C*-CONVEX SETS AND COMPLETELY POSITIVE MAPS

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Abstract. We prove a geometric version of an operator valued Hahn-Banach theorem and use it to study sets K that are A-convex over a unital C*-algebra A in the sense that $\sum_{j=1}^{n} a_j^* y_j a_j \in K$ whenever $y_j \in K$ and $a_j \in A$ with $\sum_{j=1}^{n} a_j^* a_j = 1$. We show how weak* compact such sets can be realized as concrete sets of unital completely positive maps. An application to C*-extreme points is also presented.

1. Introduction

Given a C*-algebra A, a subset K of an A-bimodule Y is called *A-convex* if $\sum_{j=1}^{n} a_j^* y_j a_j \in K$ whenever $y_j \in K$ and $a_j \in A$ are such that $\sum_{j=1}^{n} a_j^* a_j = 1$. This notion arises naturally in the context of completely bounded or completely positive maps (see e.g. [1], [4], [31]). For example, for a C*-algebra C, an operator C-system X and a Hilbert C-module $\mathcal{H}$ the set $\mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ of all unital completely positive C-bimodule maps from X into $\mathbb{B}(\mathcal{H})$ is $\mathcal{A}$-convex, where $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ is the commutant of C. Such sets can be regarded as operator valued state spaces since in the case $C = \mathbb{C} = \mathcal{H}$, $\mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ is just the usual state space $S(X)$ of X.

In operator space theory ([8], [32]) the role of the classical Hahn-Banach theorem is played by the extension theorems of Arveson [1] and Wittstock [37], but we will also need a more geometric version concerned with separation of points from closed A-convex sets. Motivated by matrix valued separation theorems of [9] and [10] (see also [16] for an algebro-geometric viewpoint) the author proved such an operator valued separation theorem in [26], but only for A-convex sets of operators containing 0. In Section 2 (Theorem 2.3) we will remove this restriction. (Although this will not be needed in the present paper, we note here that an A-convex set K need not contain any A-convex point (Proposition 9.1), so it is not always possible simply to translate K to a set containing 0.) Section 2 also presents other preliminaries.

A well-known result from classical functional analysis says that each compact convex set in a linear topological Hausdorff space can be represented as a state space of an operator system. Here we will consider an operator valued analogue of this result. Namely, in Section 3 we prove, that for every C*-algebra A each weak* compact A-convex set K of Hilbert space operators can be realized as an operator valued state space $S := \mathrm{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ for a universal Hilbert A-module $\mathcal{H}$ and an operator $\mathcal{C}$-system $\mathcal{X}$, where $\mathcal{C} = \mathbb{B}_A(\mathcal{H})$. Further, in Section 4 we show that each weak* compact $\mathcal{A}$-convex subset of general normal dual Banach bimodule over a von Neumann algebra $\mathcal{A}$ can be regarded as a $\mathcal{A}$-convex set of Hilbert space

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operators. This is surprising, since not all Banach $\mathcal{A}$-bimodules are isometric to operator $\mathcal{A}$-bimodules.

A point y in an A-convex set K is called *A-extreme* if the condition

$$y=\sum_{j=1}^{n}a_j^*y_ja_j,\ \text{where } y_j\in K,\quad a_j\in A \text{ with } \sum_{j=1}^{n}a_j^*a_j=1 \text{ and } a_j \text{ invertible},$$

implies that there exist unitary elements $u_j \in A$ such that $y_j = u_j^* y u_j$. In the case $K \subseteq A$ this notion reduces to the notion of C*-extreme points (studied e.g. in [20], [17], [12], [13], [19]). In [30], [36] and [12] various non-commutative versions of the Krein-Milman theorem are proved, where the coefficients in the definitions of convexity are taken from matrix algebras $\mathbb{M}_n(\mathbb{C})$. To allow the coefficients to be in a general C*-algebra, a new technique is needed.

If K is a convex set, realized as the state space of a closed self-adjoint subspace E of a commutative C*-algebra B so that E generates B and contains 1, then the extreme points of K are precisely those states on E which can be extended in only one way to a state on B (then the extension is necessarily multiplicative [4, 4.1.2]). In Section 5 we will consider a non-commutative analogy of this, a special kind of A-extreme points, called *Choquet A-points*, which are suggested by properties of completely positive maps with the unique extension property (u.e.p.) in the sense of [2], [6] and [7]. The relation between the concepts of Choquet A-points and maps with the u.e.p. is examined in Section 6, where it is shown that in an appropriate context these two notions coincide.

A desired form of a non-commutative Krein Milman theorem would be that each weak* compact $\mathcal{A}$-convex set K (not necessarily contained in $\mathcal{A}$) is equal to the weak* closure of the smallest $\mathcal{A}$-convex set containing $\mathcal{A}$-extreme points of K. This is an open problem, the solution is known only in special cases. Here we will consider only one such case. Thus, after a preparatory Section 7, we will prove in Section 8 for subsets contained in the self-adjoint part of an arbitrary von Neumann algebra $\mathcal{A}$ a sharper result, that they are generated already by their Choquet $\mathcal{A}$-points.

2. Preliminaries on operator bimodules and a separation theorem

All C*-algebras considered in this paper are assumed to be unital (unless explicitly stated otherwise) and are typically denoted by $A, B, C, ...$; von Neumann algebras are usually denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}, ...$

By a *representation* of A on a Hilbert space $\mathcal{H}$ we usually mean a unital representation; then $\mathcal{H}$ is a *Hilbert A-module.* The class of all Hilbert A-modules is denoted by ${}_A\mathbb{H}$. A Hilbert A-module $\mathcal{H}$ is called *faithful* if the corresponding representation $A \to \mathbb{B}(\mathcal{H})$ is injective. $\mathcal{H}$ is *cyclic* if there exists a vector $\xi \in \mathcal{H}$ such that $\mathcal{H} = [A\xi]$ (where [.] denotes the closure of the linear span). $\mathcal{H}$ is called *universal* if it contains a unitarily isomorphic copy of every cyclic Hilbert A-module. The set of all bounded A-module maps on $\mathcal{H}$ is denoted by $\mathbb{B}_A(\mathcal{H})$ (the commutant of A).

A Hilbert module $\mathcal{H}$ over a von Neumann algebra $\mathcal{A}$ is called *normal* if the underlying representation $\mathcal{A} \to \mathbb{B}(\mathcal{H})$ is normal. A normal $\mathcal{H} \in {}_{\mathcal{A}}\mathbb{H}$ is called *W^*-universal* if every normal state on $\mathcal{A}$ comes from a vector in $\mathcal{H}$.

The notion of A-convexity makes sense for subsets of A-bimodules. A *Banach A-bimodule* is a Banach space Y, which is also an A-bimodule, such that $1y = y = y1$ and $\|ayb\| \leq \|a\|\|y\|\|b\|$ for all $a, b \in A$ and $y \in Y$. If in addition Y is a dual

Banach space, A is a von Neumann algebra and all the maps $Y \ni y \mapsto ay, ya$ and $A \ni a \mapsto ay, ya$ are weak* continuous then Y is a *normal dual Banach A bimodule.*

Operator A-bimodules can be defined abstractly [4], [31], but concretely they are just norm closed A-subbimodules of $\mathbb{B}(\mathcal{H})$ for a $\mathcal{H} \in {}_A\mathbb{H}$. Thus a *normal dual operator $\mathcal{A}$-bimodule* is just a weak* closed $\mathcal{A}$-subbimodule of $\mathbb{B}(\mathcal{H})$ for a normal Hilbert $\mathcal{A}$-module $\mathcal{H}$.

$\mathrm{CB}_A(X, Y)$ denotes the space of all completely bounded (c.b.) A-bimodule maps from X to Y and $\mathrm{NCB}_A(X, Y)$ the space of all weak* continuous such maps.

If Y is a norm closed $\mathcal{A}$-subbimodule of $\mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is a normal Hilbert $\mathcal{A}$-module, then the set $\overline{Y}^{\mathcal{A}}$, consisting of those $b \in \mathbb{B}(\mathcal{H})$ for which there exist two families of projections (e_i) and (f_j) in $\mathcal{A}$ with $\sum_i e_i = 1 = \sum_j f_j$ such that $e_i b f_j \in Y$ for all i, j, is an $\mathcal{A}$-subbimodule of $\mathbb{B}(\mathcal{H})$ [24, 2.2]. If $\overline{Y}^{\mathcal{A}} = Y$, then Y is called a *strong $\mathcal{A}$-bimodule* (an alternative definition is in [23]). In general $\overline{Y}^{\mathcal{A}}$ is the smallest strong $\mathcal{A}$-bimodule containing Y.

Proposition 2.1. *If $\phi : X \to Y$ is a completely contractive homomorphism between two normal operator $\mathcal{A}$-bimodules, then ϕ extends uniquely to a completely contractive homomorphism $\overline{\phi}^{\mathcal{A}} : \overline{X}^{\mathcal{A}} \to \overline{Y}^{\mathcal{A}}$ between the corresponding smallest strong $\mathcal{A}$-bimodules. If ϕ is a completely isometric isomorphism (or just a complete isometry into Y), then so is also $\overline{\phi}^{\mathcal{A}}$ (a complete isometry into $\overline{Y}^{\mathcal{A}}$).*

Proof. Assuming that $Y \subseteq \mathbb{B}(\mathcal{H})$ for a normal Hilbert $\mathcal{A}$-module $\mathcal{H}$, by the Wittstock extension theorem ϕ extends to an $\mathcal{A}$-bimodule map $\psi : \overline{X}^{\mathcal{A}} \to \mathbb{B}(\mathcal{H})$. It follows from [24, 4.6, 5.3] that $\psi(\overline{X}^{\mathcal{A}}) \subseteq \overline{Y}^{\mathcal{A}}$ and that ψ is unique.

If ϕ is a completely isometric isomorphism, then we extend both ϕ and ϕ^{-1}, and from $\phi\phi^{-1} = \mathrm{id}_Y$ we obtain by the uniqueness of the extension that $\overline{\phi}^{\mathcal{A}}\overline{\phi^{-1}}^{\mathcal{A}} = \mathrm{id}_{\overline{Y}^{\mathcal{A}}}$ and similarly $\overline{\phi^{-1}}^{\mathcal{A}}\overline{\phi}^{\mathcal{A}} = \mathrm{id}_{\overline{X}^{\mathcal{A}}}$. If ϕ is a complete isometry into Y, we apply this argument to ϕ regarded as a map into $\phi(X)$. It shows that $\overline{\phi}^{\mathcal{A}}$ is a complete isometry of $\overline{X}^{\mathcal{A}}$ onto $\overline{\phi(X)}^{\mathcal{A}}$, but $\overline{\phi(X)}^{\mathcal{A}} \subseteq \overline{Y}^{\mathcal{A}}$ since $\phi(X) \subseteq Y$. □

A *(faithful) operator A-system* is a norm closed self-adjoint operator A-subbimodule $X \subseteq \mathbb{B}(\mathcal{H})$, where $\mathcal{H} \in {}_A\mathbb{H}$ is *faithful*, containing the identity operator. For an abstract characterization see [31, 15.13]. Faithfulness implies that the map $a \mapsto a1$ injects A into X and therefore A is regarded as a subset of X. The set of all contractive completely positive A-bimodule maps from X to Y is denoted by $\mathrm{CCP}_A(X, Y)$ and the subset of all unital such maps by $\mathrm{UCP}_A(X, Y)$. A faithful operator $\mathcal{A}$-system is called *strong* if it is such as an $\mathcal{A}$-bimodule.

Definition 2.2. For a subset G of a Banach A-bimodule Y we denote by $\mathrm{co}_A G$ the smallest A-convex set containing G. Thus $\mathrm{co}_A G$ consists of all sums of the form

$$\sum_{j=1}^{n} a_j^* y_j a_j, \text{ where } y_j \in G,\ a_j \in A \text{ and } \sum_{j=1}^{n} a_j^* a_j = 1. \tag{2.1}$$

Such sums are called *A-convex combinations.*

By a (weak* compact) *A-convex set of operators* we mean a (weak* compact) A-convex subset of a (dual) operator A-bimodule.

We denote by $\overline{G}$ the weak* closure and by $\overline{\overline{G}}$ the norm closure of a set G. Let $\operatorname{Re} b = 1/2(b+b^*)$. Here is an operator valued separation theorem.

Theorem 2.3. *Let K be a norm closed A-convex subset of an operator A-bimodule Y and $y_0 \in Y \setminus K$. Then there exist a cyclic Hilbert A-module $\mathcal{K}$, $\varphi \in \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{K}))$ and $h = h^* \in \mathcal{C} := \mathbb{B}_A(\mathcal{K})$ such that*

$$\operatorname{Re}\varphi(y) \leq h \text{ for all } y \in K \text{ and } \operatorname{Re}\varphi(y_0) \not\leq h.$$

Moreover, if A is a von Neumann algebra, Y a normal dual operator A-bimodule and K is weak closed, then $\mathcal{K}$ can be taken to be normal and φ weak* continuous. For the existence of a normal $\mathcal{K}$ it suffices that K is a strong (not necessarily weak* closed) A-subbimodule of Y (but then φ is not necessarily normal). If $0 \in K$, then in all cases h can be taken to be the identity $1 \in \mathcal{C}$.*

Proof. In the special case when $0 \in K$ this is proved in [26, 1.1, 3.8, 3.9]. Now the proof will be reduced to this special case. We will only consider the case when K is weak* closed and A is a von Neumann algebra (the proof in other cases is similar). We may assume that $Y = \mathbb{B}(\mathcal{H})$ for a faithful normal Hilbert A-module $\mathcal{H}$. Consider the set

$$G := \mathrm{co}_A((K \times \{1\}) \cup \{(0,0)\}) \subseteq \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}),$$

its weak*-closure $\overline{G}$ and the element $x_0 := (y_0, 1) \in Y \times Y$. We claim that $x_0 \notin \overline{G}$. Otherwise, since the weak* closure and the ultrastrong (called also σ-strong [35, II.2.2]) closure of a convex set coincide, there exists a net (x_k) in G converging ultrastrongly to x_0. By the definition of G each x_k is a finite sum of the form

$$x_k = \sum_j a_{j,k}^*(y_{j,k}, 1)a_{j,k}, \text{ where } y_{j,k} \in K,\ a_{j,k} \in A \text{ and } \sum_j a_{j,k}^* a_{j,k} \leq 1.$$

Thus the elements $y_k := \sum_j a_{j,k}^* y_{j,k} a_{j,k}$ converge ultrastrongly to y_0, while the elements $b_k := \sum_j a_{j,k}^* a_{j,k}$ converge to 1. So, for any fixed $z_0 \in K$ the elements

$$z_k := \sum_j a_{j,k}^* y_{j,k} a_{j,k} + \sqrt{1-b_k}\, z_0 \sqrt{1-b_k}$$

converge ultrastrongly to y_0. Since K is A-convex and $z_0, y_{j,k} \in K$, it follows that $z_k \in K$. But then also $y_0 \in K$ since K is closed, which contradicts the hypothesis of the Proposition. This proves that $x_0 \notin \overline{G}$.

Since $\overline{G}$ contains 0, it follows from [26, 3.9] that there exists a normal cyclic Hilbert A-module $\mathcal{K}$ and $\theta \in \mathrm{NCB}_A(Y \times Y, \mathbb{B}(\mathcal{K}))$ such that

$$\operatorname{Re}\theta(x) \leq 1 \text{ for all } x \in \overline{G} \text{ and } \operatorname{Re}\theta(x_0) \not\leq 1.$$

Clearly θ is of the form $\theta = (\varphi, \psi)$, where $\varphi, \psi \in \mathrm{NCB}_A(Y, \mathbb{B}(\mathcal{K}))$. Since $a\psi(1) = \psi(a1) = \psi(1a) = \psi(1)a$ for all $a \in A$, we see that $\psi(1) \in \mathbb{B}_A(\mathcal{K}) = \mathcal{C}$. Set $h = 1 - \operatorname{Re}\psi(1)$. For each $y \in K$ we have that $x := (y,1) \in \overline{G}$, hence $\operatorname{Re}\varphi(y) + \operatorname{Re}\psi(1) = \operatorname{Re}\theta(x) \leq 1$ and consequently $\operatorname{Re}\varphi(y) \leq h$. Similarly $\operatorname{Re}\varphi(y_0) \not\leq h$ since $\operatorname{Re}\theta(x_0) \not\leq 1$. □

Corollary 2.4. *Let Z be a norm closed A-subbimodule of an operator A-bimodule Y, $y_0 \in Y \setminus Z$ and $\mathcal{H} \in {}_A\mathbb{H}$ universal. Then there exists $\varphi \in \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{H}))$ such that $\varphi(Z) = 0$ and $\varphi(y_0) \neq 0$. The same conclusion holds if A is a von Neumann algebra, Y a normal dual operator A-bimodule, Z is strong and $\mathcal{H}$ is any faithful* *normal Hilbert A-module; moreover, if Z is weak* closed, then φ can be taken to be weak* continuous.*

Proof. Since Z is a complex vector space, the relation $\operatorname{Re}\varphi(Z) \leq h$ is equivalent to $\varphi(Z) = 0$. So the corollary is an immediate consequence of Theorem 2.3, except that in the von Neumann algebra case we need to apply the theorem first to the Hilbert A-module $\mathcal{H}^\infty$ (instead of $\mathcal{H}$). Namely, since $\mathcal{H}$ is faithful and normal, each normal state on A arises from a vector in $\mathcal{H}^\infty$ [18, 7.1.8], hence $\mathcal{H}^\infty$ contains a copy of each cyclic normal Hilbert A-module. Thus by Theorem 2.3 there exists $\varphi \in \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{H}^\infty))$ such that $\varphi(Z) = 0$ and $\varphi(y_0) \neq 0$. Identifying $\mathbb{B}(\mathcal{H}^\infty)$ with the space $\mathbb{M}_\infty(\mathbb{B}(\mathcal{H}))$ of all countably infinite bounded matrices with the entries in $\mathbb{B}(\mathcal{H})$ in the usual way, φ is represented by a matrix $\varphi = [\varphi_{i,j}]$ of maps $\varphi_{i,j} \in \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{H}))$ and we may replace φ by any $\varphi_{i,j}$ such that $\varphi_{i,j}(y_0) \neq 0$. □

Given $\mathcal{H} \in {}_A\mathbb{H}$ and an operator A-bimodule Y, let $\mathcal{C} = \mathbb{B}_A(\mathcal{H})$ and set

$$Y^\natural = \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{H})).$$

Observe that $Y^\natural$ is a weak* closed $\mathcal{C}$-subbimodule of $\ell^\infty_Y(\mathbb{B}(\mathcal{H}))$ (the ℓ^∞-direct sum of $\mathrm{card}(Y)$ copies of $\mathbb{B}(\mathcal{H})$), where the bimodule action of $\mathcal{C}$ on $Y^\natural$ is given by

$$(c\varphi d)(y) := c\varphi(y)d \quad (\varphi \in Y^\natural,\ y \in Y,\ c, d \in \mathcal{C}).$$

Thus $Y^\natural$ is a normal dual operator $\mathcal{C}$-bimodule.

Remark 2.5. Weak* continuous functionals on $Y^\natural$ are restrictions of such functionals on $\ell^\infty_Y(\mathbb{B}(\mathcal{H}))$, hence represented by elements of $\ell^1_Y(\mathbb{T}(\mathcal{H}))$, where $\mathbb{T}(\mathcal{H})$ denotes the predual of $\mathbb{B}(\mathcal{H})$. Thus finite sums of functionals of the form

$$Y^\natural \ni \varphi \mapsto \omega(\varphi(y)) \in \mathbb{C}\ (y \in Y,\ \omega \in \mathbb{T}(\mathcal{H}))$$

are norm dense in the predual of $Y^\natural$.

The following theorem generalizes the familiar fact that a Banach space V is equal to the predual of its dual $V^\sharp$.

Theorem 2.6. *Let $\mathcal{H}$ be a faithful normal Hilbert $\mathcal{A}$-module, $\mathcal{C} = \mathbb{B}_{\mathcal{A}}(\mathcal{H})$ and Y a strong operator $\mathcal{A}$-bimodule. Then, with $Y^\natural := \mathrm{CB}_{\mathcal{A}}(Y, \mathbb{B}(\mathcal{H}))$, the equality*

$$(Y^\natural)_\natural := \mathrm{NCB}_{\mathcal{C}}(Y^\natural, \mathbb{B}(\mathcal{H})) = Y$$

holds in the sense that the map $y \mapsto \rho_y$, where $\rho_y \in (Y^\natural)_\natural$ is the evaluation $Y^\natural \ni \varphi \overset{\rho_y}{\mapsto} \varphi(y)$, is a completely isometric $\mathcal{A}$-bimodule map from Y onto $(Y^\natural)_\natural$.

Theorem 2.6 is proved in [29, 5.1]. To show that the hypothesis, that Y is strong, is not redundant, consider $Y = \mathcal{A} \overset{h}{\otimes} \mathcal{A}$ (the Haagerup tensor product). By the bimodule property it is easy to see that $Y^\natural = \mathrm{CB}_A(Y, \mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H})$, hence, assuming that $\mathcal{A}$ is infinite dimensional, $(Y^\natural)_\natural = \mathrm{NCB}_{\mathcal{C}}(\mathbb{B}(\mathcal{H}), \mathbb{B}(\mathcal{H})) \neq Y$.

3. Weak* compact C*-convex sets as operator valued state spaces

Definition 3.1. A map $f : K \to G$ between two A-convex sets is called A-*affine* if

$$f(\sum_{j=1}^n a_j^* y_j a_j) = \sum_{j=1}^n a_j^* f(y_j) a_j \text{ whenever } y_j \in K,\ a_j \in A,\ \sum_{j=1}^n a_j^* a_j = 1.$$

We denote by $\mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$ the set of all *weak*-continuous* A-affine maps from K into $\mathbb{B}(\mathcal{H})$ (whenever this makes sense).

Remark 3.2. If Y is a dual operator A-bimodule, that is, a weak* closed A-subbimodule of $\mathbb{B}(\mathcal{K})$ for a $\mathcal{K} \in {}_A\mathbb{H}$, then Y is a normal dual operator bimodule over the universal von Neumann envelope $\mathcal{A}$ of A since $\mathcal{K}$ is a normal Hilbert $\mathcal{A}$-module. Namely, a representation π of A on $\mathcal{K}$ extends uniquely to a normal representation $\overline{\pi}$ of $\mathcal{A}$ and $\overline{\pi}(\mathcal{A})$ is just the weak* closure of $\pi(A)$, hence Y is a dual normal operator bimodule over $\overline{\pi}(\mathcal{A})$ and therefore also over $\mathcal{A}$ (by $ayb := \overline{\pi}(a)y\overline{\pi}(b)$).

Proposition 3.3. *Let Y be a dual operator A-bimodule and $\mathcal{A}$ the universal von Neumann envelope of A. Then each weak* closed A-convex subset of Y is also $\mathcal{A}$-convex. Moreover, for every $\mathcal{H} \in {}_A\mathbb{H}$ we have*

$$\mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H})) = \mathbb{A}_{\mathcal{A}}^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H})). \tag{3.1}$$

Proof. Let

$$y = \sum_{j=1}^{n} a_j^* y_j a_j,$$

where $y_j \in K$, $a_j \in \mathcal{A}$ and $\sum_{j=1}^{n} a_j^* a_j = 1$. By the Kaplansky density theorem [35, II.4.8], applied in $\mathbb{M}_n(\mathcal{A})$ to the matrix with the first column $a := [a_1, \ldots, a_n]$ and the remaining columns 0, we find nets $(a_{j,k})_k \subseteq A$ such that $a_{j,k} \xrightarrow{k} a_j$ in the strong* operator topology and $b_k := \sum_{j=1}^{n} a_{j,k}^* a_{j,k} \leq 1$. Then for any fixed $y_0 \in K$ the A-convex combinations

$$z_k := \sum_{j=1}^{n} a_{j,k}^* y_j a_{j,k} + (1 - b_k)^{1/2} y_0 (1 - b_k)^{1/2} \tag{3.2}$$

are in K and converge to y in the weak* topology by the continuity of the functional calculus ([5, I.7.2], [35, II.4.3]). Since K is weak* closed, $y \in K$.

If $f : K \to \mathbb{B}(\mathcal{H})$ is an A-affine weak* continuous function, then from (3.2)

$$f(z_k) = \sum_{j=1}^{n} a_{j,k}^* f(y_j) a_{j,k} + (1 - b_k)^{1/2} f(y_0)(1 - b_k)^{1/2}$$

and it follows by the weak* continuity that $f(y) = \sum_{j=1}^{n} a_j^* f(y_j) a_j$ since $a_{j,k} \to a_j$, $a_{j,k}^* \to a_j^*$ and $b_k \to 1$ in the strong operator topology. $\square$

Proposition 3.4. *Let $\mathcal{H}$ be a faithful Hilbert A-module, $\mathcal{C} = \mathbb{B}_A(\mathcal{H})$ and K a weak* compact A-convex set of operators. Then $\mathcal{X} := \mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$ is a faithful strong operator $\mathcal{C}$-system, where for each $n \in \mathbb{N}$ the norm on $\mathbb{M}_n(\mathcal{X}) = \mathbb{A}_A^{\mathrm{w}*}(K, \mathbb{B}(\mathcal{H}^n))$ is defined by $\|f\| = \sup_{y \in K} \|f(y)\|$.*

Proof. For each $f \in \mathcal{X}$, $f(K)$ is a weak* compact (hence norm bounded) subset of $\mathbb{B}(\mathcal{H})$, so $\|f\| < \infty$ and it follows that $\mathcal{X}$ may be regarded as an A-subbimodule of $\ell_K^\infty(\mathbb{B}(\mathcal{H})) =: D$ $(\subseteq \mathbb{B}(\mathcal{H}^K))$. Elements of $\mathcal{C}$ are regarded as constant $\mathcal{A}$-affine maps, so $\mathcal{C} \subseteq \mathcal{X}$. The only nontrivial fact to be proved is that $\mathcal{X}$ is strong. We will prove that $\mathcal{X}$ is strong as a right $\mathcal{C}$-module, a similar argument then shows that $\mathcal{X}$ is strong also as a left $\mathcal{C}$-module and then it readily follows that $\mathcal{X}$ is strong as a $\mathcal{C}$-bimodule. By [24, 2.1] it suffices to prove that for each $f \in D$ and each orthogonal set of projections $(e_i)_{i \in \mathbb{I}}$ in $\mathcal{C}$ with $\sum_i e_i = 1$ the condition $fe_i \in \mathcal{X}$ for all $i \in \mathbb{I}$ implies that $f \in \mathcal{X}$. Clearly the map f is A-affine since all fe_i are A-affine and the projections e_i commute with A. To prove that f is weak* continuous, it suffices to prove that $\omega \circ f$ is weak* continuous for each normal state ω on $\mathbb{B}(\mathcal{H})$. For each finite subset $\mathbb{F}$ of $\mathbb{I}$ let $e_{\mathbb{F}} = \sum_{i \in \mathbb{F}} e_i$ and $e_{\mathbb{F}}^\perp = 1 - e_{\mathbb{F}}$. Since $fe_i \in \mathcal{X}$, the functions $\omega\circ(fe_i)$ are weak* continuous, hence so are the finite sums $\omega\circ(fe_{\mathbb{F}})$. But the net $(\omega\circ(fe_{\mathbb{F}})_{\mathbb{F}}$ converges uniformly to $\omega\circ f$ since for each $y\in K$ we have by the Schwarz inequality for states

$$|\omega(f(y)e_{\mathbb{F}}^{\perp})|\leq\omega(f(y)^*f(y))^{1/2}\omega(e_{\mathbb{F}}^{\perp})^{1/2}\leq\|f\|\omega(e_{\mathbb{F}}^{\perp})^{1/2}\to 0.$$

□

The following generalizes a well-known result from classical convexity theory; a version for matricially convex sets has been proved by Webster and Winkler in [36].

Theorem 3.5. *Let $\mathcal{H}$ be a W^*-universal Hilbert $\mathcal{A}$-module, $\mathcal{C}=\mathbb{B}_{\mathcal{A}}(\mathcal{H})$, K a weak* compact $\mathcal{A}$-convex subset of a normal dual operator $\mathcal{A}$-bimodule Y and $\mathcal{X}=\mathbb{A}_{\mathcal{A}}^{\mathrm{w}^*}(K,\mathbb{B}(\mathcal{H}))$. Then the map*

$$\varepsilon:K\to S:=\mathrm{UCP}_{\mathcal{C}}(\mathcal{X},\mathbb{B}(\mathcal{H})),\ \varepsilon(y)(f):=f(y)\ (y\in K,\ f\in\mathcal{X})$$

is an $\mathcal{A}$-affine weak-homeomorphism of K onto S.*

Proof. Clearly $\varepsilon(y)$ is a u.c.p. $\mathcal{C}$-bimodule map for each $y\in K$, that is, $\varepsilon(y)\in S$. It is straightforward to verify that the map ε is $\mathcal{A}$-affine. To prove that ε is continuous from the weak* topology on K (inherited from Y) to the weak* topology on S (inherited from $\mathcal{X}^{\natural}=\mathrm{CB}_{\mathcal{C}}(\mathcal{X},\mathbb{B}(\mathcal{H}))\subseteq\ell^{\infty}_{\mathcal{X}}(\mathbb{B}(\mathcal{H}))$), note that since the predual of $\ell^{\infty}_{\mathcal{X}}(\mathbb{B}(\mathcal{H}))$ is $\ell^{1}_{\mathcal{X}}(\mathbb{T}(\mathcal{H}))$, we only have to prove that for each sequence $(t_j)\subset\mathbb{T}(\mathcal{H})$ satisfying $\sum_{j=1}^{\infty}\|t_j\|_1<\infty$ and each bounded sequence $(f_j)\subset\mathcal{X}$ the function $K\ni y\mapsto\sum_{j=1}^{\infty}(f_j(y))(t_j)\in\mathbb{C}$ is weak* continuous (see Remark 2.5). But this follows from the uniform convergence of the series and the continuity of the functions $y\mapsto(f_j(y))(t_j)$. The map ε is injective since the functions $f\in\mathcal{X}$ separate points of K; namely, already the restrictions $\rho|K$, where $\rho\in\mathrm{NCB}_{\mathcal{A}}(Y,\mathbb{B}(\mathcal{H}))$, separate points of K by Corollary 2.4. Thus ε is a homeomorphism of K onto $\varepsilon(K)$.

Suppose that $\varepsilon(K)\neq S$ and let $\rho\in S\setminus\varepsilon(K)$. By Theorem 2.3 there exist $\varphi\in\mathrm{NCB}_{\mathcal{A}}(\mathcal{X}^{\natural},\mathbb{B}(\mathcal{H}))$ and $h=h^*\in\mathbb{B}_{\mathcal{A}}(\mathcal{H})=\mathcal{C}$ such that $\mathrm{Re}\,\varphi(\varepsilon(y))\leq h$ for all $y\in K$ and $\mathrm{Re}\,\varphi(\rho)\not\leq h$. Since $\mathcal{X}$ is strong as $\mathcal{C}$-bimodule by Proposition 3.4, it follows from Theorem 2.6 that φ is the evaluation at some element $f\in\mathcal{X}$, so

$$\mathrm{Re}\,f(y)\leq h\text{ for all }y\in K\text{ and }\ \mathrm{Re}\,\rho(f)\not\leq h. \tag{3.3}$$

Thus $g\in\mathcal{X}$ defined by $g(y)=h-f(y)$ satisfies $\mathrm{Re}\,g\geq 0$, hence also $\mathrm{Re}\,\rho(g)=\rho(\mathrm{Re}\,g)\geq 0$ (since $\rho\in S$ preserves involution by positivity). Since $h\in\mathcal{C}$ and ρ is a unital $\mathcal{C}$-bimodule map, $\rho(h)=h\rho(1)=h$, hence $h-\mathrm{Re}\,\rho(f)=\mathrm{Re}\,\rho(h)-\mathrm{Re}\,\rho(f)=\mathrm{Re}\,\rho(g)\geq 0$, which contradicts the second relation in (3.3). Thus $\varepsilon(K)=S$. □

Theorem 3.5 and Proposition 3.3 immediately imply the following corollary.

Corollary 3.6. *Let $\mathcal{H}\in{}_A\mathbb{H}$ be universal, $\mathcal{C}:=\mathbb{B}_A(\mathcal{H})$, K a weak* compact A-convex set of operators and $\mathcal{X}=\mathbb{A}_A^{\mathrm{w}^*}(K,\mathbb{B}(\mathcal{H}))$. Then $K\cong\mathrm{UCP}_{\mathcal{C}}(\mathcal{X},\mathbb{B}(\mathcal{H}))$.*

4. $\mathcal{A}$-convex subsets of normal dual Banach $\mathcal{A}$-bimodules

We would like to prove that weak* compact $\mathcal{A}$-convex subsets of normal dual Banach $\mathcal{A}$-bimodules can be represented as such subsets of Hilbert space operators. For this we need a few preliminary results.

A set K in an A-bimodule is called *absolutely A-convex* if $\sum_{j=1}^{n}a_j^*y_jb_j\in K$ whenever $y_j\in K$, $a_j,b_j\in A$ and $\sum_{j=1}^{n}a_j^*a_j\leq 1$, $\sum_{j=1}^{n}b_j^*b_j\leq 1$.

If a Banach A-bimodule Y is isometric to an operator A-bimodule, then among all possible norms on spaces $\mathbb{M}_n(Y)$ ($n \in \mathbb{N}$) that make Y an operator A-bimodule there are minimal ones [25], which are given by

$$\|[y_{i,j}]\| = \sup\{\| \sum_{i,j=1}^{n} a_i^* y_{i,j} b_j \| : \, a_j, b_j \in A, \; \sum_{j=1}^{n} a_j^* a_j \leq 1, \; \sum_{j=1}^{n} b_j^* b_j \leq 1\}.$$

Theorem 4.1. *Let Y be a normal dual Banach $\mathcal{A}$-bimodule, $Y_\sharp$ its predual Banach space, H a weak* compact $\mathcal{A}$-absolutely convex subset of Y, $Z := \mathbb{R}^+ H$ and $\| \, \|_H$ the Minkovski seminorm on Z defined by*

$$\|z\|_H = \inf\{t \in \mathbb{R}^+ : \, z \in tH\}.$$

Then Z, equipped with the norm $\| \, \|_H$ and the corresponding minimal operator $\mathcal{A}$-bimodule norms on $\mathbb{M}_n(Z)$ ($n \in \mathbb{N}$), is a normal dual operator $\mathcal{A}$-bimodule such that on H its Banach space predual $Z_\sharp$ determines the same weak topology as $Y_\sharp$.*

Proof. Since H is absolutely $\mathcal{A}$-convex, Z is an $\mathcal{A}$-subbimodule of Y. Since H is weak* compact, hence bounded, the Minkovski seminorm $\| \, \|_H$ is indeed a norm on Z. To show completeness of $(Z, \| \, \|_H)$, let (z_k) be a Cauchy sequence in Z. Since the original norm $\| \, \|$ on Z is dominated by a constant multiple of $\| \, \|_H$ (because H is bounded) and Y is complete, there exists the limit z of (z_k) in Y. For each k let $t_k = \sup_{m \geq k} \|z_k - z_m\|_H$; then $t_k \to 0$ since the sequence (z_k) is Cauchy. Moreover, $z_k - z_m \in t_k H$ for all $m \geq k$ implies (since H is closed) that $z_k - z \in t_k H$, hence $z \in Z$ and $\|z_k - z\|_H \leq t_k \to 0$.

Since H is $\mathcal{A}$-absolutely convex, the inequality

$$\|a_1^* z_1 b_1 + a_2^* z_2 b_2\|_H \leq \|a_1^* a_1 + a_2^* a_2\|^{1/2} \max\{\|z_1\|_H, \|z_2\|_H\} \|b_1^* b_1 + b_2^* b_2\|^{1/2}$$

holds for all $z_j \in Z$ and $a_j, b_j \in \mathcal{A}$, hence $(Z, \| \, \|_H)$ is an operator $\mathcal{A}$-bimodule [25].

Now we would like to prove that $(Z, \| \, \|_H)$ is a dual Banach space. Let $Z^\sharp$ be the dual space of Z,

$$Z_\prime := \{\omega \in Z^\sharp : \, \omega | H \text{ is weak* continuous}\} \text{ and } B_{Z_\prime} := \{\omega \in Z_\prime : \, \|\omega\| \leq 1\},$$

where the weak* topology on H is inherited from Y. Denote by $H_\circ$ the (absolute) polar of H in $Y_\sharp$. Then $H = (H_\circ)^\circ$ by the bipolar theorem. Observe that the set $B_{Z_\prime}|H := \{\omega|H : \, \omega \in B_{Z_\prime}\}$ coincides with the set of all homogeneous (that is, $f(\alpha h) = \alpha f(h)$ for all $h \in H$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$) affine (in the usual sense) weak* continuous functions f on H satisfying $\sup_{z \in H} |f(z)| \leq 1$. (That is, each such f extends to a unique element of $Z_\prime$ since H is just the unit ball of Z.) Since the set of all restrictions of functionals in $Y_\sharp$ to H is uniformly dense in the set of all weak* continuous affine functions on H by [35, III, 6.3], it follows (just by restricting to functions with the sup norm on H at most 1) that the set

$$H_\circ | H := \{\omega | H : \, \omega \in H_\circ\}$$

is uniformly dense in the set

$$B_{Z_\prime} | H = \{\omega | H : \, \omega \in B_{Z_\prime}\}.$$

Consequently for each $z \in Z$ the last in following chain of equivalences hold (other equivalences are evident):

$$\|z\|_H \leq 1 \Leftrightarrow z \in H \Leftrightarrow |\langle \omega, z \rangle| \leq 1 \; \forall \omega \in H_\circ \Leftrightarrow |\langle \omega, z \rangle| \leq 1 \; \forall \omega \in B_{Z_\prime}.$$

Thus by homogeneity

$$\|z\|_H = \sup\{|\langle \omega, z\rangle| : \omega \in B_{Z_\prime}\} \text{ for all } z \in Z,$$

which proves that the natural map $Z \to (Z_\prime)^\sharp$ is isometric, so we will regard from now on Z as a subspace of $(Z_\prime)^\sharp$. Moreover, since the set $H_\circ|H$ is uniformly dense in $B_{Z_\prime}|H$, the two sets determine the same weak topology on H. Since H is bounded (both, in the original norm on Y and in the new norm $\| \ \|_H$), the topology determined by $H_0|H$ is the same as the one determined by $Y_\sharp$ (thus the original weak* topology on H), while the topology determined by $B_{Z_\prime}|H$ is the same as that determined by $Z_\prime$. We conclude that on H the topology determined by $Z_\prime$ is the same as that defined by $Y_\sharp$, hence H is compact in the topology determined by $Z_\prime$. Since H is also absolutely convex, it follows that H must coincide with the unit ball $B_{(Z_\prime)^\sharp}$. (Otherwise we could choose $z \in B_{(Z_\prime)^\sharp} \setminus H$ and then by the bipolar theorem an $\omega \in Z_\prime$ such that $|\langle \omega, y\rangle| \leq 1$ for all $y \in H$ and $|\langle \omega, z\rangle| > 1$. But the first inequality means that $\|\omega\| \leq 1$, which, together with $z \in B_{(Z_\prime)^\sharp}$ implies $|\langle \omega, z\rangle| \leq 1$, a contradiction.) Since H (the unit ball of Z) coincides with $B_{(Z_\prime)^\sharp}$, it follows that $Z = (Z_\prime)^\sharp$. So $Z_\prime$ is a predual of Z and we will therefore denote it by $Z_\sharp$.

To prove that Z is a normal Banach $\mathcal{A}$-bimodule, we have to verify that the multiplications $\mathcal{A} \ni a \mapsto az \in Z$ and $\mathcal{A} \ni a \mapsto za \in Z$ are weak* continuous for each $z \in Z$. We may assume that $\|z\| \leq 1$ and by the Krein-Smulian theorem it suffices to check the continuity on the unit ball $B_\mathcal{A}$ of $\mathcal{A}$. But for each $\omega \in Z_\sharp$ the functions $B_\mathcal{A} \ni a \mapsto \langle \omega, az\rangle \in \mathbb{C}$ and $B_\mathcal{A} \ni a \mapsto \langle \omega, za\rangle \in \mathbb{C}$ are weak* continuous since $\omega|H$ is weak* continuous in the weak* topology inherited from Y (by the definition of $Z_\prime$), $B_\mathcal{A}H, HB_\mathcal{A} \subseteq H$ and since the multiplications $\mathcal{A} \ni a \mapsto az, za \in Y$ are weak* continuous because Y is a normal dual Banach $\mathcal{A}$-bimodule. A similar argument shows that the maps $Z \ni z \mapsto az, za \in Z$ are weak* continuous for each $a \in \mathcal{A}$, hence Z is a normal dual Banach $\mathcal{A}$-bimodule. From this it follows easily that the unit ball of each $\mathbb{M}_n(Z)$ equipped with the minimal operator $\mathcal{A}$-bimodule norm is weak* closed in $\mathbb{M}_n(Z)$, hence Z is a dual operator space (see [4, 1.6.4]). By [4, 3.8.3] it follows now that Z is a normal dual operator $\mathcal{A}$-bimodule. □

Lemma 4.2. *If $(K_k)_{k=1}^m$ is a finite collection of $\mathcal{A}$-convex subsets of a Banach $\mathcal{A}$-bimodule, then the set*

$$K := \{\sum_{k=1}^m b_k y_k b_k : y_k \in K_k,\ b_k \in \mathcal{A}_+,\ \sum_{k=1}^m b_k^2 = 1\}$$

is $\mathcal{A}$-convex, hence K is the smallest $\mathcal{A}$-convex set containing the union $\cup_{k=1}^m K_k$.

Proof. We have to prove that for every finite subset $\{z_1, \ldots, z_n\}$ of K and every isometric column $a = [a_1, \ldots, a_n]^T$ (that is, $a^*a = 1$) with components $a_j \in \mathcal{A}$ the element $z := \sum_{j=1}^n a_j^* z_j a_j$ is in K. Each z_j is of the form $z_j = \sum_{k=1}^m b_{j,k} y_{j,k} b_{j,k}$, where $y_{j,k} \in K_k$, $b_{j,k} \in \mathcal{A}_+$ and $\sum_{k=1}^m b_{j,k}^2 = 1$, so

$$z = \sum_{j=1}^n \sum_{k=1}^m a_j^* b_{j,k} y_{j,k} b_{j,k} a_j. \tag{4.1}$$

Let

$$g_k := \begin{bmatrix} b_{1,k}a_1 \\ b_{2,k}a_2 \\ \vdots \\ b_{n,k}a_n \end{bmatrix} = \begin{bmatrix} u_{1,k} \\ u_{2,k} \\ \vdots \\ u_{n,k} \end{bmatrix} c_k = u_k c_k$$

be the polar decomposition of g_k, where $c_k = |g_k| = (\sum_{j=1}^n a_j^* b_{j,k}^2 a_j)^{1/2} \in \mathcal{A}$ and u_k is a partial isometry with the initial projection $p_k := u_k^* u_k \in \mathcal{A}$ (the range projection of c_k) and the final projection $q_k := u_k u_k^* \in \mathbb{M}_n(\mathcal{A})$ (the range projection of g_k). Observe that $\sum_{k=1}^m c_k^2 = 1$. If we show that in this polar decomposition each u_k can be replaced by an isometry $v_k = [v_{1,k}, \ldots, v_{n,k}]^T$ (where $v_{j,k} \in \mathcal{A}$), then we will have $g_k = v_k c_k$, that is $b_{j,k}a_j = v_{j,k}c_k$. Then rewriting (4.1) as

$$z = \sum_{k=1}^m c_k (\sum_{j=1}^n v_{j,k}^* y_{j,k} v_{j,k}) c_k$$

and noting that $\sum_{j=1}^n v_{j,k}^* y_{j,k} v_{j,k} \in K_k$ since $\sum_{j=1}^n v_{j,k}^* v_{j,k} = v_k^* v_k = 1$ and K_k is $\mathcal{A}$-convex, this will prove that $z \in K$, as required.

Let $e_1 \in \mathbb{M}_n(\mathcal{A})$ be the projection of $\mathcal{A}^n$ onto its first summand $\mathcal{A}$ and regard $\mathcal{A}$ as a subalgebra of $\mathbb{M}_n(\mathcal{A})$, embedded into $(1,1)$ position. If we show that $e_1 - p_k$ is equivalent in $\mathbb{M}_n(\mathcal{A})$ to a subprojection of $1 - q_k$, say through a partial isometry w_k, then $v_k := u_k + w_k$ is an isometry in $\mathbb{M}_n(\mathcal{A})$ with non-zero components only in the first column (since its initial projection is e_1) and $u_k c_k = v_k c_k$ (since $w_k c_k = 0$ because the initial projection of w_k is orthogonal to p_k, the range projection of c_k). Note that $e_1 - p_k$ is the projection onto $\operatorname{im} c_k{}^\perp = \ker c_k = \ker \sum_{j=1}^n a_j^* b_{j,k}^2 a_j = \ker d_k a$, where $d_k \in \mathbb{M}_n(\mathcal{A})$ is the diagonal matrix with the entries $b_{1,k}, \ldots, b_{n,k}$ along the diagonal. Thus $e_1 - p_k$ is the projection onto $\ker d_k a = a^{-1}(\ker d_k)$. On the other hand $1 - q_k$ is the projection onto $\ker g_k^* = \ker a^* d_k \supseteq \ker d_k$. Let $f_k \in \mathbb{M}_n(\mathcal{A})$ be the projection onto $a(a^{-1}(\ker d_k))$ ($\subseteq \ker d_k$). Since a is an isometry, $w_k = f_k a$ is a partial isometry in $\mathbb{M}_n(\mathcal{A})$ with the initial space $a^{-1}(\operatorname{im} f_k) = a^{-1}(\ker d_k)$ and the final space $\operatorname{im} f_k$ contained in $\ker d_k$ (hence also contained in the range of $1 - q_k$). Thus w_k implements the desired equivalence between $e_1 - p_k$ and a subprojection of $1 - q_k$. □

Corollary 4.3. *The absolutely $\mathcal{A}$-convex hull of every $\mathcal{A}$-convex bounded subset K of a normal dual Banach $\mathcal{A}$-bimodule Y is bounded.*

Proof. By considering the weak* closure of K the proof reduces to the situation when K is weak* closed. Let G be the $\mathcal{A}$-convex hull of $\{0\} \cup \cup_{k=0}^3 (-i)^k K$. Since all the sets $(-i)^k K$ are $\mathcal{A}$-convex, by Lemma 4.2

$$G = \left\{ \sum_{k=0}^3 a_k (-i)^k y_k a_k : y_k \in K,\ a_k \in \mathcal{A}_+,\ \sum_{k=0}^3 a_k^2 \le 1 \right\}. \tag{4.2}$$

Observe that $(-i)^k G \subseteq G$ for all k. The set G is not necessarily absolutely $\mathcal{A}$-convex, however, for all finite collections $(z_j)_{j=1}^n \subseteq G$ and $a_j, b_j \in \mathcal{A}$ with $\sum_{j=1}^n a_j^* a_j \le 1$ and $\sum_{j=1}^n b_j^* b_j \le 1$ we have by the polarization identity

$$a^* z b = \frac{1}{4} \sum_{k=0}^3 (-i)^k (a + i^k b)^* z (a + i^k b) \quad (\text{where } i = \sqrt{-1}),$$

that

$$\sum_{j=1}^{n} a_j^* z_j b_j = 2\sum_{j=1}^{n}\sum_{k=0}^{3} c_{j,k}^*((-i)^k z_j)c_{j,k}, \tag{4.3}$$

where $(-i)^k z_j \in G$ and $c_{j,k} := \frac{1}{2\sqrt{2}}(a_j + i^k b_j)$. Since

$$\sum_{j=1}^{n}\sum_{k=0}^{3} c_{j,k}^* c_{j,k} = \frac{1}{2}\sum_{j=1}^{n}(a_j^* a_j + b_j^* b_j) \leq 1,$$

(4.3) implies (since G is $\mathcal{A}$-convex and contains 0) that the absolutely $\mathcal{A}$-convex hull H of G satisfies $H \subseteq 2G$. From this and (4.2) we conclude that H is bounded. □

We mention that in a general Banach $\mathcal{A}$-bimodule (for example, in the predual of $\mathcal{A}$) the (absolutely) $\mathcal{A}$-convex hull of a point can be unbounded.

Corollary 4.4. *For each weak* compact $\mathcal{A}$-convex subset K of a normal dual Banach $\mathcal{A}$-bimodule Y there exists an $\mathcal{A}$-affine weak* homeomorphism from K onto a subset of a normal dual operator $\mathcal{A}$-bimodule.*

Proof. Let H be the weak* closure of the $\mathcal{A}$-absolutely convex hull of K. Since K is bounded, so must be H by Corollary 4.3, hence H is weak* compact. If Z is the $\mathcal{A}$-subbimodule of Y constructed as in Theorem 4.1 (so that H is the unit ball of Z in the new norm), then Z is a normal dual operator $\mathcal{A}$-bimodule and K, regarded as a subset of Z, is still weak* compact, since $K \subseteq H$ and on H the two preduals $Z_\sharp$ and $Y_\sharp$ define the same topology. □

5. Choquet A-points

Definition 5.1. If K is an A-convex set of operators, a point $y \in K$ is called a *Choquet A-point* if the condition

$$y = \sum_{j=1}^{n} a_j^* y_j a_j,\ y_j \in K,\ a_j \in A,\ \sum_{j=1}^{n} a_j^* a_j = 1 \tag{5.1}$$

implies that

$$|a_j|y = y|a_j|,\quad |a_j^*|y_j = y_j|a_j^*| \ \text{ and } \ a_j^* y_j a_j = |a_j|y|a_j| \ \text{ for all } j = 1, \ldots, n. \tag{5.2}$$

If we assume that the coefficients a_j in (5.1) are invertible, then by the polar decomposition $a_j = u_j|a_j|$ of a_j the last identity in (5.2) implies that y_j is unitary equivalent to y, which is just the requirement in the customary definition of a C*-extreme (that is, A-extreme) point. So Choquet A-points are special A-extreme points. Obviously Choquet A-points are also Choquet A_0-points for any C*-subalgebra A_0 of A. In particular Choquet A-points are extreme in the usual sense. In contrast, for the usual C*-extreme points it has only been proved in special situations with a nontrivial proof [20, p. 73] that they are extreme in the ordinary sense. The above definition can be reformulated as follows:

Proposition 5.2. *Let $a_j = u_j|a_j|$ be the polar decomposition and p_j (respectively q_j) the range projection of $|a_j|$ (of $|a_j^*|$) in a von Neumann algebra containing A. Then $y \in K$ is a Choquet A-point if and only if the condition (5.1) implies that*

$$|a_j|y = y|a_j|,\quad |a_j^*|y_j = y_j|a_j^*| \ \text{ and } \ u_j^* y_j u_j = y p_j \tag{5.3}$$

(where the last identity makes sense in $\mathbb{B}(\mathcal{K}) \supseteq K$). The relations (5.3) imply that

$$u_j y u_j^* = y_j q_j = q_j y_j \tag{5.4}$$

and

$$a_i a_j^* y_j = y_i a_i a_j^* \quad (i, j = 1, \ldots, n). \tag{5.5}$$

Proof. Assume that y is a Choquet A-point in K. Writing the equality $a_j^* y_j a_j = |a_j| y |a_j|$ as $|a_j| u_j^* y_j u_j |a_j| = |a_j| y |a_j|$ and noting that p_j commutes with $|y|$ if $|a_j|$ does, it follows from the well-known properties of the polar decomposition that $u_j^* y_j u_j = y p_j$. Similarly (since $q_j = u_j u_j^*$ is the range projection of $|a_j^*|$)

$$y_j q_j = q_j y_j q_j = u_j u_j^* y_j u_j u_j^* = u_j y p_j u_j^* = u_j y u_j^*.$$

These reasoning holds also in the reversed direction. The proof of (5.5) is also an easy computation:

$$\begin{aligned} a_i a_j^* y_j &= u_i |a_i| |a_j| u_j^* y_j = u_i |a_i| |a_j| u_j^* q_j y_j = \text{(by (5.4))}\ u_i |a_i| |a_j| u_j^* u_j y u_j^* \\ &= u_i |a_i| |a_j| p_j y u_j^* = \text{(by the first equality in (5.3))}\ u_i y |a_i| |a_j| u_j^* \\ &= u_i y u_i^* u_i |a_i| |a_j| u_j^* = \text{(by (5.4))}\ y_i q_i u_i |a_i| |a_j| u_j^* \\ &= y_i a_i a_j^*. \end{aligned}$$

□

Here is a very concise reformulation of the definition of a Choquet A-point.

Proposition 5.3. *A point $y \in K$ is a Choquet A-point if and only if the following condition is satisfied: for every diagonal matrix $z = \oplus_{j=1}^n y_j$, where $y_j \in K$, and for every isometric column $a = [a_1, \ldots, a_n]^T$ with components $a_j \in A$, the condition*

$$y = a^* z a \tag{5.6}$$

implies that the projection $e := aa^*$ commutes with z.

Proof. Suppose that y is a Choquet A-point in K. If $y = a^* z a$, where z and a are as in (5.6), then by Proposition 5.2 (more precisely, by (5.5)) the projection $e = aa^* = [a_i a_j^*]$ commutes with z. To prove the converse implication, suppose that y is an A-convex combination of elements of K, that is, $y = a^* z a$, where z and a are as in (5.6). Then $ez = ze$ by assumption, hence (considering the (j, j) entries of matrices) $a_j a_j^* y_j = y_j a_j a_j^*$ and so $|a_j^*| y_j = y_j |a_j^*|$. Moreover, $aya^* = aa^* z aa^* = eze = ze$, hence $a_j y a_j^* = y_j a_j a_j^* = |a_j^*| y_j |a_j^*|$. Using the well-known property $a_j = u_j |a_j| = |a_j^*| u_j$ we now have $|a_j^*| u_j y u_j^* |a_j^*| = a_j y a_j^* = |a_j^*| y_j |a_j^*|$, hence

$$u_j y u_j^* = y_j q_j. \tag{5.7}$$

Now we have $y = \sum_{j=1}^n a_j^* y_j a_j = \sum_{j=1}^n a_j^* u_j y u_j^* a_j = \sum_{j=1}^n |a_j| y |a_j|$, which can be written as $y = b^* v b$, where $v := y^{(n)}$ (the direct sum of n copies of y) and $b := [|a_1|, \ldots, |a_n|]^T$ is isometric. By the hypothesis this implies that the projection $f := bb^* = [|a_i||a_j|]$ commutes with $y^{(n)}$, that is, $|a_i||a_j| y = y |a_i||a_j|$. In particular y commutes with $|a_j|$ for all j. Thus y commutes also with the range projection p_j of $|a_j|$ and using (5.7) we finally conclude that $y p_j = p_j y p_j = u_j^* u_j y u_j^* u_j = u_j^* y_j u_j$. Thus y is a Choquet A-point of K. □

An easy consequence of the fact that A is linearly spanned by its unitaries and Proposition 5.3 (the proof will be omited) is that the notion of a Choquet A-point is an A-affine invariant in the following sense:

Proposition 5.4. *If $f : K \to H$ is an A-affine bijection and y is a Choquet A-point of K, then $f(y)$ is a Choquet A-point of H.*

Lemma 5.5. *(i) It suffices to require the conditions in Definition 5.1 of Choquet $\mathcal{A}$-points in the case $n = 2$.*

(ii) If $\mathcal{A}$ is a von Neumann algebra, K is weak compact $\mathcal{A}$-convex subset of a normal dual operator $\mathcal{A}$-bimodule and y is a Choquet $\mathcal{A}$-point of K, then for every families $(y_i)_{i\in\mathbb{I}} \subseteq K$ and $(a_i)_{i\in\mathbb{I}} \subseteq \mathcal{A}$ with $\sum_{i\in\mathbb{I}} a_i^* a_i = 1$ the condition*

$$y = \sum_{i\in\mathbb{I}} a_i^* y_i a_i \tag{5.8}$$

implies the relations (5.3) and hence also (5.4) and (5.5).

Proof. Assume that $K \subseteq \mathbb{B}(\mathcal{H})$ for a $\mathcal{H} \in \mathbb{H}_{\mathcal{A}}$ (with $\mathcal{H}$ normal in the case (ii)). The column $a := (a_i)_{i\in\mathbb{I}}$ is an isometry from $\mathcal{H}$ into $\mathcal{H}^{\mathbb{I}}$. We write (5.8) as

$$y = \frac{1}{2} a_1^* y_1 a_1 + (\frac{1}{2} a_1^* y_1 a_1 + \sum_{i\neq 1} a_i^* y_i a_i) \tag{5.9}$$

and introduce the column $b := (b_i)_{i\in\mathbb{I}}$, where $b_1 := \frac{1}{\sqrt{2}} a_1$ and $b_i := a_i$ if $i \neq 1$. Since $b^* b = \frac{1}{2} a_1^* a_1 + \sum_{i\neq 1} a_i^* a_i \geq \frac{1}{2} 1$, in the polar decomposition $b = u|b|$ the element $|b| \in \mathcal{A}$ is invertible, u is an isometry, and the components u_i of $u = b|b|^{-1}$ are in $\mathcal{A}$. If $\mathbb{I}$ is finite, this implies that $u^* K^{\mathbb{I}} u \subseteq K$ since K is $\mathcal{A}$-convex. To see that this holds also if $\mathbb{I}$ is infinite and K is weak* compact, we approximate a general element $y = \sum_{i\in\mathbb{I}} u_i^* y_i u_i$ of $u^* K^{\mathbb{I}} u$ by elements of the form

$$y_{\mathbb{F}} = \sum_{i\in\mathbb{F}} u_i^* y_i u_i + u_{\mathbb{F}} y_0 u_{\mathbb{F}},$$

where y_0 is a fixed element of K, $\mathbb{F}$ is a finite subset of $\mathbb{I}$ and $u_{\mathbb{F}} := (1 - \sum_{i\in\mathbb{F}} u_i^* u_i)^{1/2}$. Since $\sum_{i\in\mathbb{I}} u_i^* u_i = 1$, the elements $u_{\mathbb{F}}$ converge to 0 in the s.o.t. and the net $(y_{\mathbb{F}})_{\mathbb{F} \overset{\text{finite}}{\subseteq} \mathbb{I}}$ converges to y.

Denoting $z = \oplus_{i\in\mathbb{I}} y_i$, we may now write (5.9) as

$$y = \frac{1}{2} a_1^* y_1 a_1 + b^* z b = \frac{1}{2} a_1^* y_1 a_1 + |b| x |b|, \tag{5.10}$$

where $x := u^* z u = \sum_i u_i^* y_i u_i \in K$. If y satisfies the conditions for a Choquet $\mathcal{A}$-point of K in the case $n = 2$, then we conclude from (5.10) that in particular $|a_1| y = y |a_1|$, $|a_1^*| y_1 = y_1 |a_1^*|$ and $a_1^* y_1 a_1 = |a_1| y |a_1|$. Since the same argument can be applied to any $i \in \mathbb{I}$ (instead of $i = 1$), this proves that y must be a Choquet $\mathcal{A}$-point of K and also proves part (ii) of the Proposition. □

6. Choquet A-points and maps with the unique extension property

Farenick and Morenz proved in [13, 1.2] that representations of C*-algebras are C*-extreme points in the set of all u.c.p. maps from a C*-algebra into $\mathbb{B}(\mathcal{H})$. The following theorem shows that they are in fact Choquet A-points.

Theorem 6.1. *Let $C \subseteq B$ be C*-algebras with the same unit, $\mathcal{H}$ a Hilbert C-module, $\pi_C : C \to \mathbb{B}(\mathcal{H})$ the underlying representation, $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ and $Q := \mathrm{CCP}_C(B, \mathbb{B}(\mathcal{H}))$. Then every (not necessarily unital) representation $\pi : B \to \mathbb{B}(\mathcal{H})$ in Q is a Choquet $\mathcal{A}$-point of Q. Hence unital such representations π are Choquet $\mathcal{A}$-points of the subset $S := \mathrm{UCP}_C(B, \mathbb{B}(\mathcal{H}))$ of Q.*

Proof. Let $\pi : B \to \mathbb{B}(\mathcal{H})$ be any (not necessarily unital) representation in Q. The fact that π is a Choquet $\mathcal{A}$-point in Q follows from Proposition 5.3 by the following argument (already used in other contexts by several authors). Suppose that $\pi = a^*\varphi a$, where $a = [a_1, \ldots, a_n]^T$ is an isometry with $a_j \in \mathcal{A}$ and $\varphi = \oplus_{j=1}^n \varphi_j$ is a diagonal matrix with $\varphi_j \in Q$. Since φ is a c.p. contraction, $\varphi(b)^*\varphi(b) \leq \varphi(b^*b)$ for each $b \in B$, hence (since $aa^* \leq 1$)

$$a^*\varphi(b^*b)a = \pi(b)^*\pi(b) = a^*\varphi(b)^*aa^*\varphi(b)a \leq a^*\varphi(b)^*\varphi(b)a \leq a^*\varphi(b^*b)a. \tag{6.1}$$

Thus equality must hold throughout in (6.1). Let $e := aa^* = [a_ia_j^*]$. From the fact that the first inequality in (6.1) must be equality we deduce that $a^*\varphi(b)^*e^\perp\varphi(b)a = 0$, which implies that $e^\perp\varphi(b)e = 0$ and (replacing b by b^* and taking the adjoints) it follows that $e\varphi(b) = \varphi(b)e$ for all $b \in B$, thus e commutes with φ. □

Example. For all C*algebras $A \subseteq B$ the Choquet A-points of the positive part B_1^+ of the unit ball of B are just the projections. Indeed, since Choquet A-points are extreme in the usual sense and for B_1^+ these are just projections [18, 7.4.6], we only need to show that each projection $p \in B$ is a Choquet A-point. By considering the unitary $u := 2p - 1$, it follows easily that it suffices to show that each unitary $u \in B$ is a Choquet B-point in the unit ball B_1 of B. Now writing u as a B-convex combination $u = a^*ya$, where $a = [a_1, \ldots, a_n]^T$ ($a_j \in A$) and $y = \oplus_{j=1}^n y_j$ ($y_j \in B_1$), and using the identities $u^*u = 1 = uu^*$, it follows by a computation very similar to the one in (6.1) and from Proposition 5.3 that u is a Choquet B-point in B_1.

We note, however, that a proper isometry $v \in B$ is not a Choquet B-point in B_1 since $v = v^*vv$ and the projection vv^* does not commute with v. This, together with [17, 1.1], shows that C*-extreme points are not necessarily Choquet A-points.

Let $\mathcal{H} \in {}_C\mathbb{H}$, let X be a faithful operator C-system contained in a C*-algebra B so that C and B have the same unit 1 and assume that B, as a C*-algebra, is generated by X. Then, as defined in [2, 2.1] a map $\varphi \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ has *the unique extension property* (u.e.p.) if it can be extended to a representation $\pi : B \to \mathbb{B}(\mathcal{H})$ and π is the only c.p. extension of φ to B. By the well-known multiplicative domain argument [31, 3.18] any c.p. extension of φ must be a C-bimodule map (since it extends the representation $\varphi|C$), hence $\pi \in \mathrm{UCP}_C(B, \mathbb{B}(\mathcal{H}))$.

Definition 6.2. A map $\varphi \in \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$ is said to have the *u.e.p.* if $\varphi(1)$ is a projection, say p, and φ has the u.e.p. as a (unital) map into $\mathbb{B}(p\mathcal{H})$.

Remark 6.3. If $\varphi \in \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$ has the u.e.p., $p = \varphi(1)$ and and $\pi : B = C^*(X) \to \mathbb{B}(p\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ is the representation extending φ, then π is the only c.c.p. extension of φ as a map into $\mathbb{B}(\mathcal{H})$. Indeed, if $\psi : B \to \mathbb{B}(\mathcal{H})$ is any such extension, then $0 \leq \psi(b) \leq p$ for all $b \in B$ with $0 \leq b \leq 1$, hence $\psi(B) = p\psi(B)p$. So ψ maps into $\mathbb{B}(p\mathcal{H})$ and therefore must coincide with π by the u.e.p.

From Theorem 6.1 and the Arveson extension theorem (together with the fact that the C-linearity is preserved by c.p. extensions of maps) we immediately deduce:

Corollary 6.4. *If $\varphi \in Q = \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$ has the u.e.p., then φ is a Choquet $\mathcal{A}$-point in Q, where $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$. Similarly for $S = \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ instead of Q.*

Under certain conditions Choquet A-points coincide with maps with the u.e.p.. To show this, we need a simple lemma.

Lemma 6.5. *Let $\mathcal{H} \in {}_C\mathbb{H}$, $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ and let Z be a faithful operator C-system. Then for each $\psi \in \mathrm{CCP}_C(Z, \mathbb{B}(\mathcal{H}))$ there exist a positive contraction $a \in \mathcal{A}$ and $\rho \in S = \mathrm{UCP}_C(Z, \mathbb{B}(\mathcal{H}))$ such that*

$$\psi(x) = a\rho(x)a \quad (x \in Z). \tag{6.2}$$

Proof. Since ψ is a C-bimodule map, $\psi(1) \in \mathcal{A}$, hence also $a := \psi(1)^{1/2} \in \mathcal{A}$. For each $\varepsilon > 0$ the map

$$\rho_\varepsilon : Z \to \mathbb{B}(\mathcal{H}), \quad \rho_\varepsilon(x) := (a + \varepsilon 1)^{-1}\psi(x)(a + \varepsilon 1)^{-1}$$

is completely positive C-linear and $\rho_\varepsilon(1) \leq 1$, thus $\rho_\varepsilon \in \mathrm{CCP}_C(Z, \mathbb{B}(\mathcal{H}))$. If $\rho_0 \in \mathrm{CCP}_C(Z, \mathbb{B}(\mathcal{H}))$ is any weak* limit point of the net $(\rho)_{\varepsilon \to 0}$, then from $\psi = (a + \varepsilon 1)\rho_\varepsilon(a + \varepsilon 1)$ we infer that $\psi = a\rho_0 a$. Since the operators $(a + \varepsilon 1)^{-1}a^2(a + \varepsilon 1)^{-1}$ increase to the range projection p of a as $\varepsilon \searrow 0$, it follows that $\rho_0(1) = p$. Note that $\rho_0(Z) = p\rho_0(Z)p$ since $0 \leq \rho_0(z) \leq p$ for all contractions $z \in Z_+$. Let $\pi : C \to \mathbb{B}(\mathcal{H})$ be the representation for which $\mathcal{H}$ is a Hilbert C-module and let $\theta \in \mathrm{UCP}_C(Z, \mathbb{B}(p^\perp\mathcal{H}))$ be an extension of the map $C \to \mathbb{B}(p^\perp\mathcal{H})$, $c \mapsto \pi(c)p^\perp$. Then the map

$$\rho : Z \to \mathbb{B}(\mathcal{H}), \quad \rho(x) := \rho_0(x) + \theta(x)$$

is in $\mathrm{UCP}_C(Z, \mathbb{B}(\mathcal{H}))$ and satisfies (6.2). □

Theorem 6.6. *If B is a C*-algebra generated by a faithful operator C-system X, $\mathcal{H}$ is a universal Hilbert B-module, $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ and φ is a Choquet $\mathcal{A}$-point of $Q = \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$ or of $S = \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$, then φ has the u.e.p..*

Proof. We will prove the case of Q, the proof for S is similar. By Lemma 6.5 we may write φ as an $\mathcal{A}$-convex combination

$$\varphi = a\rho a + (1 - a^2)^{1/2}0(1 - a^2)^{1/2}, \tag{6.3}$$

where $\rho \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$, $a = \varphi(1)^{1/2}$ and 0 is the zero map. Let p be the range projection of a. Note that $\varphi(1) \leq p$ and consequently $\varphi = p\varphi p$ since φ is a c.p. map. Since φ is a Choquet $\mathcal{A}$-point in Q and the polar decomposition of a is just $a = pa$, it follows from (6.3) and the last equality in (5.3) (with y replaced by φ and y_j by ρ) that $\varphi p = p\rho p$. Thus (using also $\varphi = p\varphi p$) $\varphi(1) = p$. We regard B as contained in $\mathbb{B}(\mathcal{H})$. Let $\psi_0 \in \mathrm{UCP}_C(B, \mathbb{B}(\mathcal{H}))$ be an extension of φ and ψ the weak* continuous extension of ψ_0 to the universal von Neumann envelope $\overline{B}$ of B. As a normal c.p. C-bimodule map ψ is of the form

$$\psi(x) = \sum_{i \in \mathbb{I}} a_i^* x a_i + (1 - p)0(1 - p), \text{ where } a_i \in \mathcal{A}, \text{ and } \sum_i a_i^* a_i = \psi(1) = p. \tag{6.4}$$

Here $\mathbb{I}$ is some index set (see [21, 1.2] for more details) and obviously we may assume that $0 \notin \mathbb{I}$. The zero term $(1 - p)0(1 - p)$ has been added just to obtain a (generalized) $\mathcal{A}$-convex combination. Since φ is a Choquet $\mathcal{A}$-point for Q and $\varphi = \psi|X$, it follows from (6.4) by Lemma 5.5(ii) that φ satisfies the identities (5.3)-(5.5); more precisely, with $\iota : X \to \mathbb{B}(\mathcal{H}))$ the inclusion, we have

$$|a_i|\varphi = \varphi|a_i|, \ \iota q_i = q_i\iota \text{ and } u_i\varphi u_i^* = \iota q_i, \tag{6.5}$$

where $a_i = u_i|a_i|$ is the polar decomposition and p_i and q_i are the range projections of $|a_i|$ and a_i (respectively). (Here, for example, the identity $\iota q_i = q_i\iota$ means that q_i commutes with all $\iota(x) = x \in X$, hence q_i commutes also with $B = C^*(X)$.) The last equality in (6.5) means that u_i implements a unitary equivalence between φp_i and ιq_i, hence in particular φp_i is the restriction to X of the representation $\pi_i := u_i^*(\mathrm{id}\, q_i)u_i$ of B on $p_i\mathcal{H}$, where id is the identity representation of B on $\mathcal{H}$. We would like to show that φ itself is the restriction of a representation of B on $p\mathcal{H}$.

For each subset $\mathbb{J}$ of $\mathbb{I}$ let $e_{\mathbb{J}} = \bigvee_{j\in\mathbb{J}} p_j$. Consider the set $\mathcal{J}$ of all pairs $(\mathbb{J}, \sigma)$, where $\mathbb{J} \subseteq \mathbb{I}$ and $\sigma : B \to \mathbb{B}(e_{\mathbb{J}}\mathcal{H})$ is a representation extending the map $\varphi e_{\mathbb{J}}$. Introduce a partial ordering on $\mathcal{J}$ by $(\mathbb{J}_1, \sigma_1) \le (\mathbb{J}_2, \sigma_2)$ if and only if $\mathbb{J}_1 \le \mathbb{J}_2$ and $\sigma_1 = \sigma_2 e_{\mathbb{J}_1}$. Then it can be verified that each linearly ordered subset of $\mathcal{J}$ has an upper bound, hence by Zorn's lemma $\mathcal{J}$ has a maximal element $(\mathbb{J}, \sigma)$. Let $e := e_{\mathbb{J}} = \bigvee_{j\in\mathbb{J}} p_j$. If $e = p$, then σ is the required representation. Observe that $\bigvee_{i\in\mathbb{I}} p_i = p$. (From the last equality in (6.4) it follows that $p_i \le p$ for all $i \in \mathbb{I}$. If $\bigvee_{i\in\mathbb{I}} p_i \ne p$, then a nonzero vector $p\xi$ would be orthogonal to all $p_i\mathcal{H}$, hence to all $a_i^*\mathcal{H}$. Thus $p\xi \in \cap_{i\in\mathbb{I}} \ker a_i = \ker \sum_{i\in\mathbb{I}} a_i^*a_i = \ker p$, which is possible only if $p\xi = 0$.) Thus, if $e \ne p$, then there exists $i \in \mathbb{I}$ such that $p_i \not\le e$ (hence $i \notin \mathbb{J}$). Let $f = e \vee p_i$ and $e_i := p_i - (e \wedge p_i)$. Note that e_i commutes with $\varphi(X)$ (since all p_j commute with $\varphi(X)$ as follows from the first equality in (6.5)). Also $f - e$ commutes with $\varphi(X)$ since this holds for e and f. Since by the previous paragraph φp_i extends to a representation π_i of B on $p_i\mathcal{H}$ and e_i commutes with $\varphi(X)p_i = \pi_i(X)$ (hence also with $\pi_i(B)$ since X generates B), it follows that φe_i extends to the representation $\pi_i e_i$ of B on $e_i\mathcal{H}$. Since $f - e$ is equivalent in the von Neumann algebra $\varphi(X)'$ to e_i by [35, V.1.6], we deduce that $\varphi \cdot (f - e)$ extends to a representation σ_i of B on $(f - e)\mathcal{H}$. (Indeed, if $v_i \in \varphi(X)'$ is a partial isometry such that $v_i^*v_i = f - e$ and $v_iv_i^* = e_i$, then $\sigma_i(b) := v_i^*\pi_i(b)e_iv_i|(f - e)\mathcal{H}$ defines a representation of B on $(f - e)\mathcal{H}$ such that $\sigma_i|X = \varphi \cdot (f - e)$ since $\sigma_i(x) = v_i^*\pi(x)e_iv_i|(f - e)\mathcal{H} = v_i^*\varphi(x)v_i|(f - e)\mathcal{H} = \varphi(x)v_i^*v_i|(f - e)\mathcal{H} = \varphi(x)|(f - e)\mathcal{H}$ for all $x \in X$.) But then the direct sum $\sigma \oplus \sigma_i$ is a representation of B on $e\mathcal{H} \oplus (f - e)\mathcal{H} = f\mathcal{H}$. Since $(\sigma \oplus \sigma_i)e = \sigma$, it follows that $(\mathbb{J}, \sigma) < (\mathbb{J} \cup \{i\}, \sigma \oplus \sigma_i)$, which contradicts the maximality of $(\mathbb{J}, \sigma)$. Thus $e = p$ and $\varphi = \sigma|X$, where σ is a representation of B on $p\mathcal{H}$. Since X generates B as a C*-algebra, σ is the unique representation of B extending φ. Note that all $|a_i|$ commute with $\sigma(B)$ since they commute with $\sigma(X) = \varphi(X)$ by (6.5).

Now from the last equality in (6.5) we have $u_i(\sigma|X)u_i^* = u_i\varphi u_i^* = \iota q_i = \mathrm{id}|X\, q_i$, so for each i the two representations $u_i\sigma u_i^*$ and $\mathrm{id}\, q_i$ of B on $q_i\mathcal{H}$ agree on the generating subspace X of B, hence $u_i\sigma u_i^* = \mathrm{id}\, q_i$ and consequently $\sigma p_i = u_i^*(\mathrm{id}\, q_i)u_i$. But then from (6.4) and since $|a_i|$ commutes with $\sigma(B)$ we have

$$\begin{aligned}\psi(x) &= \sum_{i\in\mathbb{I}} |a_i|u_i^*\mathrm{id}(x)q_iu_i|a_i| = \sum_{i\in\mathbb{I}} |a_i|\sigma(x)p_i|a_i| \\ &= \sigma(x)\sum_{i\in\mathbb{I}} |a_i|^2 = \sigma(x)p = \sigma(x)\end{aligned}$$

for all $x \in B$. Thus every c.p. extension ψ_0 to B of φ agrees with the representation σ, hence φ has the u.e.p.. □

7. Norming and generating subsets

Definition 7.1. A set $\mathcal{F}$ of maps $\phi : X \to \mathbb{B}(\mathcal{H}_\phi)$ *norms* X if

$$\|x\| = \sup_{\phi \in \mathcal{F}} \|\phi(x)\|$$

for all $x \in X$. If the equality $\|x\| = \sup_{\phi\in\mathcal{F}} \|(\phi)_n(x)\|$ holds for all $x \in \mathbb{M}_n(X)$ and all $n = 1, 2, \ldots$ (where $(\phi)_n = \varphi \otimes 1_{\mathbb{M}_n(\mathbb{C})}$), then $\mathcal{F}$ *completely norms* X.

A subset K_0 of a weak* closed A-convex set K *generates* K if $K = \overline{\mathrm{co}}_A K_0$.

Theorem 7.2. *Let $\mathcal{H}$ be a W*-universal Hilbert module over a von Neumann algebra $\mathcal{C}$ and over $\mathcal{A} := \mathbb{B}_C(\mathcal{H})$, X a faithful strong operator $\mathcal{C}$-system, X_h the selfadjoint part of X, and let S_0 be a subset of $Q := \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ such that either $S_0 \subseteq S := \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ or $0 \in S_0$. Then S_0 in norming for X_h if and only if $\overline{\mathrm{co}}_{\mathcal{A}} S_0 \supseteq S$ (hence $\overline{\mathrm{co}}_{\mathcal{A}} S_0 = Q$ if $0 \in S_0$).*

Proof. First observe that

$$\|x\| = \sup\{\|\varphi(x)\| : \varphi \in S\} \quad \text{for all } x \in X_h. \tag{7.1}$$

(Indeed, let $\mathcal{K}$ be a normal Hilbert $\mathcal{C}$-module such that $X \subseteq \mathbb{B}(\mathcal{K})$. Given $x \in X_h$ and $\varepsilon > 0$, there exists a unit vectors $\xi \in \mathcal{K}$ such that

$$|\langle x\xi, \xi\rangle| \geq \|x\| - \varepsilon. \tag{7.2}$$

By the hypothesis $\mathcal{H}$ contains (a unitarily isomorphic copy of) $[\mathcal{C}\xi]$, so we may assume that $\xi \in \mathcal{H}$. Let $p \in \mathbb{B}_{\mathcal{C}}(\mathcal{H})$ be the projection with the range $[\mathcal{C}\xi]$ and note that the map

$$\rho_0 : X \to \mathbb{B}(p\mathcal{H}), \quad \rho_0(x) = pxp$$

is in $\mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(p\mathcal{H}))$. Now choose any $\theta \in \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(p^\perp\mathcal{H}))$ (as in the proof of Lemma 6.5) and set $\rho = \rho_0 + \theta$. Then $\rho \in \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and from (7.2) we have

$$\|\rho(x)\| \geq \|\rho_0(x)\| = \|pxp\| \geq |\langle x\xi, \xi\rangle \geq \|x\| - \varepsilon.$$

Since maps in S are contractive, this proves (7.1).)

For each $\mathcal{A}$-convex combination $\varphi = \sum_{j=1}^n a_j^* \varphi_j a_j$, $\varphi_j \in S_0$, we have

$$\|\varphi(x)\| \leq \|a^*\| \| \oplus_{j=1}^n \varphi_j(x)\| \|a\| \leq \max_j \|\varphi_j(x)\|, \quad \text{where } a := (a_1, \ldots, a_n)^T.$$

From this and (7.1) it follows that S_0 must be norming for X_h if $\overline{\mathrm{co}}_A S_0 \supseteq S$.

Assume now that S_0 is norming for X_h. First we will show that

$$K := \overline{\mathrm{co}}_{\mathcal{A}}(S_0 \cup (-S_0)) \supseteq S. \tag{7.3}$$

If (7.3) is not true choose $\varphi_0 \in S \setminus K$. Then by Theorem 2.3 (applied in the $\mathcal{A}$-bimodule $X^\natural = \mathrm{CB}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$) there exists $\rho \in \mathrm{NCB}_{\mathcal{A}}(X^\natural, \mathbb{B}(\mathcal{H}))$ such that $\mathrm{Re}\,\rho(\varphi) \leq 1$ for all $\varphi \in K$ and $\mathrm{Re}\,\rho(\varphi_0) \not\leq 1$. Since X is assumed to be strong as a $\mathcal{C}$-bimodule by Theorem 2.6 ρ must be the evaluation at an element $x \in X$. Thus, denoting $h = \mathrm{Re}\, x$, we have

$$\varphi(h) \leq 1 \text{ for all } \varphi \in K \text{ and } \varphi_0(h) \not\leq 1. \tag{7.4}$$

In particular $\varphi(h) \leq 1$ for all $\varphi \in S_0 \cup (-S_0)$, which means that $-1 \leq \varphi(h) \leq 1$ for all $\varphi \in S_0$. But then $\|\varphi(h)\| \leq 1$ and $\|h\| \leq 1$ since S_0 is norming for X_h. Thus $-1 \leq h \leq 1$, which implies that $-1 \leq \varphi_0(h) \leq 1$ since φ_0 is u.c.p.. This contradicts the second relation in (7.4), so the inclusion (7.3) must hold.

Given $\varphi \in S$, by (7.3) there exist a net (φ_k) weak* converging to φ such that each φ_k is of the form

$$\varphi_k = \sigma_k - \tau_k, \text{ where } \sigma_k = \sum_i a_{k,i}^* \psi_{k,i} a_{k,i}, \ \tau_k = \sum_j b_{k,j}^* \theta_{k,j} b_{k,j},$$
$$\psi_{k,i}, \theta_{k,j} \in S_0, \ a_{k,i}, b_{k,j} \in \mathcal{A}, \ \sum_i a_{k,i}^* a_{k,i} + \sum_j b_{k,j}^* b_{k,j} = 1.$$

Passing to suitable subnets we may assume that $\sigma_k \overset{\text{weak*}}{\longrightarrow} \sigma$ and $\tau_k \overset{\text{weak*}}{\longrightarrow} \tau$, where $\sigma, \tau \in \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Then $\varphi = \sigma - \tau$, hence $1 = \varphi(1) = \sigma(1) - \tau(1)$. Since $0 \le \tau(1), \sigma(1) \le 1$, it follows that $\tau(1) = 0$, hence $\tau = 0$ (since τ is c.p.). Thus the net (σ_k) converges to φ in the weak* topology. Replacing the maps σ_k by appropriate convex combinations, we may assume that the convergence is pointwise in the strong operator topology (s.o.t.), hence in particular $\sum_i a_{k,i}^* a_{k,i} = \sigma_k(1) \overset{k}{\longrightarrow} \varphi(1) = 1$ in the s.o.t.. Therefore for any fixed $\theta \in S_0$ the maps

$$\alpha_k := \sigma_k + (1 - \sigma_k(1))^{1/2} \theta (1 - \sigma_k(1))^{1/2}$$

converge to φ in the weak* topology. Since each α_k is an $\mathcal{A}$-convex combination of maps from S_0, this shows that $\varphi \in \overline{\mathrm{co}}_{\mathcal{A}} S_0$. Thus $\overline{\mathrm{co}}_{\mathcal{A}} S_0 \supseteq S$.

If $0 \in S_0$, it follows that $\overline{\mathrm{co}}_{\mathcal{A}} S_0$ contains the $\mathcal{A}$-convex hull of $S \cup \{0\}$, which is Q, as can be easily deduced from Lemma 6.5. □

The following example shows that Theorem 7.2 does not hold for general operator systems over C*-algebras.

Example. Let C be the C*-algebra of all continuous complex valued functions on the interval $[0,1]$ acting on $\mathcal{K} := L^2[0,1]$ in the usual way, so that $\overline{C} = L^\infty[0,1]$. Choose a Borel subset $G \subseteq [0,1]$ such that $0 < m(G \cap I) < m(I)$ for every nonempty open interval $I \subseteq [0,1]$ [33] and let η be the characteristic function of G. Each element of the dual of C is given by a unique Radon measure, hence we may regard $\overline{C} = L^\infty[0,1]$ as contained in the bidual $\mathcal{C}$ of C, in particular $\eta \in \mathcal{C}$. Let $\mathcal{H}$ be a universal Hilbert module over $\mathcal{C}$ on which $\mathcal{C}$ is maximal abelian (that is, $\mathcal{A} = \mathbb{B}_{\mathcal{C}}(\mathcal{H}) = \mathcal{C}$) and

$$X := \{c(1,1) + d\zeta : \ c, d \in C\} \subseteq \mathcal{C}^2,$$

where $\zeta = (\eta, 1-\eta)$. The operator state space $\mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2, \mathbb{B}(\mathcal{H}))$ can be naturally identified with $\{(a, 1-a) : \ a \in \mathcal{A}, 0 \le a \le 1\}$ by associating to each a the map $\rho_a \in \mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2, \mathbb{B}(\mathcal{H}))$ defined by $\rho_a(c_1, c_2) = ac_1 + (1-a)c_2$. By restriction each $\rho_a \in \mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2, \mathbb{B}(\mathcal{H}))$ acts on X as $\rho_a(c(1,1)+d\zeta) = a(c+d\eta)+(1-a)(c+d(1-\eta)) = c + db$, where $b = (1-\eta)(1-a) + \eta a$ is positive in $\mathcal{A}$. Conversely, each $\omega \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ is determined by its value $b := \omega(\zeta)$ so that $\omega(c(1,1)+d\zeta) = c+db$. Since $(1,1) \ge \zeta \ge 0$ and ω is positive unital, $1 \ge b \ge 0$. If we try to compute $a \in \mathcal{A}$ so that $\omega = \rho_a|X$, we find that the only possibility is $a := (1-\eta)(1-b) + \eta b$. Since such an a satisfies $0 \le a \le 1$ and $\rho_a|X = \omega$, we see that each $\omega \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ has a unique extension to a map in $\mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2, \mathbb{B}(\mathcal{H}))$, thus the two operator state spaces are essentially the same.

Let $\omega_0 \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ be defined by $\omega_0(c(1,1) + d\zeta) = c + d\eta$. Then the singleton $S_0 = \{\omega_0\}$ is norming for X_h. To see this, note that X is a C*-subalgebra of $\mathcal{C}^2$, ω_0 is a $*$-homomorphism and that ω_0 is injective (hence isometric) since $c + d\eta = 0$ implies that $c = 0 = d$ because η does not agree (almost everywhere with respect to the Lebesgue measure) with any continuous function on any open nonempty subinterval of $[0,1]$. On the other hand, the extension ρ_1 to $\mathcal{C}^2$ of ω_0 is not norming for $\mathcal{C}^2_h$ since $\rho_1(0\times\mathcal{C})=0$. Since $\mathcal{C}^2$ is weak* closed (hence strong) operator $\mathcal{C}$-system, it follows from Theorem 7.2 that $\mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2,\mathbb{B}(\mathcal{H}))\neq\overline{\mathrm{co}}_{\mathcal{A}}\{\rho_1\}$. Since $\mathrm{UCP}_C(X,\mathbb{B}(\mathcal{H}))\cong\mathrm{UCP}_{\mathcal{C}}(\mathcal{C}^2,\mathbb{B}(\mathcal{H}))$ (whereby ω_0 corresponds to ρ_1), we conclude that $\mathrm{UCP}_C(X,\mathbb{B}(\mathcal{H}))\neq\overline{\mathrm{co}}_{\mathcal{A}}S_0$.

8. $\mathcal{A}$-convex subsets of $\mathcal{A}_h$

A weak* compact $\mathcal{R}$-convex subsets of an injective factor $\mathcal{R}$ is generated by its $\mathcal{R}$-extreme points [28]. For such subsets of $\mathcal{A}_h$ for a general von Neumann algebra $\mathcal{A}$ we will prove (without using [28]) the following sharper result.

Theorem 8.1. *Every weak* compact $\mathcal{A}$-convex subset K of the self-adjoint part $\mathcal{A}_h$ of a von Neumann algebra $\mathcal{A}$ is generated by its Choquet $\mathcal{A}$-points.*

Proof. Let $\mathcal{A}\subseteq\mathbb{B}(\mathcal{H})$, where $\mathcal{H}\in\mathbb{H}_{\mathcal{A}}$ is standard [35, Chapter IX], hence W*-universal over $\mathcal{C}:=\mathbb{B}_{\mathcal{A}}(\mathcal{H})$. Set $\mathcal{Z}=\mathcal{A}\cap\mathcal{C}$, the center of $\mathcal{A}$. Denote by $\chi:K\to\mathbb{B}(\mathcal{H})$ the inclusion and let

$$X=\overline{\overline{\mathcal{C}1+\mathcal{C}\chi}},\qquad X_0=\overline{\overline{\mathcal{Z}1+\mathcal{Z}\chi}},$$

Regard X as a subspace of

$$\mathcal{X}_0:=\mathbb{A}^{\mathrm{w}^*}_{\mathcal{A}}(K,\mathbb{B}(\mathcal{H}))\subseteq D:=\ell^\infty_K(\mathbb{B}(\mathcal{H}))\subseteq\mathbb{B}(\mathcal{H}^K).$$

Since X is contained in $\mathcal{X}_0$ and $\mathrm{UCP}_{\mathcal{C}}(\mathcal{X}_0,\mathbb{B}(\mathcal{H}))=K$ by Theorem 3.5, the fact that X separates points of K implies that $\mathrm{UCP}_{\mathcal{C}}(X,\mathbb{B}(\mathcal{H}))=K$. (Namely, the restriction map $\mathrm{UCP}_{\mathcal{C}}(\mathcal{X}_0,\mathbb{B}(\mathcal{H}))\to\mathrm{UCP}_{\mathcal{C}}(X,\mathbb{B}(\mathcal{H}))$, $\phi\mapsto\phi|X$, is injective and since it is also surjective (by the Wittstock extension theorem), weak* continuous and $\mathcal{A}$-affine, we may identify the two sets.) Let $\mathcal{X}$ be the strong $\mathcal{C}$-subbimodule of $\mathcal{X}_0$ generated by X, thus $\mathcal{X}=\overline{X}^{\mathcal{C}}$. (It can be shown that $\mathcal{X}=\mathcal{X}_0$, but this is not needed.) From $X=\overline{\overline{\mathcal{C}X_0}}$ we also have $\mathcal{X}=\overline{\mathcal{C}X_0}^{\mathcal{C}}$. Let B be the C*-subalgebra of D generated by $\mathcal{X}$ and let B_0 the C*-subalgebra generated by X_0. Then

$$\overline{\overline{\mathcal{C}B_0}}\subseteq B\subseteq\overline{\mathcal{C}B_0}^{\mathcal{C}}, \tag{8.1}$$

where $\overline{\mathcal{C}B_0}^{\mathcal{C}}$ denotes the strong $\mathcal{C}$-bimodule generated by $\mathcal{C}B_0$.

Since B_0 commutes with $\mathcal{C}$ (because $K\subseteq\mathcal{A}$) and $\mathcal{Z}$ is the center of $\mathcal{C}$, it follows from [18, 5.5.4] that the algebra $\mathcal{C}B_0$ is isomorphic to the algebraic tensor product $\mathcal{C}\odot_{\mathcal{Z}}B_0$ (via the map $c\odot_{\mathcal{Z}}b\mapsto cb$). If $\mathcal{A}$ is a factor (that is, $\mathcal{Z}=\mathbb{C}$) then, since Abelian C*-algebras are nuclear [18, 11.3.13], this isomorphism extends to isomorphism of C*-algebras $\mathcal{C}\otimes_{\mathcal{Z}}B_0\cong\overline{\overline{\mathcal{C}B_0}}$. The same holds in general by [3, 2.7, 3.2], where $\otimes_{\mathcal{Z}}$ denotes the minimal tensor product of C*-algebras over $\mathcal{Z}$, which is by definition the completion of the algebraic tensor product $\mathcal{C}\odot_{\mathcal{Z}}B_0$ in the minimal C*-norm. For any two $\mathcal{Z}$-submodules $U\subseteq\mathcal{C}$ and $V\subseteq B_0$ we define $U\otimes_{\mathcal{Z}}V$ to be the norm closure of the algebraic tensor product $U\odot_{\mathcal{Z}}V$ inside $\mathcal{C}\otimes_{\mathcal{Z}}B_0$. Note that the isomorphism $\mathcal{C}\otimes_{\mathcal{Z}}B_0\cong\overline{\overline{\mathcal{C}B_0}}$ maps $\mathcal{C}\otimes_{\mathcal{Z}}X_0$ onto X, so $\mathcal{C}\otimes_{\mathcal{Z}}X_0\cong X$. By Proposition 2.1 we then obtain isomorphisms $\overline{\mathcal{C}\otimes_{\mathcal{Z}}B_0}^{\mathcal{C}}\cong\overline{\mathcal{C}B_0}^{\mathcal{C}}$ and $\overline{\mathcal{C}\otimes_{\mathcal{Z}}X_0}^{\mathcal{C}}\cong\mathcal{X}$.

To see that the set $\mathrm{UCP}_{\mathcal{Z}}(X_0,\mathcal{Z})$ completely norms X_0, let $\mathcal{R}$ be the commutant of $\mathcal{Z}$ in $\mathbb{B}(\mathcal{H}^K)$ and consider the Glimm decomposition of $\mathcal{R}$ along $\mathcal{Z}$ [14]. Thus denote by Δ the maximal ideal space of $\mathcal{Z}$, for each $t \in \Delta$ by $\mathcal{R}t$ the closed ideal in $\mathcal{R}$ generated by t and set $\mathcal{R}(t) = \mathcal{R}/\mathcal{R}t$. For each $x \in \mathcal{R}$ and $t \in \Delta$ let $x(t)$ be the coset of x in $\mathcal{R}(t)$ and $W(x(t))$ its (algebraic) numerical range. By [14, p. 233] the function $t \mapsto \|x(t)\|$ is continuous on Δ, hence defines an element of $\mathcal{Z}$. By [27, 3.4] for each $c \in \mathcal{Z}$ satisfying $W(c(t)) \subseteq W(x(t))$ for all $t \in \Delta$ there exists a $\rho \in \mathrm{UCP}_{\mathcal{Z}}(\mathcal{R}, \mathcal{Z})$ such that $\rho(x) = c$. Since X_0 $(\subseteq \mathcal{R})$ is contained in an abelian C*-algebra (namely in B_0), for every $x \in X_0$ the norm of $x(t)$ is equal to the numerical radius, so we may set $c := \|x(\cdot)\|$. Since $\|x\| = \sup_{t\in\Delta} \|x(t)\| = \|c\|$ by [14, p. 232], by considering $\rho|X_0$ it follows that $\mathrm{UCP}_{\mathcal{Z}}(X_0, \mathcal{Z})$ norms X_0. Then by minimality of operator space structure on abelian C*-algebras $\mathrm{UCP}_{\mathcal{Z}}(X_0, \mathcal{Z})$ completely norms X_0. Using the Krein-Milman theorem we see that the set

$$E_{\mathcal{Z}} := \mathrm{extreme}\,\mathrm{UCP}_{\mathcal{Z}}(X_0, \mathcal{Z})$$

consisting of all extreme points of $\mathrm{UCP}_{\mathcal{Z}}(X_0, \mathcal{Z})$ also norms (hence by commutativity of B_0) completely norms X_0.

Given $\rho \in E_{\mathcal{Z}}$, let S_ρ be the set of all maps in $\mathrm{UCP}_{\mathcal{Z}}(B_0, \mathcal{Z})$ that extend ρ and let $\tilde{\rho}$ be an extreme point of S_ρ; then $\tilde{\rho}$ is extreme also in $\mathrm{UCP}_{\mathcal{Z}}(B_0, \mathcal{Z})$. Since B_0 and $\mathcal{Z}$ are abelian, by [34, 3.1.6] $\tilde{\rho}$ is multiplicative. Since B_0 is generated by X_0, we see that $\tilde{\rho}$ is the only multiplicative extension of ρ to B_0. Moreover, since S_ρ is generated by its extreme points by the Krein-Milman theorem and all such extreme extensions must be multiplicative, it follows that $\tilde{\rho}$ is the unique extension of ρ to a map in $\mathrm{UCP}_{\mathcal{Z}}(B_0, \mathcal{Z})$. To see, moreover, that $\tilde{\rho}$ is the only map $\hat{\rho} \in \mathrm{UCP}_{\mathcal{Z}}(B_0, \mathbb{B}(\mathcal{H}))$ extending ρ, note that, as a $\mathcal{Z}$ bimodule map, each such extension $\hat{\rho}$ satisfies $z\hat{\rho}(b) = \hat{\rho}(zb) = \hat{\rho}(bz) = \hat{\rho}(b)z$ for all $z \in \mathcal{Z}$ and $b \in B_0$, hence $\hat{\rho}(B_0)$ is contained in the commutant $\mathcal{R}$ of $\mathcal{Z}$. Note that each $\omega \in \mathrm{UCP}_{\mathcal{Z}}(\mathcal{R}, \mathcal{Z})$ acts as the identity on $\mathcal{Z}$ (by the $\mathcal{Z}$-bimodule property and unitality), hence $\omega\hat{\rho} \in \mathrm{UCP}_{\mathcal{Z}}(B_0, \mathcal{Z})$ extends ρ and therefore $\omega\hat{\rho} = \tilde{\rho} = \omega\tilde{\rho}$ by the already proved uniqueness of such extensions. Since maps in $\mathrm{UCP}_{\mathcal{Z}}(\mathcal{R}, \mathcal{Z})$ separate points in $\mathcal{R}$ (this can be seen from the spatial description of abelian von Neumann algebras [18, 9.3.2]; or see [15] for more), it follows that $\hat{\rho} = \tilde{\rho}$.

For each $\rho \in E_{\mathcal{Z}}$ let φ_ρ be the restriction to $\mathcal{X}$ of the map $\overline{\mathrm{id} \otimes_{\mathcal{Z}} \tilde{\rho}}^{\mathcal{C}} : \overline{\mathcal{C}B_0}^{\mathcal{C}} \cong \overline{\mathcal{C} \otimes_{\mathcal{Z}} B_0}^{\mathcal{C}} \longrightarrow \overline{\mathcal{C} \otimes_{\mathcal{Z}} \mathcal{Z}}^{\mathcal{C}} = \mathcal{C}$. Then $\varphi_\rho \in \mathrm{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ extends ρ (and is the unique such extension of ρ by the $\mathcal{C}$-bimodule property and Proposition 2.1). If $\psi : B \to \mathbb{B}(\mathcal{H})$ is any c.p. extension of φ_ρ, then ψ is a $\mathcal{C}$-bimodule map (by the multiplicative domain argument, since $\varphi_\rho|\mathcal{C}$ is a $*$-homomorphism). Moreover $\psi|B_0 : B_0 \to \mathbb{B}(\mathcal{H})$ extends ρ, hence is unique and multiplicative by what we have already proved in the previous paragraph. Now it follows that $\psi|\mathcal{C}B_0$ is unique, hence by (8.1) and Proposition 2.1 ψ is unique, and multiplicative. In other words, *φ_ρ has the u.e.p.*.

Since $E_{\mathcal{Z}}$ completely norms X_0, the natural map $X_0 \mapsto \ell^\infty_{E_{\mathcal{Z}}}(\mathcal{Z})$, $\iota(x) = (\rho(x))_{\rho\in E_{\mathcal{Z}}}$, is completely isometric, consequently (since the tensor product $\otimes_{\mathcal{Z}}$ respects inclusions by [22, 3.11]) such is also the map $\mathrm{id} \otimes_{\mathcal{Z}} \iota : \mathcal{C} \otimes_{\mathcal{Z}} X_0 \to \mathcal{C} \otimes_{\mathcal{Z}} \ell^\infty_{E_{\mathcal{Z}}}(\mathcal{Z}) \subseteq \ell^\infty_{E_{\mathcal{Z}}}(\mathcal{C})$. (That the last inclusion is completely isometric is well-known in the case $\mathcal{Z} = \mathbb{C}$, for a general $\mathcal{Z}$ see Remark 8.2 below.) Then by Proposition 2.1 also the map $\overline{\mathrm{id} \otimes_{\mathcal{Z}} \iota}^{\mathcal{C}} : \overline{\mathcal{C} \otimes_{\mathcal{Z}} X_0}^{\mathcal{C}} \to \overline{\mathcal{C} \otimes_{\mathcal{Z}} \ell^\infty_{E_{\mathcal{Z}}}(\mathcal{Z})}^{\mathcal{C}} \subseteq \ell^\infty_{E_{\mathcal{Z}}}(\mathcal{C})$ is completely isometric. Since $\overline{\mathcal{C} \otimes_{\mathcal{Z}} X_0}^{\mathcal{C}} \cong \mathcal{X}$ and under this isomorphism the map $\overline{\mathrm{id} \otimes_{\mathcal{Z}} \rho}^{\mathcal{C}}$ corresponds to φ_ρ, it follows that the set S_0 of all maps in $K = \mathrm{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ of the form φ_ρ $(\rho \in E_{\mathcal{Z}})$

completely norms $\mathcal{X}$. Then by Theorem 7.2 S_0 generates K as a weak* compact $\mathcal{A}$-convex set. Since by the previous paragraph all such maps φ_ρ have the u.e.p., by Corollary 6.4 they are Choquet $\mathcal{A}$-points in K. Thus K is generated by its Choquet $\mathcal{A}$-points. □

Remark 8.2. Towards the end of the above proof we have used the following fact: if C and F are C*-algebras containing a von Neumann algebra $\mathcal{Z}$ in their centers, then the natural homomorphism $\theta : C \otimes_{\mathcal{Z}} \ell^\infty_{\mathbb{I}}(F) \to \ell^\infty_{\mathbb{I}}(C \otimes_{\mathcal{Z}} F)$ of C*-algebras is injective, hence completely isometric, for every index set $\mathbb{I}$. This follows from the minimality of the spatial tensor norm over $\mathcal{Z}$ [3], if we can show that the restriction of θ to the algebraic tensor product $C \odot_{\mathcal{Z}} \ell^\infty_{\mathbb{I}}(F)$ is injective. So suppose that $w := \sum_{j=1}^n c_j \otimes_{\mathcal{Z}} (d_{j,i})_{i\in\mathbb{I}} \in C \odot_{\mathcal{Z}} \ell^\infty_{\mathbb{I}}(F)$ is in the kernel of θ, that is, $\sum_{j=1}^n c_j \otimes_{\mathcal{Z}} d_{j,i} = 0$ for all $i \in \mathbb{I}$. Then for each $\omega \in \mathrm{CB}_{\mathcal{Z}}(C, \mathcal{Z})$ we have $\sum_{j=1}^n \omega(c_j)d_{j,i} = 0$ for all $i \in \mathbb{I}$, hence $(\omega \otimes_{\mathcal{Z}} 1)(w) = \sum_{j=1}^n \omega(c_j)(d_{j,i})_{i\in\mathbb{I}} = 0$ in $\tilde{F} := \ell^\infty_{\mathbb{I}}(F)$, where $\omega \odot_{\mathcal{Z}} 1 : C \odot_{\mathcal{Z}} \tilde{F} \to \tilde{F}$ is defined by $(\omega \odot_{\mathcal{Z}} 1)(c \otimes x) := \omega(c)x$. Since the family of all such maps $\omega \otimes_{\mathcal{Z}} 1$ ($\omega \in \mathrm{CB}_{\mathcal{Z}}(C, \mathcal{Z})$) separates points of $C \odot_{\mathcal{Z}} \tilde{F}$ (see [27, 4.4] for more), we conclude that $w = 0$, hence θ is injective.

9. Appendix: C*-convex sets containing no C*-convex points

In general a weak* compact A-convex set need not contain any A-convex point.

Proposition 9.1. *A unital C*-algebra A is nuclear if and only if every weak* compact A-convex set of operators contains an A-convex point.*

Proof. Assume $A \subseteq \mathbb{B}(\mathcal{H})$. Since A is spanned by its unitary elements, a point $y \in \mathbb{B}(\mathcal{H})$ is A-convex if and only if $y \in A' = \mathbb{B}_A(\mathcal{H})$.

If A is nuclear and $\mathcal{A}$ is its universal von Neumann envelope, then $\pi(\mathcal{A})$ is injective, hence hyperfinite for each normal representation π (if A is separable, this is in [35, Vol. III, p. 213]; for a general case see [11]). Therefore for each $\mathcal{H} \in {}_A\mathbb{H}$ and each $y \in \mathbb{B}(\mathcal{H})$ the set $\overline{\mathrm{co}}(U_{\mathcal{A}}(y))$, where $U_{\mathcal{A}}(y) = \{uyu^* : u \in \mathcal{A}, u^*u = 1 = uu^*\}$, intersects $\mathbb{B}_A(\mathcal{H})$ by [18, 8.3.11]. It follows that each weak* compact A-convex subsets K of $\mathbb{B}(\mathcal{H})$ contains a point from $\mathbb{B}_A(\mathcal{H})$, hence an A-convex point. Conversely, if each such K intersects $\mathbb{B}_A(\mathcal{H})$, then this holds in particular for $K = \overline{\mathrm{co}}_A(y)$ for each $y \in \mathbb{B}(\mathcal{H})$. Taking $\mathcal{H}$ to be universal over A, this implies, by a similar method as in [18, 8.7.24], that $\mathcal{A}$ is injective, hence A nuclear. □

References

[1] W. B. Arveson, *Subalgebras of C*-algebras,* Acta Math. **123** (1969), 141–224.

[2] W. B. Arveson, *The noncommutative Choquet boundary,* J. Amer. Math. Soc. **21** (2008), 1065–1084.

[3] E. Blanchard, *Tensor products of $C(X)$-algebras over $C(X)$*, Recent advances in operator algebras (Orléans, 1992), Astérisque **232** (1995), 81–92.

[4] D. P. Blecher and C. Le Merdy, *Operator Algebras and their Modules,* L.M.S. Monographs, New Series **30**, Clarendon Press, Oxford, 2004.

[5] K. Davidson, *C*-algebras by example,* Fields Institute Monographs **6**, Amer. Math. Soc., Providence, RI, 1996.

[6] K. R. Davidson and M. Kennedy, *The Choquet boundary of an operator system,* Duke Math. J. **164** (2015), 2989–3004.

[7] M. Dritchel and S. McCullough, *Boundary representations for families of representations of operator algebras and spaces,* J. Operator Theory **53** (2005), 159–167.

[8] E. G. Effros and Z-J. Ruan, *Operator Spaces,* London Mathematical Society Monographs. New Series, **23**, Oxford University Press, New York, 2000.

[9] G. Effros and C. Webster, *Operator analogues of locally convex spaces,* Operator algebras and applications (Samos, 1996), 163–207, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 495, Kluwer Acad. Publ., Dordrecht, 1997.
[10] G. Effros and S. Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems,* J. Funct. Anal. **144** (1997), 117–152.
[11] G. A. Elliott, *On approximately finite-dimensional von Neumann algebras, II,* Canad. Math. Bull. **21** (1978), 415–418.
[12] D. R. Farenick, *Pure matrix states on operator systems,* Linear Algebra Appl. **393** (2004), 149–173.
[13] D. R. Farenick and P. B. Morenz, *C^*-extreme points in the generalised state spaces of a C^*-algebra,* Trans. Amer. Math. Soc. 349 (1997), 1725–1748.
[14] J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras,* Annals of Math. **72** (1960), 216–244.
[15] H. Halpern, *Module homomorphisms of a von Neumann algebra into its Center,* Trans. Amer. Math. Soc. **140** (1969), 183–193.
[16] J. W. Helton and S. McCullough, *Every convex free basic semi-algebraic set has an LMI representation*, Annals of Math. **176** (2012), 979–1013.
[17] A. Hopenwasser, R. L. Moore and V. I. Paulsen, *C^*-extreme points,* Trans. Amer. Math. Soc. **266** (1981), 291–307.
[18] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Vols. **1** and **2**, Academic Press, London, 1983 and 1986.
[19] C. Kleski, *Boundary representations and pure completely positive maps,* J. Operator Theory **71** (2014), 45–62.
[20] R. I. Loebl and V. I. Paulsen, Some remarks on C*-convexity, Linear Algebra Appl. **35** (1981), 63–78.
[21] B. Magajna, *The Haagerup norm on the tensor product of operator modules,* J. Funct. Anal. **129** (1995), 325–348.
[22] B. Magajna, *Tensor products over abelian W^*–Algebras,* Trans. Amer. Math. Soc. **348** (1996), 2427–2440.
[23] B. Magajna, *Strong operator modules and the Haagerup tensor product,* Proc. London Math. Soc. **74** (1997), 201–240.
[24] B. Magajna, *A topology for operator modules over W^*-algebras,* J. Funct. Anal. **154** (1998), 17–41.
[25] B. Magajna, *The minimal operator module of a Banach module,* Proc. Edinburgh Math. Soc. **42** (1999), 191–208.
[26] B. Magajna, *C^*-convex sets and completely bounded bimodule homomorphisms,* Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 375–387.
[27] B. Magajna, *C^*-convexity and the numerical range,* Canad. Math. Bull. **43** (2000), 193–207.
[28] B. Magajna, *On C^*-extreme points,* Proc. Amer. Math. Soc. **129** (2001), 771–780.
[29] B. Magajna, *On completely bounded bimodule maps over W^*-algebras,* Studia Math. **154** (2003), 137–164.
[30] P. B. Morenz, *The structure of C^*-convex sets,* Canad. J. Math. **46** (1994), 1007–1026.
[31] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras,* Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, Cambridge, 2002.
[32] G. Pisier, *Introduction to Operator Space Theory,* London Mathematical Society Lecture Note Series **294**, Cambridge University Press, Cambridge, 2003.
[33] W. Rudin, *Well distributed measurable sets,* Amer. Math. Monthly **90** (1983), 41–42.
[34] E. Stormer, *Positive Linear Maps of Operator Algebras,* Springer Monographs in Mathematics, Springer, Berlin, 2013.
[35] M. Takesaki, *Theory of Operator Algebras,* Vols **I**, **II**, and **III**, Encyclopaedia of Math. Sciences **124**, **125** and **127**, Springer, Berlin, 2002 and 2003.
[36] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity,* Trans. Amer. Math. Soc. **351** (1999), 307–322.
[37] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz,* J. Funct. Anal. **40** (1981), 127–150.

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