} A\psi_n,$$

it is straightforward to check that $\{\varphi_n\}$ is orthonormal:

$$(\varphi_n, \varphi_m) = \frac{1}{s_n s_m}(A\psi_n, A\psi_m) = \frac{1}{s_n s_m}(A^*A\psi_n, \psi_m) = \frac{s_n}{s_m}(\psi_n, \psi_m) = 0.$$

Moreover, for every $f \in \overline{\operatorname{span}\{\psi_n\}} = \overline{\operatorname{ran}(|A|)}$, we have

$$f = \sum_n (f, \psi_n)\psi_n,$$

and hence

$$Af = A\Big(\sum_n (f, \psi_n)\psi_n\Big) = \sum_n (f, \psi_n) A\psi_n = \sum_n s_n (f, \psi_n)\varphi_n. \qquad \square$$

Remark 2.3.4. (i) The proof shows that the singular values of A are precisely the eigenvalues of $|A|$.

(ii) Exercise 2.3.4 shows that the set of compact operators is a two-sided symmetric ideal. Moreover, it is a closed ideal by Theorem 2.1.2. It turns out that singular values play the central role in the description of operator ideals in the Banach algebra of bounded operators in $\mathfrak{H}$.

2.3.3. Examples of compact operators. We start with the example of an integral operator considered in Example 2.1.1.

2.3.3.1. *Integral operators. Hilbert–Schmidt class.* In contrast to Example 2.1.1, assume that $\mathcal{K}$ acts as an operator on $L^2(0,1)$, that is, $\mathcal{K}\colon L^2(0,1) \to L^2(0,1)$.

Lemma 2.3.1. *If $K \in L^2((0,1)\times(0,1))$, then $\mathcal{K}$ is compact.*

Proof. Let $\{\varphi_n\}_{n\geq 1}$ be an orthonormal basis in $L^2(0,1)$. Then $\{\varphi_n(x)\varphi_m(y)^*\}_{n,m\geq 1}$ forms an orthonormal basis in $L^2((0,1)\times(0,1))$ and hence

Exercise: Explain why!

$$K(x,y) = \sum_{n,m\geq 1} k_{n,m}\varphi_n(x)\varphi_m(y)^*,$$

where

$$k_{n,m} = \int_0^1 \int_0^1 K(x,y)\varphi_n(x)^*\varphi_m(y)dydx, \qquad n,m \geq 1,$$

are the Fourier coefficients of K. In particular,

$$\|K\|^2_{L^2([0,1]\times[0,1])} = \sum_{n,m\geq 1} |k_{n,m}|^2 < \infty.$$

The above expansion converges in L^2, that is,

$$\|K - K_N\|^2_{L^2} \to 0$$

as $N \to \infty$, where

$$K_N(x,t) = \sum_{n,m=1}^{N} k_{n,m}\varphi_n(x)\varphi_m(y)^*.$$

Thus by the Cauchy–Schwarz inequality

$$\begin{aligned}\|(\mathcal{K} - \mathcal{K}_N)f\|^2 &= \int_0^1 \Big| \int_0^1 (K(x,y) - K_N(x,y))f(y)dy\Big|^2 dx \\ &\leq \int_0^1 |f(y)|^2 dy \int_0^1 \int_0^1 |K(x,y) - K_N(x,y)|^2 dydx \\ &= \|f\|^2\|K - K_N\|^2.\end{aligned}$$

This shows that $\|\mathcal{K} - \mathcal{K}_N\| \leq \|K - K_N\| \to 0$ as $N \to \infty$. Hence $\mathcal{K}$ is compact. ☐

Remark 2.3.5. (i) It turns out that

$$\sum_{k\geq 1} s_k(\mathcal{K})^2 = \|K\|^2_{L^2}.$$

This class of integral operators plays a very important role and it is known as the *Hilbert–Schmidt class*. The converse is also true. Namely, we shall see below that every Hilbert–Schmidt operator, that is, an operator such that its singular values belong to ℓ^2, is unitary equivalent to the integral operator with a square summable kernel.

(ii) Notice also that in contrast to Example 2.1.1 we did not use compactness of $[0,1]$. In particular, the claim remains valid if $[0,1]$ is replaced by $\mathbb{R}$ or any other locally compact Hausdorff space X equipped with a positive Borel measure.

(iii) It is worth mentioning that the Fourier coefficients $\{k_{n,m}\}$ of K can be interpreted as matrix coefficients of the operator $\mathcal{K}$ in the basis $\{\varphi_n\}_{n\geq 1}$. Namely, it is not difficult to show that every bounded operator $\mathcal{A}$ in $\mathfrak{H}$ can be identified with its matrix $A = (a_{n,m})_{n,m\geq 1}$,

$a_{n,m} = (\mathcal{A}\varphi_n, \varphi_m)$. Thus Hilbert–Schmidt operators are exactly those for which the series

$$\sum_{n,m\geq 1} |a_{n,m}|^2 < \infty.$$

For operators in finite dimensional spaces the latter is known as the *Frobenius norm* and can be defined as $\|\mathcal{A}\|_F^2 := \operatorname{tr}(\mathcal{A}^*\mathcal{A})$.

2.3.3.2. *Sturm–Liuoville operators.* Let $q\colon [0,1] \to [0,1]$ be continuous (for simplicity). Consider the following equation

$$-y'' + q(x)y = zy + f, \qquad x \in (0,1), \tag{2.3.3}$$

subject to the Dirichlet boundary conditions at the endpoints

$$y(0) = y(1) = 0.$$

Here z is a complex parameter (*spectral parameter*). It is well known that how to solve this equation. Namely, let $y_1(z,x)$ and $y_2(z,x)$ be solutions to (2.3.3) with $f \equiv 0$ and such that

$$y_1(z,0) = y_2(z,1) = 0, \qquad y_1'(z,0) = y_2'(z,1) = 1.$$

Then either there is a nontrivial solution to (2.3.3) with $f \equiv 0$ or for every suitable f there is a unique solution. The former happens exactly when $W(z) := W(y_1(z), y_2(z)) = y_1(z,x)y_2'(z,x) - y_1'(z,x)y_2(z,x) = -y_2(z,0) = y_1(z,1) = 0$. Otherwise, a solution to (2.3.3) is given by

$$y(x) = \int_0^1 G(z;x,s)f(s)ds, \quad G(z,x,s) = \frac{1}{W(z)}\begin{cases} y_1(z,x)y_2(z,s), & x \leq s \\ y_1(z,s)y_2(z,x), & x \geq s \end{cases}.$$

Since both solutions y_1, y_2 are continuous, the operator defined by this integral equation ($\mathcal{K}\colon f \mapsto y$) in L^2 is compact. Moreover, it is Hilbert–Schmidt. Notice that as in the case of matrices *the solvability of* (2.3.3) *is decided by a scalar function* W.

Remark 2.3.6. Let us also mention that for real-valued functions q, applying the Hilbert–Schmidt theorem, we arrive at the following fact: *the eigenfunctions of* (2.3.3) *form an orthogonal basis in* $L^2(0,1)$. For $q \equiv 0$, this basis consists of

$$\sin(\pi n x), \quad n \geq 1,$$

and it is related to the discrete sine transform. Before the work of D. Hilbert and E. Schmidt the above claim was considered as highly nontrivial. It was first established by V. A. Steklov and this work made him famous. In turn, if either the boundary conditions are not symmetric or $q \neq q^*$, then the analysis of basisness of eigenfunctions is still a highly nontrivial problem!

Chapter 3

Trace Ideals

3.1. Inequalities on singular values

In what follows, we shall always assume that the singular values $\{s_n\}_{n=1}^N$ of a compact operator A are arranged in the decreasing order, that is,

$$s_1(A) \geq s_2(A) \geq \dots$$

The basis of the inequalities on singular values we discuss now is the elementary equality

$$s_n(A) = s_n(A^*), \tag{3.1.1}$$

which follows from (2.3.1) (or alternatively from the fact that A^*A and AA^* have the same non-zero eigenvalues with the same multiplicities - prove it!), and the following characterization:

Theorem 3.1.1 (Min-max principle for singular values). *Let A be a compact operator on a separable Hilbert space $\mathfrak{H}$. Then*

$$s_n(A) = \min_{\phi_1,\dots,\phi_{n-1}\in\mathfrak{H}} \Big[\max_{\substack{\psi\perp\operatorname{span}\{\phi_1,\dots,\phi_{n-1}\} \\ \|\psi\|=1}} \|A\psi\| \Big]. \tag{3.1.2}$$

Proof. The proof follows from the min-max characterization of the eigenvalues of $-A^*A$ if we note that $\|A\psi\|^2 = (A^*A\psi,\psi)$ and that $-s_n(A)^2 = -\lambda_n(A^*A)$ is the n-th eigenvalue of $-A^*A$ counting from the bottom. $\square$

Exercise 3.1.1. Show that $0 \leq |B| \leq |A|$ implies

$$s_k(B) \leq s_k(A)$$

for all k.

The next result provides another definition of singular values (*approximation property*) and it turns out to be more convenient than the original definition.

Theorem 3.1.2 (Dž. É. Allakhverdiev)**.** *If A is a compact operator, then*

$$s_{n+1}(A) = \min_{\operatorname{rank}(K)\le n} \|A-K\|, \qquad n \ge 0.$$

Proof. The canonical form (2.3.1) immediately implies that

$$\min_{\operatorname{rank}(K)\le n} \|A-K\| \le s_{n+1}(A).$$

Indeed, it suffices to take $K = A_n := \sum_{k=1}^{n} s_k(\cdot,\psi_k)\varphi_k$.

Suppose $\operatorname{rank}(K) = n$. Then $\dim(\mathfrak{H}\ominus\ker(K)) = n$ and hence by (3.1.1)

$$s_{n+1}(A) \le \max_{\substack{\psi\in\ker(K)\\ \|\psi\|=1}} \|A\psi\| = \max_{\substack{\psi\in\ker(K)\\ \|\psi\|=1}} \|(A-K)\psi\| \le \|A-K\|. \qquad \square$$

Remark 3.1.1. The above characterization of singular values means that $s_{n+1}(A)$ is *the distance between A and the set of operators of rank $\le n$.*

Corollary 3.1.1. *If $s_n(A)\to 0$ as $n\to\infty$, then A is compact.*

Notice that Corollary 3.1.1 "characterizes" compact operators in terms of singular values.

Corollary 3.1.2. *If A is compact and* $\operatorname{rank}(T) = k$*, then*

$$s_{n+k}(A) \le s_n(A+T) \le s_{n-k}(A).$$

Proof. By Theorem 3.1.2,

$$s_{n+1}(A) = \|A-A_n\| = \|(A+T)-(T+A_n)\| \ge s_{n+k+1}(A+T),$$

for all $n\ge 0$. Replacing the roles of A and $A+T$, we immediately get the second inequality: $s_{n+1}(A+T) \ge s_{n+k+1}(A)$. $\square$

Proposition 3.1.3. *For any compact operator A and a bounded operator B,*

$$s_n(AB) \le \|B\| s_n(A), \qquad s_n(BA) \le \|B\| s_n(A). \tag{3.1.3}$$

Proof. The second inequality in (3.1.3) follows from the first one and (3.1.1) since

$$s_n(BA) = s_n(A^*B^*) \le \|B^*\| s_n(A^*) = \|B\| s_n(A).$$

To prove the first one, use Theorem 3.1.1 and $\|BAf\| \le \|B\|\|Af\|$. $\square$

Remark 3.1.2. The operators AB and BA have the same nonzero eigenvalues, however, they might have different singular values! Indeed, take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and hence $s_1(AB) = s_2(AB) = 0$, however, $s_1(BA) = 1$.

Remark 3.1.3. Proposition 3.1.3 is the first in the family of *Fan's inequalities*:

$$s_{n+m+1}(AB) \le s_{n+1}(A)s_{m+1}(B) \tag{3.1.4}$$

for all $n, m \ge 0$ ((3.1.4) with $m = 0$ gives (3.1.3)).

Exercise 3.1.2. Prove (3.1.4).

Proposition 3.1.4. *Let A and B be compact. Then for all $n, m \ge 0$*

$$s_{n+m+1}(A + B) \le s_{n+1}(A) + s_{m+1}(B) \tag{3.1.5}$$

Proof. Let

$$\mathfrak{t}_n(\phi_1, \dots, \phi_n; A) := \max\{\|A\psi\| \mid \|\psi\| = 1, \psi \in \operatorname{span}\{\phi_1, \dots, \phi_n\}^\perp\}.$$

Since $\|(A + B)\psi\| \le \|A\psi\| + \|B\psi\|$, we get

$$\begin{aligned} \mathfrak{t}_{n+m}(\phi_1, \dots, \phi_{n+m}; A + B) &\le \mathfrak{t}_{n+m}(\phi_1, \dots, \phi_{n+m}; A) + \mathfrak{t}_{n+m}(\phi_1, \dots, \phi_{n+m}; B) \\ &\le \mathfrak{t}_n(\phi_1, \dots, \phi_n; A) + \mathfrak{t}_m(\phi_{n+1}, \dots, \phi_{n+m}; B). \end{aligned}$$

It remains to apply the mini-max principle. □

3.2. The trace and trace class

Our main aim is to show that one can define a trace for compact operators such that their singular values belong to ℓ^1. It turns out that the class $\mathfrak{S}_1(\mathfrak{H})$ of operators such that their singular values belong to ℓ^1 is the only reasonable class on which we can define a trace (e.g., take a positive diagonal matrix in $\ell^2(\mathbb{N})$).

We begin with the following statement.

Proposition 3.2.1. *Let $A = A^* \ge 0$. Then for any two orthonormal bases $\{\varphi_n\}_{n\ge1}$ and $\{\psi_n\}_{n\ge1}$, we have*

$$\sum_{n\ge1} (A\varphi_n, \varphi_n) = \sum_{n\ge1} (A\psi_n, \psi_n). \tag{3.2.1}$$

Proof. Since $A \geq 0$, by Theorem 2.3.4, there is a unique $\sqrt{A} \geq 0$ such that $(\sqrt{A})^2 = A$. Hence

$$\sum_{n\geq 1}(A\varphi_n, \varphi_n) = \sum_{n\geq 1}(\sqrt{A}\varphi_n, \sqrt{A}\varphi_n) = \sum_{n\geq 1}\|\sqrt{A}\varphi_n\|^2.$$

However, by Parseval's equality,

$$\begin{aligned}\sum_{n\geq 1}\|\sqrt{A}\varphi_n\|^2 &= \sum_{n,m\geq 1}|(\sqrt{A}\varphi_n, \psi_m)|^2 = \sum_{n,m\geq 1}|(\varphi_n, \sqrt{A}\psi_m)|^2 \\ &= \sum_{m\geq 1}\|\sqrt{A}\psi_m\|^2 = \sum_{n\geq 1}(A\psi_n, \psi_n).\end{aligned}$$ □

Definition 3.2.1. We say that $A \in [\mathfrak{H}]$ belongs to the trace class $\mathfrak{S}_1(\mathfrak{H})$ if and only if for one (and hence for all) orthonormal basis $\{\varphi_n\}_{n\geq 1}$

$$\sum_{n\geq 1}(|A|\varphi_n, \varphi_n) < \infty. \tag{3.2.2}$$

Notice that according to the canonical decomposition (2.3.1),

$$|A| = \sum_{n\geq 1} s_n(A)(\cdot, \psi_n)\psi_n,$$

and hence

$$\sum_{n\geq 1}(|A|\varphi_n, \varphi_n) = \sum_{n,m\geq 1} s_n(A)|(\psi_m, \varphi_n)|^2 = \sum_{n\geq 1} s_n(A), \tag{3.2.3}$$

where positivity allows a rearrangement of sums.

Our aim is to show the following result.

Proposition 3.2.2. *For any bounded operator A,*

$$\sum_{n\geq 1}(|A|\varphi_n, \varphi_n) = \sup_{\{\phi\},\{\psi\}} \sum_{n\geq 1}|(A\phi_n, \psi_n)|. \tag{3.2.4}$$

Here the supremum is taken over all orthonormal sets $\{\phi\}$ and $\{\psi\}$ in $\mathfrak{H}$.

Moreover, if these quantities are finite, then A is compact and they are further equal to

$$\|A\|_{\mathfrak{S}_1} := \sum_{n\geq 1} s_n(A). \tag{3.2.5}$$

Proof. Assume first that A is compact. Then A admits a canonical form decomposition

$$A = \sum_{n\geq 1} s_n(A)(\cdot, \widetilde{\psi}_n)\widetilde{\varphi}_n.$$

Then for orthonormal sets $\{\varphi\}$ and $\{\psi\}$ in $\mathfrak{H}$, we have

$$(A\varphi_m, \psi_m) = \sum_{n\geq 1} s_n(A)(\varphi_m, \widetilde{\psi}_n)(\widetilde{\varphi}_n, \psi_m) =: \sum_{n\geq 1} s_n(A)b_{n,m}. \tag{3.2.6}$$

By the Cauchy–Schwarz and then Bessel inequality, we have

$$\Big(\sum_{n\geq 1}|b_{n,m}|\Big)^2 \leq \sum_{n\geq 1}|(\varphi_m,\widetilde{\psi}_n)|^2\sum_{n\geq 1}|(\widetilde{\varphi}_n,\psi_m)|^2 \leq 1.$$

A similar inequality holds if the summation goes over m[1]. Thus the sum on the RHS in (3.2.6) converges absolutely and hence

$$\sum_{m\geq 1}|(A\varphi_m,\psi_m)| \leq \sum_{m,n\geq 1} s_n(A)|b_{n,m}| \leq \sum_{n\geq 1} s_n(A). \tag{3.2.7}$$

Taking $\{\varphi\}=\{\widetilde{\varphi}\}$ and $\{\psi\}=\{\widetilde{\psi}\}$, we get $b_{n,m}=\delta_{n,m}$ and hence

$$\sup_{\{\varphi\},\{\psi\}}\sum_{n\geq 1}|(A\varphi_n,\psi_n)| = \sum_{n\geq 1} s_n(A). \tag{3.2.8}$$

This completes the proof of the claim if A is compact.

On the other hand, by Corollary 3.1.1, $\|A\|_{\mathfrak{S}_1}<\infty$ implies that A is compact. If the RHS in (3.2.4) is finite, then compactness of A follows from the next lemma.

Thus, if either side of (3.2.4) is finite, A is compact, and so (3.2.4) holds and equals (3.2.5). If neither is finite, both are infinite, and again equality holds in (3.2.4). □

Lemma 3.2.1. *Let A be bounded. If for all orthonormal sets $\{\varphi_n\}$ and $\{\psi_n\}$ in $\mathfrak{H}$*

$$(A\varphi_n,\psi_n)\to 0$$

as $n\to\infty$, then A is compact.

Exercise 3.2.1. Prove Lemma 3.2.1 (*Hint*: use Corollary 3.1.1).

Theorem 3.2.3. *The trace class $\mathfrak{S}_1$ is a two-sided symmetric ideal and $\|\cdot\|_{\mathfrak{S}_1}$ is a norm on it (sometimes we shall write simply $\|\cdot\|_1$ instead of $\|\cdot\|_{\mathfrak{S}_1}$). Moreover, for $A\in\mathfrak{S}_1$ and $B\in[\mathfrak{H}]$,*

$$\|A\|_1=\|A^*\|_1,\qquad \|BA\|_1\leq\|B\|\|A\|_1,\qquad \|AB\|_1\leq\|B\|\|A\|_1. \tag{3.2.9}$$

Proof. (3.1.1) and (3.1.3) immediately imply (3.2.9), which also imply that A^*, AB and BA belong to $\mathfrak{S}_1$. We only need to show that $A+B\in\mathfrak{S}_1$ if both A, $B\in\mathfrak{S}_1$. However,

$$\sum_{n\geq 1}|((A+B)\varphi_n,\psi_n)| \leq \sum_{n\geq 1}|(A\varphi_n,\psi_n)| + \sum_{n\geq 1}|(B\varphi_n,\psi_n)|$$

and hence

$$\sup_{\{\varphi\},\{\psi\}}\sum_{n\geq 1}|((A{+}B)\varphi_n,\psi_n)| \leq \sup_{\{\varphi\},\{\psi\}}\sum_{n\geq 1}|(A\varphi_n,\psi_n)| + \sup_{\{\varphi\},\{\psi\}}\sum_{n\geq 1}|(B\varphi_n,\psi_n)|.$$

[1]Sometimes matrices $B=(b_{n,m})$ having these two properties are called *doubly substochastic*

It remains to apply Proposition 3.2.2. The proof of the fact that $\|\cdot\|_1$ defines a norm is left as an exercise. □

Exercise 3.2.2. Show that $\mathfrak{S}_1$ is complete w.r.t. $\|\cdot\|_1$. Moreover, show that the closure of finite rank operators w.r.t. $\|\cdot\|_1$ coincides with $\mathfrak{S}_1$.

Now we are in position to define the trace.

Theorem 3.2.4. *For any $A \in \mathfrak{S}_1$ and any orthonormal basis $\{\varphi\}$*

$$\operatorname{tr}_\varphi(A) := \sum_{n\geq 1} (A\varphi_n, \varphi_n) \tag{3.2.10}$$

is absolutely convergent and has the same value, $\operatorname{tr}(A)$.

Moreover, for any $B \in [\mathfrak{H}]$ and A given by (2.3.1),

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) = \sum_{n=1}^{N(A)} s_n(A)(B\psi_n, \varphi_n) \tag{3.2.11}$$

Proof. Let us first prove (3.2.11). Take an orthonormal basis $\{\phi_n\}_{n\geq 1}$ and use (2.3.1),

$$\begin{aligned}\sum_{n\geq 1}(BA\phi_n, \phi_n) &= \sum_{n\geq 1}\sum_{m=1}^{N(A)} s_m(A)(\phi_n, \psi_m)(B\varphi_m, \phi_n)\\ &= \sum_{m=1}^{N(A)} s_m(A)(B\varphi_m, \psi_m).\end{aligned}$$

We were able to change the order of summation since $\sum_{n\geq 1} s_n(A) < \infty$ and by the Cauchy–Schwarz inequality

$$\sum_{n\geq 1} \big|(\phi_n, \psi_m)\big|\big|(B\varphi_m, \phi_n)\big| \leq \|\psi_m\|\|B\varphi_m\| \leq \|B\|.$$

Similar calculations for AB imply (3.2.11).

The rest of the theorem follows from taking $B = I$ and noting that the above calculations are independent of choice of basis $\{\phi_n\}_{n\geq 1}$ (take $\phi_n = \varphi_n$, $n \geq 1$ in (2.3.1) and notice that $(A\varphi_n, \varphi_n) = s_n(A)(\varphi_n, \psi_n)$ for all $n \geq 1$). □

Remark 3.2.1. A few remarks are in order.

(i) $\operatorname{tr}(A)$ defined by (3.2.10) is called the *(matrix) trace* of A.

(ii) If $B \in [\mathfrak{H}]$ is boundedly invertible, $B^{-1} \in [\mathfrak{H}]$, then (3.2.11) implies that $\operatorname{tr}(A) = \operatorname{tr}(BAB^{-1})$. Indeed, by Theorem 3.2.3, $AB^{-1} \in \mathfrak{S}_1$ and hence replacing A in (3.2.11) by AB^{-1}, we get the desired equality.

(iii) One can use $\mathrm{tr}(\cdot)$ to describe some dualities. Namely, it turns out that

$$\mathfrak{S}_\infty(\mathfrak{H})^* = \mathfrak{S}_1(\mathfrak{H}), \qquad \mathfrak{S}_1(\mathfrak{H})^* = [\mathfrak{H}]. \tag{3.2.12}$$

The proof can be found in, e.g. [**51**, Theorem 3.6.8]

The latter is reminiscent of the following well known equalities:

$$(c_0)^* = \ell_1, \qquad (\ell_1)^* = \ell_\infty.$$

(iv) The fact that *the matrix trace coincides with the spectral trace* is a nontrivial fact and is known as *Lidskii's Theorem.* We postpone this problem to Section 3.4.

3.3. Hilbert–Schmidt operators

3.3.1. Hilbert–Schmidt operators. The next class of operators that we are going to discuss is the Hilbert–Schmidt class. We already touched this class in Section 2.3.3.1.

Definition 3.3.1. An operator $A \in [\mathfrak{H}]$ is called *Hilbert–Schmidt* if

$$\mathrm{tr}(A^*A) < \infty. \tag{3.3.1}$$

The set of Hilbert–Schmidt operators is usually denoted by $\mathfrak{S}_2(\mathfrak{H})$. We shall see that this class is in some sense analogous to ℓ_2.

Arguing as in the previous section, one can prove the following results.

Theorem 3.3.1. *Let $\mathfrak{H}$ be a separable complex Hilbert space. Then:*

(i) *$\mathfrak{S}_2(\mathfrak{H})$ is a two-sided symmetric ideal.*

(ii) *If $A, B \in \mathfrak{S}_2$, then the series*

$$\sum_{n\ge 1} (B^*A\varphi_n, \varphi_n) =: (A, B)_2 \tag{3.3.2}$$

is absolutely convergent for every orthonormal basis $\{\varphi_n\}_{n\ge1}$ and, moreover, the sum does not depend on $\{\varphi_n\}_{n\ge1}$.

(iii) *$\mathfrak{S}_2$ equipped with the inner product $(\cdot,\cdot)_2$ is a Hilbert space.*

(iv) *Every $A \in \mathfrak{S}_2$ is compact. Moreover, $A \in \mathfrak{S}_\infty$ is Hilbert–Schmidt if and only if*

$$\sum_{n\ge1} s_n(A)^2 < \infty. \tag{3.3.3}$$

In particular, if we denote $\|A\|_2 := \sqrt{(A, A)_2} = \sqrt{\mathrm{tr}(A^*A)}$, then

$$\|A\| \le \|A\|_2 \le \|A\|_1 \tag{3.3.4}$$

(v) *Finite rank operators are dense in $\mathfrak{S}_2$ w.r.t. $\|\cdot\|_2$.*

(vi) *$A \in \mathfrak{S}_1$ if and only if there are $B, C \in \mathfrak{S}_2$ such that $A = BC$.*

3.3.2. Integral operators. Perhaps, the most important fact is that $\mathfrak{S}_2$ admits a concrete realization as integral operators. Namely, $\mathfrak{S}_2$ coincides with the set of integral operators having a square integrable kernel. Since every separable complex Hilbert space is isometrically isomorphic to $L^2(0,1)$, the claim follows from the next result.

Theorem 3.3.2. *Let $\mathfrak{H} = L^2(0,1)$. Then $A \in [\mathfrak{H}]$ is Hilbert–Schmidt if and only if there is $K\colon (0,1)\times(0,1) \to \mathbb{C}$ such that $K \in L^2((0,1)\times(0,1))$ and*

$$(Af)(x) = (\mathcal{K}f)(x) = \int_0^1 K(x,y)f(y)dy \tag{3.3.5}$$

for all $f \in \mathfrak{H}$ and a.e. $x \in (0,1)$. Moreover,

$$\|A\|_2^2 = \int_0^1\int_0^1 |K(x,y)|^2 dydx. \tag{3.3.6}$$

Proof. Sufficiency follows from the proof of Lemma 2.3.1. To prove necessity, take an orthonormal basis $\{\varphi_n\}_{n\ge1}$ in $L^2(0,1)$. Then $\{\varphi_n\otimes\varphi_m^*\}_{n,m\ge0}$ is an orthonormal basis of $L^2((0,1)\times(0,1))$. Set

$$K(x,y) \sim \sum_{n,m\ge1} (A\varphi_m,\varphi_n)(\varphi_n\otimes\varphi_m^*)(x,y). \tag{3.3.7}$$

The above series converges in $L^2((0,1)\times(0,1))$ since $A\in\mathfrak{S}_2$ and hence

$$\sum_{n,m\ge1} |(A\varphi_m,\varphi_n)|^2 = \sum_{m\ge1}\|A\varphi_m\|^2 = \operatorname{tr}(A^*A) < \infty.$$

It remains to show that A coincides with $\mathcal{K}$ defined by (3.3.7). However, this follows from the following equality of matrix coefficients

$$(A\varphi_m,\varphi_n) = (\mathcal{K}\varphi_m,\varphi_n), \qquad n,m\ge1.$$ □

Remark 3.3.1. Theorem 3.3.2 implies that every Hilbert–Schmidt operator on $\mathfrak{H} = L^2(X;\mu)$, where (X,Σ,μ) is a σ-finite (separable) measure space is an integral operator:

$$(\mathcal{K}f)(x) = \int_X K(x,y)f(y)\mu(dy) \tag{3.3.8}$$

for all $f\in\mathfrak{H}$ and a.e. $x\in X$. Moreover,

$$\|A\|_2^2 = \int_{X\times X} |K(x,y)|^2\mu(dy)\mu(dx). \tag{3.3.9}$$

Corollary 3.3.1. *If $\mathfrak{H} = L^2(X;\mu)$ and $A\in\mathfrak{S}_1(\mathfrak{H})$, then A is an integral operator (3.3.8) with a square integrable kernel (3.3.9).*

Proof. By Theorem 3.3.1, $A\in\mathfrak{S}_2(\mathfrak{H})$ and hence it remains to apply Remark 3.3.1. □

Remark 3.3.2. The problem whether a bounded operator A on $L^2(X;\mu)$ is an integral operator, i.e., $A = \mathcal{K}$ where the integral is understood in the Lebesgue–Stieltjes sense, was posed by J. von Neumann in 1936. The problem is not trivial (take $A = I$ and then A is not integral in $L^2((0,1))$, however, it is integral in $\ell^2(\mathbb{N})$). It was solved by A. V. Bukhvalov in 1974[2]: *A bounded operator A on $L^2(X;\mu)$ is integral if for every sequence $\{f_n\}$ in L^2 such that $0 \le f_n \le f$ for some $f \in L^2$ and $f_n \to 0$ in L^2-sense, $Af_n \to 0$ pointwise.*

Remark 3.3.3. By Theorem 3.3.1(vi), every $A \in \mathfrak{S}_1$ is a product of two Hilbert–Schmidt operators. Applying Theorem 3.3.2, we can represent A as a superposition of two integral operators. Unfortunately, this does not provide a nice characterization of $\mathfrak{S}_1$ like that of $\mathfrak{S}_2$. However, the next result shows that there is a nice description under the additional positivity assumption.

Theorem 3.3.3 (Mercer). *Let K be a continuous positive definite kernel on a compact metric space X. Then the operator $\mathcal{K}$ given by (3.3.8) belongs to $\mathfrak{S}_1$ and*

$$\operatorname{tr}(\mathcal{K}) = \int_X K(x,x)\mu(dx). \tag{3.3.10}$$

The kernel $K\colon X \times X \to \mathbb{C}$ is called *positive definite* if for all $N \in \mathbb{N}$ and $\{x_k\}_{k=1}^N \subset \operatorname{supp}(\mu)$ the matrix $\{K(x_k,x_j)\}_{k,j=1}^N$ is non-negative. It is an easy exercise to prove the next result.

Lemma 3.3.1. *The operator $\mathcal{K}$ is non-negative if K is a continuous positive definite kernel, that is,*

$$(\mathcal{K}f,f)_{L^2} = \int_{I\times I} K(x,y)f(x)f(y)^*\mu(dx)\mu(dy) \ge 0 \tag{3.3.11}$$

for every $f \in C_c(I)$.

The proof of Mercer's Theorem can be found in [**18**, Chapter III.10], [**51**, Chapter 3.11]. Let us only stress that both continuity and positivity of the kernel K are essential and cannot be dropped.

Remark 3.3.4. Let us stress that there are continuous kernels such that the corresponding integral operator $\mathcal{K}$ is not trace class. Consider the interval $\mathbb{T} = [0,2\pi]$ as a torus and $K(x,y) = F(x-y)$ for some $F \in C(\mathbb{T})$. Then $\{\mathrm{e}_n(x) = \mathrm{e}^{\mathrm{i}nx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{T})$ and, moreover,

$$(\mathcal{K}\mathrm{e}_n)(x) = \frac{1}{2\pi}\int_0^{2\pi} F(x-y)\mathrm{e}^{\mathrm{i}ny}dy = \frac{1}{2\pi}\int_0^{2\pi} F(y)\mathrm{e}^{\mathrm{i}n(x-y)}dy = \mathrm{e}^{\mathrm{i}nx}\hat{F}_n,$$

[2]See a very nice overview [**5**].

that is, every e_n is an eigenfunction and the corresponding eigenvalue is the n-th Fourier coefficient of F. Thus $\mathcal{K} = \mathcal{K}_F$ is trace class if and only if, by Definition 3.2.1,

$$\sum_{n\in\mathbb{Z}} |\hat{F}_n| < \infty,$$

that is, F belongs to the *Wiener algebra* $A(\mathbb{T})$, the space of absolutely convergent Fourier series. It is well known that $A(\mathbb{T})$ is a proper subset of $C(\mathbb{T})$, that is, $A(\mathbb{T}) \subset C(\mathbb{T})$ and $A(\mathbb{T}) \neq C(\mathbb{T})$. Functions from $A(\mathbb{T})$ cannot be characterized by smoothness conditions (some smoothness conditions are, however, sufficient to imply the absolute convergence of the Fourier series, e.g., $\mathrm{Lip}_\alpha(\mathbb{T}) \subset A(\mathbb{T})$ for all $\alpha > 1/2$, see [**24**, Chapter I.6]). The following explicit example is due to G. H. Hardy and J. E. Littlewood [**21**]. The series

$$\sum_{n\geq 1} \frac{\mathrm{e}^{\mathrm{i}c\,n\log(n)}}{n^{\alpha+1/2}}\mathrm{e}^{\mathrm{i}nx} \tag{3.3.12}$$

converges uniformly on $\mathbb{T}$ to a continuous function if $\alpha \in (0,1)$ and $c > 0$. However, it does not belong to $A(\mathbb{T})$ if $\alpha \leq 1/2$. Extensive discussion of the example and detailed proof of the claimed uniform convergence can be found in [**53**, Sect. V.4].

Exercise 3.3.1. Consider the following integral operator in $L^2(0,1)$:

$$(\mathcal{J}f)(x) = \int_0^x f(s)ds. \tag{3.3.13}$$

Is this operator bounded? compact? Hilbert–Schmidt? Trace class?

Exercise 3.3.2. Consider the following integral operator in $L^2(0,1)$:

$$(\mathcal{H}f)(x) = \frac{1}{x}\int_0^x f(s)ds. \tag{3.3.14}$$

Is this operator bounded? compact? Hilbert–Schmidt? Trace class?

3.3.3. Operators as (infinite) matrices. Let $\mathfrak{H}$ be a complex separable Hilbert space. Let $\{e_n\}$ be an orthonormal basis in $\mathfrak{H}$. Then with every bounded operator A on $\mathfrak{H}$ one can associate a matrix $A = (a_{kn})$, where $a_{kn} = (Ae_n, e_k)_{\mathfrak{H}}$.

Problem 3.3.4. Suppose we are given an infinite matrix $A = (a_{kn})$. When A defined a bounded and/or compact operator on $\mathfrak{H}$?

By definition, boundedness of A implies that $\sup_{k,n} |a_{kn}| < \infty$. Even more, applying A and A^* to e_k, we get

$$\sup_k \sum_n |a_{kn}|^2 < \infty, \qquad \sup_n \sum_k |a_{kn}|^2 < \infty.$$

Unfortunately, there is no simple criteria to decide whether or not the matrix A defines a abounded operator. The simplest condition is

$$\sum_{k,n} |a_{kn}|^2 = \|A\|_{\mathfrak{S}_2} < \infty, \tag{3.3.15}$$

which provides a criterion for A to belong to the Hilbert–Schmidt class. The latter also implies that A is bounded and compact on $\mathfrak{H}$.

Even for concrete classes of operators Problem 3.3.4 is a subtle issue. Perhaps, the simplest class when one can easily decide boundedness/compactness/trace class is the case when $a_{kn} = 0$ for all $|k - n| \geq N$ with some (fixed) $N \in \mathbb{N}$. Indeed, if $N = 1$, then A is a diagonal matrix and hence everything is obvious. If $N > 1$, then A can be written as a finite linear combination of products of diagonal matrices and shifts/backward shifts and then Problem 3.3.4 becomes an easy exercise.

Exercise: Do it!

Example 3.3.1 (Hankel matrix)**.** Let $\{s_n\}_{n\geq 0}$ be a sequence of complex numbers. The matrix

$$H = (s_{k+n})_{k,n\geq 0} \tag{3.3.16}$$

is called a *Hankel matrix*.

Notice that boundedness of H immediately implies that $\{s_n\} \in \ell^2$. However, the converse is not true! Moreover, $H \in \mathfrak{S}_2$ if and only if

$$\sum_{n\geq 0} (n+1)|s_n|^2 < \infty.$$

As we shall see the answers to Problem 3.3.4 are much more complicated.

The theory of Hankel matrices is closely connected with the classical moment problem and with the theory of functions in the unit disc (see [**40**]). Namely, the following theorem is due to H. Hamburger.

Theorem 3.3.5 (Hamburger)**.** *A Hankel matrix is non-negative if and only if there is a positive Borel measure μ on $\mathbb{R}$ such that*

$$s_k = \int_{\mathbb{R}} \lambda^k \, \mu(d\lambda) \tag{3.3.17}$$

for all $k \geq 0$.

For non-negative Hankel matrices Problem 3.3.4 was solved by H. Widom:[3]

Theorem 3.3.6 (Widom)**.** *Let $\{s_n\} \in \ell_2$ be such that the Hankel matrix H_α is non-negative. Then the following statements are equivalent:*

(i) *The Hankel matrix H is bounded (compact) on ℓ_2,*

(ii) $s_n = \mathcal{O}(n^{-1})$ ($s_n = o(n^{-1})$) *as $n \to \infty$,*

[3] H. Widom, *Hankel matrices*, Trans. Amer. Math. Soc. **121**, no. 1, 1–35 (1966).

(iii) *There exists a positive measure* μ *on* $(-1,1)$ *such that*

$$s_n = \int_{(-1,1)} \lambda^n \, \mu(d\lambda) \tag{3.3.18}$$

holds for all $n \geq 0$ *and* μ *is a* Carleson measure *(a* vanishing Carleson measure*), i.e.,*

$$\mu((-1,-t)\cup(t,1)) = \mathcal{O}(1-t) \qquad (\mu((-1,-t)\cup(t,1)) = o(1-t)) \tag{3.3.19}$$

as $t \uparrow 1$.

(iv) $H^2(\mathbb{D})$ *is continuously (compactly) embedded into* $L^2((-1,1);d\mu)$*, i.e., there is* $C > 0$ *such that* $\|f\|_{L^2(d\mu)} \leq C\|f\|_{H^2}$ *for all* $f \in H^2(\mathbb{D})$.

Exercise 3.3.3. When a non-negative Hankel matrix H belongs to the trace class? Find its trace.

Without the positivity assumptions, the answers to Problem 3.3.4 become much more complicated and less transparent comparing, e.g., to condition (ii) of Theorem 3.3.6.

Theorem 3.3.7 (Z. Nehari/P. Hartman)**.** *Let* $\{s_n\} \in \ell^2$. *The Hankel matrix* H *generates a bounded (compact) operator on* ℓ^2 *if and only if there is a function* $\varphi \in L^\infty(\mathbb{T})$ *(*$\varphi \in C(\mathbb{T})$*) such that*

$$s_n = \hat{\varphi}_n, \quad n \geq 0. \tag{3.3.20}$$

Remark 3.3.5. Sufficiency is easy. Indeed, one simply needs to note that the operator $H_\varphi : H^2 \to H^2_-$ defined by

$$H_\varphi : f \mapsto P_-(\varphi f), \tag{3.3.21}$$

has the Hankel matrix representation in the canonical basis. Here

$$H^2 = \{f \in L^2(\mathbb{T}) |\, \hat{f}_n = 0,\ n < 0\}, \qquad H^2_- = L^2(\mathbb{T}) \ominus H^2, \tag{3.3.22}$$

and P_- is the orthogonal projection in L^2 onto H^2_-. Indeed, if $\varphi = \sum_{n\in\mathbb{Z}} \hat{\varphi}_n \mathrm{e}_n$, where $\mathrm{e}_n = \mathrm{e}^{in\theta}$, then

$$H_\varphi \mathrm{e}_k = P_-(\varphi \mathrm{e}_k) = P_-\Big(\sum_{n\in\mathbb{Z}} \hat{\varphi}_{n-k}\mathrm{e}_n\Big) = \sum_{n\in\mathbb{N}} \hat{\varphi}_{-(n+k)}\mathrm{e}_{-n} \tag{3.3.23}$$

for all $k \in \mathbb{Z}_+$. Therefore, the matrix representation of H_φ is given by the Hankel matrix H with coefficients $s_n = \hat{\varphi}_{-(n+1)}$, $n \geq 0$.

Example 3.3.2 (The Hilbert matrix)**.** The Hankel matrix with the coefficients $\alpha_n = \frac{1}{n+1}$, $n \geq 0$ is called the *Hilbert matrix*:

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.3.24}$$

The Hilbert inequality [**22**, Chapter IX] states that the bilinear form generated by H (and hence the operator H) is bounded on ℓ_2:

$$|(Hf,g)_{\ell_2}| = \Big| \sum_{k,n\geq 0} \frac{f_k g_n^*}{k+n+1} \Big| \leq \pi \, \|f\|_2 \|g\|_2. \tag{3.3.25}$$

By Theorem 3.3.7, there is a function $\varphi \in L^\infty(\mathbb{T})$ such that $\hat{\varphi}_n = \frac{1}{n+1}$ for all $n \geq 0$. However, it is easy to see that the function

$$\varphi_+(\theta) = \sum_{n\geq 0} \frac{\mathrm{e}^{in\theta}}{n+1}, \quad \theta \in \mathbb{T},$$

does not belong to $L^\infty(\mathbb{T})$. On the other hand, the function $\varphi(\theta) = \sum_{n\in\mathbb{Z}} \frac{\mathrm{e}^{in\theta}}{|n|+1}$ belongs to L^∞.

Remark 3.3.6. Notice that by Fefferman's Theorem (resp., Sarason's Theorem) [4], Nehari's Theorem (resp., Hartman's Theorem) is equivalent to the fact that the function

$$f(z) = \sum_{n\geq 0} s_n z^n, \qquad z \in \mathbb{D}, \tag{3.3.29}$$

belongs to the space $BMOA := BMO \cap H^1$ (resp., $VMOA := VMO \cap H^1$).

A reasonably simple characterization of finite rank Hankel matrices is known as *Kronecker's theorem*: *a symbol φ must be a rational function.* A characterization of von Neumann–Schatten classes $\mathfrak{S}_p$ with $p \neq 2$ was

[4] Let $f \in L^1(\mathbb{T})$. We shall say that $f \in BMO(\mathbb{T})$, the space of function of *bounded mean oscillation*, if

$$\sup_I \frac{1}{|I|} \int_I |f - f_I| d\theta =: \|f\|_* < \infty. \tag{3.3.26}$$

Here I is any arc on $\mathbb{T}$, $|I| = \int_I d\theta$ and

$$f_I := \frac{1}{2\pi} \int_I f d\theta \tag{3.3.27}$$

is the average of f over I. The BMO function f is said to have *vanishing mean oscillation*, $f \in VMO(\mathbb{T})$ if

$$\lim_{\varepsilon\to 0} \sup_{|I|<\varepsilon} \frac{1}{|I|} \int_I |f - f_I| d\theta = 0. \tag{3.3.28}$$

The *Fefferman duality theorem* states that *$f \in BMO$ if and only if $f = u + \tilde{v}$, where u, $v \in L^\infty(\mathbb{T})$ and $\tilde{v}$ is a conjugate function.* Sarason's theorem provides a similar characterization of the VMO space: *$f \in VMO$ if and only if $f = u + \tilde{v}$, where u, $v \in C(\mathbb{T})$.* For further details see [**17, 44**].

obtained by V. V. Peller in 1980 and for this one needs *Besov spaces* B_p^s. In particular, $H_\varphi \in \mathfrak{S}_1$ iff $P_-\varphi \in B_1^1$ (see [40, Chapter VI] for further details).

3.4. Lidskii's theorem

3.4.1. Antisymmetric tensor products. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be separable Hilbert spaces. For each $\phi_1 \in \mathfrak{H}_1$ and $\phi_2 \in \mathfrak{H}_2$, denote by $\phi_1 \otimes \phi_2$ a bilinear form acting on $\mathfrak{H}_1 \times \mathfrak{H}_2$ as

$$(\phi_1 \otimes \phi_2)(f_1, f_2) := (f_1, \phi_1)_{\mathfrak{H}_1}(f_2, \phi_2)_{\mathfrak{H}_2}. \tag{3.4.1}$$

Let $\mathcal{L}$ be a linear span of all such forms. We define an inner product on $\mathcal{L}$ by

$$(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2)_{\mathfrak{H}_1 \otimes \mathfrak{H}_2} := (\phi_1, \psi_1)_{\mathfrak{H}_1}(\phi_2, \psi_2)_{\mathfrak{H}_2}. \tag{3.4.2}$$

and then extend by linearity on all of $\mathcal{L}$. It is straightforward to check that $(\cdot, \cdot)_{\mathfrak{H}_1 \otimes \mathfrak{H}_2}$ is well defined and positive definite. The Hilbert space obtained after completion of $\mathcal{L}$ with respect to this inner product is called *the tensor product* of Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and is denoted by $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. The next result is an easy exercise.

Lemma 3.4.1. *If $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal bases in, respectively, $\mathfrak{H}_1$ and $\mathfrak{H}_2$, then $\{\varphi_n \otimes \psi_k\}_{n,k}$ is an orthonormal basis in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$.*

Remark 3.4.1. Tensor products arise naturally in the following situations:
(a) the Hilbert space $L^2(\mathbb{R}^2)$ is isometrically isomorphic to $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$;
(b) the Hilbert space $L^2(\mathbb{R}; \mathbb{C}^n)$ is isometrically isomorphic to $L^2(\mathbb{R}) \otimes \mathbb{C}^n$.

The next example plays a very important role in quantum mechanics.

Example 3.4.1 (Fock space)**.** Let $\mathcal{H}$ be a separable complex Hilbert space. Let $\mathcal{H}^{\otimes n} := \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{n \text{ times}}$ for all $n \ge 1$ and $\mathcal{H}^0 := \mathbb{C}$. Set

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n \ge 0} \mathcal{H}^{\otimes n}. \tag{3.4.3}$$

The Hilbert space $\mathcal{F}(\mathcal{H})$ is called *the Fock space*. It is separable. For example, if $\mathcal{H} = L^2(\mathbb{R})$, then every $\psi \in \mathcal{F}(\mathcal{H})$ can be considered as a sequence of functions

$$\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \dots, \psi_n(x_1, \dots, x_n), \dots\}$$

such that

$$|\psi_0|^2 + \sum_{n \ge 1} \int_{\mathbb{R}^n} |\psi_n(x_1, \dots, x_n)|^2 dx_1 \dots dx_n < \infty.$$

Usually in quantum mechanics two subspaces of the Fock space $\mathcal{F}$ are used. Let S_n be a symmetric group (the group of all permutations of n

"letters"). Also, let $\{\varphi_n\}$ be a basis in $\mathcal{H}$. For every $\sigma \in S_n$ define the operator on the basis elements in $\mathcal{H}^{\otimes n}$ (also denoted by σ) by

$$\sigma(\varphi_{k_1} \otimes \cdots \otimes \varphi_{k_n}) := \varphi_{\sigma(k_1)} \otimes \cdots \otimes \varphi_{\sigma(k_n)}. \tag{3.4.4}$$

By linearity this operator extends to a bounded operator on all of $\mathcal{H}^{\otimes n}$ (find its norm!). Now we set

$$\mathcal{S}_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma, \qquad \mathcal{A}_n := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^\sigma \sigma. \tag{3.4.5}$$

Exercise 3.4.1. Show that both $\mathcal{S}_n$ and $\mathcal{A}_n$ are orthogonal projections, that is, $\mathcal{S}_n = \mathcal{S}_n^*$, $\mathcal{S}_n^2 = \mathcal{S}_n$ and $\mathcal{A}_n = \mathcal{A}_n^*$ and $\mathcal{A}_n^2 = \mathcal{A}_n$.

The subspaces

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n \ge 0} \mathcal{S}_n(\mathcal{H}^{\otimes n}), \qquad \mathcal{F}_a(\mathcal{H}) := \bigoplus_{n \ge 0} \mathcal{A}_n(\mathcal{H}^{\otimes n}), \tag{3.4.6}$$

are called *symmetric (bosonic)* and, respectively, *antisymmetric (phermionic) Fock spaces*.

If $\mathcal{H} = L^2(\mathbb{R})$, then $\mathcal{F}_s$ consists of *symmetric* functions, e.g., functions which are invariant under permutations of their arguments. Also, $\mathcal{F}_a$ consists of functions which are odd under interchange of two coordinates.

In what follows, we shall use the following notation

$$\psi_1 \wedge \cdots \wedge \psi_n := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^\sigma \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}. \tag{3.4.7}$$

Notice that in quantum mechanics (usually, the case $\mathcal{H} = L^2$), the latter is known as the *Slater determinant*. The next result sheds some light on connections with determinants.

Lemma 3.4.2.

$$(\phi_1 \wedge \cdots \wedge \phi_n, \psi_1 \wedge \cdots \wedge \psi_n)_{\mathcal{H}^{\otimes n}} = \det\big((\phi_i, \psi_j)\big)_{i,j=1}^n \tag{3.4.8}$$

In particular, $\{\phi_{k_1} \wedge \cdots \wedge \phi_{k_n}\}_{k_1 < \cdots < k_n}$ *is an orthonormal basis in* $\mathcal{F}_a(\mathcal{H})$ *if* $\{\phi_k\}$ *is an orthonormal basis in* $\mathcal{H}$.

Proof. Straightforward calculations show that

$$\begin{aligned}(\phi_1 \wedge \cdots \wedge \phi_n, \psi_1 \wedge \cdots \wedge \psi_n)_{\mathcal{H}^{\otimes n}} &= \frac{1}{n!} \sum_{\sigma, \pi \in S_n} (-1)^\sigma (-1)^\pi \prod_{k=1}^n (\phi_{\sigma(k)}, \psi_{\pi(k)}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{k=1}^n (\phi_k, \psi_{\sigma(k)})\end{aligned}$$

due to the commutativity of multiplication in $\mathbb{C}$. $\square$

Finally, set $\Lambda^n(\mathcal{H}) = \mathcal{A}_n(\mathcal{H}^{\otimes n})$. For a bounded operator on A on $\mathcal{H}$, let us define the operator $A^{\otimes n} = A \otimes \cdots \otimes A$ acting on $\mathcal{H}^{\otimes n}$ by

$$(A \otimes \cdots \otimes A)(\phi_1 \otimes \cdots \otimes \phi_n) := (A\phi_1) \otimes \cdots \otimes (A\phi_n), \tag{3.4.9}$$

and then extends by linearity onto $\mathcal{H}^{\otimes n}$. It is straightforward to check that both $\mathcal{S}_n(\mathcal{H}^{\otimes n})$ and $\mathcal{A}_n(\mathcal{H}^{\otimes n})$ are invariant subspaces for $A \otimes \cdots \otimes A$. Denote by $\Lambda^n(A)$ the restriction of $A \otimes \cdots \otimes A$ onto $\Lambda^n(\mathcal{H})$. Clearly, $\Lambda^n(A) = (A \otimes \cdots \otimes A)\mathcal{A}_n$ and hence

$$\begin{aligned}\Lambda^n(A)(\phi_1 \wedge \cdots \wedge \phi_n) &= (A\phi_1) \wedge \cdots \wedge (A\phi_n) \\ &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^\sigma (A\psi_{\sigma(1)}) \otimes \cdots \otimes (A\psi_{\sigma(n)}),\end{aligned} \tag{3.4.10}$$

Exercise 3.4.2. (i) Show that $\Lambda^n(AB) = \Lambda^n(A)\Lambda^n(B)$ for each bounded operators A and B on $\mathcal{H}$.

(ii) If $\mathcal{H} = \mathbb{C}^N$ for some $N \in \mathbb{N}$ and A is a linear operator in $\mathbb{C}^N$, show that

$$\Lambda^N(A) = \det(A).$$

In particular, this implies that $\det(AB) = \det(A)\det(B)$ for $A, B \in \mathbb{C}^{N \times N}$.

Exercise 3.4.3. Show that

$$\Lambda^n(A)^* = \Lambda^n(A^*), \qquad \left|\Lambda^n(A)\right| = \Lambda^n(|A|).$$

This machinery will play a major role in the definition of determinants.

3.4.2. The Weyl and Horn inequalities.

The above machinery is also useful for the following results.

Theorem 3.4.1 (Horn). *For compact operators A and B and any $N \in \mathbb{N}$,*

$$\prod_{n=1}^{N} s_n(AB) \le \prod_{n=1}^{N} s_n(A)s_n(B). \tag{3.4.11}$$

In what follows we shall use the following notation: for a compact operator A, its eigenvalues are ordered so that $|\lambda_1| \ge |\lambda_2| \ge \dots$ and each eigenvalue is counted according to its *algebraic* multiplicity.

Theorem 3.4.2 (H. Weyl). *For a compact operator A and any $N \in \mathbb{N}$,*

$$\prod_{n=1}^{N} |\lambda_n(A)| \le \prod_{n=1}^{N} s_n(A). \tag{3.4.12}$$

Proof of Theorems 3.4.1 and 3.4.2. Let C be a non-negative compact operator. Also, let $\{\phi_n\}$ be a complete orthonormal set of eigenvectors of C.

Then $\{\phi_{k_1} \wedge \cdots \wedge \phi_{k_n}\}_{k_1 < \cdots < k_m}$ is a complete orthonormal set of eigenvectors for $\Lambda^m(C)$. Moreover,

$$\|\Lambda^m(C)\| = \prod_{n=1}^{m} s_n(C). \tag{3.4.13}$$

Next observe that $|\Lambda^m(C)| = \Lambda^m(|C|)$ for any C, and hence the latter equality holds for any C. Now Horn's inequality follows simply by noting that $\|\Lambda^m(AB)\| \le \|\Lambda^m(A)\| \|\Lambda^m(B)\|$.

To prove Weyl's inequality, one notices that by using a Jordan normal form in $\mathrm{ran}(P_\lambda)$ for $\lambda \in \{\lambda_1, \ldots, \lambda_N\}$, we find a set $\{\eta_n\}$ of independent vectors so that $A\eta_n = \lambda_n \eta_n + x_n \eta_{n-1}$, where $x_n \in \{0,1\}$. Clearly,

$$\Lambda^N(A)(\eta_1 \wedge \cdots \wedge \eta_N) = \Big(\prod_{k=1}^{N} \lambda_k\Big) \eta_1 \wedge \cdots \wedge \eta_N.$$

This means that $\prod_{k=1}^{N} \lambda_k$ is an eigenvalue of $\Lambda^N(A)$ since $\eta_1 \wedge \cdots \wedge \eta_N \neq 0$ due to their linear independence. Now (3.4.12) follows from (3.4.13) since norm dominates every eigenvalue. $\square$

Remark 3.4.2. Inequalities (3.4.11) and (3.4.12) are special cases[5] of the following inequalities. Let $\phi\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a nondecreasing function such that $x \mapsto \phi(\mathrm{e}^x)$ is convex. Then

$$\sum \phi(|\lambda_n(A)|) \le \sum \phi(s_n(A)), \tag{Weyl}$$

$$\sum \phi(s_n(AB)) \le \sum \phi(s_n(A) s_n(B)). \tag{Horn}$$

Notice that for $\phi(x) = x^p$ with $p > 0$ this implies

$$\sum |\lambda_n(A)|^p \le \sum s_n(A)^p. \tag{3.4.14}$$

The case $p = 2$ was first proved by I. Schur in 1909:

$$\sum_{n=1}^{N} |\lambda_n|^2 \le \sum_{i,j} |a_{ij}|^2 = \sum_{n=1}^{N} s_n(A)^2.$$

Sometimes (3.4.14) is called the *Lalesco–Schur–Weyl inequality*. For further details we refer to [**18**], [**50**, Chapter 1] and also Appendix A.

[5] Indeed, for every fixed N we can rescale both sides in (3.4.11) and (3.4.12). Namely, choose $\gamma = \gamma(N) > 0$ such that $s_n(\gamma \cdot) = \gamma s_n(\cdot) \ge 1$ and $|\lambda_n(\gamma \cdot)| = \gamma |\lambda_n(\cdot)| \ge 1$ for all $n \in \{1, \ldots, N\}$. Hence we can assume that all first N singular values are greater than 1. Now take $\phi(x) = \log(x)$. On the other hand, one can obtain these inequalities as a corollary of (3.4.11)–(3.4.12) and the rearrangement inequalities, see Appendix A.

3.4.3. Determinants. The key idea is the following equality: if A is a $\mathbb{C}^{N\times N}$ matrix, then

$$\begin{aligned}\det(I+A) &= \prod_{n=1}^{N}\lambda_n(I+A) = \prod_{n=1}^{N}(1+\lambda_n(A))\\ &= 1+\sum_{n=1}^{N}\sum_{i_1<\dots<i_n}\lambda_{i_1}(A)\dots\lambda_{i_n}(A)\\ &= \sum_{n=0}^{N}\operatorname{tr}(\Lambda^n(A)).\end{aligned}$$

This suggests the definition

$$\det(I+A) := \sum_{n\geq 0}\operatorname{tr}(\Lambda^n(A)). \tag{3.4.15}$$

Of course, we need to prove the convergence of this sum for a suitable class of operators. We begin with the following result.

Proposition 3.4.3. *Let $A\in\mathfrak{S}_1$. Then for all $n\geq 0$:*

(i) *$\Lambda^n(A)\in\mathfrak{S}_1(\Lambda^n(\mathfrak{H}))$, where $\Lambda^n(\mathfrak{H}) := \mathcal{A}_n(\mathfrak{H}^{\otimes n})$.*
(ii) *$\|\Lambda^n(A)\|_1 = \sum_{i_1<\dots<i_n} s_{i_1}(A)\dots s_{i_n}(A)$.*
(iii) *$\|\Lambda^n(A)\|_1 \leq \|A\|_1^n/n!$.*

Proof. By Exercise 3.4.3, $\left|\Lambda^n(A)\right| = \Lambda^n(|A|)$ and hence singular values of $\Lambda^n(A)$ are given by $s_{i_1}(A)\dots s_{i_n}(A)$ with $i_1<\dots<i_n$. Hence

$$\|\Lambda^n(A)\|_1 = \sum_{i_1<\dots<i_n} s_{i_1}(A)\dots s_{i_n}(A),$$

which proves (ii). Moreover, if $A\in\mathfrak{S}_1$, then

$$\begin{aligned}\operatorname{tr}(|A|)^n &= \Big(\sum_{k\geq 0}s_k(A)\Big)^n = \sum_{i_1,\dots,i_n}s_{i_1}(A)\dots s_{i_n}(A)\\ &\geq n!\sum_{i_1<\dots<i_n}s_{i_1}(A)\dots s_{i_n}(A) = n!\|\Lambda^n(A)\|_1,\end{aligned}$$

which proves (i) and (iii). □

Corollary 3.4.1. *Let $A\in\mathfrak{S}_1$. Then the function*

$$D(z) := \det(I+zA) = \sum_{n\geq 0} z^n\operatorname{tr}(\Lambda^n(A)), \qquad z\in\mathbb{C}, \tag{3.4.16}$$

is entire. Moreover, for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|D(z)| \leq C_\varepsilon \mathrm{e}^{\varepsilon|z|} \tag{3.4.17}$$

for all $z\in\mathbb{C}$.

Proof. The first claim immediately follows from Proposition 3.4.3(iii). Moreover, it implies the estimate

$$|D(z)| \le \mathrm{e}^{|z|\|A\|_1}, \qquad z \in \mathbb{C}. \tag{3.4.18}$$

To prove the second claim, observe that

$$|D(z)| \le \prod_{n\ge1}(1+|z|s_n(A))$$

for all $z \in \mathbb{C}$ (cf. Proposition 3.4.3(ii)). Choosing N so that $\sum_{n\ge N+1} s_n(A) < \varepsilon/2$ and using the trivial inequality $1+x \le \mathrm{e}^x$ for all $x \ge 0$, we see that

$$\begin{aligned}|D(z)| &\le \prod_{n=1}^{N}(1+|z|s_n(A)) \prod_{n\ge N+1}(1+|z|s_n(A)) \\ &\le \prod_{n=1}^{N}(1+|z|s_n(A)) \prod_{n\ge N+1} \mathrm{e}^{|z|s_n(A)} \\ &\le \mathrm{e}^{\frac12\varepsilon|z|}\prod_{n=1}^{N}(1+|z|s_n(A)) \le C_\varepsilon \mathrm{e}^{\varepsilon|z|}. \qquad \square\end{aligned}$$

Theorem 3.4.4. *The function* $A \mapsto \det(I+A)$ *is continuous on* $\mathfrak{S}_1$. *Explicitly,*

$$|\det(I+A)-\det(I+B)| \le \|A-B\|_1 \mathrm{e}^{1+\|A\|_1+\|B\|_1}. \tag{3.4.19}$$

Proof. If $\|A-B\|_1 = 0$, then there is nothing to prove. Set

$$f(z) := \det\Big(I+\frac12(A+B)+z(A-B)\Big). \tag{3.4.20}$$

Clearly,

$$f(1/2)-f(-1/2) = \det(I+A)-\det(I+B).$$

Moreover, the function f is entire and hence we can apply the following estimate:

$$\big|f(1/2)-f(-1/2)\big| \le R^{-1}\max_{|z|=R+1/2}|f(z)|, \qquad R>0. \tag{3.4.21}$$

Indeed, by the Cauchy integral formula for every $t \in [-1/2, 1/2]$ we have

$$f'(t) = \frac{1}{2\pi\mathrm{i}}\int_{|z-t|=R}\frac{f(z)}{(z-t)^2}dz,$$

and hence using the maximum principle we get

$$|f'(t)| \le R^{-1}\max_{|z-t|=R}|f(z)| \le R^{-1}\max_{|z|=R+1/2}|f(z)|.$$

It remains to notice that

$$\big|f(1/2)-f(-1/2)\big| = \Big|\int_{-1/2}^{1/2} f'(t)dt\Big| \le \int_{-1/2}^{1/2}|f'(t)|dt.$$

Finally, applying (3.4.21) to (3.4.20) with $R^{-1} = \|A - B\|_1$, we get by employing (3.4.18)

$$\begin{aligned}\big|\det(I+A) - \det(I+B)\big| &\le \|A-B\|_1 \max_{|z|=R+1/2} |f(z)| \\ &\le \|A-B\|_1 \mathrm{e}^{\frac{1}{2}\|A\|_1 + \frac{1}{2}\|B\|_1 + (R+\frac{1}{2})\|(A-B)\|_1} \\ &\le \|A-B\|_1 \mathrm{e}^{1+\|A\|_1+\|B\|_1}. \qquad \square\end{aligned}$$

Corollary 3.4.2. *For any $A, B \in \mathfrak{S}_1$,*

$$\det(I + A + B + AB) = \det(I+A)\det(I+B). \tag{3.4.22}$$

Proof. By continuity (Theorem 3.4.4), it suffices to prove the claim for finite rank operators, where it is essentially that $\det(CD) = \det(C)\det(D)$. □

3.4.4. Lidskii's theorem. Estimate (3.4.18) shows that D is of *exponential type*. However, Corollary 3.4.1 shows that D is of *minimal exponential type*. Entire functions of this class admit a unique factorization, or in other words, they are determined uniquely (up to a constant multiple) by the set of its zeros. The latter is known as the *Hadamard product formula* (see [**33**, Lecture 4.2]).

Theorem 3.4.5 (J. Hadamard)**.** *Let f be an entire function with $f(0) = 1$ and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(z)| \le C_\varepsilon \mathrm{e}^{\varepsilon |z|}, \qquad z \in \mathbb{C}.$$

If its zeros $\{z_n\}$ satisfy

$$\sum \frac{1}{|z_n|} < \infty, \tag{3.4.23}$$

then

$$f(z) = \prod \Big(1 - \frac{z}{z_n}\Big), \qquad z \in \mathbb{C}. \tag{3.4.24}$$

Now we need to relate the zeros of $\det(I + zA)$ with the spectrum of A.

Theorem 3.4.6. *Let $A \in \mathfrak{S}_1$. Then $\det(I + zA) \neq 0$ if and only if $I + zA$ is boundedly invertible. Moreover, if λ is an eigenvalue of A of algebraic multiplicity n, then $z_0 = -1/\lambda$ is an n-th order zero of $\det(I + zA)$.*

Proof. First, let $I + zA$ be invertible, that is, $-z^{-1} \notin \sigma(A)$. Then for $B = -zA(I + zA)^{-1}$, we get

$$\begin{aligned}(I+zA)(I+B) &= I + zA + B + zAB \\ &= I + zA - zA(I+zA)^{-1} - z^2A^2(I+zA)^{-1} \\ &= I + zA - zA(I+zA)(I+zA)^{-1} = I,\end{aligned}$$

and hence we infer from (3.4.22)

$$1 = \det(I + zA)\det(I + B).$$

This implies that $\det(I + zA) \neq 0$.

Let $\lambda = -1/z_0$ be an eigenvalue of A and let P_λ be the corresponding spectral projection:

$$P_\lambda = \frac{1}{2\pi \mathrm{i}} \int_{|z-\lambda|=\varepsilon} (A - z)^{-1} dz.$$

Here $\varepsilon > 0$ is sufficiently small such that $B_\varepsilon(\lambda) \cap \sigma(A) = \lambda$. One can verify that P_λ is a projection commuting with A. Moreover, its range coincides with the kernel subspace of A corresponding to λ. Then

$$(P_\lambda A)((I - P_\lambda)A) = 0.$$

Hence by (3.4.22),

$$\det(I + zA) = \det(I + zP_\lambda A)\det(I + z(I - P_\lambda)A).$$

Notice that $I + z_0(I - P_\lambda)$ is invertible by construction (indeed, $\sigma((I - P_\lambda)A) = \sigma(A) \setminus \{\lambda\})$ and hence $\det(I + z_0(I - P_\lambda)A) \neq 0$. On the other hand, $P_\lambda A$ is finite rank and, by construction, $\det(I + zP_\lambda A) = (1 - z/z_0)^n$, where $n = \dim(\mathrm{rank}(P_\lambda))$. This completes the proof. □

Now we are ready to finish the proof of Lidskii's theorem.

Theorem 3.4.7. *Let $A \in \mathfrak{S}_1$. Then*

$$D(z) = \det(I + zA) = \prod_n (1 + z\lambda_n(A)). \tag{3.4.25}$$

In particular,

$$\mathrm{tr}(A) = \sum_n \lambda_n(A). \tag{3.4.26}$$

Proof. Combining Corollary 3.4.1 with Theorem 3.4.6 and the Hadamard product formula, we arrive at (3.4.25) since $D(0) = 1$. We only need to mention that

$$\sum_n |\lambda_n(A)| < \infty,$$

by the Lalesco–Schur–Weyl inequality (3.4.14) with $p = 1$.

Finally, the RHS in (3.4.25) has the form

$$1 + z\sum_n \lambda_n + \mathcal{O}(z^2)$$

as $z \to 0$. Comparing it with (3.4.16), we arrive at (3.4.26). □

Corollary 3.4.3. *If A, $B \in \mathfrak{S}_\infty(\mathfrak{H})$ have the property that both AB, $BA \in \mathfrak{S}_1(\mathfrak{H})$, then*

$$\mathrm{tr}(AB) = \mathrm{tr}(AB). \tag{3.4.27}$$

Proof. It is well known that AB and BA have the same non-zero eigenvalues, including identical algebraic multiplicities. Hence Lidskii's theorem implies (3.4.27). □

Let us finish this section by giving explicit formulas for $\det(A)$ for some simple operators. Suppose that $A = \mathcal{K}$ is an integral operator on $L^2([a, b])$ with continuous kernel K,

$$(\mathcal{K}f)(x) = \int_a^b K(x, y)f(y)dy. \tag{3.4.28}$$

Theorem 3.4.8. *Let $\mathcal{K}$ be given by (3.4.28) with $K \in C([a, b] \times [a, b])$. If $\mathcal{K} \in \mathfrak{S}_1$, then*

$$\mathrm{tr}(\mathcal{K}) = \int_a^b K(x, x)dx, \tag{3.4.29}$$

and

$$\det(I + \mathcal{K}) = \sum_{n \geq 0} \frac{\alpha_n(K)}{n!}, \tag{3.4.30}$$

where

$$\alpha_n(K) := \int_a^b \dots \int_a^b K \begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} dx_1 \dots dx_n, \tag{3.4.31}$$

and

$$K \begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} := \det[K(x_i, y_j)]_{1 \leq i,j \leq n}. \tag{3.4.32}$$

Proof. Let, for simplicity, $[a, b] = [0, 1]$. For each $n \geq 1$, let $\{\phi_{m,n}\}_{m=1}^{2^n}$ be the functions

$$\phi_{m,n} := \begin{cases} 2^{n/2}, & \frac{m-1}{2^n} \leq x < \frac{m}{2^n} \\ 0, & \text{otherwise} \end{cases}.$$

Denote by P_n the orthogonal projection in $L^2([a, b])$ onto $\mathrm{span}\{\phi_{m,n}\}$. Then

$$\mathrm{tr}(\mathcal{K}) = \lim_{n \to \infty} \mathrm{tr}(P_n \mathcal{K} P_n),$$

Exercise! since one can construct an orthonormal basis $\{\psi_k\}$ such that $\mathrm{span}\{\psi_k\}_{k=1}^{2^n} = \mathrm{ran}(P_n)$ for all $n \in \mathbb{N}$.

Now we compute by noting that $\{\phi_{m,n}\}$ form an orthonormal basis in $\operatorname{ran}(P_n)$:

$$\begin{aligned}\operatorname{tr}(P_n\mathcal{K}P_n) &= \sum_{m=1}^{2^n}(\mathcal{K}\phi_{m,n},\phi_{m,n})\\ &= 2^n\sum_{m=1}^{2^n}\iint_{\frac{m-1}{2^n}\le x,y<\frac{m}{2^n}} K(x,y)dxdy.\end{aligned}$$

Using the uniform continuity of K, one easily concludes that the limit converges to the RHS in (3.4.29).

Let Q_n be the orthogonal projection in $\otimes^n L^2([a,b])$ onto $\Lambda^n(L^2([a,b]))$. Then

$$\Lambda^n(\mathcal{K}) = Q_n\mathcal{K}^{\otimes n}Q_n.$$

Let us show that $\Lambda^n(\mathcal{K})$ is an integral operator with the kernel

$$\frac{1}{n!}K\begin{pmatrix}x_1 & \dots & x_n\\ y_1 & \dots & y_n\end{pmatrix}.$$

Indeed, we have[6]

$$\Lambda^n(\mathcal{K})(f_1\wedge\dots\wedge f_n) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\mathcal{K}f_{\sigma(1)}\otimes\dots\otimes\mathcal{K}f_{\sigma(n)}$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\int_0^1 K(x_1,y_1)f_{\sigma(1)}(y_1)dy_1\dots\int_0^1 K(x_n,y_n)f_{\sigma(n)}(y_n)dy_n$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\int_0^1 K(x_1,y_{\sigma(1)})f_{\sigma(1)}(y_{\sigma(1)})dy_{\sigma(1)}\dots\int_0^1 K(x_n,y_{\sigma(n)})f_{\sigma(n)}(y_{\sigma(n)})dy_{\sigma(n)}$$

$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\int_0^1\dots\int_0^1 K(x_1,y_{\sigma(1)})\dots K(x_n,y_{\sigma(n)})f_{\sigma(1)}(y_{\sigma(1)})\dots f_{\sigma(n)}(y_{\sigma(n)})dy_{\sigma(1)}\dots dy_{\sigma(n)}$$

[6] For $n=2$,

$$\begin{aligned}&2!\Lambda^2(\mathcal{K})(f_1\wedge f_2) = \mathcal{K}f_1\otimes\mathcal{K}f_2 - \mathcal{K}f_2\otimes\mathcal{K}f_1\\ &\quad= \int_0^1 K(x_1,y_1)f_1(y_1)dy_1\int_0^1 K(x_2,y_2)f_2(y_2)dy_2 - \int_0^1 K(x_1,y_1)f_2(y_1)dy_1\int_0^1 K(x_2,y_2)f_1(y_2)dy_2\\ &\quad= \int_0^1 K(x_1,y_1)f_1(y_1)dy_1\int_0^1 K(x_2,y_2)f_2(y_2)dy_2 - \int_0^1 K(x_1,y_2)f_2(y_2)dy_2\int_0^1 K(x_2,y_1)f_1(y_1)dy_1\\ &\quad= \int_0^1\int_0^1 K(x_1,y_1)K(x_2,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 - \int_0^1\int_0^1 K(x_1,y_2)K(x_2,y_1)f_1(y_1)f_2(y_2)dy_1dy_2\\ &\quad= \int_0^1\int_0^1(K(x_1,y_1)K(x_2,y_2) - K(x_1,y_2)K(x_2,y_1))f_1(y_1)f_2(y_2)dy_1dy_2.\end{aligned}$$

Notice that in the second equality we changed the integration variable and then in the third one we changed the order of integration

$$= \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\int_0^1\dots\int_0^1 K(x_1,y_{\sigma(1)})\dots K(x_n,y_{\sigma(n)})f_1(y_1)\dots f_n(y_n)dy_1\dots dy_1$$
$$= \int_0^1\dots\int_0^1\Big(\frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma K(x_1,y_{\sigma(1)})\dots K(x_n,y_{\sigma(n)})\Big)f_1(y_1)\dots f_n(y_n)dy_1\dots dy_n$$
$$= \frac{1}{n!}\int_0^1\dots\int_0^1 K\begin{pmatrix}x_1 & \dots & x_n\\ y_1 & \dots & y_n\end{pmatrix}f_1(y_1)\dots f_n(y_n)dy_1\dots dy_n.$$

Finally, using the same argument as in the proof of (3.4.29), one shows that

$$\operatorname{tr}(\Lambda^n(\mathcal{K})) = \frac{\alpha_n(\mathcal{K})}{n!}.$$

It remains to apply (3.4.15). □

Remark 3.4.3. Formulas (3.4.30)–(3.4.32) provide the original expansion of Fredholm [**15**].

Remark 3.4.4. That Lidskii's theorem is subtle is shown by an example of A. Grothendieck [**19**]. Namely, any Banach space has a natural class of operators, the nuclear operators, $\mathcal{N}_1$, which have a trace.[7] So one can ask if Lidskii's theorem extends to arbitrary Banach spaces. What Grothendieck does is finds $A\in\mathcal{N}_1(\ell^1)$ so that $\operatorname{tr}(A)=1$ but $A^2=0$ (so the only eigenvalue is 0). Thus Lidskii's theorem does not extend to ℓ^1!

Another subtle issue with general Banach spaces is the fact that one can define a matrix trace only if X has a Schauder basis, which is further equivalent to the fact that X has the approximation property (every compact operator can be approximated by finite rank operators).

Setting

$$s_{n+1}(A) = \int_{\operatorname{rank}(K)=n}\|A-K\|_X, \qquad n\in\mathbb{Z}_{\geq 0},$$

let $\mathfrak{S}_1(X) := \{A\in[X]|\ \sum_n s_n(A)<\infty\}$ Let us also mention that

Remark 3.4.5. In fact, Lidskii's theorem appeared first in the work of A. Grothendieck [**20**] in 1956. Because this paper was on Banach space theory and Lidskii's theorem an aside, his work on this result was not widely known to those working on Hilbert space operator theory. In 1959, V. B. Lidskii [**34**], unaware of Grothendieck [**20**], rediscovered the theorem, and in a case of Arnold's principle, got the theorem named after him.
In these lectures we follow B. Simon [**49, 50**], who rediscovered the approach of A. Grothendieck in [**49**].

[7] In a Banach space X, for a finite rank operator A one can define the norm $\|A\|_1 := \inf\sum_n\|\ell_n\|\|\phi_n\|$, where inf is taken over all representations $A=\sum_n\ell_n(\cdot)\phi_n$. Then the class of *nuclear operators* $\mathcal{N}_1(X)$ is defined as a closure of finite rank operators w.r.t. $\|\cdot\|_1$. Defining $\operatorname{tr}(A)=\sum_n\ell_n(\phi_n)$ for a finite rank A and noting that $\operatorname{tr}(A)$ is independent of the decomposition $A=\sum_n\ell_n(\phi_n)$, the trace extends by continuity onto $\mathcal{N}_1(X)$.

3.5. Fredholm theory

By "Fredholm theory" we mean explicit formulas for $(1 + zA)^{-1} =: F(z)$, which work for all z with $-z^{-1} \notin \sigma(A)$. Notice that $f(\cdot)$ is not entire for general A, but it is meromorphic, so one would like to write it as a ratio of entire functions: $C(z)/B(z)$. $B(z)$ must have zeros where A has poles, that is, at points where $(1 + zA)$ does not have an inverse. This suggests that one take $B(z) := \det(I + zA)$. In fact, a detailed analysis shows that $(1 + zA)^{-1}$ has a pole at $-z_0^{-1} = \lambda_0 \in \sigma(A)$ — of order not more than the algebraic multiplicity, so that $\det(I + zA) \cdot (I + zA)^{-1}$ is indeed an entire function. Rather than go through this analysis or follow the conventional analysis, we use a slightly different approach. Observe that

$$\begin{aligned}\det(I + A + \lambda B) &= \det(I + A)\det(I + \lambda(I + A)^{-1}B)\\ &= \det(I + A)(1 + \lambda \mathrm{tr}((I + A)^{-1}B) + \mathcal{O}(\lambda^2)),\end{aligned}$$

and $\mathrm{tr}((I+A)^{-1}B)$ is a linear functional on $\mathfrak{S}_1$. Thus, $(I+A)^{-1}\det(I+A)$ is the derivative of $A \mapsto \det(I + A)$. To make it more precise, recall the definition of the Fréchet derivative of a map between Banach spaces.

Definition 3.5.1. Let X, Y be Banach spaces. A function $f\colon X \to Y$ is called *Fréchet differentiable* at $x_0 \in X$ if there is a bounded linear map $T \in [X, Y]$ such that

$$\|f(x_0 + x) - f(x_0) - Tx\| = o(\|x\|). \tag{3.5.1}$$

The operator T is called *the Fréchet derivative* and it will be denoted by Df_{x_0}.

Notice that in the case when $Y = \mathbb{C}$ the derivative Df_{x_0} is a bounded linear functional on X and hence $Df_{x_0} \in X^*$.

Theorem 3.5.1. *Let $f\colon \mathfrak{S}_1(\mathfrak{H}) \to \mathbb{C}$ be given by $f\colon A \mapsto \det(I+A)$. Then f is Fréchet differentiable at any $A \in \mathfrak{S}_1(\mathfrak{H})$ with $-1 \notin \sigma(A)$ and its derivative is given by*

$$D(A) := Df_A = (1 + A)^{-1}\det(I + A). \tag{3.5.2}$$

Proof. Let A, $B \in \mathfrak{S}_1(\mathfrak{H})$ and $-1 \notin \sigma(A)$, that is $(I + A)^{-1} \in [\mathfrak{H}]$. Then

$$\begin{aligned}\det(I + A + zB) &= \det(I + A)\det(I + z(I + A)^{-1}B)\\ &= \det(I + A)\big(1 + z\,\mathrm{tr}((I + A)^{-1}B) + o(z^2)\big).\end{aligned}$$

More carefully, in view of (3.4.16) and Proposition 3.4.3(iii),

$$\det(I + C) = 1 + \mathrm{tr}(C) + o(\|C\|_1^2).$$

Thus

$$\det(I+A+B)-\det(I+A)-\det(I+A)\mathrm{tr}((I+A)^{-1}B)=\mathcal{O}(\|B\|_1^2). \tag{3.5.3}$$

Upon identifying $\mathfrak{S}_1(\mathfrak{H})^* = [\mathfrak{H}]$ (see Remark 3.2.1(iii); every bounded linear functional on $\mathfrak{S}_1(\mathfrak{H})$ has the form $\ell_A\colon B \mapsto \mathrm{tr}(AB)$, where $A \in [\mathfrak{H}]$), we arrive at (3.5.2). □

Corollary 3.5.1 (Cramer's rule)**.** *If $A \in \mathfrak{S}_1(A)$ and $-z^{-1} \notin \sigma(A)$, then*

$$(I+zA)^{-1} = \frac{D(zA)}{\det(I+zA)}. \tag{3.5.4}$$

Before stating the final set of formulas, we need the following fact.

Lemma 3.5.1. *Let f and g be analytic near 0 with*

$$f(z) = 1 + \sum_{n\ge1} a_n \frac{z^n}{n!}, \qquad g(z) = \sum_{n\ge1}(-1)^{n+1} b_n \frac{z^n}{n},$$

and

$$f(z) = \mathrm{e}^{g(z)}.$$

Then

$$a_n = \det\begin{pmatrix} b_1 & n-1 & 0 & \dots & 0 \\ b_2 & b_1 & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & 1 \\ b_n & b_{n-1} & b_{n-2} & \dots & b_1 \end{pmatrix}, \quad n \ge 1. \tag{3.5.5}$$

Proof. Observe that

$$f'(z) = g'(z) f(z),$$

and hence equating the coefficients in z^n, we get that a_n are defined inductively

$$\frac{a_{n+1}}{n!} = \sum_{k=0}^{n} (-1)^k b_{k+1} \frac{a_{n-k}}{(n-k)!} = \sum_{k=1}^{n+1} (-1)^{k+1} b_k \frac{a_{n+1-k}}{(n+1-k)!}$$

for all $n \ge 0$ with the convention $a_0 \equiv 1$. Expanding the determinant (3.5.5) in the first column, we get

$$a_n = \sum_{k=1}^{n} (-1)^{k-1} b_k a_{n-k} \frac{(n-1)!}{(n-k)!},$$

which proves the claim. □

Now the idea is to apply Lemma 3.5.1 to the following well known from the linear algebra course formula:

$$\det(I+zA)=\mathrm{e}^{\operatorname{tr}(\log(I+zA))},$$

which holds for all sufficiently small $z\in\mathbb{C}$ (for instance, for $|z|<1/\|A\|$). In fact, one can extend it by continuity for all $A\in\mathfrak{S}_1$:

$$\det(I+zA)=\exp\Big\{\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}z^n\operatorname{tr}(A^n)\Big\}. \tag{3.5.6}$$

Exercise 3.5.1. Prove (3.5.6) by using Lidskii's theorem.
Hint: Notice that the LHS has the form (3.4.25)

Applying Lemma 3.5.1 to (3.5.6), we get the following formulas.

Theorem 3.5.2 (Plemelj–Smithies formulas)**.** *Define $\alpha_n(A)$ and $\beta_n(A)$ for $A\in\mathfrak{S}_1(A)$ by*

$$\det(I+zA)=1+\sum_{n\geq 1}\alpha_n(A)\frac{z^n}{n!},$$

$$D(zA)=\det(I+zA)I+\sum_{n\geq 1}\beta_n(A)\frac{z^n}{(n-1)!}.$$

Then

$$\alpha_n(A)=\det\begin{pmatrix}\operatorname{tr}(A) & n-1 & 0 & \dots & 0\\ \operatorname{tr}(A^2) & \operatorname{tr}(A) & n-2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \operatorname{tr}(A^{n-1}) & \operatorname{tr}(A^{n-2}) & \operatorname{tr}(A^{n-3}) & \dots & 1\\ \operatorname{tr}(A^n) & \operatorname{tr}(A^{n-1}) & \operatorname{tr}(A^{n-2}) & \dots & \operatorname{tr}(A)\end{pmatrix} \tag{3.5.7}$$

and

$$\beta_n(A)=\det\begin{pmatrix}A & n-1 & 0 & \dots & 0\\ A^2 & \operatorname{tr}(A) & n-2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A^{n-1} & \operatorname{tr}(A^{n-2}) & \operatorname{tr}(A^{n-3}) & \dots & 1\\ A^n & \operatorname{tr}(A^{n-1}) & \operatorname{tr}(A^{n-2}) & \dots & \operatorname{tr}(A)\end{pmatrix}. \tag{3.5.8}$$

Proof. (3.5.7) is straightforward. To prove (3.5.8) observe that by (3.5.2)

$$D(zA)=(I+zA)^{-1}\det(I+zA)=\det(I+zA)\sum_{n\geq 0}z^nA^n.$$

Taking into account (3.5.7), we get after equating the coefficients in z^n:

$$\frac{\beta_{n+1}(A)}{n!}=\sum_{k=1}^{n}(-1)^{k+1}A^k\frac{\alpha_{n+1-k}(A)}{(n+1-k)!}. \tag{3.5.9}$$

□

Remark 3.5.1. Most of this section is an abstract version of the concrete Fredholm theory that solved

$$f(x) = g(x) + \int_a^b K(x,y)f(y)dy \tag{3.5.10}$$

for f. The Plemelj–Smithies formulae were first found in 1904 by J. Plemelj [**41**] and rediscovered and popularized by F. Smithies [**52**].

3.6. Regularized determinants

There are several approaches how to extend the notion of *determinant* to compact operators not belonging to the trace class. For example, assume that A is Hilbert–Schmidt, $A \in \mathfrak{S}_2(\mathcal{H})$. As we know, both $\det(I + A)$ and $\mathrm{tr}(A)$ may be singular, however, it turns out that

$$\det(I + A)\mathrm{e}^{-\mathrm{tr}(A)}$$

can be extended by continuity from $\mathfrak{S}_1(\mathcal{H})$ onto $\mathfrak{S}_2(\mathcal{H})$! More specifically, once $A^n \in \mathfrak{S}_1(\mathcal{H})$ for some $n \in \mathbb{N}$, then eliminating the first n terms in the expansion on the RHS in (3.5.6) allows to extend it to the whole class $\mathfrak{S}_n(\mathcal{H})$. These ideas trace back to D. Hilbert and T. Carleman. H. Poincaré suggested to consider the following

$$(I + A) = (I - A)^{-1}(I - A^2),$$

which also allows to introduce the determinant for $A \in \mathfrak{S}_2$ and also to extend Fredholm's formulas for the Hilbert–Schmidt class. However, one of the drawbacks is, for example, the situation when $1 \in \sigma(A)$. A very convenient approach was suggested by E. Seiler [**48**], which is a mixture of Poincaré and Hilbert–Carleman approaches.

Lemma 3.6.1. *For $A \in [\mathcal{H}]$, set*

$$\mathcal{R}_2(A) := (I + A)\mathrm{e}^{-A} - I. \tag{3.6.1}$$

Then $\mathcal{R}_2(A) \in \mathfrak{S}_1(\mathcal{H})$ if $A \in \mathfrak{S}_2(\mathcal{H})$.

Proof. Let $g(z) = (1 + z)\mathrm{e}^{-z} - 1$. Then $h(z) := g(z)/z^2$ is entire since $g(z) = -z^2/2 + \mathcal{O}(z^3)$ near zero. Hence $h(A) \in [\mathcal{H}]$ if $A \in [\mathcal{H}]$ and, moreover, $g(A) = A^2h(A)$. In particular, $g(A) \in \mathfrak{S}_1(\mathcal{H})$ whenever $A^2 \in \mathfrak{S}_1(\mathcal{H})$. It remains to notice that by Theorem 3.3.1(iv), $A^2 \in \mathfrak{S}_1(\mathcal{H})$ if $A \in \mathfrak{S}_2(\mathcal{H})$. $\square$

Remark 3.6.1. The same approach allows to consider operators belonging to the Schatten–von Neumann class $\mathfrak{S}_n(\mathcal{H})$ with an arbitrary $n \in \mathbb{N}$ by setting

$$\mathcal{R}_n(A) := (I + A)\exp\Big(\sum_{j=1}^{n-1}\frac{(-A)^j}{j}\Big) - I. \tag{3.6.2}$$

Definition 3.6.1. For $A \in \mathfrak{S}_2(\mathcal{H})$,

$$\det_2(I + A) := \det(I + \mathcal{R}_2(A)). \tag{3.6.3}$$

Theorem 3.6.1. *Let* $A \in \mathfrak{S}_2(\mathcal{H})$. *Then:*

(i)
$$\det_2(I + zA) = \prod_n (1 + z\lambda_n(A))\mathrm{e}^{-z\lambda_n(A)}. \tag{3.6.4}$$

(ii) $|\det_2(I + A)| \le \mathrm{e}^{2\|A\|_2^2}$.

(iii)
$$|\det_2(I + A) - \det_2(I + B)| \le \|A - B\|_2 \mathrm{e}^{C(1+\|A\|_2+\|B\|_2)^2}. \tag{3.6.5}$$

(iv) *If* $A \in \mathfrak{S}_1(\mathcal{H})$, *then*
$$\det_2(I + A) = \det(I + A)\mathrm{e}^{-\mathrm{tr}(A)}. \tag{3.6.6}$$

(v) $-1/z \in \sigma(A)$ *only if* $\det_2(I + zA) = 0$.

Proof. (i) By the spectral mapping theorem,

$$\lambda_k(\mathcal{R}_2(A)) = g(\lambda_k(A)) = (1 + \lambda_k(A))\mathrm{e}^{-\lambda_k(A)} - 1,$$

and hence (3.6.4) follows from (3.4.25).

(ii) Noting that $|1 + g(z)| \le \mathrm{e}^{2|z|^2}$ for all $z \in \mathbb{C}$. Indeed, since $1 + g(z) = (1 + z)\mathrm{e}^{-z}$, the estimate is obvious for $|z|$ large enough, say for all $|z| > 1/2$. If $|z| \le 1/2$, then

$$|\log(1 + g(z))| = |\log(1 + z) - z| \le \sum_{n \ge 2} \frac{|z|^n}{n} \le 2|z|^2.$$

It remains to use (3.6.4) and apply the Schur–Lalesco–Weyl inequality (3.4.14) with $p = 2$.

(iii) Follows from (ii) by following the arguments of the proof of Theorem 3.4.4 by taking

$$f(z) = \det_2(I + (A + B)/2 + z(A - B))$$

and $R^{-1} = \|A - B\|_2$

(iv) is obvious from (i) and Lidskii's theorem.

(v) is obvious from (iv) or the fact that $I + A$ is invertible exactly when so is $I + \mathcal{R}_2(A)$. □

Remark 3.6.2. Let us mention that the only important property lost is (3.4.22). However, instead we have

$$\det_2(I + A + B + AB) = \det_2(I + A)\det_2(I + B)\mathrm{e}^{-\mathrm{tr}(AB)}. \tag{3.6.7}$$

We are not going to bother with developing Fredholm's theory for $\mathfrak{S}_n$ with $n \geq 2$, however, notice that the integral operator $\mathcal{K}$ with continuous, or more general with an L^2 kernel, is Hilbert–Schmidt.

Historical remarks

Modern operator theory has its roots in a seminal paper [**15**] of Ivar Fredholm (1866–1927). Fredholm was a Swedish mathematician and student of G. Mittag-Leffler. In 1901, E. Holmgren lectured on Fredholm's work and David Hilbert was so struck by the work that he totally shifted his attention to analysis, having made his reputation in algebra and geometry. According to E. Hellinger, at that time a student of Hilbert, when Hilbert announced his seminar would be devoted to the study of integral equations, he declared he expected to be able to use them to prove the Riemann hypothesis! Apparently, he hoped to realize the zeta function as a Fredholm determinant — something he did not succeed at. However, in 1972 Ludwig Faddeev and Boris Pavlov observed that the zeta function appears in the scattering matrix for the automorphic wave equation [**32**, f-la (7.68)],

$$u_{tt} = y^2 \Delta u + \frac{1}{4} u$$

on the Poincaré plane. In particular, poles of the scattering matrix coincide with the zeros of the zeta function. Moreover, they noticed that Riemann's hypothesis is equivalent to certain properties of the scattering matrix [**32**, Theorem 7.23], however, the analysis of those properties is also a highly nontrivial problem (see [**32**, Sect. 7] and also [**31**, Chapter 37.9][8]).

[8] The title of Appendix 2 to Section 7 in [**32**] is "How not to prove the Riemann hypothesis".

Applications

4.1. Bound state problems

In this section we want to apply trace ideal methods to study the eigenvalues of $-\Delta + V$. We shall assume for simplicity that $V = V^* \in C_c^\infty(\mathbb{R}^d)$ in order to avoid some technical issues (e.g., the definition and self-adjoinness of $-\Delta + V$ if $V \notin L^\infty$) since $-\Delta$ is an unbounded operator. We set

$$V^{1/2}(x) := V(x)/|V|^{1/2}(x), \qquad x \in \mathbb{R}^d. \tag{4.1.1}$$

The problem is to estimate the number of negative eigenvalues of $H_V := -\Delta + V$ $(H := -\Delta)$. We begin with the following simple fact.

Lemma 4.1.1. *A negative number $E < 0$ is an eigenvalue of H_V if and only if* 1 *is an eigenvalue of*

$$K_E(V) := -V^{1/2}(-\Delta - E)^{-1}|V|^{1/2}, \tag{4.1.2}$$

in which case their multiplicities are equal.

Proof. First observe that

$$(H_V - E)^{-1} = (-\Delta + V - E)^{-1} = (-\Delta - E)^{-1}(I + V(-\Delta - E)^{-1})^{-1}.$$

Thus $E < 0$ is an eigenvalue of H_V if and only if -1 is an eigenvalue of $V(-\Delta - E)^{-1}$. However, for bounded operators A and B, the nonzero eigenvalues (including multiplicities) of AB and BA coincide. Since $V = V^{1/2}|V|^{1/2}$, -1 is an eigenvalue of $V(-\Delta - E)^{-1}$ if and only 1 is an eigenvalue of $K_E(V)$. The multiplicity question is left as an exercise. $\square$

For later purposes, notice that the absolute value of the *Birman–Schwinger operator* $K_E(V)$ is simply

$$|K_E(V)| = |V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2} = -K_E(|V|), \tag{4.1.3}$$

since

$$\begin{aligned} K_E(V)^* K_E(V) &= |V|^{1/2}(-\Delta - E)^{-1}V^{1/2} \cdot V^{1/2}(-\Delta - E)^{-1}|V|^{1/2} \\ &= (|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2})^2 \end{aligned}$$

and $|V|^{1/2}(-\Delta - E)^{-1}|V|^{1/2}$ is clearly a non-negative self-adjoint operator.

Next define

$$N_E(V) := \dim P_{H_V}((-\infty, E]), \tag{4.1.4}$$

where P_{H_V} is a spectral projection (e.g., resolution of identity of H_V). In our case, $N_E(V)$ is exactly the number of eigenvalues (counting multiplicities) of H_V not greater than E.

The general procedure for relating $N_E(V)$ to eigenvalues of $K_E(V)$ is given by the following result obtained independently by M. Sh. Birman [**2**] and J. Schwinger [**47**].

Proposition 4.1.1 (The Birman–Schwinger principle)**.** *For $E < 0$, $N_E(V)$ is exactly the number of eigenvalues of $K_E(V)$ which are larger than or equal to* 1*. Moreover,*

$$N_E(V) \leq \#\{\textit{eigenvalues of } |K_E(V)| \geq 1\} \tag{4.1.5}$$

for all $E \leq 0$.

In particular,

$$N_E(V) \leq \min\{\mathrm{tr}|K_E(V)|,\ \|K_E(V)\|_2^2\}. \tag{4.1.6}$$

Proof. Let $\lambda \in \mathbb{R}$. Let also $\varepsilon_n(\lambda)$ be the n-th eigenvalue of $H_{\lambda V} = -\Delta + \lambda V$ counting from $\inf \sigma(H_{\lambda V})$ and counting multiplicities. If $H_{\lambda V}$ has only m eigenvalues below 0, we set $\varepsilon_n(\lambda) = 0$ for all $n > m$. Moreover, once $\varepsilon_n(\lambda) < 0$, they are strictly monotone (follows from the minmax principle). Thus, for $E < 0$,

$$\begin{aligned} N_E(V) &= \#\{\lambda \leq 1 \,|\, \varepsilon_n(\lambda) = E \text{ for some } n\} \\ &= \#\{\lambda \leq 1 \,|\, \lambda K_E(V) \text{ has eigenvalue } 1\} \\ &= \#\{\text{eigenvalues of } K_E(V) \geq 1\}. \end{aligned}$$

This also implies (4.1.5) for all $E < 0$. To obtain the $E = 0$ result, notice that $|K_E(V)| \leq |K_{E=0}(V)|$ (monotonicity of $(-\Delta - E)^{-1}$) and $N(V) = \lim_{E\uparrow 0} N_E(V)$.

Finally, observe the following simple estimate which proves (4.1.6)

$$\operatorname{tr}|K_E(V)| \geq \sum_{\lambda \in \sigma(|K_E(V)|)\cap[1,\infty)} \lambda \geq \sum_{\lambda \in \sigma(|K_E(V)|)\cap[1,\infty)} 1. \qquad \square$$

Remark 4.1.1. In our considerations it is hidden that the Birman–Schwinger operator $K_E(V)$ is compact if $V \in C_c^\infty(\mathbb{R}^d)$. All the above proofs remain true under the compactness assumption. In fact, compactness of $K_E(V)$ is a rather nontrivial task since it is closely connected with compactness of embeddings of certain functional spaces. In our case, it is not difficult to see that $K_E(V)$ is compact (it even belongs to some ideals as we shall see this below).

Now we are ready to state some bound state estimates for the Schrödinger operator H_V.

Theorem 4.1.2. *Let $V_\pm := (|V| \pm V)/2$. Then*

(i) **Birman–Schwinger***: For $d = 3$,*

$$N(V) \leq \frac{1}{(4\pi)^2} \iint \frac{V_-(x)V_-(y)}{|x-y|^2} dxdy. \tag{4.1.7}$$

(ii) **Bargmann***: Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$, $\ell > -1/2$ and let $H_{V,\ell}$ be the operator defined in $L^2(\mathbb{R}_{\geq 0})$ by*

$$\tau_{V,\ell} := -\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + V(x).$$

In the case $|\ell| < 1/2$ boundary conditions $x = 0$ parameterizing the Friedrichs extension are assumed (for $\ell = 0$ it is simply the Dirichlet b.c.$f(0) = 0$). Then

$$N(V) \leq \frac{1}{2\ell+1} \int_0^\infty xV_-(x)dx. \tag{4.1.8}$$

(iii) **Cwikel–Lieb–Rozenblum***: For $d \geq 3$, there is $C_d > 0$ (independent of V!) such that*

$$N(V) \leq C_d \int_{\mathbb{R}^d} V_-(x)^{d/2}dx. \tag{4.1.9}$$

(iv) *If $d = 1$ or $d = 2$ and $V = -V_- \leq 0$ is not identically zero, then H_V has at least one negative eigenvalue.*

(v) **Bargmann***: If $d = 1$, then*

$$N(V) \leq 1 + \int_{\mathbb{R}} |x|V_-(x)dx \tag{4.1.10}$$

Proof. It is an immediate consequence of the minimax principle that $N_E(V) \leq N_E(-V_-)$ for all $E \leq 0$ and hence it suffices to show the above estimates for $V = -V_- \leq 0$.

(i) Notice that for $d = 3$, $(-\Delta - E)^{-1}$ is the integral operator

$$(-\Delta - E)^{-1} f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\sqrt{|E|}|x-y|}}{|x-y|} f(y) dy, \qquad E < 0. \tag{4.1.11}$$

Next notice that

$$|K_E(V)| f(x) = \frac{1}{4\pi} |V|^{1/2}(x) \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\sqrt{|E|}|x-y|}}{|x-y|} |V|^{1/2}(y) f(y) dy,$$

and the Hilbert–Schmidt norm of this operator is exactly (see (3.3.9))

$$\|K_E(V)\|_2^2 = \frac{1}{(4\pi)^2} \iint \frac{|V(x)||V(y)|}{|x-y|^2} \mathrm{e}^{-2\sqrt{|E|}|x-y|} dx dy.$$

Moreover, the latter holds for all $E \le 0$. It remains to apply the Birman–Schwinger principle.

(ii) We shall prove (4.1.8) only for $\ell = 0$ (just in order to avoid the use of Bessel functions). For $E < 0$,

$$(H_{V,0} - E)^{-1} f(x) = \int_0^\infty G(x, y; E) f(y) dy,$$

where the Green's function is given by

$$G(x, y; E) = \frac{1}{\sqrt{|E|}} \begin{cases} \sinh(\sqrt{|E|}x) \mathrm{e}^{-\sqrt{|E|}y}, & x \le y \\ \sinh(\sqrt{|E|}y) \mathrm{e}^{-\sqrt{|E|}x}, & x \ge y \end{cases}.$$

The corresponding Birman–Schwinger operator $K_E(V)$ is also an integral operator

$$K_E(V) f(x) = \int_0^\infty V_-^{1/2}(x) G(x, y; E) V_-^{1/2}(y) f(y) dy.$$

Moreover, it is non-negative and its kernel is positive definite. Hence by Mercer's theorem 3.3.3, $K_E(V)$ is trace class and

$$\mathrm{tr} K_E(V) = \int_0^\infty G(x, x; E) V(x) dx = \int_0^\infty \frac{\sinh(\sqrt{|E|}x)}{\sqrt{|E|}} \mathrm{e}^{-\sqrt{|E|}x} V(x) dx.$$

Applying the Birman–Schwinger principle and sending $E \uparrow 0$, we end up with the Bargmann bound.

(iv) Since $V \le 0$, $K_E(V)$ is a non-negative self-adjoint operator and hence $\|K_E(V)\|$ is an eigenvalue (if, e.g., $K_E(V)$ is compact). Thus according to the Birman–Schwinger principle, it suffices to show that

$$\lim_{E \uparrow 0} \|K_E(V)\| = +\infty. \tag{4.1.12}$$

In fact, (4.1.12) is also necessary since $\|K_E(V)\| \le L$ would imply that $H_{\lambda V}$ with $\lambda < \frac{1}{L}$ has no eigenvalues.

Since $K_E(V)$ is self-adjoint, it suffices to find $f \in L^2$ such that

$$(K_E(V)f, f)_{L^2} \to \infty$$

as $e \uparrow 0$. However, using the Fourier transform $\mathcal{F}$, which is unitary and sends $-\Delta$ to the multiplication operator, we get

$$(K_E(V)f, f)_{L^2} = ((-\Delta - E)^{-1}|V|^{1/2}f, |V|^{1/2}f) = \int \frac{(\widehat{|V|^{1/2}f})(\lambda)}{\lambda^2 - E} d\lambda.$$

By choosing f with $|V|^{1/2}f \in L^1(\mathbb{R}^d)$ and $(\widehat{|V|^{1/2}f})(0) \neq 0$ for all $\lambda \in \mathbb{R}^d$, we see that the RHS diverges as $E \uparrow 0$. $\square$

A few remarks are in order.

Remark 4.1.2. (i) is due to Birman [**2**] and Schwinger [**47**] independently. (ii) is originally due to V. Bargmann [**1**] whose proof was very different from the above proof due to Birman and Schwinger. (iii) is due independently to G. V. Rozenblum [**43**], E. Lieb [**35**], and M. Cwikel [**10**]. Lieb, who gets the smallest value for C_d, uses Wiener path integrals. Rosenblum uses approximation methods in Sobolev spaces pioneered by M. Sh. Birman and M. Z. Solomjak. Cwikel uses the Birman–Schwinger principle, however, he needs to estimate the Birman–Schwinger operator in the norm slightly different than $\|\cdot\|_1$ or $\|\cdot\|_2$ (see [**50**, p.62]).

The Cwikel–Lieb–Rozenblum (CLR) bound is the most subtle and interesting of these results. It is impossible to overview its important connections in one single remark.

4.2. Scattering theory in 1D

In the Hilbert space $L^2(\mathbb{R})$, consider the 1D Schrödinger operator

$$H_{\lambda V} = -\frac{d^2}{dx^2} + \lambda V(x) \tag{4.2.1}$$

assuming for simplicity that $V = V^* \in C_c^\infty(\mathbb{R})$ and λ is a complex parameter, $\lambda \in \mathbb{C}$. The domain of definition is $\operatorname{dom}(H_V) = H^2(\mathbb{R})$. Since V is compactly supported, the differential equation (spectral problem)

$$-f'' + \lambda V(x)f = k^2 f, \qquad \operatorname{Im}(k) \geq 0, \tag{4.2.2}$$

has two solutions $f_\pm(k, x; \lambda)$, the so-called *Jost solutions*, such that

$$f_\pm(k, x; \lambda) = \mathrm{e}^{\pm ikx} \tag{4.2.3}$$

for $\pm x$ large enough. Clearly, these solutions behave at infinity as solutions of the unperturbed ($V \equiv 0$) equation. Moreover, $f_\pm$ are entire in k for every fixed $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Remark 4.2.1. One can establish the existence of the Jost solutions if V is an integrable function having a finite first moment,

$$\int_{\mathbb{R}}(1+|x|)|V(x)|dx<\infty.$$

Indeed, one can define f_+ as a solution of the integral equation

$$f_+(k,x;\lambda)=\mathrm{e}^{\mathrm{i}kx}-\lambda\int_x^\infty\frac{\sin(k(x-y))}{k}V(y)f_+(k,y;\lambda)dy,$$

and then employ the standard iteration procedure

$$f_+(k,x;\lambda)=\sum_{n=0}^\infty\lambda^n f_n^+(k,x;\lambda),\quad f_0^+(k,x):=\mathrm{e}^{\mathrm{i}kx}, \tag{4.2.4}$$

$$f_{n+1}^+(k,x):=\int_x^\infty\frac{\sin(k(x-y))}{k}V(y)f_n^+(k,y)dy, \tag{4.2.5}$$

for all $n\geq 0$, to prove the existence of solutions and their analyticity in k in $\mathbb{C}_+$ and continuity up to the real line. Moreover, both $f_\pm$ are entire functions in λ for each fixed x and k satisfying the following bounds

$$|f_+(k,x;\lambda)-\mathrm{e}^{\mathrm{i}kx}|\leq C\frac{\mathrm{e}^{-\mathrm{Im}(k)x}}{1+|k|}\int_x^\infty(1+|x|)|q(x)|dx,\qquad x\geq 0.$$

For further details see, e.g., [11].

Next we have

$$f_+(k,x;\lambda)=s_{11}(k)\mathrm{e}^{\mathrm{i}kx}+s_{12}(k)\mathrm{e}^{-\mathrm{i}kx}, \tag{4.2.6}$$

$$f_-(k,x;\lambda)=s_{21}(k)\mathrm{e}^{\mathrm{i}kx}+s_{22}(k)\mathrm{e}^{-\mathrm{i}kx}. \tag{4.2.7}$$

The first equality holds as $x\to-\infty$ and the second one hold as $x\to+\infty$. The solution $\frac{1}{s_{11}}f_+$ describes the plane wave $\mathrm{e}^{\mathrm{i}kx}$ sent in from $-\infty$, transmitting $\frac{1}{s_{11}}\mathrm{e}^{\mathrm{i}kx}$ to $+\infty$ and reflecting $\frac{s_{12}}{s_{11}}\mathrm{e}^{-\mathrm{i}kx}$ to $-\infty$. Thus $T:=\frac{1}{s_{11}}$ is called *the transmission coefficient* and $R_+:=\frac{s_{12}}{s_{11}}$ is *the reflection coefficient*.

Remark 4.2.2. One can show that

$$s_{11}(k)=s_{22}(k)=:a(k),\qquad s_{12}(k)=-s_{21}(-k)=:b(k).$$

Moreover,

$$a(k^*)=a(-k),\quad b(k)^*=b(-k),\qquad |a(k)|^2=1+|b(k)^2|$$

and hence

$$|T(k)|^2+|R(k)|^2=1.$$

The unitary matrix

$$S(k)=\begin{pmatrix}T(k)&R_+(k)\\R_-(k)&T(k)\end{pmatrix} \tag{4.2.8}$$

is called *the scattering matrix*.

Taking into account the fact that $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{-\mathrm{i}kx}$ is a fundamental system of solutions away of the support of V, it is straightforward to show that

$$T(k;\lambda) := \frac{1}{s_{11}(k)} = \frac{1}{s_{22}(k)} = \frac{1}{2\mathrm{i}k} W(f_+(k,x;\lambda), f_-(k,x;\lambda))$$
$$= \frac{\mathrm{e}^{-\mathrm{i}kx}}{2\mathrm{i}k}(f_+'(k,x;\lambda) + \mathrm{i}kf_+(k,x;\lambda)).$$

Consider now the integral operator

$$\mathcal{K}(k)f(x) = \frac{1}{2\mathrm{i}k}\int_{\mathbb{R}} |V|^{1/2}(x)\mathrm{e}^{\mathrm{i}k|x-y|}V^{1/2}(y)f(y)dy, \qquad \mathrm{Im}(k) > 0. \quad (4.2.9)$$

Clearly, $-\mathcal{K}(k)$ is the Birman–Schwinger operator $K_E(V)$ for $d = 1$ when $k = \sqrt{E} \in \mathrm{i}\mathbb{R}_{>0}$. In particular, one can show that for all $\mathrm{Im}(k) > 0$ this operator is trace class.

Theorem 4.2.1.

$$T(k;\lambda) = \det(I + \lambda\mathcal{K}(k)). \quad (4.2.10)$$

Proof. Let $D(k;\lambda) := \det(I + \lambda\mathcal{K}(k))$. We need to show that $T = D$. We know that $T(\cdot;\lambda)$ is analytic in $\mathbb{C}_+$ and continuous up to $\mathbb{R}$ except $k = 0$. One can show that so is $D(\cdot;\lambda)$ and hence it suffices to consider purely imaginary k.

So, fix $k \in \mathrm{i}\mathbb{R}_{>0}$. First of all, $D(k;\lambda) = 0$ for some λ if and only if

$$\phi = -\lambda\mathcal{K}(k)\phi$$

has a nontrivial solution $\phi \in L^2(\mathbb{R})$. Arguing as in the proof of Lemma 4.1.1, one can show that this happens exactly when

$$-f'' + \lambda V(x)f = k^2 f$$

has a nontrivial solution $f \in L^2(\mathbb{R})$. However, by (4.2.3) there is exactly one (up to a scalar multiple) solution which is square integrable near $+\infty$, f_+, and exactly one (again up to a scalar multiple) solution which is square integrable near $-\infty$, f_-. Hence the above equation has an L^2 solution if and only if the Wronskian of f_+ and f_- equals zero, that is, $T(k;\lambda) = 0$. Therefore, for a fixed k, $D(k;\cdot)$ and $T(k;\cdot)$ have the same zeros. Moreover, both are exponentially bounded and hence, by the Hadamard factorization theorem,

$$D(k;\lambda) = T(k;\lambda)\mathrm{e}^{c+d\lambda},$$

where $c, d \in \mathbb{C}$ depend only on k. However, by the normalization at $\lambda = 0$, $D(k;0) = 1$ and $T(k;0) = 1$, which implies that $c = 0$. It remains to notice that

$$D(k;\lambda) = T(k;\lambda) = 1 + \frac{\lambda}{2\mathrm{i}k}\int_{\mathbb{R}} V(x)dx + \mathcal{O}(\lambda^2)$$

as $\lambda \to 0$. The formula for T follows from the integral equations for $f_\pm$, and the formula D comes from (3.5.6) after computing the corresponding trace of $\mathcal{K}(k)$. □

4.3. Conservation laws for the KdV equation

The Korteweg–de Vries equation

$$u_t = -u_{xxx} + 6uu_x \tag{4.3.1}$$

is one of the most studied nonlinear equations. It was introduced in 1895[1] to model the behavior of long waves on shallow water with $u(t,x)$ representing the wave height above a flat bottom. Now it serves as an important effective model for a diverse range of physical phenomenas (see, e.g., [**9**]). Moreover, this equation is the first example of a completely integrable infinite dimensional Hamiltonian system [**14**]. In particular, it has infinitely many conserved quantities and the first three conservation laws are

$$I_1 = \int_{\mathbb{R}} u\,dx, \qquad I_2 = \int_{\mathbb{R}} u^2\,dx, \qquad I_3 = \int_{\mathbb{R}} \frac{1}{2}u_x^2 + u^3\,dx. \tag{4.3.2}$$

These are, respectively, the mass, momentum and energy (the Hamiltonian of the system).

A connection between the KdV equation and the scattering problem for 1-D Schrödinger equation (4.2.2) on the line was discovered in [**16**] and a very elegant formulation of this relationship was found by P. Lax in [**30**]. Namely, introducing the Lax pair

$$L(t) := H_{u(t)} = -\frac{d^2}{dx^2} + u(x,t), \tag{4.3.3}$$

$$A(t) := 4\frac{d^3}{dx^3} - 3\frac{d}{dx}u(x,t) - 3u(x,t)\frac{d}{dx}, \tag{4.3.4}$$

where $L(\cdot)$ and $A(\cdot)$ are understood as functions in t whose values are unbounded operators on the Hilbert space $L^2(\mathbb{R})$, it is straightforward to verify that u is a solution of the KdV equation exactly when

$$\frac{d}{dt}L(t) = [L(t), A(t)] := L(t)A(t) - A(t)L(t). \tag{4.3.5}$$

If u is real-valued and, for example, belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$, then $L(t)$ is a self-adjoint operator and $A(t)$ is a skew-self-adjoint operator for all t. Moreover, one has

$$L(t) = U(t)L(0)U(t)^{-1}, \tag{4.3.6}$$

[1] It appeared earlier in J. Boussinesq, *J. Math. Pures et Appl.* **17**, 55–108 (1872)

where unitary $U(t)$ is defined via

$$\frac{d}{dt}U(t) = -A(t)U(t), \quad U(0) = I. \tag{4.3.7}$$

Indeed, differentiating (4.3.6), we get

$$\begin{aligned}\frac{d}{dt}L(t) &= U_t(t)L(0)U(t)^{-1} - U(t)L(0)U^{-1}(t)U_t(t)U^{-1}(t)\\ &= -A(t)U(t)L(0)U(t)^{-1} + U(t)L(0)U^{-1}(t)A(t)U(t)U^{-1}(t)\\ &= -A(t)L(t) + L(t)A(t)\\ &= [L(t), A(t)].\end{aligned}$$

This says that $L(t)$ is unitarily equivalent to $L(0)$, that is *all spectral properties of the Schrödinger operator $L(0)$ are conserved under the KdV flow.* Another manifestation of this fact is the following observation made by C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura in 1967.

Theorem 4.3.1 ([**16**])**.** *Let u be a solution to the KdV equation (4.3.1) with the initial data $u(\cdot,0) := u_0 \in \mathcal{S}(\mathbb{R})$. Let also $T(k,t)$ and $R(k,t)$ be the transmission and reflection coefficients of $H_{u(t)} = -\frac{d^2}{dx^2} + u(x,t)$. Then*

$$T(k,t) = T(k,0), \qquad R(k,t) = R(k,0)\mathrm{e}^{8ik^3t} \tag{4.3.8}$$

for all $t \in \mathbb{R}$.

Remark 4.3.1. The above result provides a way to solve the nonlinear equation by essentially linear methods. More specifically, the evolution of scattering data is *linear*! Another manifestation of this fact is the following observation: Take a solution $w(x,t)$ of a linear part of (4.3.1), that is, $w_t = -w_{xxx}$. Then use w to make an operator on the half-line $[0,\infty)$ (which is a Hankel operator!),

$$W(t,x) : f \in C([0,\infty)) \mapsto \int_0^\infty w(t, x + \cdot + s) f(s)ds. \tag{4.3.9}$$

Now let $\Delta(t,x) := \det(I + W(t,x))$, which was introduced by F. Dyson [**12**]. Then miraculously

$$u(x,t) := \frac{1}{2}\partial_x^2 \log \Delta(t,x)$$

solves (4.3.1)! For further details we refer to [**37**, §3.4].

Denote $\mathcal{R}_0(k) := (H_0 + k^2)^{-1}$, $k > 0$ and set[2]

$$\begin{aligned}\alpha(k;V) &:= -\log\det\nolimits_2(I + \sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)})\\ &= \sum_{n\geq 2}\frac{(-1)^n}{n}\mathrm{tr}\Big(\big(\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)}\big)^n\Big).\end{aligned} \tag{4.3.10}$$

[2]We follow here the recent elegant work of R. Killip, M. Visan and X. Zhang [**27**]. For a different approach see [**29**].

Let us first show that the above quantity is correctly defined.

Lemma 4.3.1. *For any $V \in H^{-1}(\mathbb{R})$,*

$$\|\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)}\|^2_{\mathfrak{S}_2} = \frac{1}{k}\int_{\mathbb{R}} \frac{|\widehat{V}(\lambda)|^2}{\lambda^2+4k^2}d\lambda \tag{4.3.11}$$

Proof. Let us proof (4.3.11) for $V \in \mathcal{S}(\mathbb{R})$ to avoid the discussion of quadratic forms (however, notice that $\sqrt{\mathcal{R}_0(k)}$ maps L^2 into H^1 and also H^{-1} into L^2, and $V \in H^{-1}$ serves as a multiplicator from H^1 to H^{-1}). The free resolvent $\mathcal{R}_0$ is the integral operator with the kernel

$$R_0(x,y;-k^2) = \frac{1}{2k}\mathrm{e}^{-k|x-y|}, \quad x,y\in\mathbb{R}.$$

Then

$$\begin{aligned}
\|\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)}\|^2_{\mathfrak{S}_2} &= \operatorname{tr}(\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)}\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)})\\
&= \operatorname{tr}(V\mathcal{R}_0(k)V\mathcal{R}_0(k))\\
&= \frac{1}{4k^2}\iint V(x)\mathrm{e}^{-k|x-y|}V(y)\mathrm{e}^{-k|x-y|}dxdy\\
&= \frac{1}{k}\iint V(x)\mathcal{R}_0(x,y;-4k^2)V(y)dxdy\\
&= \frac{1}{k}(V,\mathcal{R}_0(2k)(V))_{L^2} = \mathrm{RHS}(4.3.11).
\end{aligned}$$

Finally, observe that (4.3.11) extends to all of H^{-1} by continuity. □

Corollary 4.3.1. *Let $V \in H^{-1}(\mathbb{R})$. Then for $k \geq 1$*

$$\frac{1}{4k^3}\|V\|^2_{H^{-1}} \leq \|\sqrt{\mathcal{R}_0(k)}V\sqrt{\mathcal{R}_0(k)}\|^2_{\mathfrak{S}_2} \leq \frac{1}{k}\|V\|^2_{H^{-1}} \tag{4.3.12}$$

Proof. Just recall that $\|V\|^2_{H^{-1}} = \int_{\mathbb{R}} \frac{|\hat{V}(\lambda)|^2}{\lambda^2+1}d\lambda$ and then use the simple estimate

$$\frac{1}{4k^2}\frac{1}{\lambda^2+1} \leq \frac{1}{\lambda^2+4k^2} \leq \frac{1}{\lambda^2+1}, \quad \lambda\in\mathbb{R},$$

which holds true for all $k \geq 1$. □

Proposition 4.3.2. *Let $u(t)$ be a solution to KdV with $u_0 \in \mathcal{S}(\mathbb{R})$. Then*

$$\frac{d}{dt}\alpha(k;u(t)) = 0 \tag{4.3.13}$$

for all $k \geq 1 + 9\|u(t)\|^2_{H^{-1}}$.

Remark 4.3.2. As the perturbation determinant is an analytic function of k in the right half-plane, constancy extends to this whole region. In particular, this fact together with Theorem 4.2.1 implies the first equality in (4.3.8).

Before proving this statement we need the following fact.

Lemma 4.3.2. *Let $t \mapsto A(t)$ be a C^1 curve in $\mathfrak{S}_2(\mathcal{H})$. If $\|A(t_0)\| < 1/3$, then in a sufficiently small neighbourhood I of t_0 the series*

$$\alpha(t) := \sum_{n\geq 2} \frac{(-1)^n}{n} \mathrm{tr}(A(t)^n)$$

converges and defines a C^1 function with

$$\frac{d}{dt}\alpha(t) := \sum_{n\geq 2} (-1)^n \mathrm{tr}(A(t)^{n-1}A'(t)).$$

Moreover, if $A(t)$ is self-adjoint, then

$$\frac{1}{3}\|A(t)\|_2^2 \leq \alpha(t) \leq \frac{2}{3}\|A(t)\|_2^2, \quad t \in I.$$

Proof. Take I to be the interval containing t_0 and such that $\|A(t)\|_2 \leq 1/3$ on I. Since

$$|\mathrm{tr}(A^n)| \leq \mathrm{tr}(|A|^n) \leq \|A\|^{n-2}\mathrm{tr}(|A|^2) \leq \|A\|_2^n,$$

both series converge whenever $\|A(t)\|_2 < 1$. Because of the stronger hypothesis, we have

$$\Big|\alpha(t) - \frac{1}{2}\mathrm{tr}(A(t)^2)\Big| \leq \sum_{n\geq 3} \frac{1}{n}\|A\|_2^n \leq \frac{1}{6}\|A\|_2^2,$$

which gives the desired estimate. The uniform convergence also implies that we can change the order of summation and differentiation. □

Proof of Proposition 4.3.2. The bounds (4.3.12) show that Lemma 4.3.2 applies. Thus, taking into account that $u(t) \in \mathcal{S}(\mathbb{R})$ as well as boundedness of $\mathcal{R}_0(k)$, we can cycle the trace:

$$\begin{aligned}
\frac{d}{dt}\alpha(k; u(t)) &= \sum_{n\geq 2} (-1)^n \mathrm{tr}\Big(\big(\sqrt{\mathcal{R}_0}u(t)\sqrt{\mathcal{R}_0}\big)^{n-1}\sqrt{\mathcal{R}_0}u_t(t)\sqrt{\mathcal{R}_0}\Big)\\
&= \sum_{n\geq 2} (-1)^n \mathrm{tr}\Big(\big(\mathcal{R}_0 u(t)\big)^{n-1}\mathcal{R}_0 u_t(t)\Big)\\
&= \sum_{n\geq 2} (-1)^n \mathrm{tr}\Big(\big(\mathcal{R}_0 u(t)\big)^{n-1}\mathcal{R}_0(-u_{xxx}(t) + 6u(t)u_x(t))\Big).
\end{aligned}$$

Thus, the claim would follow once we could prove that

$$\mathrm{tr}\big(\big(\mathcal{R}_0 u(t)\big)^{n-1}\mathcal{R}_0(-u_{xxx}(t))\big) = \mathrm{tr}\big(\big(\mathcal{R}_0 u(t)\big)^{n-2}\mathcal{R}_0(6u(t)u_x(t))\big), \quad (4.3.14)$$

for all $n \geq 2$, and

$$\mathrm{tr}\big(\mathcal{R}_0(6u(t)u_x(t))\big) = 0. \quad (4.3.15)$$

To prove (4.3.15), it suffices to notice that

$$\begin{aligned}\operatorname{tr}\big(\mathcal{R}_0(6u(t)u_x(t))\big) &= \int_{\mathbb{R}} R_0(x,x;-k^2)6u(x,t)u_x(x,t)dx \\ &= \frac{3}{2k}\int_R \partial_x u^2(x,t)dx = 0.\end{aligned}$$

Next, observe that $u_x = [\partial_x, u]$[3] and hence we get

$$\begin{aligned}-u_{xxx} &= -[\partial_x,[\partial_x,[\partial_x,u]]] = -[\partial_x,[\partial_x,\partial_x u - u\partial_x]] \\ &= -[\partial_x, \partial_x^2 u + u\partial_x^2 - 2\partial_x u \partial_x] \\ &= (-\partial_x^2+k^2)u_x + u_x(-\partial_x^2+k^2) - 2(-\partial_x^2+k^2)u\partial_x + 2\partial_x u(-\partial_x^2+k^2) \\ &\quad - 4k^2[\partial_x,u].\end{aligned}$$

Substituting this into (4.3.14) and cycling the trace gives:

$$\begin{aligned}\text{LHS}(4.3.14) &= \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-1}\mathcal{R}_0((-\partial_x^2+k^2)u_x + u_x(-\partial_x^2+k^2))\big) \\ &\quad + \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-1}\mathcal{R}_0(-2(-\partial_x^2+k^2)u\partial_x + 2\partial_x u(-\partial_x^2+k^2))\big) \\ &\quad \underbrace{-4k^2\operatorname{tr}\big(\big(\mathcal{R}_0u(t)\big)^{n-1}\mathcal{R}_0[\partial_x,u]\big)}_{=0} \\ &= \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-1}2u_x\big) + \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-1}(-2u\partial_x + 2\partial_x u)\big) \\ &= \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-2}\mathcal{R}_0(2uu_x)\big) + \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-2}\mathcal{R}_0(-2u^2\partial_x + 2u\partial_x u)\big) \\ &= \operatorname{tr}\big((\mathcal{R}_0u(t))^{n-2}\mathcal{R}_0\underbrace{(2uu_x + 2[\partial_x,u^2])}_{=6uu_x}\big) = RHS(4.3.14). \qquad \square\end{aligned}$$

Now we are in position to state and prove the main result, which provides asymptotic low regularity conservation laws for the KdV equation.

Theorem 4.3.3 ([27])**.** *Let $u(t)$ be a solution to the KdV equation with $u_0 \in \mathcal{S}(\mathbb{R})$. Then*

$$\|u_0\|_{H^s}(1+\|u_0\|_{H^s}^2)^{-\frac{|s|}{1+2|s|}} \lesssim \|u(\cdot,t)\|_{H^s} \lesssim \|u_0\|_{H^s}(1+\|u_0\|_{H^s}^2)^{|s|}, \quad (4.3.16)$$

for all $s \in [-1,0)$.

Proof. We shall prove (4.3.16) only for $s=-1$. Take $k \geq 1 + 9\|u_0\|_{H^{-1}}^2$. Then (4.3.12) implies that

$$\|\sqrt{\mathcal{R}_0(k)}u_0\sqrt{\mathcal{R}_0(k)}\|_{\mathfrak{S}_2}^2 < \frac{1}{9}$$

[3] Both the LHS and u on the RHS are considered as operators of multiplication, that is, $u_x f = [\partial_x, u]f = \partial_x(uf) - u(\partial_x f) = u_x f + uf_x - uf_x = u_x f$.

and hence Lemma 4.3.2 applies. Moreover, by conservation of α, we then have

$$\|\sqrt{\mathcal{R}_0(k)}u(t)\sqrt{\mathcal{R}_0(k)}\|_{\mathfrak{S}_2}^2 \leq 3\alpha(k;u(t)) \leq 2\|\sqrt{\mathcal{R}_0(k)}u_0\sqrt{\mathcal{R}_0(k)}\|_{\mathfrak{S}_2}^2 < \frac{2}{9}$$

in a neighbourhood of $t = 0$. A simple continuity argument implies that

$$\|\sqrt{\mathcal{R}_0(k)}u(t)\sqrt{\mathcal{R}_0(k)}\|_{\mathfrak{S}_2}^2 \leq 2\|\sqrt{\mathcal{R}_0(k)}u_0\sqrt{\mathcal{R}_0(k)}\|_{\mathfrak{S}_2}^2 < \frac{2}{9} \tag{4.3.17}$$

for all t whenever $k \geq k_0 := 1 + 9\|u_0\|_{H^{-1}}^2$. Using (4.3.12) we immediately get the upper bound in (4.3.16) with $s = -1$.

The lower bound in (4.3.16) follows directly from the upper bound by exploiting the time translation symmetry of the KdV equation. Given a solution $\widetilde{u}$ to (4.3.1) and a time $t_0 \in \mathbb{R}$, then $u(t) = \widetilde{u}(t + t_0)$ is also a solution to (4.3.1) and so by the upper bound in (4.3.16),

$$\|\widetilde{u}_0\|_{H^{-1}}^2 \leq \sup_{t\in\mathbb{R}} \|\widetilde{u}(t)\|_{H^{-1}}^2 \lesssim \|\widetilde{u}_0\|_{H^s}(1 + \|\widetilde{u}_0\|_{H^s}^2)^{|s|}.$$

Rearranging this inequality yields the lower bound for $\widetilde{u}$ at time t_0 and since time was arbitrary this give (4.3.16) in full generality for $s = -1$. $\square$

Remark 4.3.3 (Wellposedness of KdV)**.** The KdV equation is probably the most studied nonlinear equation, which serves as an important effective model for a diverse range of physical phenomenas [**9**]. It attracted a considerable interest after the discovery of *solitons* by Zabusky and Kruskal in 1965: the pulselike solitary waves solution to the KdV keep their shape and size after interaction. They termed these solutions *solitons*. Shortly after, Gardner, Greene, Kruskal and Miura gave a method of solution for the KdV equation by making use of the ideas of direct and inverse scattering (the core of this approach is Theorem 4.3.1). In 1968, Lax considerably generalized these ideas, and later, in 1971, V. E. Zakharov and L. D. Faddeev observed that the KdV is a completely integrable infinite dimensional Hamiltonian system and the scattering data of the corresponding isospectral operator H_u are its action-angle variables.

One of the most basic mathematical questions one may ask of (4.3.1): *whether is it wellposed?* That is, whether the problem has a solution? Is this solution unique or not? Does this solution depend continuously on time and initial data?

In the whole line case (an excellent account on nonlinear dispersive PDEs can be found at [**8**]), the fact that the Schwartz class initial data $u_0 \in \mathcal{S}(\mathbb{R})$ gives rise to a unique global solution u is known since the end of the 1960s. Moreover, in this case the solution map is not only continuous but also smooth in both variables.

The wellposedness in L^2 or, more generally, in Sobolev spaces H^s with $s \in \mathbb{R}$ is a much harder problem. Using different approaches, the wellposedness in H^s with $s \geq 2$ was established in [3, 45, 46]. The case $s \leq 1$ turned out to be an extremely difficult task requiring new tools and ideas. Global wellposedness for finite energy initial data was proved in [25] by utilizing local smoothing and maximal function estimates. Local wellposedness of (4.3.1) in L^2 was proved by J. Bourgain in [4]. The key ingredient is the Bourgain space $X^{s,b}$, which efficiently captures the dispersive nature of (4.3.1) and controls the deviation of the KdV dynamics from solutions to the Airy equation $u_t = -u_{xxx}$. Further development and refinement of Bourgain's approach ultimately led to a proof of wellposedness for (4.3.1) in $H^s(\mathbb{R})$ for $s \geq -3/4$ [26, 7]. Notice that these ranges of s are sharp if one requires the data to solution map to be uniformly continuous on bounded sets [6]. Moreover, it was shown in [38] that wellposedness cannot persist for any $s < -1$. Finally, the gap was closed in the recent work of R. Killip and M. Vişan [28], who proved a (global) wellposedness of (4.3.1) in $H^{-1}(\mathbb{R})$.[4]

[4] The analog of this result on the circle was proved by Kappeler and Topalov in 2006 [23].

Rearrangement inequalities

Definition A.0.1. Let $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ be such that $a_n \to 0$ as $n \to \infty$. The sequence $\{\tilde{a}_n\}_{n\geq 1}$ is defined as follows

$$\tilde{a}_1 := \max_{k\geq 1} |a_k|, \qquad \tilde{a}_1 + \tilde{a}_2 := \max_{1\leq k<j} (|a_k| + |a_j|), \quad \ldots \tag{A.0.1}$$

Lemma A.0.1.

$$\sum_{n\geq 1} |a_n b_n| \leq \sum_{n\geq 1} \tilde{a}_n \tilde{b}_n. \tag{A.0.2}$$

Proof. Clearly, it suffices to prove the claim for finite series. So, suppose that $a_n = 0$ for all $n \geq N$ with some $N \in \mathbb{Z}_{\geq 1}$. Without loss of generality we can enumerate them such that $|a_1| \geq |a_2| \geq \cdots \geq |a_N|$. Then we get

$$\begin{aligned}
\sum_{n=1}^{N} |a_n b_n| &= |a_N| \sum_{n=1}^{N} |b_n| + (|a_N| - |a_{N-1}|) \sum_{n=1}^{N-1} |b_n| + \cdots + (|a_2| - |a_1|)|b_1| \\
&\leq |a_N| \sum_{n=1}^{N} \tilde{b}_n + (|a_{N-1}| - |a_N|) \sum_{n=1}^{N-1} \tilde{b}_n + \cdots + (|a_1| - |a_2|)\tilde{b}_1 \\
&= \sum_{n=1}^{N} |a_n| \tilde{b}_n = \sum_{n=1}^{N} \tilde{a}_n \tilde{b}_n.
\end{aligned}$$

□

Theorem A.0.1 (A. S. Markus)**.** *Let* $\mathbf{a} = \{a_n\}_{n=1}^{N}$, $\mathbf{b} = \{b_n\}_{n=1}^{N} \in \mathbb{C}^N$ *be such that* $a_1 \geq a_2 \geq \cdots \geq a_N \geq 0$ *and*

$$\sum_{n=1}^{k} \tilde{b}_n \leq \sum_{n=1}^{k} a_n, \quad k \in \{1, \ldots, N\}. \tag{A.0.3}$$

Then there are points $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(m)} \in \mathbb{C}^N$ *with* $\widetilde{\mathbf{a}}^{(l)} = \mathbf{a}$ *for all* $l \in \{1, \dots, m\}$ *and* $\{\lambda_l\}_1^m \in [0,1]$ *with* $\sum \lambda_l = 1$ *such that*

$$\mathbf{b} = \sum_{l=1}^{m} \lambda_l \, \mathbf{a}^{(l)}. \tag{A.0.4}$$

In particular, if (A.0.3) *holds and* $\Phi\colon \mathbb{R}^N_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is such that* $\phi(\mathbf{c}) := \Phi(\tilde{c}_1, \dots, \tilde{c}_N)$ *is convex on* $\mathbb{C}^N$*, then*

$$\phi(\mathbf{b}) \leq \phi(\mathbf{a}). \tag{A.0.5}$$

The proof can be found in, e.g., [**50**, Chapter 1.4].

We need the following corollary.

Corollary A.0.1. *Let* $a_1 \geq a_2 \geq \dots a_N \geq 0$ *and* $b_1 \geq b_2 \geq \dots b_N \geq 0$ *be such that*

$$\prod_{k=1}^{n} a_k \geq \prod_{k=1}^{n} b_k, \tag{A.0.6}$$

for all $n \in \{1, \dots, N\}$*. Then for any continuous, monotone increasing* $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that* $t \mapsto \phi(\mathrm{e}^t)$ *is convex, we have*

$$\sum_{k=1}^{n} \phi(b_k) \leq \sum_{k=1}^{n} \phi(a_k), \qquad n \in \{1, \dots, N\}. \tag{A.0.7}$$

Proof. Clearly, we can assume that all a's and b's are non-zero. Then replacing a_k's by γa_k, b_k's by γb_k and $\phi(\cdot)$ by $\phi(\gamma^{-1}\cdot)$ for $\gamma > 0$ sufficiently large, we can assume that all a's and b's are greater than 1. Taking $a_k' := \log(a_k)$ and $b_k' := \log(b_k)$, the new variables satisfy (A.0.3). Setting

$$\Phi(x_1, \dots, x_n) := \sum_{k=1}^{n} \phi(\mathrm{e}^{x_k}),$$

and applying Theorem A.0.1, we prove the claim. □

As another corollary we obtain Weyl's and Horn's inequalities (Remark 3.4.2).

Corollary A.0.2. *Let* $\phi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *be a nondecreasing function such that* $x \mapsto \phi(\mathrm{e}^x)$ *is convex. Then for any* A*,* $B \in \mathfrak{S}_\infty(\mathfrak{H})$

$$\sum \phi(|\lambda_n(A)|) \leq \sum \phi(s_n(A)), \qquad \textit{(Weyl)} \tag{A.0.8}$$

$$\sum \phi(s_n(AB)) \leq \sum \phi(s_n(A) s_n(B)). \qquad \textit{(Horn)} \tag{A.0.9}$$

Proof. Clearly, it suffices to prove

$$\sum_{n=1}^{N} \phi(|\lambda_n(A)|) \leq \sum_{n=1}^{N} \phi(s_n(A))$$

for each $N \in \mathbb{N}$. However, this follows from (3.4.12) and Corollary A.0.2. $\square$

Analytic functions with values in Banach spaces

Definition B.0.1. Suppose $\mathcal{D} \subseteq \mathbb{C}$ is open and $f\colon \mathcal{D} \to X$, where X is a Banach space.

(i) f is called *(strongly) analytic* at $z_0 \in \mathcal{D}$ if the limit

$$\frac{f(z_0 + z) - f(z_0)}{z - z_0}$$

exists in X as z goes to 0.

(ii) f is called *weakly analytic* at $z_0 \in \mathcal{D}$ if the function $\ell(f(\cdot))\colon \mathcal{D} \to \mathbb{C}$ is analytic at z_0 for each $\ell \in X^*$.

Although weak analyticity is a priori weaker than the strong one, *the two definitions are equivalent.*

Theorem B.0.1. *Every weak analytic function is strongly analytic.*

The proof of this result is based on the uniform boundedness principle and we refer to, e.g., [**42**, Theorem VI.4]. In the following, we shall use analyticity without specifying it is weak or strong. This is very important, since weak analyticity is often much easier to check. Moreover, starting from this point one can develop a theory of vector-valued analytic functions which is almost exactly parallel to the usual theory; in particular, a strongly analytic function has a norm-convergent Taylor series:

$$f(z) = \sum_{n\geq 0} T_n (z-z_0)^n,$$

which converges for all $|z-z_0| < r$, where $r^{-1} := \limsup_{n\to\infty} \|T_n\|^{1/n}$.

The most important function is the *resolvent* of T,

$$\mathcal{R}_T(z) = (T-z)^{-1}. \tag{B.0.1}$$

The set of all $z \in \mathbb{C}$ for which $(T-z)$ is a bijection and $\mathcal{R}_T(z)$ is a bounded operator is called the *resolvent set* of T and is denoted by $\rho(T)$. Its complement is called the *spectrum* of T and is denoted by $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Theorem B.0.2. *Let T be a bounded operator on a Banach space X. Then $\rho(T)$ is an open subset of $\mathbb{C}$ and $\mathcal{R}_T$ is an analytic $[X]$-valued function on each connected component of $\rho(T)$. Moreover, for each $z, \zeta \in \rho(T)$,*

$$\mathcal{R}_T(z) - \mathcal{R}_T(\zeta) = (\zeta - z)\mathcal{R}_T(z)\mathcal{R}_T(\zeta). \tag{B.0.2}$$

In particular, $\mathcal{R}_T(z)$ and $\mathcal{R}_T(\zeta)$ commute.

Proof. Let $z_0 \in \rho(T)$. Then (at least formally) we have

$$\frac{1}{T-z} = \frac{1}{(T-z_0)-(z-z_0)} = \frac{1}{T-z_0}\frac{1}{I-\frac{z-z_0}{T-z_0}} = \mathcal{R}_T(z_0)\sum_{n\geq 0}\Big(\frac{z-z_0}{T-z_0}\Big)^n.$$

The latter suggest to define $\mathcal{R}_T(z)$ by

$$\widetilde{\mathcal{R}}_T(z) := \mathcal{R}_T(z_0)\Big(I + \sum_{n\geq 1}(z-z_0)^n \mathcal{R}_T(z_0)^n\Big). \tag{B.0.3}$$

Since $\|\mathcal{R}_T(z_0)^n\| \leq \|\mathcal{R}_T(z_0)\|^n$ for all $n \geq 1$, the series on the RHS in (B.0.3) converges in the norm topology whenever

$$|z-z_0| < \frac{1}{\|\mathcal{R}_T(z_0)\|}.$$

Hence $\widetilde{\mathcal{R}}_T$ is well defined for these values of z. Moreover, it is straightforward to check that

$$(T-z)\widetilde{\mathcal{R}}_T(z) = I = \widetilde{\mathcal{R}}_T(T-z).$$

This shows that $z \in \rho(T)$ whenever $|z-z_0| < 1/\|\mathcal{R}_T(z_0)\|$ and also $\widetilde{\mathcal{R}}_T(z) = \mathcal{R}_T(z)$. Thus $\rho(T)$ is open. Moreover, $\mathcal{R}_T$ is analytic since we can expand it in a Taylor series.

To prove the *first resolvent formula* (B.0.2), it suffices to notice that

$$\mathcal{R}_T(z) - \mathcal{R}_T(\zeta) = \mathcal{R}_T(z)(T-\zeta)\mathcal{R}_T(\zeta) - \mathcal{R}_T(z)(T-z)\mathcal{R}_T(\zeta).$$

Commutativity is obvious. □

Corollary B.0.1. *Let T be a bounded operator on a Banach space X. Then $\sigma(T)$ is not empty.*

Proof. If $|z| > \|T\|$, then

$$\mathcal{R}_T(z) = \frac{1}{z}\frac{1}{\frac{T}{z} - I} = \frac{1}{z}\Big(I + \sum_{n\geq 1}\frac{T^n}{z^n}\Big), \tag{B.0.4}$$

and the *Neumann series* converges in the norm topology. Moreover, it is easy to see that

$$\|\mathcal{R}_T(z)\| \to 0$$

as $|z| \to \infty$. Thus, $\sigma(T) =$ would imply that $\mathcal{R}_T$ is a bounded entire function. Applying Liouville's theorem, we get a contradiction. □

Definition B.0.2. The number

$$r(T) := \sup_{z\in\sigma(T)} |z| \tag{B.0.5}$$

is called the *spectral radius* of T.

We finish this section with the following result.

Theorem B.0.3. *Let T be a bounded operator on a Banach space X. Then*

$$r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}. \tag{B.0.6}$$

Remark B.0.1. Clearly, $r(T) \leq \|T\|$. However, it might happen that $r(T) < \|T\|$! On the other hand, for normal operators in Hilbert spaces it is always true that $r(T) = \|T\|_{\mathfrak{H}}$.

Appendix C

Exercises

Exercise C.0.1. Complete the details in Example 2.1.2.

Exercise C.0.2. Prove that $T\colon X \to Y$ is a finite rank operator if and only if there are vectors $\{\ell_n\}_{n=1}^N \in X^*$ and $\{\varphi_n\}_{n=1}^N \in Y$ such that

$$Tf = \sum_{k=1}^{N} \ell_n(f)\varphi_n \tag{C.0.1}$$

for all $f \in X$. Find the rank of F.

Exercise C.0.3. Find $T^\times$ of (C.0.1).

Exercise C.0.4. Prove (2.1.4).

Exercise C.0.5. Using the Weierstrass theorem, show that every integral operator $\mathcal{K}$ from Example 2.1.1 with a continuous kernel can be approximated by finite-rank operators.

Exercise C.0.6. Let $A \in [\mathfrak{H}]$ be self-adjoint. Let also ϕ_1 and ϕ_2 are eigenfunctions of A corresponding to eigenvalues λ_1 and λ_2. Show that $\phi_1 \perp \phi_2$ if $\lambda_1 \neq \lambda_2$.

Exercise C.0.7. Show that for a bounded self-adjoint operator A,

$$\|A\| = \sup_{\|f\| \leq 1} |(Af, f)|.$$

Hint: By the polarization identity

$$\operatorname{Re}(Af, g) = \frac{1}{4}(A(f+g), f+g) - (A(f-g), f-g).$$

Then using the inequality

$$|(Af,f)| \le \|f\|^2 \sup_{\|g\|=1} |(Ag,g)|$$

and the parallelogram law ($\|f+g\|^2+\|f-g\|^2 = 2\|f\|^2+2\|g\|^2$), prove that

$$|(Af,g)| \le \sup_{\|\phi\|=1} |(A\phi,\phi)|$$

whenever $\|f\|, \|g\| \le 1$.

Exercise C.0.8. If A is a non-negative bounded operator, show that

- A^n is non-negative for all $n \in \mathbb{N}$.
- $I - A$ is non-negative if $\|A\| \le 1$.

Exercise C.0.9. For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, find $|A|$ and $|A^*|$. Is $|A| = |A^*|$?

Exercise C.0.10. Take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$. Is $|A+B| \le |A| + |B|$?

Exercise C.0.11. Show that V^* is a partial isometry if so is V. Find its initial and final subspaces. Show that V^*V and VV^* are the projections onto the initial and final subspaces of V, respectively. Find initial and final subspaces for the shift operator.

Exercise C.0.12. Prove uniqueness in Theorem 2.3.5.

Exercise C.0.13. Show that if S is bounded linear operator in $\mathfrak{H}$, then ST is compact if either T or S is compact.

Exercise C.0.14. Verify that (φ_n) given in the proof of Theorem 2.3.3 is an orthonormal set.

Exercise C.0.15. Let A be a compact operator on $\mathfrak{H}$. Show that A^*A and AA^* have the same non-zero eigenvalues with the same multiplicities.

Exercise C.0.16. Show that $0 \le |B| \le |A|$ implies

$$s_k(B) \le s_k(A)$$

for all k.

Exercise C.0.17. For $A \in \mathfrak{S}_\infty(\mathfrak{H})$, show that

$$\min_{\mathrm{rank}(K)\le n} \|A - K\| \le s_{n+1}(A).$$

Exercise C.0.18. Prove Fan's inequalities (3.1.4).

Exercise C.0.19. Show that $A \in \mathfrak{S}_1$ if and only if there are $B, C \in \mathfrak{S}_2$ such that $A = BC$.

Exercise C.0.20. Prove Lemma 3.2.1.

Exercise C.0.21. Show that $\|\cdot\|_1$ defines a norm and $\mathfrak{S}_1$ is complete w.r.t. $\|\cdot\|_1$. Moreover, show that the closure of finite rank operators w.r.t. $\|\cdot\|_1$ coincides with $\mathfrak{S}_1$.

Exercise C.0.22. Complete the details of the proof of Theorem 3.3.1.

Exercise C.0.23. Consider the following integral operator in $L^2((0,1))$:

$$(\mathcal{J}f)(x) = \int_0^x f(s)ds. \tag{C.0.2}$$

Is this operator bounded? compact? Hilbert–Schmidt? Trace class?

Exercise C.0.24. Consider the following integral operator in $L^2((0,1))$:

$$(\mathcal{H}f)(x) = \frac{1}{x}\int_0^x f(s)ds. \tag{C.0.3}$$

Is this operator bounded? compact? Hilbert–Schmidt? Trace class?

Exercise C.0.25. When a non-negative Hankel matrix $H = (s_{k+j})_{j,k\ge 0}$ belongs to the trace class? Find its trace.

Exercise C.0.26. Let $A = (a_{k,j})_{k,j\ge 0}$ be an infinite matrix such that $a_{k,j} = 0$ for all $|k-j| \ge N$ with some (fixed) $N \in \mathbb{N}$. When A defines a bounded/compac/trace class/Hilbert–Schmidt class operator on $\ell^2(\mathbb{Z}_{\ge 0})$? (*Hint:* Show that A can be written as a finite linear combination of products of diagonal matrices and shifts/backward shifts).

Exercise C.0.27. Show that $\{\varphi_n \otimes \psi_k\}_{n,k}$ is an orthonormal basis in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ if $\{\varphi_n\}$ and $\{\psi_n\}$ are orthonormal bases in, respectively, $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

Exercise C.0.28. Show that both $\mathcal{S}_n$ and $\mathcal{A}_n$ (see (3.4.5)) are orthogonal projections on $\mathcal{F}(\mathcal{H})$, that is, $\mathcal{S}_n = \mathcal{S}_n^*$, $\mathcal{S}_n^2 = \mathcal{S}_n$ and $\mathcal{A}_n = \mathcal{A}_n^*$ and $\mathcal{A}_n^2 = \mathcal{A}_n$.

Exercise C.0.29. (i) Show that $\Lambda^n(AB) = \Lambda^n(A)\Lambda^n(B)$ for each bounded operators A and B on $\mathcal{H}$.

(ii) If $\mathcal{H} = \mathbb{C}^N$ for some $N \in \mathbb{N}$ and A is a linear operator in $\mathbb{C}^N$, show that

$$\Lambda^N(A) = \det(A).$$

In particular, this implies that $\det(AB) = \det(A)\det(B)$ for $A, B \in \mathbb{C}^{N\times N}$.

Exercise C.0.30. Show that

$$\Lambda^n(A)^* = \Lambda^n(A^*), \qquad\qquad \big|\Lambda^n(A)\big| = \Lambda^n(|A|).$$

Exercise C.0.31. For $A \in \mathfrak{S}_\infty(\mathcal{H})$, find the spectrum of $\Lambda_n(A)$ and also its singular values.

Exercise C.0.32. Let $J_n \in \mathbb{C}^{n\times n}$ be a Jordan block of size $n \in \mathbb{N}$,

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and set $J = J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$. Find $(z - J)^{-1}$ and compute then

$$P_0 = \frac{1}{2\pi \mathrm{i}} \int_{|z|=\varepsilon} (J - z)^{-1} dz, \qquad \varepsilon > 0.$$

Exercise C.0.33. Construct an orthonormal basis $\{\psi_k\}$ in the proof of Theorem 3.4.8.

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