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Duality and operator algebras: automatic weak* continuity and applications

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Abstract

We investigate some subtle and interesting phenomena in the duality theory of operator spaces and operator algebras, and give several applications of the surprising fact that certain maps are always weak*-continuous on dual spaces. In particular, if X is a subspace of a C^*-algebra A, and if $a \in A$ satisfies $aX \subset X$, then we show that the function $x \mapsto ax$ on X is automatically weak* continuous if either (a) X is a dual operator space, or (b) $a^*X \subset X$ and X is a dual Banach space. These results hinge on a generalization to Banach modules of Tomiyama's famous theorem on contractive projections onto a C^*-subalgebra. Applications include a new characterization of the σ-weakly closed (possibly nonunital and nonselfadjoint) operator algebras, and a generalization of the theory of W^*-modules to the framework of modules over such algebras. We also give a Banach module characterization of σ-weakly closed spaces of operators which are invariant under the action of a von Neumann algebra.

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1. Introduction

Functional analytic questions about spaces of operators often boil down to considerations involving dual, or weak*, topologies. In many such calculations, the key point is to prove that certain linear functions are weak* continuous. In the present paper, we offer a couple of results which ensure that a linear map is automatically continuous with respect to such topologies. In the right situation, these results can be extremely useful. We will illustrate this with several applications.

By definition, an operator space is a subspace of a C^*-algebra. A *left multiplier* of an operator space X is a linear map $T : X \to X$ such that there exists a *completely isometric* (this term is defined below) embedding $\sigma : X \to A$, for a C^*-algebra A, and an element $a \in A$, such that $\sigma(Tx) = a\sigma(x)$ for all $x \in X$. These operator space multipliers have been useful in various ways in the last several years (see e.g. [5,10] or [7, Chapter 4] and references therein). We prove that left multipliers of dual operator spaces are automatically w^*-continuous, which is surprising in the light of any of the known alternative definitions of left multipliers (see [7, Theorem 4.5.2] or [5]). We give several sample applications of this, mostly to *operator algebras* (that is, possibly nonselfadjoint subalgebras of C^*-algebras), and to modules of operators. We are also able to relax the restriction above that σ be a complete isometry, and allow X to be a dual Banach space and σ an isometry, provided that $a^*\sigma(X) \subset \sigma(X)$ too.

Generally, our paper is concerned with some subtle and interesting phenomena in the duality theory of operator spaces and operator algebras. For example, in Section 2, we give perhaps the simplest example of an operator space which is a dual Banach space but not a dual operator space. In Section 3, we establish a ‘Banach module variant’ of Tomiyama’s famous theorem on contractive projections onto a C^*-subalgebra. We use this to prove our first ‘automatic continuity’ result, and as an application we give a new proof of the Zettl/Effros–Ozawa–Ruan characterization of W^*-modules/corners in a von Neumann algebra. In Section 4, we prove our main result, that left multipliers of dual operator spaces are automatically w^*-continuous. We also give numerous corollaries and complementary results. For example, we give a new characterization of σ-weakly closed algebras of operators (a characterization which we were able to show, a few months after we had submitted the present paper for publication, to be sharp—see [8]), and also a Banach module characterization of σ-weakly closed spaces of operators which are invariant under the action of a von Neumann algebra. Finally, in Section 5, we illustrate again the power of our main result, by using it to generalize key aspects of the theory of the ‘selfdual modules’ of Paschke [27] and Rieffel [29], also known as W^*-modules, to nonselfadjoint algebras.

We now turn to the definitions, and background facts. The basic source text used for background information is [7]; some of this information may be found in [28] too. Throughout H and K are Hilbert spaces. Operator spaces, defined above, may also be thought of as the linear subspaces of $B(K, H)$. Equivalently, by a theorem of Ruan, an operator space is a vector space X with a norm defined on each of the spaces $M_n(X)$ of $n \times n$ matrices over X satisfying two compatibility conditions which we shall not spell out here. A linear map $T : X \to Y$ between operator spaces clearly induces a map $T_n : M_n(X) \to M_n(Y)$. We say that T is *completely isometric* (resp., *completely* *contractive*, *a complete quotient map*) if T_n is isometric (resp., contractive, takes the open ball of $M_n(X)$ onto the open ball of $M_n(Y)$), for all $n \in \mathbb{N}$. We say that T is *completely bounded* if

$$\|T\|_{\mathrm{cb}} \overset{\mathrm{def}}{=} \sup_{n\in\mathbb{N}} \|T_n\| \ < \ \infty.$$

We write $CB(X,Y)$ for the space of completely bounded maps, with this norm, and $CB(X) = CB(X,X)$. If X and Y are left (resp., right) A-modules then ${}_ACB(X,Y)$ (resp., $CB_A(X,Y)$) denotes the subspace of $CB(X,Y)$ consisting of module maps. The reader can guess the meaning of $B_A(X,Y)$, ${}_ACB(X)$, etc. We say that an operator space X is *unital* if it has a distinguished element 1, such that there exists a complete isometry $T: X \to A$ into a unital C^*-algebra with $T(1) = 1_A$. Examples of these include *operator systems*, namely linear selfadjoint subspaces of a C^*-algebra A with $1_A \in X$.

Although we shall not need this explicitly in our paper, operator algebras (defined in the second paragraph of our paper) may also be defined purely abstractly in terms of matrix norms (e.g. see [7, Theorem 2.3.2] or [28, p. 252]). We say that an operator algebra is *approximately unital* if it has a contractive approximate identity. If X is an operator space, then the set $\mathcal{M}_l(X)$ of left multipliers (also defined in the second paragraph) of X, turns out to be such an abstract operator algebra. There are several equivalent definitions of $\mathcal{M}_l(X)$ in the literature (e.g. see [5] or [7, Sections 4.5 and 8.4]). For example, for a unital operator space X, one may define $\mathcal{M}_l(X)$ to be the image in $CB(X)$ of the subalgebra $\{a \in D : aX \subset X\}$, via the canonical map taking a to the map $x \mapsto ax$ on X. Here D is a certain 'extremal' C^*-algebra containing X, with $1_X = 1_D$. In fact, D may be taken to be either Hamana's *injective envelope*, or Arveson's *noncommutative Shilov boundary* (also known as the *C^*-envelope*), of X. See e.g. [1,15], [7, Sections 4.2–4.4, and 8.3] or [28, Chapter 15] for a thorough discussion of the latter objects.

A dual Banach space is a Banach space linearly isometric to the dual of another Banach space (the latter is called a predual). We abbreviate the word 'weak*' to 'w^*'. The w^*-topology on $B(H)$ is often called the σ-weak topology. The product of $B(H)$ (and hence of any w^*-closed subalgebra of $B(H)$) is a separately w^*-continuous bilinear map. A *W^*-algebra* is a C^*-algebra which is a dual Banach space; by a well-known theorem of Sakai, these are 'exactly' the von Neumann algebras. Indeed, the methods of our paper owe enormously to Tomiyama's quick proof of Sakai's theorem, and adaptions of this method by others, e.g. [32], [30, Theorem 9.1], [11,23]. The second dual of a C^*-algebra is well known to be a W^*-algebra, a fact for which there do exist simple proofs in the literature. A consequence of the well-known Krein–Smulian theorem, is that a linear bounded map $u: E \to F$ between dual Banach spaces is w^*-continuous if and only if whenever $x_t \to x$ is a bounded net converging in the w^*-topology in the domain space, then $u(x_t) \to u(x)$ in the w^*-topology. If this holds, and if moreover u is a w^*-continuous isometry, then u has w^*-closed range, and u is a w^*–w^*-homeomorphism onto $Ran(u)$. See e.g. [12] or [4] for proofs. These facts will be used silently very often.

We write $M(A)$ for the multiplier algebra of A. If X and Y are sets (in a C^*-algebra say) then we write XY for the *norm closure* of the span of terms of the form xy, for $x \in X$, $y \in Y$. Similar conventions hold for products of three subsets.

2. Dual operator spaces

An important and basic fact from operator space theory, is that the Banach space dual Y^* of an operator space Y is again an operator space, in a canonical way. Namely, for $n \geqslant 2$ we assign $M_n(Y^*)$ the norm pulled back via the canonical algebraic isomorphism $M_n(Y^*) \cong CB(Y, M_n)$ (e.g. see [7, Section 1.4]). This is usually called the *operator space dual*. We recall that X is a *dual operator space* if X is completely isometric to the operator space dual Y^*, for an operator space Y. Le Merdy gave a beautiful characterization of dual operator spaces (see [22, 1.6.4], [7]); and he also showed that an operator space which is a dual Banach space need not be a dual operator space. Simpler examples were later found by Peters–Wittstock, Effros–Ozawa–Ruan [11], [26, Remark 7.9]. This phenomenon will play an important role in this paper and its sequel [8]; for example we will often ask when results valid for dual operator spaces are also valid for an operator space which is a dual Banach space. Indeed, the following example, which will play a role later in the paper, may be the simplest example of this phenomenon:

Proposition 2.1. *There is an operator space structure on* $B(\ell_2)$*, for which there exists a predual Banach space, but not a predual operator space.*

Proof. Let $H = \ell_2$, let S^∞ denote the compact operators on ℓ_2, let Q be the Calkin algebra $B(H)/S^\infty$, and let Q^{op} denote its 'opposite C^*-algebra'. Let X be the subspace of $B(H) \oplus^\infty Q^{\mathrm{op}}$ consisting of pairs $(x, \dot{x})$ for $x \in B(H)$, where $\dot{x}$ is the class of x in the quotient. Then X is a unital operator space (in fact it is even an operator system) which is linearly isometric to $B(H)$. Thus X has a predual Banach space, the trace class S^1, which is even a unique predual. However, X is not a dual operator space. The reason for this is that the canonical embedding $\iota : S^\infty \hookrightarrow X$ is a complete isometry. Thus, if there were an operator space structure on S^1 such that the canonical map $(S^1)^* \to X$ was a complete isometry, then the unique w^*-continuous contraction $B(H) \to X$ extending ι, would be a complete contraction (see [7, 1.4.8]). This unique extension must be the canonical 'identity' map from $B(H)$ to X. The fact that it is completely contractive forces the canonical quotient map $B(H) \to Q^{\mathrm{op}}$ to be a complete contraction, which in turn implies that the 'identity map' $Q \to Q^{\mathrm{op}}$ is a complete contraction. However, it is well known that the 'identity map' from a C^*-algebra A to its opposite algebra A^{op} is a complete contraction if and only if the C^*-algebra is commutative (indeed this is clear if one applies a 'noncommutative Banach–Stone theorem' such as [7, Corollary 1.3.10] to the canonical map from A to A^{op}). □

In our paper, we shall be quite concerned with the multiplier algebras of a dual operator space X. As we mentioned in Section 1 for unital operator spaces (and a similar thing is true for general operator spaces), $\mathcal{M}_l(X)$ may be defined in terms of either the injective envelope or the C^*-envelope. If either of the latter two objects were a W^*-algebra, then many of the technical difficulties which we will need to overcome in this paper, would disappear. Unfortunately, this is not generally the case. To show that the methods of our paper are not gratuitous, it seems worthwhile to take the time to exhibit a simple explicit example of this phenomena.

Proposition 2.2. *There exists a finite-dimensional unital operator algebra M, such that neither its injective envelope, nor its C^*-envelope, are W^*-algebras.*

Proof. Let X be the span of $\{1, x, x^2\}$ in $C([0,1])$. This is a unital operator space, and it generates $C([0,1])$ by the Stone–Weierstrass theorem. It is easy to see that $[0,1]$ is the Shilov boundary for X in $[0,1]$ (because for any nontrivial closed subset $C \subset [0,1]$, there is a function in X that peaks outside of C). Thus the C^*-envelope of X is $C([0,1])$ (by e.g. [7, 4.3.4]). Let D be an injective envelope of X as mentioned in Section 1, thus D is a unital C^*-algebra with $1_D = 1$. In fact D is also the injective envelope of $C([0,1])$ (this follows from the basic theory of the injective envelope from e.g. [7, Section 4.2 and 4.3] or [28, Chapter 15], since the C^*-envelope of X may be defined to be the C^*-subalgebra of D generated by X, and any minimal $C([0,1])$-projection on D is an X-projection and is consequently the identity). Next, let M be the subalgebra of $M_2(C([0,1]))$ with 0 in the 2-1 entry, scalars on the main diagonal, and an element from X in the 1-2 entry. This is a five dimensional unital operator algebra. Note that $M + M^*$ is the Paulsen system $\mathcal{S}(X)$ (see [28, Lemma 8.1] or [7, p. 21]). By 4.2.7, 4.3.6, and 4.4.13 in [7], $I(M) = I(M + M^*) = I(\mathcal{S}(X)) = M_2(D)$. Now $M \subset M_2(C([0,1]))$, and the latter is a $*$-subalgebra of $M_2(D)$. Thus $C^*_e(M)$, the C^*-algebra generated by M in $M_2(D)$, is also the C^*-algebra generated by M in $M_2(C([0,1]))$. Hence $C^*_e(M) = M_2(C([0,1]))$. The second assertion of the theorem is now clear. The first follows too, if D is not a W^*-algebra. Claim 1: The injective envelope of $C([0,1])$ is the well-known Dixmier algebra. This is the algebra $C(Y)$ in [19, Exercise 5.7.21]), which is shown there not to be a W^*-algebra. Since we are not aware of a proof of Claim 1 in the literature, we provide one (the paper [14] proves the same fact, but in a different category). We first note that the Dixmier algebra $C(Y)$ is injective in the category of Banach spaces (see [19, Exercise 5.7.20 (viii)] and [31, Exercise III.1.5]). Hence it is injective as an operator space (by e.g. [7, 4.2.11]). We identify $C([0,1])$ with its image under the (completely isometric) canonical injection $C([0,1]) \to C(Y)$. Claim 2: Every selfadjoint element k in $C(Y)$ is the least upper bound of the functions $h \in C([0,1])$ with $h \leqslant k$. If Claim 2 holds, then it is easy to see that $C(Y)$ has the 'rigidity property', and hence is the injective envelope (see [7, Section 4.2] or [28, Chapter 15]). Indeed, suppose that $T : C(Y) \to C(Y)$ is a (complete) contraction extending the identity map on $C([0,1])$. Since $T(1) = 1$, T is positive. If k, h are as above, then $h = T(h) \leqslant T(k)$. By the claim, $k \leqslant T(k)$. Similarly, $-k \leqslant T(-k)$, and so $T(k) = k$. This proves Claim 1.

Claim 2 is no doubt well known, but again we are not aware of a reference for it. To prove it, we may assume that $k \geqslant 0$ (by adding a scalar multiple of the identity, if necessary). Hence k is the equivalence class of a nonnegative bounded Borel function g, modulo functions which are zero except on a meager Borel set. By basic measure theory we may write $g = \sum_{i=1}^{\infty} c_i \chi_{A_i}$, where $c_i > 0$ and A_i are Borel sets in $[0, 1]$. This sum converges pointwise to g. By [19, Exercise 5.7.20 (iii)], there is an open set U_i in $[0, 1]$, and a meager set N_i such that $\chi_{A_i} = \chi_{U_i}$ outside of N_i. Let $E = \cup_i N_i$, another meager set, then $\sum_{i=1}^{\infty} c_i \chi_{U_i}$ converges pointwise outside E to g. For each i, let f_n^i be an increasing sequence of continuous nonnegative functions converging to χ_{U_i}. Let $s_n^N = \sum_{i=1}^{N} c_i f_n^i$, let f be the least upper bound in $C(Y)$ of the (doubly indexed) sequence s_n^N, and let v be a bounded Borel function on $[0, 1]$ whose equivalence class is f. Outside of E, $s_n^N \leqslant g$, so that $s_n^N \leqslant k$ in $C(Y)$, hence $f \leqslant k$ and, consequently $v \leqslant g$ outside a meager set. It suffices now to show that in fact $v = g$ outside a meager set (for then $f = k$). If this was not true, then $g - v$ would be positive on a nonmeager set and, since s_n^N converge pointwise to g outside E, it would follow that $s_n^N - v$ is positive on a nonmeager set for some n, N. But this would contradict the fact that $f \geqslant s_n^N$ in $C(Y)$. $\square$

Remark. Other such examples, at least in the operator space case, are probably known privately to experts. One may deduce such examples from the intricate [16, Corollary 3.8] (there X is the span of the generators in the reduced C^*-algebra of the free group on n generators).

3. Modules, and a result of Zettl

A *concrete left operator module* over an operator algebra A, is a subspace $X \subset B(H)$ such that $\pi(A)X \subset X$ for a completely contractive representation $\pi : A \to B(H)$. An *(abstract)* operator A-module is an operator space X which is also an A-module, such that X is completely isometrically isomorphic, via an A-module map, to a concrete operator A-module. Note that in this case it is then clear from the definitions that the map $x \mapsto ax$ on X is in $\mathcal{M}_l(X)$, for any $a \in A$. Many of the most important modules over operator algebras are operator modules, such as Hilbert C^*-modules. There is an elegant characterization of operator A-modules, due to Christensen–Effros–Sinclair (cf. [7, Theorem 3.3.1]), but we shall not need this. Instead, we will use a Banach module variant of the latter characterization. To state this, we will need the following definition:

Definition 3.1. If A is a C^*-algebra, and if X is a nondegenerate A-module possessing a norm, then we say that X is *representable* if

$$\|a_1 x_1 + a_2 x_2\| \leqslant \sqrt{\|a_1 a_1^* + a_2 a_2^*\|}\sqrt{\|x_1\|^2 + \|x_2\|^2}\,, \quad a_1, a_2 \in A, x_1, x_2 \in X\,.$$

The requirement in the last definition is relatively mild, and in fact is satisfied by most of the important modules over C^*-algebras. Indeed, the representable A-modules are precisely the Banach A-modules which are isometrically A-isomorphic to an operator A-module (e.g. see [25]).

The main trick in the following comes from a well-known proof on Tomiyama's theorem of conditional expectations [30, Theorem 9.1]. This trick is also used in [11, Theorem 2.5]:

Lemma 3.2. *Suppose that X is a representable module over a C^*-algebra A. Suppose, further, that Y is a submodule of X for which there exists a contractive linear projection Φ from X onto Y. Then Φ is an A-module map.*

Proof. It is possible to prove this directly from the inequality in Definition 3.1, but this route is a little longer. Instead, we will use the fact stated after that definition, which allows us to assume that X is a nondegenerate left operator A-module. Without loss of generality, we may assume that A is a W^*-algebra. Indeed, by 3.8.9 in [7], X^{**} is a nondegenerate left operator A^{**}-module, and by routine arguments Y^{**} may be viewed canonically as an A^{**}-submodule of X^{**}. Then Φ^{**} is a contractive linear projection from X^{**} onto Y^{**}. If the lemma is true in the W^*-algebra case then Φ^{**} is an A^{**}-module map, so that Φ is an A-module map.

We next claim that it suffices to show that

$$p^{\perp}\Phi(px) = 0 \tag{1}$$

for all $x \in X$, and orthogonal projections $p \in A$. For if (1) holds then we have

$$p\Phi(x) = p\Phi(px) + p\Phi(p^{\perp}x) = p\Phi(px) = (p + p^{\perp})\Phi(px) = \Phi(px),$$

using (1) twice (once with p replaced by $p^{\perp}$). Since A is densely spanned by its projections, we conclude that $a\Phi(x) = \Phi(ax)$ for all $a \in A$.

To prove (1), let $y = px + tp^{\perp}\Phi(px)$, where $t \in \mathbb{R}$. By the inequality in 3.1,

$$\|y\|^2 \leqslant \|p + p^{\perp}\|(\|px\|^2 + t^2\|p^{\perp}\Phi(px)\|^2) = \|px\|^2 + t^2\|p^{\perp}\Phi(px)\|^2.$$

On the other hand, since $Ran(\Phi)$ is an A-submodule, $p^{\perp}\Phi(px) \in Ran(\Phi)$, so that

$$p^{\perp}\Phi(y) = p^{\perp}\Phi(px) + tp^{\perp}p^{\perp}\Phi(px) = (1+t)p^{\perp}\Phi(px).$$

Since Φ is a contraction, it follows that

$$(1+t)^2\|p^{\perp}\Phi(px)\|^2 \leqslant \|px\|^2 + t^2\|p^{\perp}\Phi(px)\|^2.$$

Thus $2t\|p^{\perp}\Phi(px)\|^2 \leqslant \|px\|^2$ for all $t > 0$, which forces $p^{\perp}\Phi(px) = 0$. $\square$

Remark. The Lemma fails if A is a nonselfadjoint operator algebra. Indeed, if H is a Hilbert space on which such A is completely contractively represented, and if K

is a closed A-invariant subspace of H, then the orthogonal projection onto K is not necessarily an A-module map.

Theorem 3.3. *Suppose that X is a representable module over a C^*-algebra A, and suppose that X is also a dual Banach space. Then the map $x \mapsto ax$ on X is w^*-continuous for all $a \in A$.*

Proof. Let u be the map $x \mapsto ax$. The adjoint of the canonical inclusion of the predual of X into its second dual, is a w^*-continuous contractive projection $q: X^{**} \to X$, which induces an isometric map $v: X^{**}/Ker(q) \to X$. By basic duality principles, v is w^*-continuous. By the Krein–Smulian theorem (see Section 1), it is a w^*-homeomorphism. We claim that

$$q(u^{**}(\eta)) = u\,q(\eta), \quad \eta \in X^{**}. \tag{2}$$

If (2) holds, then it follows that u^{**} induces a map $\dot{u}$ in $B(X^{**}/Ker(q))$, namely $\dot{u}(\dot{\eta}) = (u^{**}(\eta))^{\cdot}$, where $\dot{\eta}$ is the equivalence class of $\eta \in X^{**}$ in the quotient. Since u^{**} is w^*-continuous, so is $\dot{u}$, by basic Banach space duality principles. Using (2) it is easy to see that $\dot{u} = v^{-1}uv$. Since $\dot{u}, v$, and v^{-1} are w^*-continuous, so is u, and thus the result is proved.

In fact (2) follows from Lemma 3.2. Indeed, as in the proof of that result, X^{**} may be regarded as an operator A^{**}-module. Therefore it is an A-module, with module action $a\eta = u^{**}(\eta)$ in the notation above, and X is an A-submodule. $\square$

Corollary 3.4. *Let T be a map on a dual Banach space X, and suppose that there exists a C^*-algebra A, a linear isometry $\sigma : X \to A$, and an element $a \in A$, with $\sigma(Tx) = a\sigma(x)$ for all $x \in X$. Suppose also that $a^*\sigma(X) \subset \sigma(X)$. Then T is w^*-continuous.*

Proof. We may assume that A is unital. Then $\sigma(X)$ is a left operator module over the C^*-algebra generated by 1 and a. By the previous result, the map L_a of left multiplication by a is continuous in the w^*-topology of $\sigma(X)$ induced by the predual of X. It is then clear that T is w^*-continuous on X. $\square$

As one application of the last results, we give alternative proofs of some important results about Hilbert C^*-modules from [33,11]. These particular proofs will also be needed later in Section 5 (they generalize to the nonselfadjoint situation). The reader who is unfamiliar with basic C^*-module theory, can skip the proof of the next result. In fact we will not even give the basic definitions, which may be found in any of the standard C^*-module texts, or [7, Chapter 8]. We refer to Section 8.7 (and its Notes) in the latter reference for W^*-modules, their history, complementary results, and references. We recall that *ternary rings of operators* (or *TROs*) are a simple example of C^*-modules. They may be taken to be spaces of the form $Z = pA(1-p)$, for a unital C^*-algebra A and a projection $p \in A$ (e.g. see [11] or (8.3) and the line above it in [7]). For any TRO Z, ZZ^* and Z^*Z are C^*-algebras, and Z is a

ZZ^*–Z^*Z-bimodule. In fact every C^*-module may be represented canonically as a TRO (e.g. see [7, 8.1.19]). A *ternary morphism* is an operator T between TRO's satisfying $T(xy^*z) = T(x)T(y)^*T(z)$ for all x, y, z.

Corollary 3.5 (*Zettl, Effros–Ozawa–Ruan*). *Let Z be a full right Hilbert C^*-module over a C^*-algebra B, and suppose that Z has a Banach space predual. If $N = M(B)$ then N and $\mathbb{B}_N(Z)$ are W^*-algebras, the 'inner product' on Z is separately w^*-continuous as a map into N, and Z is a w^*-full selfdual C^*-module over N (see e.g.* [7, Section 8.5] *for definitions). Moreover, Z has a unique Banach space predual, and this predual is also an operator space predual. If Z is a TRO with a Banach space predual, then Z is ternary isomorphic and w^*-homeomorphic to a 'corner' $qM(1-q)$, for a W^*-algebra M and a projection $q \in M$.*

Proof. We assume that Z is a TRO, and $B = Z^*Z$. This is purely for notational simplicity, the general case is essentially identical. The subalgebra $B_B(Z)$ of $B(Z)$ is w^*-closed in the natural w^*-topology of $B(Z)$. To see this, suppose that (T_t) is a net in $B_B(Z)$ converging in this topology to $T \in B(Z)$. Thus $T_t(yb) = T_t(y)b$ converges to $T(yb)$ in the w^*-topology of Z, for all $y \in Z, b \in B$. On the other hand, $T_t(y)$ converges to $T(y)$. Thus $T_t(y)b \to T(y)b$, by Theorem 3.3. It follows that $T(yb) = T(y)b$, and so $T \in B_B(Z)$. Thus $B_B(Z)$ is w^*-closed in $B(Z)$.

Let $u: Z \to B$ be a bounded B-module map. It is well known (e.g. see [7, 8.1.23]) that we may choose a contractive approximate identity $(e_t)_t$ for ZZ^*, with terms of the form $\sum_{k=1}^n x_k x_k^*$ for some $x_k \in Z$. Set $w_t = \sum_{k=1}^n x_k u(x_k)^*$ (which depends on t). For $x \in Z$,

$$u(e_t x) = \sum_{k=1}^{n} u(x_k)x_k^* x = w_t^* x. \tag{3}$$

It follows that $\|w_t\|^2 = \|u(e_t(w_t))\| \leqslant \|u\|\|w_t\|$. Thus $(w_t)_t$ is a bounded net in Z, and so it has a w^*-convergent subnet, with limit w say. Replace the net with the subnet. By Theorem 3.3, $zx^*w_t \to zx^*w$, for all $x, z \in Z$. Since $u(e_t(x)) \to u(x)$ in norm, by (3) we have $zu(x)^* = zx^*w$, for all $x, z \in Z$. Thus $u(x) = w^*x$, and so Z is selfdual over B. It follows immediately that $B_B(Z)$ is the C^*-algebra of 'adjointable' maps on Z (e.g. see [7, 8.5.1 (2)]). Equivalently, by a result of Kasparov (e.g. see [7, 8.1.16]), $B_B(Z) = M(ZZ^*)$. Since $B_B(Z)$ has a predual, it is a W^*-algebra in its natural w^*-topology (that is, a bounded net of maps converges if and only if they converge as maps on Z, in the point-w^*-topology). By symmetry, $N = M(Z^*Z)$ is a W^*-algebra in a topology for which a bounded net (n_t) converges to n if and only if $zn_t \to zn$ for all $z \in Z$.

Claim: the natural Z^*Z-valued inner product on Z is separately w^*-continuous. Suppose that (y_t) is a bounded net in Z converging in the w^*-topology of Z to $y \in Z$, and that $w \in Z$ is fixed. Suppose that $(w^*y_{t_\mu})$ is a subnet of (w^*y_t), with w^*-limit $n \in N$. If $z \in Z$, then $(zw^*y_{t_\mu})$ converges both to zw^*y and to zn, by Theorem 3.3 and the fact at the end of the last paragraph. It follows that $n = w^*y$, which proves the claim since the subnet was arbitrary (using also the Krein–Smulian theorem as mentioned in Section 1).

The other assertions of the Theorem now all follow immediately from standard facts about selfdual modules (e.g. see [7, Section 8.5, 8.5.1–8.5.4, 8.5.10]). These facts are also all mentioned in Section 5 below, in a more general setting (see particularly Lemma 5.1). ☐

The objects characterized in the last theorem are often called W^*-modules.

4. Multipliers and duality

Theorem 4.1. *Every left multiplier of a dual operator space is* w^**-continuous.*

Proof. If $u \in \mathcal{M}_l(X)$, then $u^{**} \in B(X^{**})$. As in Theorem 3.3, let $q : X^{**} \to X$ be the canonical projection, which now is completely contractive if X is a dual operator space. As in that result, it suffices to show that

$$q(u^{**}(\eta)) = u\,q(\eta), \quad \eta \in X^{**}. \tag{4}$$

In order to prove (4), we let Z be an injective envelope of X, viewed as a TRO $pD(1-p)$, for a unital C^*-algebra D and a projection $p \in D$ (see e.g. [7, Sections 4.2 and 4.4] or [28, Chapter 16]). If $E = Z^{**}$ then $E = pD^{**}(1-p)$ is also a TRO. Clearly X^{**} may be regarded as a w^*-closed subspace of E, and thus by injectivity of Z, we can extend the map q above, to a completely contractive map $\theta : E \to Z$. Since $\theta_{|X} = I_X$, by the rigidity property of the injective envelope we must have $\theta_{|Z} = I_Z$. Thus θ is a completely contractive projection from E onto Z. By Lemma 3.2, θ is a left pDp-module map. Let $a \in pDp$ be such that $ax = ux$ for all $x \in X$, as in [7, Theorem 4.5.2] or [28, Chapter 16]. Since θ is a left pDp-module map,

$$\theta(a\eta) = a\theta(\eta) = aq(\eta), \quad \eta \in X^{**}. \tag{5}$$

On the other hand, we claim that

$$a\,\eta = u^{**}(\eta), \quad \eta \in X^{**}. \tag{6}$$

To see this, view both sides as functions from X^{**} into E. Then both functions are w^*-continuous (note that since $E = pD^{**}(1-p)$, left multiplication by the element $a \in pDp \subset pD^{**}p$ is w^*-continuous). On the other hand, (6) certainly holds if $\eta \in X$, and by w^*-density it must therefore hold for $\eta \in X^{**}$. By (6), we have that $\theta(a\eta) = \theta(u^{**}(\eta)) = q(u^{**}(\eta))$. This together with (5) proves that $q(u^{**}(\eta)) = aq(\eta) = uq(\eta)$, which is (4). ☐

The last theorem has very many applications. For example, it answers a question that has also been open for many years:

Corollary 4.2. *If B is an operator algebra which is also a dual operator space, then the product on B is separately w^*-continuous.*

Proof. If $a \in B$, then the map $b \mapsto ab$ on A is clearly a left multiplier, and therefore is w^*-continuous by Theorem 4.1. Similarly the product is w^*-continuous in the first variable. $\square$

Putting Corollary 4.2 together with the main result in [23], we obtain the following improved characterization of σ-weakly closed operator algebras:

Corollary 4.3. *If B is an operator algebra which is also a dual operator space, then B is completely isometrically isomorphic, via a w^*-homeomorphic homomorphism, to a σ-weakly closed subalgebra of $B(H)$, for some Hilbert space H. Conversely, every σ-weakly closed subalgebra of $B(H)$ is a dual operator space.*

Remark. Corollaries 4.2 and 4.3 were obtained in [4] in the case that B also has an identity of norm 1.

In the sequel paper [8], we show that Results 4.1–4.3 are sharp, in the sense that they fail if we replace the hypothesis of 'dual operator space' by 'dual Banach space'. This illustrates the subtlety of the duality considerations at hand. In particular, we show in [8] that there exists an operator algebra with an identity of norm 1, which is a dual Banach space, but which is not w^*-homeomorphic via a homomorphism, to any σ-weakly closed subalgebra of $B(H)$. Thus the 'correct nonselfadjoint analogue' of a W^*-algebra, seems to reside naturally in the category of dual operator spaces, as opposed to dual Banach spaces. More precisely, these are the operator algebras which are also dual operator spaces. In any case, we will henceforth call the latter objects *dual operator algebras*.

The following was noticed together with Le Merdy, and is a nonselfadjoint variant of a beautiful result from [13]:

Corollary 4.4. *The Arens product is the only operator algebra product on the second dual of an operator algebra A, which extends the product on A.*

Proof. By Corollary 4.2, any such product is separately w^*-continuous and therefore must be the Arens product. $\square$

Corollary 4.5. *Every quasimultiplier* (*in the sense of* [21]) *of a dual operator space, is separately w^*-continuous.*

Proof. This follows from Corollary 4.2, and the correspondence between contractive quasimultipliers and operator algebra products [20,21]. $\square$

Corollary 4.6. *Suppose that B is a operator algebra with a bounded approximate identity, and with an operator space predual. Then B has an identity* (*of norm possibly* > 1).

Proof. If e is a w^*-limit of a bounded approximate identity, then e is an identity by Corollary 4.2. □

Historically, the one-sided multipliers of an operator space were developed as a 'generalization' of an older theory of 'Banach space multipliers' (e.g. see [17, Section I.3] and references therein). These objects have several equivalent definitions, we will just list one: a *multiplier* of a Banach space E is a linear map $T : E \to E$ such that there exists an isometric embedding $\sigma : E \to C(\Omega)$ for a compact space Ω, and a function $a \in C(\Omega)$, such that $\sigma(Tx) = a\sigma(x)$ for all $x \in E$ (e.g. see [17, Section I.3] and [7, Theorem 3.7.2]). The following result was known for 'centralizers' of Banach spaces (e.g. see [17, Theorem 1.3.14]), but seems not to be known for the larger class of Banach space multipliers (cf. [2]). We thank E. Behrends for confirming this. We imagine that it may be useful in that theory too.

Corollary 4.7. *If X is a dual Banach space, then every multiplier of X* (*in the sense above*) *is w^*-continuous.*

Proof. This follows from Theorem 4.1 applied to $\mathrm{Min}(X)$, which is a dual operator space by 1.4.12 in [7]. By 4.5.10 in [7], $\mathcal{M}_l(\mathrm{Min}(X))$ is the set of multipliers in the sense of [17, Section I.3]. □

We recall that $\mathcal{M}_l(X)$ contains the C^*-algebra $\mathcal{A}_l(X)$ of *left adjointable multipliers*. These are the left multipliers as defined in the second paragraph of our paper in terms of a complete isometry σ, but also satisfying $a^*\sigma(X) \subset \sigma(X)$ in the language of that definition. It is shown in [4,6] that if X is a dual operator space then $\mathcal{M}_l(X)$ is a dual operator algebra, $\mathcal{A}_l(X)$ is a W^*-algebra, and every $T \in \mathcal{A}_l(X)$ is w^*-continuous. These results have been key to work on one-sided multipliers and M-ideals following [6] (see e.g. [5,10]). We now show that one of these results is true, and the others false, if X merely has a Banach space predual. Indeed from Corollary 3.4 we have

Corollary 4.8. *Let X be an operator space, which is a dual Banach space. Then every $T \in \mathcal{A}_l(X)$ is w^*-continuous.*

Proposition 4.9. *There exists an operator system X which is a dual Banach space, but for which $\mathcal{A}_l(X)$ is not a W^*-algebra* (*or even an AW^*-algebra*), *and $\mathcal{M}_l(X)$ is not a dual Banach space.*

Proof. Let X be the operator system in Proposition 2.1. We will show that $\mathcal{A}_l(X) = \mathcal{M}_l(X) \cong S^\infty + \mathbb{C}I$. To this end, we first claim that $D = B(H) \oplus^\infty Q^{\mathrm{op}}$ is the C^*-envelope of X. Let B be the C^*-algebra generated by X in D. If $(a, \dot{a}), (b, \dot{b}) \in X$, then $(a, \dot{a})(b, \dot{b}) = (ab, \dot{a}\dot{b}) \in B$. Since $(ba, \dot{a}\dot{b}) \in X$, we have $(ab - ba, 0) \in B$. If $(c, \dot{c}), (d, \dot{d}) \in X$, then $(c, \dot{c})(ab - ba, 0)(d, \dot{d}) = (c(ab - ba)d, 0) \in B$. However, $\{c(ab - ba)d : a, b, c, d \in B(H)\}$ densely spans $B(H)$, and so B contains $B(H) \oplus 0$. Since also $X \subset B$, we have $0 \oplus Q^{\mathrm{op}} \subset B$, and it follows that $B = D$. Thus X generates D as a C^*-algebra. To see that D is the C^*-envelope of X, suppose that J were a nontrivial ideal in D such that the canonical map $D \to D/J$ is completely isometric on X. Since $B(\ell^2)$ has only one nontrivial closed ideal, and therefore Q has none, J must be one of the four spaces $B(H) \oplus 0, S^\infty \oplus 0, 0 \oplus Q^{\mathrm{op}}, S^\infty \oplus Q^{\mathrm{op}}$. Thus D/J is of the form $0 \oplus Q^{\mathrm{op}}, Q \oplus Q^{\mathrm{op}}, B(H) \oplus 0$, or $Q \oplus 0$. In any of these cases we obtain a contradiction. For example, the third case yields a contradiction, because, by the discussion in the proof of Proposition 2.1, the map $(a, \dot{a}) \to (a, 0)$ from X to $B(H) \oplus 0$ is not a complete isometry. This proves the claim.

As explained in Section 1, $\mathcal{M}_l(X) = \{a \in D : aX \subset X\}$. Clearly $(b + \lambda 1, \lambda \dot{1}) \in \mathcal{A}_l(X)$, for any $b \in S^\infty$. Since $1 \in X$, if $a \in \mathcal{M}_l(X)$ then $a = (b, \dot{b})$ for a $b \in B(H)$ such that $(bc, \dot{b}\dot{c}) \in X$ for all $c \in B(H)$. That is, $\dot{b}\dot{c} = \dot{c}\dot{b}$, so that $\dot{b}$ is in the center of Q. However, the latter is trivial (see e.g. [18], we thank V. Zarikian for this reference). Thus $b \in S^\infty + \mathbb{C}1$. Hence $\mathcal{M}_l(X) = \mathcal{A}_l(X) \cong S^\infty + \mathbb{C}1$. $\square$

Theorem 4.1 will also be an useful tool for future work on *operator modules*. For example, in [4] the first author was able to refine, in several ways, a theorem of Effros and Ruan characterizing certain operator modules over von Neumann algebras [12]. Theorem 4.1 allows precisely the same improvements for 'normal dual operator modules' over unital dual operator algebras. In particular, Theorem 4.1 shows that the *left normal* hypothesis used in [4] is automatic, and may therefore be removed. We state a sample of other consequences:

Corollary 4.10 (*cf. [6, Corollary 5.6]*). *Suppose that X is a left operator A-module, where A is approximately unital, and suppose that X is also a dual operator space. Then for any $a \in A$, the map $x \mapsto ax$ is automatically w^*-continuous. This is also true, if X merely is a dual Banach space, providing that A is a C^*-algebra.*

This corollary allows one to eliminate one of the hypotheses in the well-known definition of a *normal dual operator bimodule* (e.g. see [12]). Thus we may define, for example, a left *normal dual operator module* to be a left operator module X over a dual operator algebra M, such that X is a dual operator space, and the module action $M \times X \to X$ is w^*-continuous in the first variable. Similar definitions hold for right modules and bimodules.

Corollary 4.11. *Let X be a dual operator space. Then X is a normal dual $\mathcal{M}_l(X)$-$\mathcal{M}_r(X)$-bimodule.*

Conversely, any normal dual operator module (or bimodule) action on a dual operator space X, 'factors through' the one in Corollary 4.11; and moreover there is a tidy 'representation theorem' for such modules. For details, see [4] or 4.7.6 and 4.7.7 in [7].

The following is a Banach module characterization of w^*-closed subspaces X of $B(K, H)$ which are invariant under the action of two W^*-algebras M and N on H and K, respectively (that is, $\pi(M)X \subset X$ and $X\theta(N) \subset X$, where π and θ are normal $*$-representations of M and N). Our theorem is the Banach module variant of an earlier operator module characterization of such bimodules due to Effros and Ruan [12]; however it has potential to be even more useful in certain contexts since our condition

(ii) is easier to verify. Before we state the theorem we give a definition: to say that the unit ball $Ball(X)$ is *M–N-absolutely convex* is to say that $\| \sum_{k=1}^{m} m_k x_k n_k \| \leqslant 1$ whenever $x_1, \ldots, x_m \in Ball(X)$ and $m_1, \ldots, m_m \in M, n_1, \ldots, n_m \in N$ with $\| \sum_{k=1}^{m} m_k m_k^* \| \leqslant 1$ and $\| \sum_{k=1}^{m} n_k^* n_k \| \leqslant 1$. It is not hard to see that without loss of generality, $m = 2$ in the above; and that if $B = \mathbb{C}$ then this definition coincides with Definition 3.1 (see [25]).

Theorem 4.12. *Let M and N be W^*-algebras, and let X be an M–N-bimodule* (*we assume that $1_M x = x 1_N = x$ for all $x \in X$*). *Suppose that X is also a dual Banach space. The following are equivalent*:

(i) *There exist Hilbert spaces K and H, a w^*-continuous isometry $\Phi : X \to B(K, H)$, and normal $*$-representations π and θ of M and N on H and K, respectively, such that $\Phi(mxn) = \pi(m)\Phi(x)\theta(n)$ for $x \in X, m \in M, n \in N$;*
(ii) *The unit ball of X is M–N-absolutely convex, and for all $x \in X$ the canonical maps $M \to X$ and $N \to X$ given by $m \mapsto mx$ and $n \mapsto xn$, are w^*-continuous;*
(iii) *The unit ball of X is M–N-absolutely convex, and the bimodule action $M \times X \times N \to X$ is separately w^*-continuous.*

Proof. (ii) $\Rightarrow$ (iii) The condition implies by e.g. [25, Theorem 2.1] that there is an operator space structure on X for which X becomes an operator M–N-bimodule. Now (iii) is clear from Corollary 3.4.

(iii) $\Rightarrow$ (i) As in the lines above, X may be viewed as an operator M–N-bimodule. By e.g. 3.8.9 in [7], X^{**} is an operator M–N-bimodule too. As in the first few lines of Theorem 3.3, there is a canonical w^*-continuous contractive projection $q: X^{**} \to X$, which induces an isometric w^*-homeomorphism $v: X^{**}/Ker(q) \to X$. By Lemma 3.2, q is an M–N-bimodule map, and therefore so also is v. Indeed, $X^{**}/Ker(q)$ is an operator M–N-bimodule isometrically M–N-isomorphic to X, via v. We assign X a new operator space structure so that v becomes a complete isometry. Since $X^{**}/Ker(q)$ has an operator space predual (namely $Ker(q)_\perp$), so now does X. Moreover, we have not changed the w^*-topology on X, since v was a w^*-homeomorphism originally. Now X is a normal dual operator M–N-bimodule, and hence we obtain the desired representation from e.g. [12] or [7, Theorem 3.8.3].

(i) $\Rightarrow$ (ii) This is the routine ‘easy direction’ of such theorems, and is left here as an exercise. □

The next section will continue to demonstrate that Theorem 4.1 should play a significant role in future studies of operator modules.

5. Nonselfadjoint generalization of W*-modules

Notions of Morita equivalence appropriate to nonselfadjoint operator algebras, and of ‘rigged modules’, were developed in the last 10 years in [9,3]. These notions generalize the ‘strong Morita equivalence’ of C^*-algebras due to Rieffel, and the ‘C^*-modules’ used heavily in that theory. There is a parallel theory (e.g. see [29] or [7, Section 8.5]) appropriate to W^*-algebras: the corresponding notions are sometimes called ‘W^*-algebra Morita equivalence’, and ‘W^*-modules’, and they are due to Rieffel and Paschke, respectively. Hitherto there has been no attempt in the literature to generalize this ‘weak’ version of the theory, to nonselfadjoint dual operator algebras. One main reason for this, we believe, is that the technical tools were not all available or fully developed. It seems that operator space multipliers and Theorem 4.1, were one of the missing ingredients in getting this theory started. We can show that with the addition of this ingredient, one can obtain a theory that generalizes several important aspects of the W^*-algebra case. At the same time, this will illustrate how Theorem 4.1 may be powerfully used in practice. Our intention is to be very brief; the reader will need to consult the papers [9,3] for additional definitions and details.

In the following discussion, Y is a right M-rigged module, in the sense of [3], over an approximately unital operator algebra M. Then there is a canonical left M-rigged module $X = \tilde{Y}$, and a canonical pairing $(\cdot,\cdot) : X \times Y \to M$ (see [3] or [9, Chapter 4]). In our case, M will usually be a dual operator algebra. We say that Y is *selfdual* over M, if every completely bounded M-module map $f : Y \to M$ is of the form $(x,\cdot)$ for a fixed $x \in X$, and every completely bounded M-module map $g : X \to M$ is of the form $(\cdot, y)$ for a fixed $y \in Y$. If Y is selfdual then every completely bounded M-module map from Y into another rigged M-module Z is *adjointable*. This follows by considering the M-valued M-module map $(w, u(\cdot))$ on Y, for fixed $w \in \tilde{Z}$, just as in the C^*-module case (e.g. see [7, 8.5.1 (2)]). Indeed, the proofs of the next two results are also essentially just as in Section 8.5 of [7], simply replacing appeals to C^*-module facts by appeals to the matching results for rigged modules from [3,9]. Thus we omit essentially all of these proofs.

Lemma 5.1. *Let Y be a right rigged M-module over a unital dual operator algebra M. Then*:

(1) *Y is a selfdual rigged M-module if and only if X and Y have Banach space preduals with respect to which $(\cdot,\cdot)$ is separately w^*-continuous.*

If Y is a selfdual rigged M-module, then:

(2) *X and Y have unique Banach space preduals with respect to which $(\cdot,\cdot)$ is separately w^*-continuous.*
(3) *With respect to the w^*-topology induced by the predual in* (2), *a bounded net $(y_t)_t$ converges to y in Y if and only if $(x, y_t) \to (x, y)$ in the w^*-topology of M, for all $x \in X$. Similarly for bounded nets in X.*
(4) *Let $W = M_* \widehat{\otimes}_M X$ and $Z = Y \widehat{\otimes}_M M_*$* (*see* [7, Section 3.4]). *Then W and Z are operator space preduals of Y and X, respectively, inducing the w^*-topology in* (2) *and* (3) *above.*
(5) *The canonical map $m \mapsto ym$ from M to Y is w^*-continuous in the topology in* (3), *for all fixed $y \in Y$.*

Proof. We will simply prove (5), which was not mentioned in the matching result from [7]. If (m_t) is a bounded net converging to m in the w^*-topology of M, and if

$x \in X$, $y \in Y$, then we have $(x, ym_t) = (x, y)m_t \to (x, y)m = (x, ym)$, by the separate w^*-continuity of the product in M. Thus by (3), $ym_t \to ym$. The result follows by the Krein–Smulian theorem (see Section 1). ☐

We will henceforth use the phrase *the w^*-topology* of a selfdual M-rigged module, for the (unique) topology in (2)–(4) above.

Corollary 5.2. *Suppose that Y is a selfdual right M-rigged module over a unital dual operator algebra M. Then*:

(1) $CB_M(Y) = \mathbb{B}_M(Y)$, *the operator algebra of 'adjointable' M-module maps, and this is a dual operator algebra.*

(2) *A bounded net $(T_i)_i$ in $CB_M(Y)$ converges in the w^*-topology to $T \in B_M(Y)$ if and only if $T_i(y) \to T(y)$ in the w^*-topology of Y, for all $y \in Y$. Indeed, $Y \widehat{\otimes}_M W$ is a predual for $CB_M(Y)$, where W is as in Lemma 5.1* (4).

Similarly it follows, as in [7, Corollary 8.5.8], that any bounded M-module map between selfdual right rigged M-modules, is w^*-continuous.

For a right rigged module Y over an operator algebra A, we will consistently write $\mathcal{I}$ for the closed span of the range of the canonical pairing $(\cdot, \cdot)$ in A. We say that Y is *full* over A, if $A = \mathcal{I}$. If A is a dual operator algebra, we write $\mathcal{I}^w$ for the w^*-closure of this span, and say that Y is *w^*-full* if $\mathcal{I}^w = A$. In general though, $\mathcal{I}$ and $\mathcal{I}^w$ are both ideals in A. Henceforth, we say that a right rigged module Y is *a (right) rigged-equivalence module*, if $\mathcal{I}$ has a contractive approximate identity, and the canonical map $X \otimes_h Y \to \mathcal{I}$ is a complete quotient map. This is equivalent to saying that $\mathcal{I}$ possesses a contractive approximate identity of a certain special form, or to saying that Y is a strong Morita equivalence $\mathbb{K}_A(Y)$-$\mathcal{I}$-bimodule. For example, see [9,3] for more details. In this case, and if also A is a dual operator algebra, then by considering a w^*-limit of the contractive approximate identity, it follows that $\mathcal{I}^w$ is unital.

The property of selfduality defined earlier does not depend essentially on M: that is, Y is selfdual over M if and only if Y is selfdual over $\mathcal{I}$ or over the multiplier algebra $M(\mathcal{I})$. The proof of this is identical to [7, Lemma 8.5.2].

We will write $LM(A)$ and $RM(A)$ for the left and right multiplier algebras of A. For example, $LM(A)$ may be identified with $CB_A(A)$ (see e.g. [7, Section 2.6]).

Lemma 5.3. *Let J be a w^*-dense norm-closed two-sided ideal in a dual operator algebra M, and suppose that J is approximately unital. Then M is the multiplier algebra $M(J)$, and the latter equals $LM(J)$ and $RM(J)$.*

Proof. In fact this works more generally in the setting of Banach algebras, provided that the product on M is separately w^*-continuous. There is a canonical complete contractive homomorphism $M \to CB_J(J)$, and the latter space is just $LM(J)$. This map is 1-1 by the w^*-density of J, that it is completely isometric and surjective is easily seen by considering, for any $T \in CB_J(J)$, a w^*-limit point of $(T(e_t))$ in M, where (e_t) is the approximate identity for J. The other assertions are now easy. ☐

Corollary 5.4. *Let Y be a rigged-equivalence module, over a dual unital operator algebra M. In the notation above, $\mathcal{I}^w$ is the multiplier algebra of $\mathcal{I}$.*

Proof. Clearly $\mathcal{I}$ is a w^*-dense ideal in $\mathcal{I}^w$. $\Box$

We now seek to generalize Zettl's theorem (cf. Corollary 3.5) to rigged modules. It is natural to assume in our context that Y and $X = \tilde{Y}$ both have an operator space predual. Our main theorem, Theorem 4.1, then yields the following corollary, which in turn will yield the nonselfadjoint analogue of Zettl's theorem.

Corollary 5.5. *Let Y be a right rigged module over an approximately unital operator algebra A, and suppose that Y and $X = \tilde{Y}$ both have an operator space predual. Then the maps $y \mapsto y'(x', y)$, $x \mapsto (x, y')x'$, $y \mapsto ya$, and $x \mapsto ax$, are w^*-continuous on Y and X, respectively, for all fixed $x', x'' \in X$, $y', y'' \in Y$, $a \in A$.*

Proof. From the theory of rigged modules, it is clear that these maps are operator space multipliers. For example, the first of these maps belongs to $B = \mathbb{K}_A(Y)$, and Y is an operator B–A-bimodule. Thus we can appeal to Corollary 4.10 to see that this map, and also the map $y \mapsto ya$, are w^*-continuous on Y. Similarly for X. $\Box$

Theorem 5.6. *Let Y be a full right rigged-equivalence module over an approximately unital operator algebra A, and suppose that Y and $X = \tilde{Y}$ are dual operator spaces. If $M = M(A)$ then M and $\mathbb{B}_A(Y)$ are dual operator algebras, and Y is a w^*-full selfdual M-rigged module.*

Proof. We follow the proof of Corollary 3.5. As in that proof, but also using Corollary 5.5, $CB_A(Y)$ is a w^*-closed subalgebra of $CB(Y)$. From the theory of strong Morita equivalence (see e.g. [9, Theorem 4.9]), $CB_A(Y)$ is an operator algebra, hence it is a dual operator algebra, by Corollary 4.3. Similarly for ${}_ACB(X)$.

If $B = \mathbb{K}_A(Y)$, then from the theory of strong Morita equivalence $LM(A) \cong CB_B(X)$, and $RM(A) \cong {}_BCB(Y)$, completely isometrically. This may be seen from the fact that such equivalence implements a ‘completely isometric’ equivalence between the categories of right modules over A and B (see [9, p. 25]), thus

$$LM(A) = CB_A(A) \cong CB_B(A \otimes_{hA} X) \cong CB_B(X).$$

The map here from $LM(A)$ into $CB(X)$ may be checked to be the canonical one: if $\eta \in LM(A), a \in A, x \in X$, then (the image in $CB_B(X)$ of) η takes ax to $(\eta a)x$. Similarly, for the map from $RM(A)$ into $CB(Y)$. Thus, we identify the operator algebras $LM(A)$ and $RM(A)$ with $CB_B(X)$ and ${}_BCB(Y)$, respectively; and by the argument above, these subspaces are dual operator algebras, and w^*-closed subspaces of $CB(X)$ and $CB(Y)$, respectively. Let $u \in CB_A(Y, A)$. Following Corollary 3.5, we choose a contractive approximate identity $(e_t)_t$ for $\mathbb{K}_M(Y)$, of the form $\sum_{k=1}^n [y_k, x_k]$ for some

$x_k \in X, y_k \in Y$ as in e.g. [3, Theorem 5.2]. We have

$$u(e_t(y)) = \sum_{k=1}^{n} u(y_k)(x_k, y) = \left(\sum_{k=1}^{n} u(y_k)x_k, y\right) = (w_t, y), \quad y \in Y, \tag{7}$$

where $w_t = \sum_{k=1}^{n} u(y_k)x_k$. Using the fact that the canonical map $X \to CB(Y, A)$ is an isometry (see e.g. [9, Theorem 4.1]), we have $\|w_t\| = \|u \circ e_t\|_{cb} \leqslant \|u\|_{cb}$. Thus $(w_t)_t$ is bounded in X, and we can proceed as in Corollary 3.5, but also using Corollary 5.5, to find $w \in X$ with $u(y)x' = (w, y)x'$ for all $x' \in X, y \in Y$, so that $u(y) = (w, y)$. Similarly, any A-valued A-module map on X, is given by $(\cdot, y)$ for a fixed $y \in Y$. Thus Y is selfdual as an A-module. It follows, as asserted earlier, that $CB_A(Y) = \mathbb{B}_A(Y)$. As in the first centered equation of the present proof, we have $CB_A(Y) \cong LM(\mathbb{K}_A(Y))$. This isomorphism carries $\mathbb{B}_A(Y)$ onto $M(\mathbb{K}_A(Y))$, as one may check somewhat analogously to the proof of 8.1.16 in [7] (see [3, Theorem 3.8]). We have now shown that $M(\mathbb{K}_A(Y)) = LM(\mathbb{K}_A(Y))$, and it follows by symmetry that $M(A) = RM(A)$. Similar arguments involving X show that $M(A) = LM(A)$.

Now Y is selfdual over $M(A)$ too, as remarked above Lemma 5.3. By Corollary 5.4, it is clear that Y is a w^*-full module over $M(A)$. ☐

Corollary 5.7. *Let Y be a right rigged-equivalence module over a dual operator algebra M. Then Y is a selfdual M-rigged module if and only if Y and* $X = \tilde{Y}$ *possess operator space preduals.*

Proof. By Lemma 5.1 we need only prove one direction. Assuming the existence of operator space preduals, by the previous result and Lemma 5.4, Y is selfdual over $\mathcal{I}^w = M(\mathcal{I})$. It follows as in [7, Lemma 8.5.2] that Y is selfdual over M. ☐

Remark. The wary reader may wonder whether the given preduals in Corollary 5.7 induce the w^*-topology mentioned after Lemma 5.1. Unlike the W^*-algebra case, in fact they may not, if these preduals were chosen poorly. This is clear by considering the simplest example: $A = B = X = Y$, a unital operator algebra with several unrelated preduals (e.g. see [4] or [7, Corollary 2.7.8]). There are other conditions one may impose, that will alleviate this situation. For example, if one also insists in Corollary 5.7 that X and Y be normal dual A-modules (defined above Corollary 4.11). We leave this as an exercise for the interested reader (see the ideas in the proof of Proposition 5.8 below).

Let M and N be two unital dual operator algebras, and suppose that there exist w^*-dense norm closed ideals of M and N, respectively, which are strongly Morita equivalent in the sense of [9], via equivalence bimodules X and Y. By Lemma 5.3, M and N are the multiplier algebras of these ideals. Hence, Y is canonically an operator N–M-bimodule too (see [7, 3.1.11]), and similarly for X. We claim that Y is selfdual as a right module if and only Y is selfdual as a left module. Indeed, if Y is selfdual as a right M-module, then X and Y are dual operator spaces by Lemma 5.1. Hence, using the left version of Corollary 5.7, we see that Y is selfdual as a left N-module. Since N is the multiplier algebra of the appropriate ideal, it follows from Lemma 5.3 that Y is w^*-full as a left N-module.

Proposition 5.8. *Let X and Y be as in the last paragraph. Then X and Y have operator space preduals and are normal dual operator bimodules over M and N, if and only if Y is selfdual and its canonical w^*-topology as a selfdual right module* (*mentioned after Lemma 5.1*), *agrees with its canonical w^*-topology as a selfdual left module*, *and similarly for X.*

Proof. ($\Leftarrow$) Follows from Lemma 5.1, and the method of proof of (5) of that result.

($\Rightarrow$) Assuming X and Y have operator space preduals, we will refer to the associated w^*-topologies as *the original w^*-topologies* of X and Y. By Theorem 5.6, Y is selfdual as a right M-module. To say that a bounded net (y_t) converges to $y \in Y$ in the w^*-topology mentioned after Lemma 5.1, is to say that $(x', y_t) \to (x', y)$ in the w^*-topology of M, for all $x' \in X$. By hypothesis, this implies that

$$y'(x', y_t) \to y'(x', y) \text{ in the original } w^*\text{-topology of } Y \quad \text{for all } x' \in X, y' \in Y. \tag{8}$$

In fact it is equivalent to (8), since if (8) holds, and if $((x', y_{t_\mu}))_\mu$ is a w^*-convergent subnet of $((x', y_t))_t$ with limit $m \in M$, then by hypothesis, $y'(x', y_{t_\mu})$ converges to $y'm$. This implies that $y'm = y'(x', y)$ for all $y' \in Y$, so that $m = (x', y)$. Hence $(x', y_t) \to (x', y)$ in the w^*-topology of M. By Corollary 5.5, if $y_t \to y$ in the original w^*-topology of Y, then (8) holds. Conversely, if (8) holds, then $y_t \to y$ in the original w^*-topology, by a w^*-convergent subnet argument similar to the one we just used above. (For if a subnet of (y_t) converges with limit y'', say, then using Corollary 5.5 as above shows that $y'(x', y'') = y'(x', y)$ for all such x', y'. This implies that $y'' = y$.)

We have now shown that the canonical w^*-topology (mentioned after Lemma 5.1) of Y as a right module agrees with its original w^*-topology. By a symmetrical argument, this agrees with the canonical w^*-topology as a left module. Similarly, for X. $\square$

The equivalent conditions in the last result are automatic in the W^*-algebra case, but not more generally. If these conditions are satisfied, then we call Y a *tight w^*-equivalence N–M-bimodule*, and we say that M and N are *tightly Morita w^*-equivalent*. It then follows as in the second paragraph of the proof of Theorem 5.6, using also Lemma 5.3, that $N \cong CB_M(Y)$ completely isometrically. The isomorphism here takes $n \in N$ to the map $y \mapsto ny$ on Y. It is easy to argue, as in Lemma 5.1 (5), that this isomorphism is w^*-continuous. Hence by the Krein–Smulian theorem it is a w^*-homeomorphism. Thus, just as in the selfadjoint theory, we can forget about N, and instead work with $CB_M(Y)$ (which equals $\mathbb{B}_M(Y)$), when convenient.

Conversely, we have

Theorem 5.9. *Let Y be a selfdual right rigged-equivalence module over a unital dual operator algebra M. Then Y is a left w^*-full selfdual $CB_M(Y)$-rigged module. Also, Y implements a tight Morita w^*-equivalence between $CB_M(Y)$ and $\mathcal{I}^w$. In particular, if*

Y is also a right w^*-full M-module then Y implements a tight Morita w^*-equivalence between $CB_M(Y)$ and M.

Proof. By Lemma 5.1, Y and X are dual operator spaces. By Corollary 5.2, we have $CB_M(Y) = \mathbb{B}_M(Y)$, and this is a dual operator algebra. As we said at the end of the proof of Theorem 5.6, this space also equals $M(\mathbb{K}_M(Y))$. The 'left-hand variant' of Theorem 5.6 says that Y is a selfdual left $CB_M(Y)$-rigged module, and it is w^*-full by Lemma 5.4. The other assertions follow immediately from the definition of tight Morita w^*-equivalence, and Lemma 5.1 (5). □

Examples. Examples of tight Morita w^*-equivalence, and therefore of selfdual right rigged-equivalence modules, are not hard to find. We list just three, omitting details:

(1) W^*-algebras are Morita equivalent in the sense of [29], if and only if they are tightly Morita w^*-equivalent. This follows from the definition in e.g. 8.5.12 of [7], and the fact that for C^*-algebras, Rieffel's notion of strong Morita equivalence coincides with the one in [9] (see Chapter 6 of that reference).

(2) If A and B are any two unital operator algebras which are strongly Morita equivalent in the sense of [9], then A^{**} and B^{**} are tightly Morita w^*-equivalent. We omit the proof, which uses the method of 8.5.32 in [7].

(3) Let η be a fixed vector in a Hilbert space H. The set of bounded operators on H which have η as an eigenvector, is a unital dual operator algebra which is tightly Morita w^*-equivalent to the upper triangular 2×2 matrices. The associated w^*-equivalence bimodules may be taken to be the set of operators from $\mathbb{C}^2$ to H taking the vector e_1 to a scalar multiple of η, and the set of operators from H to $\mathbb{C}^2$ taking η to a scalar multiple of e_1.

We next show that any selfdual right rigged-equivalence module X over a unital dual operator algebra N, occurs as a 'corner' of a unital dual operator algebra $\mathcal{L}$. Note that the 'right N-rigged sum' $X \oplus_c N$ is a right N-rigged module, which is clearly selfdual. The conjugate left N-rigged module is $Y \oplus_r N$, where $Y = \tilde{X}$ (see [3, Section 4]). Therefore, by Lemma 5.1, $X \oplus_c N$ is a dual operator space, and it is easy to check using Lemma 5.1 (3) that the containments of X and N in this latter space are w^*-homeomorphisms. Let p be the projection from $X \oplus_c N$ onto $X \oplus 0$. Thus $\mathcal{L} = M_l(X \oplus_c N) = CB_N(X \oplus_c N)$ is a dual operator algebra, by [4, Corollary 3.2]. The four corners of $\mathcal{L}$ are X, Y, N, and $M \cong CB_N(X)$; indeed $X = p\mathcal{L}(1-p)$. By Lemma 5.1 (2) and Corollary 5.2 (2), the w^*-topologies on X and Y inherited from $\mathcal{L}$ coincide with the original ones. Similarly for the other corners.

It is now clear that one has a theory that is simultaneously the appropriate 'w^*-topology version' of much of the theory in [9], and a generalization of much of the C^*-algebraic theory of weak Morita equivalence and W^*-modules (see [7, Section 8.5]). Moreover, it is clear that operator space multipliers play an important role in this theory. Generalizing many of the other results in the selfadjoint variant of the theory, is now essentially a routine exercise. For example, one may show, analogously to a result due to Rieffel in the W^*-algebra case, that any selfdual rigged-equivalence module over a unital dual operator algebra M is of the form ${}_RB(K, H)$, for a suitable Hilbert module

K over M, and a Hilbert R-module H, where R is the commutant of M in $B(K)$. The argument follows the lines of that of 8.5.37 and 8.5.32 in [7], but using also the double commutant theorem for nonselfadjoint operator algebras of Blecher and Solel (e.g. see [7, 3.2.14]). We will not prove it here, since this result would take us away from the main themes of the present paper.

The main obstacles to the nonselfadjoint variant of weak Morita equivalence presented here, that we see at this point, are twofold. First, it is not clear, and probably is not true in general, that a dual unital operator algebra M is always tightly w^*-Morita equivalent to $M\bar{\otimes}B(H)$, if H is an infinite-dimensional Hilbert space. This is because the space $Y = M\bar{\otimes}H^c$, the 'first column' of $M\bar{\otimes}B(H)$, is not a rigged module over M, in general, unlike the W^*-algebra case. Presumably this latter deficiency may be fixed by considering weaker forms of the rigged module definition. However, this will not really help: this very natural M-module Y is not even selfdual—there may exist completely bounded M-module maps from Y to M which are not given by 'left multiplication with a row in $M\bar{\otimes}H^r$', where the latter space is the 'first row' of $M\bar{\otimes}B(H)$. An example of such is easy to construct in the case that M is the subalgebra of $M_2(B(\ell^2))$ with 0 in the 2-1 entry, scalars on the main diagonal, and an element from $B(\ell^2)$ in the 1-2 entry. This shows that any decent theory of selfdual modules over nonselfadjoint algebras has to either exclude such examples, or replace completely bounded M-module maps by w^*-continuous ones (which somewhat defeats the point of 'selfduality'), or perhaps by multipliers in the sense of the second part of [5]. The second obstacle is it seems not to be true in full generality, that the second dual of a strong Morita equivalence A–B-bimodule in the sense of [9], is a tight w^*-equivalence A^{**}–B^{**}-bimodule in the sense above.

Some of these problems are easily resolvable, at the expense of introducing other problems, if one instead uses a different approach to w^*-Morita theory. In fact there are several such alternative approaches. First, one could vary the theory above by allowing the 'special approximate identities' found in the theory to converge in the point-w^* topology as opposed to the point-norm topology: for example $\sum_{k=1}^{n_\alpha} y_k^\alpha(x_k^\alpha, y) \to y$ in the w^* topology for all $y \in Y$. Second, another completely different approach is to base the entire theory on a (not yet developed) nonselfadjoint dual operator algebra variant of the Haagerup module tensor product (cf. [24]). However, both of these approaches seems to present other, different, problems. For example, it seems certain that one cannot obtain, by such approaches, analogues of many of our results here.

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