

# Dual operator systems

David P. Blecher and Bojan Magajna

## ABSTRACT

We characterize weak\* closed unital vector spaces of operators on a Hilbert space  $H$ . More precisely, we first show that an operator system, which is the dual of an operator space, can be represented completely isometrically and weak\* homeomorphically as a weak\* closed operator subsystem of  $B(H)$ . An analogous result is proved for unital operator spaces. Finally, we give some somewhat surprising examples of dual unital operator spaces.

## 1. Introduction

The notion of *complete positivity* plays a profound role in the theory of operator algebras, and also in mathematical physics. The natural setting for this notion is the category of *operator systems*, namely self-adjoint vector spaces of operators on a Hilbert space  $H$  containing the identity operator  $I_H$  (see, for example, [5, Section 1.3; 10; 12, 18]). The main topic of our paper is how operator systems behave with respect to duality. Also, we shall investigate the duality theory of *unital operator spaces* (defined similarly to operator systems, but dropping the requirement of self-adjointness). Unital operator spaces constitute an important area of investigation too, for example, because this class includes most interesting operator algebras, self-adjoint or otherwise, and also includes most interesting function spaces.

Let  $X$  be an operator system which, as an operator space, is the dual of some operator space. By a basic result in operator space theory,  $X$  can be represented completely isometrically and weak\* homeomorphically as a weak\* closed subspace, say  $X'$ , of some  $B(H)$ , but this theorem fails to guarantee that  $X'$  is also an operator system, or even unital. In this note we will show the existence of completely isometric and weak\* continuous representations of  $X$  as a weak\* closed operator subsystem of  $B(H)$ . Using notation that will be defined more fully later, we have the following theorem.

**THEOREM 1.1.** *If  $X$  is an operator system such that  $X$  is a dual as an operator space, then there exists a completely isometric weak\* homeomorphism  $\Phi$  from  $X$  onto a weak\* closed operator subsystem in  $B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ , which maps the identity into the identity and hence is also a complete order isomorphism.*

This is the ‘operator system’ variant of Sakai’s fundamental characterization of von Neumann algebras precisely as the  $C^*$ -algebras with a predual. To prove the theorem, we will first show that the weak\* continuous states on  $X$  are weak\* dense in the states on  $X$ . This will provide enough weak\* continuous states on each  $M_n(X)$  to enable us to construct the

---

Received 9 October 2009; published online December 24, 2010.

2000 *Mathematics Subject Classification* 46L07, 47L05, 47L07, 47L25.

Blecher is partially supported by a grant from the National Science Foundation. Magajna is partially supported by the Ministry of Science and Education of Slovenia.

Some results in this paper were presented at GPOTS 2008 at University of Cincinnati.

desired representation by a more intricate variant of the proof of Ruan's theorem characterizing operator spaces [12].

We also give a simple metric characterization of dissipative contractions in unital operator spaces. This will enable us to prove a variant of Theorem 1.1 for unital operator spaces: a unital operator space, which is also a dual operator space, can be represented completely isometrically and weak\* homeomorphically as a weak\* closed unital subspace of  $B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ . Again, the key tool for this is the weak\* density of the weak\* continuous states in the set of all states.

In the last section, we will consider various examples, some of them surprising. For example, the extended and the normal Haagerup tensor products (see, for example, [5, Section 1.6]) of unital dual operator algebras and spaces are shown to have natural weak\* continuous unital completely isometric representations on a Hilbert space. As a consequence of this, it will follow, for example, that many of the algebras met in modern 'abstract harmonic analysis', such as the Fourier algebra  $B(G)$  for an amenable group  $G$ , are weak\* closed unital subspaces of a von Neumann algebra, or unital subspaces of a  $C^*$ -algebra.

Turning to notation, we write  $X^\sharp$  for the dual of  $X$ , and  $X_\sharp$  for a predual of  $X$ . Weak\* continuous maps will often be referred to as 'normal', as is customary. If  $X$  has a 'unit' 1, we write  $S(X)$  for the set of all states on  $X$ , that is, the set of all functionals  $\rho \in X^\sharp$  with  $\rho(1) = \|\rho\| = 1$ . We write  $X^h$  for the hermitian elements of  $X$ , namely the elements  $x$  with  $\rho(x) \in \mathbb{R}$  for all states  $\rho$  (see [9, 22]). If  $X$  is an operator system, then we write  $X^+$  for the positive cone of  $X$ . We also write  $X^{\sharp+}$  and  $X_\sharp^+$  for the positive cone of  $X^\sharp$  and  $X_\sharp$ : these are the functionals and normal functionals satisfying  $\rho(x) \geq 0 \forall x \in X^+$ , respectively. In this case the state space  $S(X)$  consists of all  $\rho \in X^{\sharp+}$  with  $\rho(1) = 1$ . The hermitian elements also have several interesting alternative characterizations in this case, for example,  $x \in X^h$  if and only if  $\|1 + itx\|^2 = 1 + t^2\|x\|^2$  for all  $t \in \mathbb{R}$ , and if and only if  $\|1 + itx\| = 1 + o(t)$  (see, for example, [9, 22]).

An abstract characterization of unital operator spaces may be found in [7]. The reader may also find metric characterizations of operator systems there, and the fact that the involution on a dual operator system is weak\* continuous. The famous order theoretic characterization of operator systems may be found in [10, 18]. By a *subsystem* of an operator system  $X$  we mean a self-adjoint subspace containing the identity of  $X$ . A *complete order isomorphism* is a linear isomorphism  $T$  between operator systems such that  $T$  and  $T^{-1}$  are completely positive (and thus necessarily preserve the involution). If  $T$  is unital (that is,  $T(1) = 1$ ), then this turns out to be equivalent to  $T$  being a surjective complete isometry. For this and any additional details or definitions concerning operator spaces and systems, the reader is referred to the texts cited in our bibliography.

## 2. Weak\* density of normal states and dissipative elements

In the proofs below we use, without explicitly mentioning it, the theorem asserting that a convex subset of a dual Banach space  $V$  is weak\* closed if and only if its intersection with each closed ball in  $V$  is weak\* closed. This is known as the Krein–Smulian theorem in operator theory, but in [13, 4.44] it is attributed to Banach and Dieudonné.

The fact that  $X_\sharp^+$  is weak\* dense in  $X^{\sharp+}$  turns out to be a consequence of the fact that the cone  $X^+$  of positive elements of  $X$  is weak\* closed.

LEMMA 2.1. *Let  $X$  be an operator system which is a dual Banach space. Then:*

- (i)  $X^+$  is weak\* closed;
- (ii)  $X^+$  is the dual cone to  $X_\sharp^+$ ; that is, denoting  $(X_\sharp^+)^{\circ} = \{x \in X : \langle x, \omega \rangle \geq 0 \forall \omega \in X_\sharp^+\}$ , we have that  $(X_\sharp^+)^{\circ} = X^+$ ;

- (iii)  $X_{\sharp}^+$  is weak\* dense in  $X^{\sharp+}$ ;
- (iv)  $X_{\sharp}^+ \cap S(X)$  is weak\* dense in  $S(X)$ .

*Proof.* (i) By [7, 3.7], the set  $X^h$  of all hermitian elements in  $X$  is weak\* closed. Denoting by  $B_X$  the unit ball of  $X$ , we have that an element  $x \in X^h \cap B_X$  is in  $X^+$  if and only if  $\|1 - x\| \leq 1$ . From this we deduce that  $X^+ \cap B_X$  is weak\* closed, hence  $X^+$  is weak\* closed (since  $X^+$  is convex and  $X^+ \cap rB_X = r(X^+ \cap B_X)$  for all  $r > 0$ ).

(ii) Suppose that  $z \in (X_{\sharp}^+)^{\circ}$ . Then it is easy to see that  $\operatorname{Re}(z)$  and  $\pm \operatorname{Im}(z)$  are in  $(X_{\sharp}^+)^{\circ}$ . Let  $x$  be any one of these three, and suppose  $x \notin X^+$ . Since  $X^+$  is weak\* closed and convex, by a geometric form of the Hahn–Banach theorem there exist an  $\alpha \in \mathbb{R}$  and an  $\omega \in X_{\sharp}$  such that  $\operatorname{Re} \langle y, \omega \rangle \leq \alpha$  for all  $y \in X^+$  and  $\operatorname{Re} \langle x, \omega \rangle > \alpha$ . We may replace  $\omega$  by  $(1/2)(\omega + \omega^*)$ , where  $\omega^*(v) := \overline{\omega(v^*)}$  for all  $v \in X$ . This is a weak\* continuous functional, since the involution is weak\* continuous on  $X$  by [7]. Since  $X^+$  is a cone, we may take  $\alpha = 0$ . Then  $\langle y, \omega \rangle \leq 0$  for all  $y \in X^+$ , while  $\langle x, \omega \rangle > 0$ . But the first relation means that  $-\omega \in X_{\sharp}^+$ , hence  $\langle x, -\omega \rangle \geq 0$  since  $x \in (X_{\sharp}^+)^{\circ}$ , which contradicts the second relation. Hence,  $z = z^* \in X^+$ .

(iii) Suppose that there exists a  $\rho \in X^{\sharp+} \setminus \overline{X_{\sharp}^+}^{w*}$ . By a geometric form of the Hahn–Banach theorem, similarly to the proof of (ii), there exists an  $x \in X^h$  such that  $\langle x, \omega \rangle \leq 0$  for all  $\omega \in X_{\sharp}^+$  and  $\langle x, \rho \rangle > 0$ . The first relation implies that  $-x \in (X_{\sharp}^+)^{\circ}$ , and hence  $-x \in X^+$  by (ii). Consequently  $\langle -x, \rho \rangle \geq 0$  since  $\rho \in X^{\sharp+}$ , which contradicts the second relation.

(iv) Let  $\rho \in S(X)$ . Since  $\rho \in X^{\sharp+}$ , by (iii) there exists a net  $\psi_{\alpha} \in X_{\sharp}^+$  weak\* converging to  $\rho$ . In particular  $\psi_{\alpha}(1) \rightarrow \rho(1) = 1$ , hence the net of states  $\phi_{\alpha} := \psi_{\alpha}(1)^{-1} \psi_{\alpha} \in X_{\sharp}^+ \cap S(X)$  also weak\* converges to  $\rho$ . □

REMARK 1. Parts (iii) and (iv) of the above lemma can be proved via a result from the theory of order unit and base normed spaces (see, for example, [3]), but we preferred the more ‘elementary’ proof given above. Indeed since the self-adjoint part of  $X$  is an order unit space, it follows that the self-adjoint part of  $X_{\sharp}$  is a base normed space, and hence it has all the useful properties of such spaces.

REMARK 2 (Added in revision). The appropriate variant of the above lemma holds also in dual Banach spaces. This follows from results of [1] (see also the survey [20]). At the time of writing the first version of our paper, we were not aware of the reference [1] and we are grateful to Professor Angel Rodrigues Palacios for bringing this fact to our attention.

*Proof of Theorem 1.1.* Let  $n \in \mathbb{N}$ , set  $Y = M_n(X)$ , let  $x^* = x \in Y$  be a fixed element with  $\|x\| = 1$  and choose a state  $\rho$  on  $Y$  such that  $|\rho(x)| = \|x\| = 1$ . Given  $\varepsilon > 0$ , by Lemma 2.1(iv) there exists a state  $\phi \in Y_{\sharp} \cap S(Y)$  such that  $|\phi(x)| > 1 - \varepsilon$ .

Put  $A = M_n(\mathbb{C})$ . Inspired by the proof of Ruan’s theorem [12, pp. 30–34], we claim that there exists a state  $\omega$  on  $A$ , such that, for all  $m \in \mathbb{N}$

$$|\phi(avb)| \leq \omega(aa^*)^{1/2} \omega(b^*b)^{1/2}, \quad v \in M_{nm}(X), \|v\| \leq 1, a, b^* \in M_{1,m}(A). \tag{1}$$

Assuming the claim (which will be proved below), denoting by  $\pi$  the cyclic representation on the Hilbert space  $H$  constructed from  $\omega$  (necessarily equal to the multiple  $k \cdot \operatorname{id}$  of the identity representation up to a unitary equivalence,  $k \leq n$ ,  $H \subseteq (\mathbb{C}^n)^n$ ) and by  $\xi \in H$  the corresponding unit cyclic vector, (1) may be rewritten as

$$|\phi(avb)| \leq \|\pi_{1,m}(a)^* \xi\| \|\pi_{m,1}(b) \xi\|, \quad a, b^* \in M_{1,m}(A), v \in M_{nm}(X), \|v\| \leq 1.$$

Taking  $m = 1$ , we see that, for each fixed contraction  $v \in M_n(X)$ , the map

$$(\pi(b)\xi, \pi(a^*)\xi) \mapsto \phi(avb),$$

is a contractive sesquilinear form on  $[\pi(A)\xi] \times [\pi(A)\xi] = H \times H$ . Hence, there is a contraction  $S(v)$  on  $H$  such that

$$\phi(avb) = \langle S(v)\pi(b)\xi, \pi(a^*)\xi \rangle, \quad a, b \in A. \quad (2)$$

We may extend  $S$  to a linear map  $S : M_n(X) \rightarrow B(H)$  satisfying (2) for all  $v \in M_n(X)$ . Since  $\xi$  is cyclic for  $\pi(A)$ , it follows from (2) that  $S$  is a bimodule map over  $A = M_n(\mathbb{C})$ ; that is,

$$S(cvd) = \pi(c)S(v)\pi(d), \quad v \in M_n(X), \quad c, d \in A.$$

Thus,  $S$  must be the amplification  $T_n$  of a map  $T : X \rightarrow B(\mathbb{C}^k)$ , and  $S$  (hence  $T$ ) is automatically completely contractive by [21] since  $\pi$  is cyclic.

Further,  $S$  (hence  $T$ ) is unital. To show this, first note that  $1 = \phi(1) = \langle S(1)\xi, \xi \rangle$  (by (2)), which implies (since  $\|S(1)\| \leq 1$  and  $\|\xi\| = 1$ ) that  $S(1)\xi = \xi$ . The bimodule property implies that  $\pi(c)S(1) = S(c1) = S(1c) = S(1)\pi(c)$  for all  $c \in A$ . Hence,  $S(1)\pi(c)\xi = \pi(c)S(1)\xi = \pi(c)\xi$ , implying that  $S(1) = 1$  since  $\xi$  is cyclic for  $\pi(A)$ .

Moreover, (2) implies that

$$\phi(avb) = \langle \pi(a)T_n(v)\pi(b)\xi, \xi \rangle, \quad v \in M_n(X), \quad a, b \in A. \quad (3)$$

By a standard argument, it follows that  $T_n$  (hence  $T$ ) is weak\* continuous (since  $\phi$  is weak\* continuous and  $\xi$  is cyclic for  $\pi(A)$ , and using also the theorem mentioned in the first paragraph of this section). Since  $|\langle T_n(x)\xi, \xi \rangle| = |\phi(x)| > 1 - \varepsilon$  implies that  $\|T_n(x)\| > 1 - \varepsilon$ , taking the direct sum of all such maps  $T$  (over all self-adjoint  $x$  in the unit sphere of  $M_n(X)$ , all  $n \in \mathbb{N}$  and  $\varepsilon \in (0, 1/2)$ ), we obtain a weak\* continuous unital complete isometry  $\Phi$  from  $X$  into  $B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ . Then  $\Phi(X)$  is automatically weak\* closed and  $\Phi$  is a weak\* homeomorphism onto its range (this is a well-known consequence of the weak\* compactness of closed balls and the result mentioned in the beginning of the section). By facts mentioned in Section 1,  $\Phi$  is a complete order isomorphism, and preserves the involution.

It remains to prove the claim, that is, to show the existence of a state  $\omega$  on  $A$  satisfying (1). For this, it suffices to show that there exists a state  $\omega$  on  $A$  such that, for every  $m \in \mathbb{N}$ , we have

$$\phi(cvc^*) \leq \omega(cc^*), \quad \text{where } c \in M_{1,m}(A), \quad v \in M_{nm}(X), \quad v = v^*, \quad \|v\| \leq 1. \quad (4)$$

Namely, using (4), with  $c = a + b^*$  and with  $c = a - b^*$ , and the fact that  $\phi^* = \phi$  and  $v^* = v$ , we obtain

$$\begin{aligned} \operatorname{Re} \phi(avb) &= \frac{1}{2} \phi(avb + b^*va^*) \\ &= \frac{1}{4} \phi((a + b^*)v(a + b^*)^* - (a - b^*)v(a - b^*)^*) \\ &\leq \frac{1}{4} [\omega((a + b^*)(a + b^*)^*) + \omega((a - b^*)(a - b^*)^*)] \\ &= \frac{1}{2} \omega(aa^* + b^*b). \end{aligned}$$

Replacing  $a$  by  $ta$  and  $b$  by  $t^{-1}b$  in this estimate, and taking the infimum over all  $t \in (0, \infty)$ , we get

$$\operatorname{Re} \phi(avb) \leq \omega(aa^*)^{1/2} \omega(b^*b)^{1/2}, \quad (5)$$

for all self-adjoint contractions  $v$ . However, for a general  $v$  we may write

$$avb = [a, 0] \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix},$$

hence (5) must hold in general. Replacing  $v$  by a suitable multiple  $zv$ , where  $|z| = 1$ , (1) follows.

Finally, to prove the existence of a state  $\omega$  satisfying (4), consider the cone  $C$  consisting of all functions  $f_{c,v}$  on the state space of  $A$  of the form  $f_{c,v}(\omega) = \omega(cc^*) - \phi(cvc^*)$ , where  $c \in M_{1,m}(A)$  (for all  $m \in \mathbb{N}$ ) and  $v \in M_{nm}(X)$  is self-adjoint with  $\|v\| \leq 1$ . A standard argument shows that  $C$  is indeed a cone and that each function in  $C$  takes a nonnegative value at some point of the state space  $S(A)$  of  $A$ . Thus, by [12, 2.3.1] there exists a point  $\omega \in S(A)$  at which all the functions of  $C$  are nonnegative, which proves (4).  $\square$

Using Lemma 2.1, we can also prove a version of Theorem 1.1 for function systems.

**COROLLARY 2.2.** *Let  $X$  be a function system (by which we mean a subsystem of  $C(K)$  for  $K$  compact) with a Banach space predual. Then there exists a weak\*-homeomorphic unital order isomorphism and isometry of  $X$  onto a weak\* closed subsystem of a commutative  $W^*$ -algebra. The  $W^*$ -algebra may be taken to be  $\ell^\infty(\Omega)$ , where  $\Omega$  is the set of normal states of  $X$ .*

*Proof.* Let  $\Omega$  be as above and let  $\theta : X \rightarrow \ell^\infty(\Omega)$  be the canonical contraction. This is easily seen to be weak\* continuous. Given  $x \in X$ , let  $\varphi$  be a state of  $X$  with  $|\varphi(x)| = \|x\|$  (this is possible by the definition of a function system). Let  $(\varphi_t)$  be a net of normal states on  $X$  converging weak\* to  $\varphi$ . Since  $|\varphi_t(x)| \rightarrow |\varphi(x)|$ , it follows that  $\theta$  is an isometry. The proof is now completed by the general functional analytic principles in the paragraph before equation (4).  $\square$

Comparing Theorem 1.1 to Sakai’s characterization of von Neumann algebras mentioned in Section 1, it is natural to ask if there exists an operator system  $X$  with a Banach space predual, but for which there exists no weak\*-homeomorphic unital complete isometry of  $X$  onto a weak\* closed subsystem of a  $W^*$ -algebra? Indeed, an example of such was exhibited in [6, Proposition 2.1]. There the involution was even weak\* continuous, but  $X$  had no operator space predual.

The following result settles another natural question that arises when considering the famous ‘unicity of predual’ of von Neumann algebras.

**PROPOSITION 2.3.** *An operator system may have more than one predual.*

*Proof.* This follows by a routine technique (see, for example, [5, Corollary 2.7.8]). Let  $X$  be an operator space with two distinct operator space preduals. With respect to each of these two preduals, there are two complete isometries from  $X$  onto weak\* closed subspaces of  $B(H)$  and of  $B(K)$ , respectively. The corresponding ‘Paulsen systems’ in  $M_2(B(H))$  and  $M_2(B(K))$  are weak\* closed operator systems, and are completely isometric to each other via a unital complete order isomorphism by ‘Paulsen’s lemma’ (for example, in the form in [5, Lemma 1.3.15]). However, if this isomorphism was a weak\*-homeomorphism, then so is its restriction to the copies of  $X$ , which gives a contradiction as in, for example, [5, Corollary 2.7.8].  $\square$

It is easy to see that the normal states span the predual. Indeed we have the following proposition.

**PROPOSITION 2.4.** *If  $X$  is a weak\* closed unital operator space, then any normal functional on  $X$  of norm less than 1 may be written as  $\varphi_1 + i\varphi_2$ , where each of  $\varphi_1, \varphi_2$  is of the form  $s\xi - t\psi$  for normal states  $\xi, \psi$  on  $X$  and  $s, t \in [0, 1], s + t < 1$ .*

*Proof.* If  $X$  is weak\* closed in  $B(H)$ , then  $X_{\sharp}$  is a quotient of  $B(H)_{\sharp}$ . Thus, we may assume that  $X = B(H)$ , by considering preimages of functionals. However, the von Neumann algebra case of the result is well known.  $\square$

To prove a version of Theorem 1.1 for unital operator spaces, we need a suitable variant of Lemma 2.1. Since there are usually not enough hermitian elements in such spaces, we will consider dissipative elements instead.

LEMMA 2.5. *An element  $x$  with  $\|x\| \leq 1$  in a unital operator space  $X$  is dissipative (has numerical range contained in the closed left half-plane) if and only if*

$$\|1 + tx\|^2 \leq 1 + t^2 \quad \text{for all } t \in [0, 1]. \tag{6}$$

Consequently, the set  $D_X$  of all dissipative elements in a unital operator space  $X$  with a Banach space predual is weak\* closed.

*Proof.* Suppose that  $\|x\| \leq 1$ . If  $x$  is dissipative, then, for all  $t \in [0, 1]$ ,

$$\|1 + tx\|^2 = \|1 + 2t \operatorname{Re} x + t^2 x^* x\| \leq \|1 + 2t \operatorname{Re} x\| + t^2 \leq 1 + t^2.$$

Conversely, if the numerical range  $W(x)$  of  $x$  contains some  $\alpha$  with  $s := \operatorname{Re} \alpha > 0$ , then  $s \in W(\operatorname{Re} x)$ , and hence

$$\|1 + tx\|^2 \geq \|1 + 2t \operatorname{Re} x\| - t^2 \|x\|^2 \geq 1 + 2ts - t^2 > 1 + t^2, \quad \text{if } t \in (0, s).$$

It follows that the intersection of  $D_X$  with the unit ball of  $X$  is weak\* closed, hence (since  $D_X$  is convex and closed under multiplication by positive scalars) the same holds for any ball around 0. Consequently  $D_X$  is weak\* closed.  $\square$

COROLLARY 2.6. *If  $X$  is a unital operator space with a Banach space predual, then the set of all weak\* continuous states on  $X$  is weak\* dense in  $S(X)$ .*

*Proof.* Denote by  $X^{\sharp+}$  the set of all nonnegative multiples of states on  $X$  and by  $(D_X)^{\circ}$  the set of all functionals  $\rho \in X^{\sharp}$  such that  $\operatorname{Re} \rho(x) \leq 0$  for all  $x \in D_X$ . Clearly  $X^{\sharp+} \subseteq (D_X)^{\circ}$ , but we claim that the two sets are equal. To prove this, let  $\rho \in (D_X)^{\circ}$ . Note that  $it1 \in D_X$  for all  $t \in \mathbb{R}$  implies that  $t \operatorname{Re} (i\rho(1)) \leq 0$ , hence  $\rho(1) \in \mathbb{R}$ . Since  $-1 \in D_X$ , it follows that  $\rho(1) \geq 0$ . Since  $x - t1 \in D_X$  for each  $x \in X$  and  $t \geq \|x\|$ , we have that  $\operatorname{Re} \rho(x) \leq \|x\| \rho(1)$ . Replacing in this inequality  $x$  by  $zx$  for all  $z \in \mathbb{C}$  with  $|z| = 1$ , it follows that  $|\rho(x)| \leq \|x\| \rho(1)$ , hence  $\rho \in X^{\sharp+}$ . Set  $X_{\sharp}^+ = X_{\sharp} \cap X^{\sharp+}$  and  $(D_X)_{\circ} = (D_X)^{\circ} \cap X_{\sharp}$ . Then  $X_{\sharp}^+ = (D_X)_{\circ}$ .

From now on the proof is similar to the proof of Lemma 2.1. Since  $D_X$  is weak\* closed by Lemma 2.5, a bipolar type argument (as in the proof of Lemma 2.1) shows that  $D_X = ((D_X)_{\circ})^{\circ}$  (where the polar  $A^{\circ}$  of a subset  $A$  of  $X_{\sharp}$  is defined as  $A^{\circ} = \{x \in X : \operatorname{Re} \langle a, x \rangle \leq 0 \ \forall a \in A\}$ ) and that  $(D_X)_{\circ}$  is weak\* dense in  $(D_X)^{\circ}$ . By the previous paragraph this means that  $X_{\sharp}^+$  is weak\* dense in  $X^{\sharp+}$  and it follows (as in the proof of Lemma 2.1(iv)) that  $X_{\sharp} \cap S(X)$  is weak\* dense in  $S(X)$ .  $\square$

Similarly to Corollary 2.2, we now have the following corollary.

COROLLARY 2.7. *A unital function space  $X$  with a Banach space predual is weak\*-homeomorphic, order isomorphic and isometric (via the same unital map), with a weak\* closed unital subspace of  $\ell^{\infty}(\Omega)$ , where  $\Omega$  is the set of normal states of  $X$ .*

We will study a dual unital operator space  $X$  in terms of its ‘canonical operator system’  $X + X^*$ ; that is, we may assume that  $X \subseteq B(H)$  for a Hilbert space  $H$  as a unital operator space, and consider the subsystem  $X + X^* = \{x + y^* : x, y \in X\}$  in  $B(H)$ . We use some of Arveson’s results concerning  $X + X^*$  (see [2] or [5, 1.3.6 and 1.3.7]), such as that this system is independent of  $H$ . If  $X$  is also a dual operator space, then one might at first hope that  $X + X^*$  is also a dual space containing  $X$  as a weak\* closed subspace, and we could directly apply Theorem 1.1. However, this is false. Indeed we recall the result from Banach space theory [13] that for weak\* closed subspaces  $E$  and  $F$  of a dual space,  $E + F$  is weak\* closed if and only if it is norm closed (that is, complete). Thus,  $X + X^*$  is weak\* closed with respect to one (or every) unital weak\* continuous completely isometric representation of  $X$  if and only if  $X + X^*$  is norm complete. If, for example,  $H^\infty(\mathbb{D}) + H^\infty(\mathbb{D})^*$  were a dual space, then it is weak\* closed in  $L^\infty(\mathbb{T})$ . Since  $H^\infty(\mathbb{D}) + H^\infty(\mathbb{D})^*$  is weak\* dense in  $L^\infty(\mathbb{T})$ , we would have  $H^\infty(\mathbb{D}) + H^\infty(\mathbb{D})^* = L^\infty(\mathbb{T})$ , which is false (see the discussion on [14, pp. 181–182]).

**THEOREM 2.8.** *If  $X$  is unital operator space, which is also a dual operator space, then there is a completely isometric unital weak\* homeomorphism from  $X$  onto a weak\* closed unital subspace of  $B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ .*

*Proof.* Set  $Y = X + X^*$  as discussed above. Extend each  $\phi \in X_\# \cap S(X)$  to a state  $\tilde{\phi}$  on  $Y$ . Since  $\tilde{\phi}(x + y^*) = \phi(x) + \overline{\phi(y)}$  for all  $x, y \in X$ , it follows from Corollary 2.6 that the set  $\tilde{S}$  of all such extensions is weak\* dense in  $S(Y)$ , and similarly on all matrix levels  $M_n(Y) = M_n(X) + M_n(X^*)$ . Now, using the states in  $\tilde{S}$  for constructing representations of  $Y$  as in the proof of Theorem 1.1, we get a unital complete isometry  $\Phi$  from  $Y$  onto a subspace of some  $B(\mathcal{H})$  such that  $\Phi|_X$  is weak\* continuous, hence a weak\* homeomorphism from  $X$  onto a weak\* closed subspace of  $B(\mathcal{H})$ . □

**REMARK.** Every unital dual operator space  $X$  has a canonical ‘dual operator system cover’  $\tilde{X}$ ; that is,  $X$  is weak\* closed in  $\tilde{X}$  and  $\tilde{X}$  is the weak\* closure of  $X + X^*$ , and for any weak\* continuous complete isometry  $T : X \rightarrow B(H)$  there exists a weak\* continuous completely positive unital map  $\Phi$  from  $\tilde{X}$  onto the weak\* closure of  $X + X^*$  in  $B(H)$ , which extends  $T$ . This may be constructed in a standard way by setting  $\mathcal{F}$  to be the set of all (or enough) weak\* continuous unital complete isometries  $T : X \rightarrow B(H)$ , where  $H$  may be any Hilbert space (of cardinality small enough). Write  $H$  here as  $H_T$ , and let  $i(x) = (T(x))_{T \in \mathcal{F}} \in B(\bigoplus_{T \in \mathcal{F}} H_T)$  for  $x \in X$ . Set  $\tilde{X}$  equal to the weak\* closure of  $i(X) + i(X)^*$ ; this will clearly have the desired property. It would be interesting to investigate the spaces  $X$  for which  $\Phi$  above is always completely isometric; equivalently, for which  $X$  is the weak\* closure of  $T(X) + T(X)^*$ , as an operator system, independent of the representation  $T$  of  $X$ . If this were always true for dual unital function spaces, for example, then it is easy to see that every normal state on a weak\* closed unital subspace  $X$  of  $L^\infty(\Omega)$  extends to a normal state on the weak\* closure of  $X + X^*$  in  $L^\infty(\Omega)$ . This seems to be known to usually be false (see, for example, [15]).

### 3. Examples

First, we indicate how the normal Haagerup tensor product of dual unital operator algebras can be represented faithfully and weak\* homeomorphically into  $B(H)$ .

If  $M$  is a von Neumann algebra, then we recall from [11] that  $M \otimes^{\text{sh}} M \cong CB_{M'}(B(H))$  isometrically and weak\* homeomorphically. In fact it is not difficult to show that this is a complete isometry (see [17, 4.4] for a more general result). As usual,  $CB$  denotes the completely bounded maps.



If  $M$  and  $N$  are von Neumann algebras, then we write  $M * N$  for the ‘universal von Neumann algebra amalgamated free product’, amalgamating over the scalars. This is a von Neumann algebra with the appropriate free product universal property, appropriate to ‘extending to  $M * N$  every pair of normal  $*$ -representations of  $M$  and  $N$  on a common Hilbert space’. This universal property forces the object to be unique. That it exists may be proved by a routine ‘soft’ argument using large direct sums (somewhat in the spirit of [4, 3.1] or the last remark), or by taking the direct sum of all cyclic representations  $\pi$  of the  $C^*$ -algebraic free product of  $M$  and  $N$  (see [19, p. 98]) such that  $\pi|_M$  and  $\pi|_N$  are normal.

**THEOREM 3.1.** (1) *If  $M$  and  $N$  are unital dual operator algebras, then  $M \otimes^{\sigma h} N$  may be identified (completely isometrically and weak\* homeomorphically), with the weak\* closure of  $M \otimes N$  in  $W_{\max}^*(M) \otimes^{\sigma h} W_{\max}^*(N)$ . Here  $W_{\max}^*(\cdot)$  is the ‘maximal von Neumann algebra’ of [8].*

(2) *If  $M$  and  $N$  are von Neumann algebras, then  $M \otimes^{\sigma h} N$  may be identified (completely isometrically and weak\* homeomorphically), with the weak\* closure of  $MN$  in the universal von Neumann algebra amalgamated free product  $M * N$ .*

(3) *If  $M$  and  $N$  are unital dual operator algebras, then  $M \otimes^{\sigma h} N$  is a Banach algebra with product  $(a \otimes b)(a' \otimes b') = aa' \otimes b'b$ .*

(4) *If  $M$  and  $N$  are unital dual operator algebras, then  $M \otimes^{\sigma h} N$  is (completely isometrically and weak\* homeomorphically) a weak\* closed unital subspace of some von Neumann algebra.*

*Proof.* (1) To prove this, let  $u : M \times N \rightarrow B(H)$  be separately weak\* continuous and completely contractive. As Effros and Ruan showed (see, for example, [5, 1.6.10]), we may write  $u(x, y) = R(x)S(y)$  for weak\* continuous complete contractions  $R, S$ . By [5, Theorem 2.7.10] we may rewrite this as  $u(x, y) = a\pi(x)b\rho(y)c$  for unital weak\* continuous completely contractive homomorphisms  $\pi, \rho$ , and contractive operators  $a, b, c$ . As in the proof of, for example, [19, Theorem 5.13], we may assume that  $b = 1$ . Let  $\tilde{\pi}$  and  $\tilde{\rho}$  be the canonical extensions to  $W_{\max}^*(M)$  and  $W_{\max}^*(N)$ , respectively. The map  $a\tilde{\pi}(x)\tilde{\rho}(y)c$  on  $W_{\max}^*(M) \times W_{\max}^*(N)$  induces a weak\* continuous complete contraction on  $W_{\max}^*(M) \otimes^{\sigma h} W_{\max}^*(N)$ . Let  $\tilde{u}$  be the restriction of the latter map to the weak\* closure  $E$  of  $M \otimes N$ . Clearly  $\tilde{u}(x \otimes y) = u(x, y)$  for  $x \in M, y \in N$ . Hence,  $E$  has the universal property of  $M \otimes^{\sigma h} N$ , and thus it follows that the canonical map  $M \otimes^{\sigma h} N \rightarrow E$  is a weak\* continuous completely isometric surjection.

(2) This is proved analogously to the same fact for the Haagerup tensor product (see [19, Theorem 5.13]). One only needs the free product’s universal property, and the methods in the previous paragraph, to show that the appropriate subspace of  $M * N$  has the universal property of  $M \otimes^{\sigma h} N$ .

(3) By (1) we may suppose that  $M, N$  are von Neumann algebras. Also we may suppose that  $M = N$  by the trick of letting  $R = M \oplus N$ . It is easy to argue that  $M \otimes^{\sigma h} N \subset R \otimes^{\sigma h} R$ , since  $M$  and  $N$  are appropriately complemented in  $R$ . However, by [11] we have  $M \otimes^{\sigma h} M \cong CB_{M'}(B(H))$ , a Banach algebra.

(4) This follows by combining (1) and (2).  $\square$

**COROLLARY 3.2.** *For any Hilbert space  $H$ , the space  $CB(B(H))$  is a weak\* closed unital subspace of a von Neumann algebra.*

*Proof.* Apply the above-mentioned complete isometry from [11], and Theorem 3.1(4).  $\square$



It follows from the last corollary that many of the algebras met in modern ‘abstract harmonic analysis’, such as the Fourier algebra  $B(G)$  for an amenable group  $G$ , are weak\* closed unital subspaces of a von Neumann algebra, or unital subspaces of a  $C^*$ -algebra. Indeed there has been quite a lot of work in recent years, in papers by Neufang and coauthors, and others (see, for example, [16] and references therein), showing that many such algebras are completely isometric to unital subalgebras of  $CB(B(H))$ . We are indebted to Neufang for discussions on the state of this topic.

**COROLLARY 3.3.** *For any Hilbert space  $H$ , the ‘completely bounded norm’ of the row  $[\text{Id } T]$  in  $R_2(CB(B(H)))$  is  $\sqrt{1 + \|T\|_{\text{cb}}^2}$  for any  $T \in CB(B(H))$ . Here  $\text{Id}$  is the identity map on  $B(H)$ . Thus, for any  $T \in CB(B(H))$  of norm 1, one can find a matrix  $[x_{ij}]$  in the unit ball of  $M_n(B(H))$  with the norm of  $[[x_{ij}] [T(x_{ij})]]$  arbitrarily close to  $\sqrt{2}$ . Similarly with  $R_2$  replaced by  $C_2$ .*

**REMARK.** Note that  $CB(X)$  is not in general a unital operator space, even if  $X$  is finite-dimensional. For example, take a unital finite-dimensional Banach algebra  $A$  that is not unittally isometric to a unital subspace of a  $C^*$ -algebra (for example, there exist 4-dimensional Banach algebras of numerical index  $1/e$  (see, for example, [9, p. 112]), whereas any unital operator space is easily seen to have index at least  $1/2$ ). The assertion follows since the matrix normed algebra  $\text{Max}(A)$  is a subspace of  $CB(\text{Max}(A))$  via the regular representation.

On the other hand, if  $M, N \subseteq B(H)$  are von Neumann algebras containing a von Neumann subalgebra  $A$  in their intersection, then the space of all  $M', N'$ -bimodule maps  $CB_{M'}(A', B(H))_{N'}$  is a weak\* closed unital subspace of a von Neumann algebra. Indeed, this space is completely isometric and weak\* homeomorphic to  $M \otimes_A^{\sigma^h} N$  by [17, 4.4]. However, using ideas of Ozawa, one can show that  $M \otimes_A^{\sigma^h} N$  may be embedded into the (maximal) von Neumann amalgamated free product  $M *_A N$ . We will perhaps present the details elsewhere.

**THEOREM 3.4.** *If  $X, Y$  are unital operator spaces, then so is  $X \otimes_h Y$ . Indeed  $X \otimes_{w^*h} Y$  is a weak\* closed unital subspace of a von Neumann algebra, if  $X, Y$  are both weak\* closed unital subspaces of von Neumann algebras.*

*Proof.* If  $X, Y$  are subspaces of  $C^*$ -algebras  $A, B$ , respectively, then  $X \otimes_h Y \subset A \otimes_h B$ , and it follows from [19, Theorem 5.13] that  $X \otimes_h Y$  is a unital subspace of the full amalgamated free product  $C^*$ -algebra  $A *_C B$ .

Since  $\otimes_h$  is projective, we have that  $\otimes_{w^*h}$  is ‘weak\*-injective’. Thus, if  $X$  and  $Y$  are unital weak\* closed subspaces of  $B(H)$  and  $B(K)$ , respectively, then  $X \otimes_{w^*h} Y$  is a unital weak\* closed subspace of  $B(H) \otimes_{w^*h} B(K)$ . However,  $B(H) \otimes_{w^*h} B(K)$  is a unital subspace of the unital dual operator space  $B(H) \otimes^{\sigma^h} B(K)$ . The proof is completed by an appeal to Theorem 2.8, since  $X \otimes_{w^*h} Y$  is a dual operator space. (We remark that one may in this case also construct an explicit weak\* continuous unital complete isometry from  $B(H) \otimes_{w^*h} B(K)$  into a von Neumann algebra.) □

Once one knows that the spaces above are unital operator spaces, it is of interest to compute their ‘noncommutative Shilov boundary’, or  $C^*$ -envelope [2, 5, 18]. It is easy to show, for example, that the  $C^*$ -envelope of (even a module) Haagerup tensor product of unital  $C^*$ -algebras is their full amalgamated free product  $C^*$ -algebra (and we thank Junge for showing us a related fact). Thus, the  $C^*$ -envelope of  $CB(M_n)$  is  $M_n *_C M_n$ .

## References

1. C. APARACIO, F. G. OCAÑA, R. PAYÁ and A. RODRIGUES, 'A non-smooth extension of Frechet differentiability of the norm with applications to numerical ranges', *Glasg. Math. J.* 28 (1986) 121–137.
2. W. B. ARVESON, 'Subalgebras of  $C^*$ -algebras', *Acta Math.* 123 (1969) 141–224.
3. L. ASIMOW and A. J. ELLIS, *Convexity theory and its applications in functional analysis*, London Mathematical Society Monographs 16 (Academic Press, London, 1980).
4. B. BLACKADAR, 'Weak expectations and nuclear  $C^*$ -algebras', *Indiana Univ. Math. J.* 27 (1978) 1021–1026.
5. D. P. BLECHER and C. LE MERDY, *Operator algebras and their modules: an operator space approach* (Oxford University Press, Oxford, 2004).
6. D. P. BLECHER and B. MAGAJNA, 'Duality and operator algebras: automatic weak\* continuity and applications', *J. Funct. Anal.* 224 (2005) 386–407.
7. D. P. BLECHER and M. NEAL, 'Metric characterizations of isometries and of unital operator spaces and systems', *Proc. Amer. Math. Soc.* 139 (2011) 985–998.
8. D. P. BLECHER and B. SOLEL, 'A double commutant theorem for operator algebras', *J. Operator Theory* 51 (2004) 435–453.
9. F. F. BONSALL and J. DUNCAN, *Numerical ranges. II*. London Mathematical Society Lecture Notes Series 10 (Cambridge University Press, New York, 1973).
10. M.-D. CHOI and E. G. EFFROS, 'Injectivity and operator spaces', *J. Funct. Anal.* 24 (1977) 156–209.
11. E. G. EFFROS and A. KISHIMOTO, 'Module maps and Hochschild–Johnson cohomology', *Indiana Univ. Math. J.* 36 (1987) 257–276.
12. E. G. EFFROS and Z.-J. RUAN, *Operator spaces* (Oxford University Press, New York, 2000).
13. M. FABIAN *et al.*, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics (Springer, New York, 2001).
14. K. HOFFMAN, *Banach spaces of analytic functions* (Dover Publications, Inc., New York, 1988).
15. K. HOFFMAN and H. ROSSI, 'Extensions of positive weak\*-continuous functionals', *Duke Math. J.* 34 (1967) 453–466.
16. M. JUNGE, M. NEUFANG and Z.-J. RUAN, 'A representation for locally compact quantum groups', *Int. J. Math.* 20 (2009) 377–400.
17. B. MAGAJNA, 'Duality and normal parts of operator modules', *J. Funct. Anal.* 219 (2005) 306–339.
18. V. I. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics 78 (Cambridge University Press, Cambridge, 2002).
19. G. PISIER, *Introduction to operator space theory*, London Mathematical Society Lecture Note Series 294 (Cambridge University Press, Cambridge, 2003).
20. A. RODRIGUES PALACIOS, 'Banach space characterizations of unitaries: a survey,' *J. Math. Anal. Appl.* 369 (2010) 168–178.
21. R. R. SMITH, 'Completely bounded module maps and the Haagerup tensor product', *J. Funct. Anal.* 102 (1991) 156–175.
22. I. VIDAV, 'Eine metrische Kennzeichnung der selbstadjungierten Operatoren', *Math. Z.* 66 (1956) 121–128 (German).

David P. Blecher  
 Department of Mathematics  
 University of Houston  
 Houston, TX 77204-3008  
 USA  
 dblecher@math.uh.edu

Bojan Magajna  
 Department of Mathematics  
 University of Ljubljana  
 Jadranska 21  
 Ljubljana 1000  
 Slovenia  
 Bojan.Magajna@fmf.uni-lj.si