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## ON C\*-EXTREME POINTS

BOJAN MAGAJNA

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ABSTRACT. Each weak\* compact C\*-convex set in a hyperfinite factor (in particular in  $B(\mathcal{H})$ ) is the weak\* closure of the C\*-convex hull of its C\*-extreme points.

### 1. INTRODUCTION AND MAIN RESULTS

A subset  $K$  of a unital C\*-algebra  $R$  is called  $R$ -convex (or  $C^*$ -convex if  $R$  is clear from the context) if  $\sum_{j=1}^n a_j^* x_j a_j \in K$  whenever  $x_j \in K$  and  $a_j \in R$  for all  $j$  and  $\sum_{j=1}^n a_j^* a_j = 1$ . As defined in [10], a point  $x \in K$  is  $C^*$ -extreme for  $K$  if the condition

$$(1.1) \quad x = \sum_{j=1}^n a_j^* x_j a_j, \quad \sum_{j=1}^n a_j^* a_j = 1, \quad x_j \in K, \quad a_j \text{ invertible in } R, \quad n \in \mathbb{N},$$

implies that all  $x_j$  are unitarily equivalent to  $x$  in  $R$ . C\*-extreme points for subsets  $K$  of  $R = M_n := M_n(\mathbb{C})$  are extreme in the usual sense by [10], but the converse is not true (see [8] or [6]). It was conjectured already in [10] that a variant of the Krein-Milman theorem should hold for compact C\*-convex sets, and indeed, much later for subsets of  $M_n$  such a theorem was established by Morenz [12] using some previous work of Farenick and Morenz (see [4], [5] and [6]). Another Krein-Milman type theorem for the so-called matricially convex sets in locally convex spaces has been proved recently by Webster and Winkler [15]. Methods in [12] and [15] (although different) both used in an essential way the finite dimensionality of  $M_n$ . In this note we shall prove a variant of the Krein-Milman theorem for C\*-convex subsets of  $B(\mathcal{H})$  where  $\mathcal{H}$  is a separable Hilbert space. But since the same proof is actually valid for subsets in hyperfinite factors, we shall formulate the result for such factors. For this we need a special kind of C\*-extreme points.

A point  $x \in K$ , where  $K$  is a C\*-convex subset in a C\*-algebra  $R$ , is  $R$ -extreme for  $K$  if the condition

$$(1.2) \quad x = \sum_{j=1}^n a_j x_j a_j, \quad \sum_{j=1}^n a_j^2 = 1, \quad x_j \in K, \quad a_j \text{ invertible and positive in } R,$$

implies that  $x_j = x$  and  $a_j x = x a_j$  for all  $j$ . Using the polar decomposition of the coefficients  $a_j$  in (1.1) it is easy to show that each  $R$ -extreme point is C\*-extreme.

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It suffices to check the condition (1.2) for an  $R$ -extreme point in the case  $n = 2$ . (Indeed, suppose that  $x$  satisfies the condition for  $n = 2$ . For a general  $n$  put  $y = \bigoplus_{j=2}^n x_j$ ,  $a = (a_2, \dots, a_n)^T$  and let  $a = u|a|$  be the polar decomposition of the column  $a$ . Then, writing (1.2) as  $x = a_1 x_1 a_1 + |a|(u^* y u)|a|$ , it follows that  $x_1 = x$  and  $a_1$  commutes with  $x$ . The same argument can be applied to  $j = 2, \dots, n$ .) If  $R$  is commutative, then it is not difficult to show that the  $R$ -extreme points (and the  $C^*$ -extreme points) are just the usual extreme points. Also, for a general  $R$ , all  $R$ -extreme points are extreme in the usual sense.

Now we can state the main result of this note.

**Theorem 1.1.** *Each weak\* compact  $C^*$ -convex set  $K$  in a hyperfinite factor  $R$  is the weak\* closure of the  $C^*$ -convex hull of the set  $\text{ext}_R(K)$  of all  $R$ -extreme points of  $K$ .*

To prove Theorem 1.1 we shall first consider for each  $x \in R$  the weak\* closure  $\overline{\text{co}}_R(x)$  of the  $C^*$ -convex hull  $\text{co}_R(x)$  of  $x$ , where

$$\text{co}_R(x) = \left\{ \sum_{j=1}^n a_j^* x a_j : a_j \in R, \sum_{j=1}^n a_j^* a_j = 1, n \in \mathbb{N} \right\}.$$

Recall also that for each  $n \in \mathbb{N}$  the matricial range  $W_n(x)$  of an element  $x \in R$  is the set of all  $\phi(x)$ , where  $\phi$  is any unital completely positive map from  $R$  to  $\mathbb{M}_n$  (see [1] or [14]).

The following two results will be used in the proof of Theorem 1.1.

**Theorem 1.2.** *Let  $R$  be any factor and  $A \subseteq R$  a subfactor (containing the unit of  $R$ ) isomorphic to  $\mathbb{M}_n$  for some  $n \in \mathbb{N}$ . Then*

$$\overline{\text{co}}_R(x) \cap A = W_n(x)$$

for each  $x \in R$ , where  $W_n(x)$  is regarded as a subset of  $A$  by identifying  $A$  with  $\mathbb{M}_n$  (using any  $C^*$ -isomorphism). Moreover,  $\phi(K) \subseteq K$  for each unital completely positive map  $\phi : R \rightarrow A$  and each weak\* compact  $C^*$ -convex subset  $K$  of  $R$ .

**Lemma 1.3.** *Let  $R$  be a unital  $C^*$ -algebra and  $A$  a  $C^*$ -subalgebra containing the unit of  $R$  such that for each non-zero  $x \in R$  there exists a conditional expectation  $E : R \rightarrow A$  satisfying  $E(x) \neq 0$ . If  $K$  is a  $C^*$ -convex subset of  $R$  such that  $\phi(K) \subseteq K$  for each unital completely positive map  $\phi : R \rightarrow A$ , then  $\text{ext}_A(K \cap A) \subseteq \text{ext}_R(K)$ .*

Theorem 1.2 and Lemma 1.3 will be proved in Section 2. In the case  $R = \mathbb{M}_n$  Theorem 1.1 will be proved in Section 3 by using some results of [5] and [12] (or [15]). Assuming this, we can now deduce Theorem 1.1 for a general hyperfinite factor  $R$ .

*Proof of Theorem 1.1.* Since  $R$  is hyperfinite there exists a sequence of finite dimensional  $C^*$ -subalgebras  $B_0 \subseteq B_1 \subseteq B_2 \dots$  of  $R$  and (normal) conditional expectations  $E_n : R \rightarrow B_n$  such that the sequence  $(E_n x)$  converges to  $x$  in the strong operator topology for each  $x \in R$ . Moreover, except in the case  $R$  is of type  $III_0$ ,  $B_n$  can be chosen to be factors since by the classification of injective factors ([3] and [7]) all such factors are infinite tensor products of finite dimensional factors (see also [9, 13.1.15]). In case  $R$  is of type  $III_0$  the algebras  $B_n$  cannot always be chosen to be factors, but their existence together with the expectations  $E_n$  follows by [2]. In any case (since all non-zero projections in a countably decomposable type  $III$

factor are equivalent) there exists for each  $n$  a subfactor  $A_n \cong \mathbb{M}_{m_n}$  ( $m_n \in \mathbb{N}$ ) in  $R$  such that  $A_n \supseteq B_n$  for all  $n$ .

Let  $K$  be as in the statement of the theorem and  $x \in K$ . Regarding  $E_n$  as a unital completely positive map from  $R$  into  $A_n$ , we have  $E_n(x) \in K \cap A_n$  by Theorem 1.2. Hence, assuming the finite dimensional version of Theorem 1.1 (proved in Section 3) it follows that

$$(1.3) \quad E_n(x) \in \overline{\text{co}}_{A_n}(\text{ext}_{A_n}(K \cap A_n)).$$

Since  $R \cong A_n \otimes R_n$  for some subfactor  $R_n \subseteq R$ , the (normal) conditional expectations from  $R$  to  $A_n$  (namely the slice maps) separate points of  $R$ , hence by Lemma 1.3  $\text{ext}_{A_n}(K \cap A_n) \subseteq \text{ext}_R(K)$ . Now (1.3) implies that  $E_n(x) \in \overline{\text{co}}_R(\text{ext}_R(K))$ , hence  $x \in \overline{\text{co}}_R(\text{ext}_R(K))$  since  $E_n(x)$  converges to  $x$ .  $\square$

We remark that the existence of C\*-extreme points in a weak\* compact C\*-convex subset  $K$  of a general von Neumann algebra is proved in [11], but the extreme points obtained in [11] are in general not sufficient to generate  $K$ . So for general von Neumann algebras the problem if each weak\* compact C\*-convex subset is generated by its C\*-extreme points remains open.

## 2. PROOFS OF THEOREM 1.2 AND LEMMA 1.3

To prove Theorem 1.2 we need the following result from [11].

**Lemma 2.1.** *Let  $A$  be a unital C\*-algebra,  $a_1, \dots, a_m$  elements of  $A$  and  $\rho$  a state on  $A$  in the weak\* closure of the pure states. Then for each  $\varepsilon > 0$  there exists an element  $h \in A$  such that  $\|h\| = 1$  and  $\|h^*(a_i - \rho(a_i))h\| < \varepsilon$  for  $i = 1, \dots, m$ .*

*Proof of Theorem 1.2.* Let  $y \in \overline{\text{co}}_R(x) \cap A$  and choose a net  $(y_\nu) \subseteq \text{co}_R(x)$  converging to  $y$  in the strong operator topology. Since the map  $w \mapsto \sum_{j=1}^n a_j^* w a_j$  on  $R$  is completely positive and unital for any  $a_1, \dots, a_n$  in  $R$  satisfying  $\sum_{j=1}^n a_j^* a_j = 1$ , it follows that  $W_n(y_\nu) \subseteq W_n(x)$  for all  $\nu$  and consequently  $W_n(y) \subseteq W_n(x)$  (since unital completely positive maps into  $\mathbb{M}_n$  can be approximated by normal such maps in the point-norm topology). But  $y \in W_n(y)$  since  $y \in A \cong \mathbb{M}_n$ , hence  $y \in W_n(x)$ . This proves the inclusion  $\overline{\text{co}}_R(x) \cap A \subseteq W_n(x)$ .

To prove the reverse inclusion, let  $y \in W_n(x)$  and let  $\phi : R \rightarrow A = \mathbb{M}_n$  be a unital completely positive map such that  $\phi(x) = y$ . By the Stinespring representation theorem (see [13])  $\phi(w) = V^* \sigma(w) V$  ( $w \in R$ ), where  $\sigma$  is a representation of  $R$  on a Hilbert space  $\mathcal{K}$  and  $V : \mathbb{C}^n \rightarrow \mathcal{K}$  is an isometry. Since  $R \cong \mathbb{M}_n(B)$ , where  $B = A' \cap R$ , there is a representation  $\pi$  of  $B$  on a Hilbert space  $\mathcal{H}$  such that  $\mathcal{K} = \mathcal{H}^n$  and  $\sigma$  is (unitarily equivalent to)  $\pi_n : \mathbb{M}_n(B) \rightarrow \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ , where  $\pi_n([b_{ij}]) = [\pi(b_{ij})]$  for each  $[b_{ij}] \in \mathbb{M}_n(B)$ . Let  $\varepsilon_j$  ( $j = 1, \dots, n$ ) be the standard basic vectors in  $\mathbb{C}^n$  and put  $\xi_j = V \varepsilon_j \in \mathcal{H}^n$ . Let  $y_{ij} \in \mathbb{C}$  be the entries of  $y$ . Then

$$(2.1) \quad y_{ij} = \langle y \varepsilon_j, \varepsilon_i \rangle = \langle \phi(x) \varepsilon_j, \varepsilon_i \rangle = \langle V^* \pi_n(x) V \varepsilon_j, \varepsilon_i \rangle = \langle \pi_n(x) \xi_j, \xi_i \rangle.$$

Suppose first that  $\sigma$  (and hence  $\pi$ ) is irreducible and let  $\xi \in \mathcal{H}$  be any unit vector. Then by the Kadison transitivity theorem there exists a unitary  $u \in R$  such that  $\pi_n(u)(\varepsilon_j \otimes \xi) = \xi_j$  for  $j = 1, \dots, n$ , where  $\varepsilon_j \otimes \xi \in \mathcal{H}^n$  denotes the vector with the  $j$ -th component equal to  $\xi$  and the remaining components equal to 0. Now (2.1) can be rewritten as

$$y_{ij} = \langle \pi_n(u^* x u)(\varepsilon_j \otimes \xi), \varepsilon_i \otimes \xi \rangle = \langle \pi((u^* x u)_{ij}) \xi, \xi \rangle = \omega((u^* x u)_{ij}),$$

where  $(u^*xu)_{ij}$  is the entry of  $u^*xu \in \mathbb{M}_n(B)$  on the position  $(i, j)$  and  $\omega$  is the state  $b \mapsto \langle \pi(b)\xi, \xi \rangle$  on  $B$ . Thus, denoting by  $\omega_n : \mathbb{M}_n(B) \rightarrow \mathbb{M}_n$  the map induced by  $\omega$ , we have  $y = \omega_n(z)$ , where  $z = u^*xu$ . Since  $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$ , it suffices now to prove that  $y \in \overline{\text{co}}_R(z)$  (for then we will have  $y \in \overline{\text{co}}_R(x) \cap A$ ). Let  $\epsilon > 0$ . Since  $\omega$  is pure and  $\omega(z_{ij}) = y_{ij}$  (where  $z_{ij}$  are the entries of  $z$ ), by Lemma 2.1 there exists a positive element  $e (= hh^*)$  in  $B$  such that  $\|e\| = 1$  and

$$(2.2) \quad \|e(z_{ij} - y_{ij})e\| < \epsilon \quad (i, j = 1, \dots, n).$$

Moreover, replacing  $e$  by a suitable spectral projection of  $e$  in  $R$ , we may assume that  $e$  itself is a projection. Since  $B$  is a factor, there is a family of partial isometries  $u_k \in B$  such that  $u_k u_k^* \leq e$  and  $\sum_k u_k^* u_k = 1$ . Now (2.2) implies that

$$\left\| \sum_k u_k^* z_{ij} u_k - y_{ij} \right\| = \left\| \sum_k u_k^* (z_{ij} - y_{ij}) u_k \right\| \leq \epsilon \quad (i, j = 1, \dots, n),$$

hence  $\left\| \sum_k u_k^{*(n)} z u_k^{(n)} - y \right\| \leq n\epsilon$ , where  $u_k^{(n)} \in \mathbb{M}_n(B)$  is the diagonal matrix with  $u_k$  on the main diagonal. This implies that the distance of  $y$  to  $\overline{\text{co}}_R(z) = \overline{\text{co}}_R(x)$  is dominated by  $n\epsilon$  for all  $\epsilon > 0$ , hence  $y \in \overline{\text{co}}_R(x)$ .

Consider now a more general situation, when  $\sigma$  is a direct sum of a family of irreducible representations  $\sigma_k : R \rightarrow \mathcal{B}(\mathcal{K}_k)$ . Then, with respect to the decomposition  $\mathcal{K} = \bigoplus_k \mathcal{K}_k$ ,  $V$  can be represented as a (possibly infinite) column  $V = (v_k)$ , where each  $v_k \in \mathcal{B}(\mathbb{C}^n, \mathcal{K}_k)$  has the polar decomposition  $v_k = u_k |v_k|$  with  $|v_k| = \sqrt{v_k^* v_k}$  and  $u_k \in \mathcal{B}(\mathbb{C}^n, \mathcal{K}_k)$  an isometry. (This is possible since  $\dim \mathcal{K}_k \geq n$  even if  $R$  is finite dimensional, since  $R \supseteq A \cong \mathbb{M}_n$  and  $\sigma_k$  is injective if the factor  $R$  is finite dimensional.) Now  $y = V^* \sigma(x) V = \sum_k |v_k| (u_k^* \sigma_k(x) u_k) |v_k|$ . Since each  $\sigma_k$  is irreducible and  $u_k$  an isometry,  $u_k^* \sigma_k(x) u_k \in \overline{\text{co}}_R(x)$  by the argument from the previous paragraph, hence it follows easily that  $y \in \overline{\text{co}}_R(x)$  since  $\sum_k |v_k|^2 = V^* V = 1$ .

Finally, in general we can approximate the map  $\phi(\cdot) = V^* \sigma(\cdot) V$  in the point-norm topology by the maps of the form  $V_\nu^* \sigma_\nu(\cdot) V_\nu$ , where each  $\sigma_\nu$  is a direct sum of irreducible representations. To get such an approximation one can use the fact that each state on  $\mathbb{M}_n(R)$  can be approximated in the weak\* topology by convex combinations of pure states and the well known connection between completely positive maps from  $R$  to  $\mathbb{M}_n$  and positive functionals on  $\mathbb{M}_n(R)$  (see [13, Chapter 5], we leave the details to the reader). Thus, in any case we have  $y \in \overline{\text{co}}_R(x)$ , hence  $W_n(x) \subseteq \overline{\text{co}}_R(x) \cap A$  (since by definition  $W_n(x) \subseteq A \cong \mathbb{M}_n$ ).

Let now  $K$  be any weak\* compact C\*-convex subset of  $R$  and  $\phi : R \rightarrow A$  a unital completely positive map. Then  $\phi(x) \in W_n(x) \subseteq \overline{\text{co}}_R(x) \subseteq K$  for each  $x \in K$ . Thus  $\phi(K) \subseteq K$ . □

*Proof of Lemma 1.3.* Let  $x \in \text{ext}_A(K \cap A)$  and suppose that

$$x = \sum_{j=1}^n a_j x_j a_j,$$

where  $x_j \in K$ ,  $a_j \in R$  is positive and invertible for each  $j$  and  $\sum_{j=1}^n a_j^2 = 1$ . Then for each conditional expectation  $E : R \rightarrow A$  we have

$$(2.3) \quad x = \sum_{j=1}^n E(a_j x_j a_j) = \sum_{j=1}^n E(a_j^2)^{1/2} \phi_j(x_j) E(a_j^2)^{1/2},$$

where  $\phi_j : R \rightarrow A$ ,  $\phi_j(w) := E(a_j^2)^{-1/2} E(a_j w a_j) E(a_j^2)^{-1/2}$  are unital completely positive mappings. Since  $\phi_j(x_j) \in K \cap A$  by the hypothesis and  $x \in \text{ext}_A(K \cap A)$ ,

(2.3) implies that  $\phi_j(x_j) = x$  and  $E(a_j^2)x = xE(a_j^2)$  for all  $j$ . By the definition of  $\phi_j$  it follows now that  $E(a_jx_ja_j) = E(a_j^2)^{1/2}xE(a_j^2)^{1/2} = E(a_j^2)x = xE(a_j^2)$ , hence

$$(2.4) \quad E(a_jx_ja_j - a_j^2x) = 0 \quad \text{and} \quad E(a_j^2x - xa_j^2) = 0$$

since a conditional expectation from  $R$  to  $A$  is  $A$ -linear. By hypothesis the conditional expectations from  $R$  to  $A$  separate points of  $R$ , hence (2.4) implies that  $a_jx_ja_j = a_j^2x$  and  $a_j^2x = xa_j^2$ , but since  $a_j$  is invertible and positive it follows that  $a_jx = xa_j$  and  $x_j = x$  for all  $j = 1, \dots, n$ . Thus  $x \in \text{ext}_R(K)$ .  $\square$

### 3. THE FINITE DIMENSIONAL CASE

In this section we shall prove Theorem 1.1 in the case  $R = M_n$  ( $n \in \mathbb{N}$ ). For convenience we recall now some results from [12] in a form suitable for our application.

Observe first that if  $x \in \text{ext}_R(K)$  for some  $C^*$ -convex set  $K$  in a unital  $C^*$ -algebra  $R$  and if

$$x = \sum_{j=1}^n a_jx_ja_j, \text{ where } x_j \in K, a_j \geq 0, \sum_{j=1}^n a_j^2 = 1,$$

then  $x_1 = x$  and  $a_1$  commutes with  $x$  if  $a_1$  is invertible. (To see this, note that  $x = \frac{1}{2}a_1x_1a_1 + a^*(\bigoplus_{j=1}^n x_j)a$ , where  $a$  is the column  $(\frac{1}{\sqrt{2}}a_1, a_2, \dots, a_n)^T$ , and use the polar decomposition of  $a$  to write  $x$  in the form  $x = \frac{1}{2}a_1x_1a_1 + |a|y|a|$ , where  $y \in K$ ,  $a_1$  and  $|a|$  are invertible and  $\frac{1}{2}a_1^2 + |a|^2 = 1$ .)

The following lemma is a variation on [6, Theorem 4.1], but we do not require finite dimensionality of  $R$ .

**Lemma 3.1.** *Let  $K$  be a norm closed  $C^*$ -convex subset in any von Neumann algebra  $R$ . Suppose that*

$$x = \sum_{j=1}^n a_jx_ja_j,$$

where  $x_j \in K$ ,  $a_j \in R$ ,  $a_j \geq 0$  and  $\sum_{j=1}^n a_j^2 = 1$ . If  $x \in \text{ext}_R(K)$ , then the range projection  $p$  of  $a_1$  reduces  $x$  and  $xp = px_1p$ . In particular, if  $x$  is irreducible in  $R$  (that is, the commutant of  $\{x, x^*\}$  in  $R$  consists of scalars only) and  $a_1 \neq 0$ , then  $x_1 = x$ .

*Proof.* By the Dixmier approximation theorem ([9, 8.3.5])  $K$  contains a central element, so translating by a central element, we may assume that  $0 \in K$ . Then  $a^*ya \in K$  for all  $y \in K$  and  $a \in R$  with  $\|a\| \leq 1$ . Using the polar decomposition of the column  $(a_2, \dots, a_n)^T$  we may write  $x$  in the form  $x = a_1x_1a_1 + aya$ , where  $y \in K$  and  $a = \sqrt{1 - a_1^2}$ . For each  $\alpha \in [0, 1]$  we have

$$(3.1) \quad x = (1 - \alpha)aya + (\alpha ya + a_1x_1a_1) = (1 - \alpha)aya + |t(\alpha)|u(\alpha)^*(y \oplus x_1)u(\alpha)|t(\alpha)|,$$

where  $t(\alpha) = (\alpha^{1/2}a)$  and  $t(\alpha) = u(\alpha)|t(\alpha)|$  is the polar decomposition of  $t(\alpha)$ . Since  $|t(\alpha)|^2 = \alpha a^2 + a_1^2$  is invertible if  $\alpha > 0$ , (3.1) implies that  $u(\alpha)^*(y \oplus x_1)u(\alpha) = x$  if  $\alpha \in (0, 1]$  by the observation preceding the lemma. But from

$$u(\alpha) = t(\alpha)|t(\alpha)|^{-1} = \begin{pmatrix} \alpha^{1/2}a(\alpha a^2 + a_1^2)^{-1/2} \\ a_1(\alpha a^2 + a_1^2)^{-1/2} \end{pmatrix} \quad (\alpha \in (0, 1])$$

we see that  $\lim_{\alpha \rightarrow 0} u(\alpha) = \begin{pmatrix} p \\ p \end{pmatrix}$  in the strong operator topology, hence by continuity  $x = \lim_{\alpha \rightarrow 0} u(\alpha)^*(y \oplus x_1)u(\alpha) = px_1p + p^\perp y p^\perp$  (where the convergence is in the weak operator topology). This implies that  $px = xp = px_1p$ . If  $x$  is irreducible and  $a_1 \neq 0$ , then  $p = 1$ , hence  $x = x_1$ .  $\square$

It can be proved [11] that each weak\* compact C\*-convex subset  $K$  of a factor  $R$  contains a scalar multiple of 1 as an  $R$ -extreme point (in the case  $R = \mathbb{M}_n$ , which is the only case that will be needed in this section, this also follows from [4]). Thus, translating, we may assume without loss of generality that  $0 \in \text{ext}_R(K)$ .

**From now on let  $K$  be a C\*-convex compact subset of  $\mathbb{M}_n$  with  $0 \in \text{ext}_{\mathbb{M}_n}(K)$ .**

*Notation.* For each  $k \leq n$  let  $P_k \in \mathbb{M}_n$  be the orthogonal projection onto the first  $k$  coordinates. Put

$$K_k = P_k K P_k$$

and regard  $K_k$  as a subset of  $\mathbb{M}_k$  by identifying  $\mathbb{M}_k$  with  $P_k \mathbb{M}_n P_k$ . It is easy to see that  $K_k$  is C\*-convex. For each  $x \in \mathbb{M}_k$  we shall denote by  $\tilde{x}$  the element  $x \oplus 0 \in \mathbb{M}_n$ .

*Remark 3.2.* If  $x$  is an irreducible C\*-extreme point of  $K$ , then  $x \in \text{ext}_{\mathbb{M}_n}(K)$  by [12, Corollary 1.8]. In our present situation this can also be seen in a more straightforward way as follows. Suppose that  $x = \sum_{j=1}^2 a_j x_j a_j$ , where  $x_j \in K$  and all  $a_j \neq 0$  are positive with  $\sum_{j=1}^2 a_j^2 = 1$ . Then by Lemma 3.1  $x_j = x$  for all  $j$ , hence

$$(3.2) \quad x = \sum_{j=1}^2 a_j x a_j.$$

Writing  $a_1$  (and hence also  $a_2 = \sqrt{1 - a_1^2}$ ) as a diagonal matrix,  $a_1 = \bigoplus_{i=1}^m \alpha_i 1_{n_i}$ , where  $\alpha_i \neq \alpha_k$  if  $i \neq k$  and writing  $x$  correspondingly as a block matrix,  $x = [x_{ik}]$  ( $x_{ik} \in \mathbb{M}_{n_i, n_k}$ ), it follows from (3.2) by a straightforward computation that  $x_{ik} = 0$  if  $i \neq k$ . But, since  $x$  is irreducible, this implies that  $m = 1$  and  $a_1, a_2$  are scalar multiples of 1.

In general,  $K$  may have no irreducible C\*-extreme points and one has to consider also the possible irreducible C\*-extreme points of the compressions  $K_k$ . Note that if  $x$  is an irreducible C\*-extreme point of  $K_k$  ( $k < n$ ) such that  $x$  is a compression of an irreducible C\*-extreme point  $y$  of  $K_j$  for some  $k < j \leq n$ , say  $\tilde{x} = p\tilde{y}p$  for a projection  $p \in \mathbb{M}_n$ , then  $\tilde{x}$  can be expressed as the C\*-convex combination  $\tilde{x} = p\tilde{y}p + p^\perp 0 p^\perp$ .

*Notation.* Denote by  $S_n$  the set of all irreducible C\*-extreme points of  $K$  and (for  $k < n$ ) by  $S_k$  the set of all irreducible C\*-extreme points of  $K_k$  which are not compressions of irreducible C\*-extreme points of  $K_j$  for  $k < j \leq n$ . (The elements of  $K_k$  are called structural elements in [12].)

From [12, Theorem 4.5] or [15, Theorem 4.3] and [5] we have the following.

**Theorem 3.3** ([12]). *Each  $x \in K$  can be expressed as a finite sum*

$$x = \sum_{i=1}^l t_i^* x_i t_i,$$

where  $x_i \in S_{n_i}$  ( $n_i \in \mathbb{N}$ ,  $n_i \leq n$ ),  $t_i \in \mathbb{M}_{n_i, n}$  and  $\sum_{i=1}^l t_i^* t_i = 1$ . Thus  $K$  is equal to the  $C^*$ -convex hull in  $\mathbb{M}_n$  of the set  $\tilde{S} := \bigcup_{k=1}^n \tilde{S}_k$ , where  $\tilde{S}_k = \{\tilde{y} : y \in S_k\}$ .

We will show (Proposition 3.5) that elements of the form  $\bigoplus_{i=1}^r x_i$ , where  $x_i \in S_{n_i}$  ( $\sum_{i=1}^r n_i = n$ ) are  $\mathbb{M}_n$ -extreme in  $K$ . Since each  $\tilde{x}_i$  can obviously be expressed as a  $C^*$ -convex combination of  $\bigoplus_{i=1}^r x_i$  and 0, it follows then from Theorem 3.3 (since  $0 \in \text{ext}_{\mathbb{M}_n}(K)$ ) that  $K = \text{co}_{\mathbb{M}_n}(\text{ext}_{\mathbb{M}_n}(K))$ . This will prove Theorem 1.1 in the case  $R = \mathbb{M}_n$ .

We need the following slight variation of [12, Lemma 4.6]. For convenience of the reader we shall present a simple proof.

**Lemma 3.4** ([12]). *Let  $x \in K_k$  ( $k \leq n$ ) be of the form  $x = \bigoplus_{i=1}^r x_i$ , where  $x_i \in S_{m_i}$  ( $m_i \in \mathbb{N}$ ). Suppose that  $x = v^* y v$  for some  $y \in K$  and some isometry  $v \in \mathbb{M}_{n, k}$ . Then there exist a unitary  $U \in \mathbb{M}_n$  and a matrix  $z \in K_{n-k}$  such that  $y = U(x \oplus z) U^*$  and  $v = U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , where  $1 \in \mathbb{M}_k$  is the identity matrix.*

*Proof.* Assume first that  $r = 1$  (that is,  $x \in S_k$ ) and  $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (that is,  $v$  is the inclusion of  $C^k$  into  $C^n$ ). By Theorem 3.3

$$(3.3) \quad y = \sum_{j=1}^l t_j^* y_j t_j,$$

where  $y_j \in S_{n_j}$ ,  $t_j \in \mathbb{M}_{n_j, n}$  and  $\sum_{j=1}^l t_j^* t_j = 1$ . Let  $t_j v = u_j |t_j v|$  be the polar decomposition of  $t_j v$  with  $u_j \in \mathbb{M}_{n_j, k}$  an isometry or a coisometry. Then

$$x = v^* y v = \sum_{j=1}^l |t_j v| (u_j^* y_j u_j) |t_j v|,$$

where  $u_j^* y_j u_j \in K$  (since  $K$  is  $C^*$ -convex and  $0 \in K$ ). Put  $\alpha_j = |t_j v|$ . Since  $x \in S_k$  (hence  $x$  is irreducible and  $C^*$ -extreme in  $K_k$ ),  $x \in \text{ext}_{\mathbb{M}_k}(K_k)$  by Remark 3.2, hence if  $\alpha_j \neq 0$ , then Lemma 3.1 and the irreducibility of  $x$  imply that  $u_j^* y_j u_j = x$  and  $\alpha_j$  is a scalar. Decompose the set  $\mathbb{L} := \{1, \dots, l\}$  as  $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$ , where  $\mathbb{L}_1 = \{j \in \mathbb{L} : \alpha_j \neq 0\}$  and  $\mathbb{L}_2 = \mathbb{L} \setminus \mathbb{L}_1$ . Since  $u_j$  is an isometry or a coisometry, the identity  $u_j^* y_j u_j = x$  means that one of  $x, y_j$  is unitarily equivalent to the compression of the other; but since  $y_j \in S_{n_j}$  and  $x \in S_k$  this is possible only if  $y_j \cong x$  (where  $\cong$  denotes unitary equivalence). Hence  $u_j$  must be unitary and  $n_j = k$  for all  $j \in \mathbb{L}_1$ . The identity  $t_j v = \alpha_j u_j$  then implies (since  $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ) that  $t_j = (\alpha_j u_j, b_j)$  for some  $b_j \in \mathbb{M}_{k, n-k}$ . If  $j \in \mathbb{L}_2$ , then  $t_j v = u_j \alpha_j = 0$  implies that  $t_j$  is of the form  $t_j = (0, b_j)$ . Thus from (3.3) we have (since  $\sum_{j \in \mathbb{L}_1} \alpha_j^2 = \sum_{j=1}^l |t_j v|^2 = v^* \sum_{j=1}^l t_j^* t_j v = 1$ )

$$y = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} u_j x u_j^* (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} y_j (0, b_j) = \begin{pmatrix} x & g \\ h & z \end{pmatrix},$$

where  $z \in \mathbb{M}_{n-k}$ ,

$$(3.4) \quad g = \sum_{j \in \mathbb{L}_1} \alpha_j x u_j^* b_j \quad \text{and} \quad h = \sum_{j \in \mathbb{L}_1} \alpha_j b_j^* u_j x.$$

But, from

$$1 = \sum_{j=1}^l t_j^* t_j = \sum_{j \in \mathbb{L}_1} \begin{pmatrix} \alpha_j u_j^* \\ b_j^* \end{pmatrix} (\alpha_j u_j, b_j) + \sum_{j \in \mathbb{L}_2} \begin{pmatrix} 0 \\ b_j^* \end{pmatrix} (0, b_j)$$



we see (looking at the entries on the position (1,2)) that  $\sum_{j \in \mathbb{L}_1} \alpha_j u_j^* b_j = 0$ , hence (3.4) implies that  $g = 0$ . Similarly  $h = 0$  and  $y$  is of the form  $x \oplus z$ .

Assume now that  $x = \bigoplus_{i=1}^r x_i$ , where  $r > 1$ , but  $v$  is still of the form  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , so that  $y$  can be represented as a matrix of the form

$$y = \begin{pmatrix} x & g \\ h & z \end{pmatrix}.$$

Applying (to elements  $x_i$ ) what we have just proved above, it follows inductively that the successive block rows and columns of the matrices  $g$  and  $h$  (respectively) corresponding to the summands  $x_i$  of  $x$  are all 0. Thus  $g = 0, h = 0$  and  $y = x \oplus z$ .

Finally, a general isometry  $v : \mathbb{C}^k \rightarrow \mathbb{C}^n$  can be extended to a unitary operator  $U$  on  $\mathbb{C}^n$ , so that  $v = U \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and the proof is completed by applying the above argument to  $U^*yU$  in place of  $y$ . □

**Proposition 3.5.** *If  $x \in K$  is of the form  $x = \bigoplus_{i=1}^r x_i$ , where  $x_i \in S_{n_i}$  ( $n_i \in \mathbb{N}$ ), then  $x \in \text{ext}_{\mathbb{M}_n}(K)$ .*

*Proof.* Suppose first that  $x \cong y^{(r)}$  (= the direct sum of  $r$  copies of  $y$ ) for some  $y \in S_k$ , where  $rk = n$ . Since  $y$  is an irreducible  $C^*$ -extreme point of  $K_k$ ,  $y$  is  $\mathbb{M}_k$ -extreme in  $K_k$  (see Remark 3.2). Observe that the canonical isomorphism  $\mathbb{M}_k \cong \mathbb{M}_k^{(r)} \subseteq \mathbb{M}_n$  identifies  $K_k$  with  $K \cap \mathbb{M}_k^{(r)}$ . (Note that  $w^{(r)} = \sum_{j=1}^r e_{j1} w e_{1j} \in K$  for each  $w \in K_k$  since  $K$  is  $C^*$ -convex and  $0 \in K$ , where  $\{e_{ij}\}_{i,j=1}^r$  is the appropriate matrix unit in  $\mathbb{M}_r(\mathbb{M}_k)$ , namely the standard matrix unit in  $\mathbb{M}_r$  tensored with  $1 \in \mathbb{M}_k$ .) It follows then by Lemma 1.3 (and Theorem 1.2 which justifies the application of Lemma 1.3, as in Section 1) that  $y^{(r)} \in \text{ext}_{\mathbb{M}_n}(K)$ .

Thus we may assume now that not all the summands  $x_i$  of  $x$  are mutually unitarily equivalent, hence the set  $\{1, \dots, r\}$  can be partitioned into two parts  $\mathbb{I}$  and  $\mathbb{J}$  such that  $x_i \not\cong x_j$  if  $i \in \mathbb{I}$  and  $j \in \mathbb{J}$ . Put

$$y = \bigoplus_{i \in \mathbb{I}} x_i \quad \text{and} \quad z = \bigoplus_{j \in \mathbb{J}} x_j$$

and let  $k = \sum_{i \in \mathbb{I}} n_i, l = \sum_{j \in \mathbb{J}} n_j$ . By induction on the dimension we may assume that  $y \in \text{ext}_{\mathbb{M}_k}(K_k)$  and  $z \in \text{ext}_{\mathbb{M}_l}(K_l)$ . To prove that  $x \in \text{ext}_{\mathbb{M}_n}(K)$ , suppose that

$$(3.5) \quad x = \sum_{j=1}^m t_j^* w_j t_j,$$

where  $w_j \in K, \sum_{j=1}^m t_j^2 = 1$  and each  $t_j \in \mathbb{M}_n$  is positive and invertible. According to the decomposition  $x = y \oplus z$ , decompose each  $t_j$  as  $t_j = (a_j, b_j)$ , where  $a_j \in \mathbb{M}_{n,k}, b_j \in \mathbb{M}_{n,l}$ , and let  $a_j = u_j |a_j|, b_j = v_j |b_j|$  be the polar decompositions. Then (3.5) implies (by considering the two diagonal blocks of sizes  $k \times k$  and  $l \times l$ , respectively) that

$$y = \sum_{j=1}^m |a_j| (u_j^* w_j u_j) |a_j| \quad \text{and} \quad z = \sum_{j=1}^m |b_j| (v_j^* w_j v_j) |b_j|.$$

Since  $y \in \text{ext}_{\mathbb{M}_k}(K_k)$  and  $z \in \text{ext}_{\mathbb{M}_l}(K_l)$  it follows now that  $|a_j|y = y|a_j|, |b_j|z = z|b_j|$  and

$$(3.6) \quad u_j^* w_j u_j = y, \quad v_j^* w_j v_j = z \quad (j = 1, \dots, m).$$

Since  $t_j$  is invertible,  $a_j$  is injective, hence  $|a_j|$  is invertible and  $u_j$  is an isometry. Similarly  $v_j$  is an isometry. Thus (3.6) implies by Lemma 3.4 that there exist unitaries  $U_j, V_j \in \mathbb{M}_n$  such that  $u_j = U_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $v_j = V_j \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  and

$$(3.7) \quad w_j = U_j(y \oplus h_j)U_j^*, \quad w_j = V_j(g_j \oplus z)V_j^*$$

for some matrices  $g, h$ . Putting  $W_j = U_j^*V_j$ , we have from (3.7) that  $(y \oplus h_j)W_j = W_j(g_j \oplus z)$  and  $(y \oplus h_j)^*W_j = W_j(g_j \oplus z)^*$ . Since no irreducible direct summand  $x_i$  of  $y$  is unitarily equivalent to any irreducible direct summand  $x_j$  of  $z$ , the only operator intertwining the identity representations of  $C^*(y)$  and  $C^*(z)$  is 0, hence  $W_j = c_j \oplus d_j$  for some unitary operators  $c_j \in \mathbb{M}_k$  and  $d_j \in \mathbb{M}_l$ . Then  $g_j = c_j^*y c_j$ ,  $h_j = d_j z d_j^*$ ,  $w_j = U_j(y \oplus d_j z d_j^*)U_j^*$  (from (3.7)),  $u_j = U_j \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $v_j = V_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j W_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j(c_j \oplus d_j) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = U_j \begin{pmatrix} 0 \\ d_j \end{pmatrix}$ . Thus  $a_j = u_j |a_j| = U_j \begin{pmatrix} |a_j| \\ 0 \end{pmatrix}$ ,  $b_j = v_j |b_j| = U_j \begin{pmatrix} 0 \\ |b_j| \end{pmatrix}$  and therefore

$$t_j = (a_j, b_j) = U_j(|a_j| \oplus |b_j|) = U_j(1 \oplus d_j)(|a_j| \oplus |b_j|).$$

Since  $t_j$  and  $|a_j| \oplus |b_j|$  are positive, while  $U_j(1 \oplus d_j)$  is unitary, it follows now by the uniqueness of the polar decomposition that  $U_j(1 \oplus d_j) = 1$  and  $t_j = |a_j| \oplus |b_j|$ . This clearly implies that  $t_j$  commutes with  $x = y \oplus z$  (since  $|a_j|$  and  $|b_j|$  commute with  $y$  and  $z$ , respectively) and  $w_j = U_j(y \oplus d_j z d_j^*)U_j^* = U_j(1 \oplus d_j)(y \oplus z)(1 \oplus d_j^*)U_j^* = y \oplus z = x$ .  $\square$

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