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Operator systems and C*-extreme points

by

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Abstract. We study operator systems over a von Neumann algebra $\mathcal{C}$ that can be represented as concrete systems of operator-valued weak* continuous $\mathcal{C}'$-affine maps. In the case $\mathcal{C} = \mathbb{B}(\mathcal{H})$ such systems are shown to be equivalent to the usual operator systems, and an application of this equivalence to C*-convex sets is presented.

1. Introduction. By the Kadison function representation each *operator system* (that is, each closed unital self-adjoint subspace V of a C*-algebra) can be represented as the space of all continuous complex-valued affine functions on a compact convex set, namely on the state space $S(V)$ [15], [27, 8.5.4]. Choi and Effros [5] gave an abstract characterization of such systems. In operator space theory [3], [9], [23], there often appear sets that are convex over a unital C*-algebra A, that is, subsets K of $\mathbb{B}(\mathcal{K})$ for a Hilbert A-module $\mathcal{K}$, which are A-*convex* in the sense that $\sum_{j=1}^{n} a_j^* y_j a_j \in K$ whenever $y_j \in K$ and $a_j \in A$ are such that $\sum_{j=1}^{n} a_j^* a_j = 1$. (By a *Hilbert A-module* we mean a Hilbert space on which A is represented as a unital C*-algebra.) For such a set K and a Hilbert A-module $\mathcal{H}$ it is natural to consider weak* continuous maps $f : K \to \mathbb{B}(\mathcal{H})$ that are A-*affine* in the sense that

$$f\Big(\sum_{j=1}^{n} a_j^* y_j a_j\Big) = \sum_{j=1}^{n} a_j^* f(y_j) a_j \text{ for all } y_j \in K \text{ and } a_j \in A \text{ with } \sum_{j=1}^{n} a_j^* a_j = 1.$$

The operator system $\mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$ consisting of all such maps is also an operator bimodule over $\mathcal{C} := \mathbb{B}_A(\mathcal{H})$ (the commutant of the image of A in $\mathbb{B}(\mathcal{H})$), thus $\mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$ is an *operator $\mathcal{C}$-system* in the sense of [23, p. 215].

After some preparatory results in Section 2, we characterize abstractly in Section 3 all operator systems over a von Neumann algebra $\mathcal{C}$ that can be represented in the form $\mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$. In the special case when $\mathcal{C} = \mathbb{B}(\mathcal{L})$

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for a Hilbert space $\mathcal{L}$, we show that the category of such systems (with unital completely positive $\mathbb{B}(\mathcal{L})$-bimodule maps as morphisms) is equivalent to the category of usual operator systems. In Section 4 we apply this equivalence to study operator convex sets.

The Krein–Milman theorem (that each compact convex set in a locally convex linear topological space is equal to the closure of the convex hull of its extreme points) is one of the fundamental results of classical functional analysis. It is natural to try to find an appropriate generalization of this theorem to the wider context of A-convex sets for a general unital C*-algebra A, where the notion of extreme point is suitably modified. So far such attempts have been successful (see [10], [11], [12], [22], [28]) mainly in the cases when the coefficients are taken from finite-dimensional W*-algebras $\mathcal{A}$. For an infinite-dimensional abelian $\mathcal{A}$ without minimal projections, however, there exist $\mathcal{A}$-convex weak* compact sets which have no $\mathcal{A}$-extreme points [21]. Therefore, in trying to formulate a possible non-commutative version of the Krein–Milman theorem, some restrictions on the C*-algebras of coefficients (or on the sets considered) are needed. Here we will prove such a theorem for weak* compact A-convex sets when A contains a large ideal of compact operators.

A convex set K in a vector space V is *in basic position* if for $x, y \in K$ and any positive real t the equality $tx = ty$ implies that $x = y$. It can always be achieved that K is in basic position (for example, by replacing K with $K \times 1 \subset V \times \mathbb{C}$). Then, if $\tilde{K}$ is the convex envelope of $K \cup \{0\}$, the extreme points of $\tilde{K}$ are just the extreme points of K and the point 0. If S is the state space of an operator system V, then $\tilde{S}$ is the quasi-state space of V, and passage from S to $\tilde{S}$ introduces only one additional extreme point. In operator-valued state spaces the situation turns out to be rather different. The Krein–Milman type problem for operator-valued quasi-state spaces turns out to be somewhat more tractable than for operator-valued state spaces. In the last section we characterize operator-valued quasi-state spaces among weak* compact $\mathcal{A}$-convex sets.

2. Preliminaries. In this paper, C*-algebras are usually assumed to be unital and are typically denoted by $A, B, C, \ldots$, while von Neumann algebras by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$.

A Hilbert A-module $\mathcal{H}$ is called *faithful* if the underlying representation $A \to \mathbb{B}(\mathcal{H})$ is injective. Further, $\mathcal{H}$ is *cyclic* if there exists a vector $\xi \in \mathcal{H}$ such that $\mathcal{H} = [A\xi]$ (where $[\cdot]$ denotes the closure of the linear span). The set of all bounded A-module maps on $\mathcal{H}$ is denoted by $\mathbb{B}_A(\mathcal{H})$.

A Hilbert module $\mathcal{H}$ over a von Neumann algebra $\mathcal{A}$ is called *normal* if the corresponding representation $\mathcal{A} \to \mathbb{B}(\mathcal{H})$ is weak* continuous. A normal Hilbert $\mathcal{A}$-module $\mathcal{H}$ is *W^*-universal* if $\mathcal{H}$ contains a copy of each cyclic normal Hilbert $\mathcal{A}$-module. If in addition $\mathcal{H}$ is W*-universal also over $\mathbb{B}_{\mathcal{A}}(\mathcal{H})$, then $\mathcal{H}$ will be called *standard*. (The existence of even more specific Hilbert modules is well-known [26, Section IX.1].)

By an *operator C-bimodule* we mean a norm closed subspace Y of $\mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert C-module, such that $CYC \subseteq Y$. If $\mathcal{C}$ is a von Neumann algebra and Y is a weak* closed $\mathcal{C}$-subbimodule of $\mathbb{B}(\mathcal{H})$ for a normal Hilbert $\mathcal{C}$-module $\mathcal{H}$, then Y is called a *normal dual operator $\mathcal{C}$-bimodule*. (For abstract characterizations of operator bimodules see e.g. [3] or [23].) We denote by $\mathrm{CB}_C(X, Y)$ the space of all completely bounded (c.b.) C-bimodule maps from X to Y, and by $\mathrm{NCB}_C(X, Y)$ the space of all weak* continuous such maps. (If X and Y are Hilbert C-modules, it is known that these are just bounded bimodule maps.)

For a normal Hilbert $\mathcal{C}$-module $\mathcal{H}$ and a norm closed $\mathcal{C}$-subbimodule $X \subseteq \mathbb{B}(\mathcal{H})$ let $\overline{X}^{\mathcal{C}}$ be the set consisting of those $b \in \mathbb{B}(\mathcal{H})$ for which there exist two (infinite) families (e_i) and (f_j) of projections in $\mathcal{C}$ with $\sum_i e_i = 1 = \sum_j f_j$ such that $e_i b f_j \in X$ for all i, j. Then $\overline{X}^{\mathcal{C}}$ is a $\mathcal{C}$-subbimodule of $\mathbb{B}(\mathcal{H})$ containing X ([17, 2.2], [18, 3.10]). If $\overline{X}^{\mathcal{C}} = X$, then X is called a *strong $\mathcal{C}$-bimodule*; in general $\overline{X}^{\mathcal{C}}$ is the smallest strong operator $\mathcal{C}$-bimodule containing X. (The class of strong $\mathcal{C}$-bimodules includes, for example, all weak* closed $\mathcal{C}$-bimodules.)

A *faithful operator C-system* is a self-adjoint operator C-bimodule $X \subseteq \mathbb{B}(\mathcal{H})$ containing the identity operator 1, where $\mathcal{H}$ is a faithful Hilbert C-module. (For an abstract characterization see [23, 15.13].) Faithfulness of the representation $C \to \mathbb{B}(\mathcal{H})$ implies that the map $c \mapsto c1$ is injective, hence C is regarded as a subset of X. The set of all contractive completely positive (c.c.p.) C-bimodule maps from X to Y is denoted by $\mathrm{CCP}_C(X, Y)$, and the subset of all unital such maps by $\mathrm{UCP}_C(X, Y)$. If $\mathcal{C}$ is a von Neumann algebra, then a faithful operator $\mathcal{C}$-system is called *strong* if it is strong as an operator $\mathcal{C}$-bimodule. The proof of the following proposition is simple (details are in [20, 3.4]).

PROPOSITION 2.1. *Let $\mathcal{H}$ be a faithful Hilbert A-module, $\mathcal{C} = \mathbb{B}_A(\mathcal{H})$ and K a weak* compact A-convex subset of $\mathbb{B}(\mathcal{K})$ for some Hilbert A-module $\mathcal{K}$. Then $X := \mathbb{A}_A^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H}))$ is a faithful strong operator $\mathcal{C}$-system, where for each $n \in \mathbb{N}$ the norm on $\mathbb{M}_n(X) = \mathbb{A}_A^{\mathrm{w}*}(K, \mathbb{B}(\mathcal{H}^n))$ is defined by $\|f\| = \sup_{y \in K} \|f(y)\|$.*

Let $\mathcal{H}$ be a faithful normal Hilbert $\mathcal{A}$-module, $\mathcal{C} = \mathbb{B}_{\mathcal{A}}(\mathcal{H})$, Y any operator $\mathcal{A}$-bimodule, and consider the following analogy of the dual space of Y:

$$Y^{\natural} := \mathrm{CB}_{\mathcal{A}}(Y, \mathbb{B}(\mathcal{H})).$$

Denote by $\mathbb{T}(\mathcal{H})$ the space of all trace class operators on $\mathcal{H}$ with the usual trace norm $\|\cdot\|_1$. (Here $\mathcal{H}$ may be regarded merely as a Hilbert space, thus an operator $a \in \mathbb{B}(\mathcal{H})$ belongs to $\mathbb{T}(\mathcal{H})$ if and only if a is compact and the sum of the eigenvalues of $|a|$, denoted by $\|a\|_1$, is finite. In our context, where $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module, $\mathbb{T}(\mathcal{H})$ is also a Banach $\mathcal{A}$-bimodule as a predual of $\mathbb{B}(\mathcal{H})$, but this bimodule structure will not be important in our considerations.) Let $\ell^1_Y(\mathbb{T}(\mathcal{H}))$ be the space of all maps f from Y into the trace class operators $\mathbb{T}(\mathcal{H})$ such that $\|f\| := \sum_{y\in Y}\|f(y)\|_1 < \infty$. Observe that $Y^\natural$ is a weak* closed $\mathcal{C}$-subbimodule of $\ell^\infty_Y(\mathbb{B}(\mathcal{H})) = (\ell^1_Y(\mathbb{T}(\mathcal{H})))^\sharp$, where the bimodule action of $\mathcal{C}$ on $Y^\natural$ is given by

$$(c\varphi d)(y) := c\varphi(y)d \quad (\varphi \in Y^\natural,\ y \in Y,\ c,d \in \mathcal{C}).$$

Thus $Y^\natural$ is a normal dual operator $\mathcal{C}$-bimodule. If Y is strong as an $\mathcal{A}$-bimodule, then by [19, 5.1],

$$(Y^\natural)_\natural := \mathrm{NCB}_{\mathcal{C}}(Y^\natural, \mathbb{B}(\mathcal{H})) = Y \tag{2.1}$$

in the sense that the map $y \mapsto \varepsilon_y$, where $\varepsilon_y \in (Y^\natural)_\natural$ is the evaluation $Y^\natural \ni \varphi \overset{\varepsilon_y}{\longmapsto} \varphi(y)$, is a completely isometric $\mathcal{A}$-bimodule map from Y onto $(Y^\natural)_\natural$.

We will also need the following simple variation of [5, 2.2] (proved in [20, 6.6]).

Lemma 2.2. *Let $\mathcal{H}$ be a Hilbert C-module, $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ and let X be a faithful operator C-system. Then for each $\psi \in \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$ there exist a positive contraction $a \in \mathcal{A}$ and $\rho \in \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$ such that $\psi(x) = a\rho(x)a$ $(x \in X)$.*

We denote by $\overline{G}$ the weak* closure and by $\overline{\overline{G}}$ the norm closure of a set G.

3. Operator $\mathcal{C}$-systems as spaces of C*-affine maps. Let X be an operator system over a von Neumann algebra $\mathcal{C}$, $\mathcal{H}$ a W*-universal Hilbert $\mathcal{C}$-module, $\mathcal{A} = \mathbb{B}_{\mathcal{C}}(\mathcal{H})$, $S := \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and $X^\natural := \mathrm{CB}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Note that S is an $\mathcal{A}$-convex weak* compact set in $X^\natural$.

Lemma 3.1. *Each weak* continuous $\mathcal{A}$-affine function $f : S \to \mathbb{B}(\mathcal{H})_h$ can be extended uniquely to an element $\tilde{f} \in \mathrm{NCB}_{\mathcal{A}}(X^\natural, \mathbb{B}(\mathcal{H}))$.*

Proof. The extension is unique since S spans $X^\natural$ by Lemma 2.2 and the fact that completely bounded bimodule maps are linear combinations of c.p. maps [23]. To prove the existence, we first extend f to a map $\check{f} : \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H})) \to \mathbb{B}(\mathcal{H})_h$ as follows. By Lemma 2.2 we can represent each map $\varphi \in \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ as

$$\varphi = \sum_{i=1}^m a_i^*\varphi_i a_i, \qquad \text{where } \varphi_i \in S,\ a_i \in \mathcal{A},\ \sum_{i=1}^m a_i^* a_i \le 1. \tag{3.1}$$

(Here we could take $m = 1$, but it is more convenient to admit a general m.) Set

$$\check{f}(\varphi) := \sum_{i=1}^{m} a_i^* f(\varphi_i) a_i. \tag{3.2}$$

To show that $\check{f}$ is well-defined, let $\varphi = \sum_{j=1}^{n} b_j^* \psi_j b_j$ be another such representation. Then

$$a := \sum_{i=1}^{m} a_i^* a_i = \varphi(1) = \sum_{j=1}^{n} b_j^* b_j \leq 1.$$

Set $b = (1-a)^{1/2}$, choose any $\theta \in S$ and note that

$$\sum_{i=1}^{m} a_i^* \varphi_i a_i + b\theta b = \varphi + b\theta b = \sum_{j=1}^{n} b_j^* \psi_j b_j + b\theta b. \tag{3.3}$$

Since $\sum_{i=1}^{m} a_i^* a_i + b^2 = 1 = \sum_{j=1}^{n} b_j^* b_j + b^2$, we have in (3.3) two representations of the map $\varphi + b\theta b$ as $\mathcal{A}$-convex combinations of elements of S. Since f is $\mathcal{A}$-affine, it follows that

$$\sum_{i=1}^{m} a_i^* f(\varphi_i) a_i + bf(\theta)b = \sum_{j=1}^{n} b_j^* f(\varphi_j) b_j + bf(\theta)b.$$

Canceling out the term $bf(\theta)b$ now shows that the definition (3.2) is unambiguous.

It is easy to verify that $\check{f}(0) = 0$ and that $\check{f}$ is $\mathcal{A}$-affine, hence affine and homogeneous. This enables us to extend $\check{f}$ unambiguously to a map $\tilde{f} : X^\natural \to \mathbb{B}(\mathcal{H})$ by complex linearity, since each element of $X^\natural$ is of the form $\sum_{k=0}^{3} i^k t_k \varphi_k$, where $\varphi_k \in \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and $t_k \in \mathbb{R}_+$. Then $\tilde{f}$ is $\mathbb{C}$-linear, involution preserving and

$$\tilde{f}\Big(\sum_{j=1}^{n} a_j^* \varphi_j a_j\Big) = \sum_{j=1}^{n} a_j^* \tilde{f}(\phi_j) a_j \quad \text{for all } \varphi_j \in X^\natural,\ a_j \in \mathcal{A} \text{ and } n \in \mathbb{N}.$$

It follows from this and the polarization identity

$$a^* \varphi b = \frac{1}{4} \sum_{k=0}^{3} (-i)^k (a + i^k b)^* \varphi (a + i^k b) \quad (\text{where } i = \sqrt{-1}) \tag{3.4}$$

that $\tilde{f}(a^* \varphi b) = a^* \tilde{f}(\varphi) b$ for all $a, b \in \mathcal{A}$ and $\varphi \in X^\natural$, thus $\tilde{f}$ is an $\mathcal{A}$-bimodule map.

To prove that $\tilde{f}$ is weak* continuous, by the Krein–Shmul'yan theorem it suffices to show that the restriction of $\tilde{f}$ to the unit ball $\mathrm{CC}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ of $X^\natural$ is weak* continuous. By linearity and continuity of the involution, this reduces to the weak* continuity at 0 of the restriction of $\tilde{f}$ to the self-adjoint part $\mathrm{CC}^h_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ of the unit ball. So, let $(\varphi_k) \subseteq \mathrm{CC}^h_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ be a net weak* converging to 0. We must show that $\tilde{f}(\varphi_k) \to 0$ in the weak* topology. Suppose the contrary; then (by the weak* compactness of closed balls in $\mathbb{B}(\mathcal{H})$) passing to a subnet we may assume that the net $(\tilde{f}(\varphi_k))$ converges to an operator $b \in \mathbb{B}(\mathcal{H})$, $b \neq 0$. Let $\varphi_k = \psi_k - \sigma_k$ be a decomposition of φ_k into maps $\psi_k, \sigma_k \in \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Using the weak* compactness of $\mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and passing to subnets again we may assume that ψ_k and σ_k converge to some maps $\psi, \sigma \in \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Since $\lim \varphi_k = 0$, we have $\psi = \sigma$. If the restriction $\tilde{f}|\mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ is weak* continuous, then we get $b = \lim \tilde{f}(\varphi_k) = \lim \tilde{f}(\psi_k) - \lim \tilde{f}(\sigma_k) = \psi - \sigma = 0$, a contradiction. Thus it suffices to prove that $\tilde{f}|\mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ is weak* continuous.

Assume, on the contrary, that there exists a net $(\varphi_k) \subseteq \mathrm{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ weak* converging to a map φ such that the net $(\tilde{f}(\varphi_k))$ does not converge to $\tilde{f}(\varphi)$. Passing to a subnet, we may assume that $(\tilde{f}(\varphi_k))$ converges to an operator $d \in \mathbb{B}(\mathcal{H})$, $d \neq \tilde{f}(\varphi)$. Recall that on bounded subsets of $X^\natural$ the weak* topology coincides with the point weak operator topology (which is defined by seminorms $\varphi \mapsto |\omega(\varphi(x))|$, where $x \in X$ and ω is a weak operator continuous linear functional on $\mathbb{B}(\mathcal{H})$), and that the latter topology has the same continuous functionals as the point strong operator topology (p.s.o.t.). Therefore, replacing the maps φ_k by suitable convex combinations of the form $\sum_{j \geq k} t_j \varphi_j$ ($\sum_{j \geq k} t_j = 1$, $t_j \geq 0$, only finitely many t_j non-zero), we may assume that (φ_k) converges to φ in the p.s.o.t. and $(\tilde{f}(\varphi_k))$ still converges to d in the weak* topology (since the weak* topology is locally convex). Let $a_k = \varphi_k(1)^{1/2}$ and $a = \varphi(1)^{1/2}$; note that these are elements of $\mathcal{A}$ since φ and φ_k are $\mathcal{C}$-bimodule maps. (Namely, $\varphi(1)c = \varphi(c) = c\varphi(1)$ for all $c \in \mathcal{C}$, hence $\varphi(1)$ is in the commutant $\mathcal{A}$ of $\mathcal{C}$ in $\mathbb{B}(\mathcal{H})$.) Then in particular a_k converges to a in the s.o.t. and $b_k := (1 - a_k^2)^{1/2}$ converges to $b := (1 - a^2)^{1/2}$. As in Lemma 2.2 we may write

$$\varphi_k = a_k \psi_k a_k \quad \text{and} \quad \varphi = a\psi a,$$

where $\psi_k, \psi \in S = \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Passing to a subnet, we may assume that the weak* limit $\sigma := \lim \psi_k$ exists. Then $\lim \varphi_k = \lim a_k \psi_k a_k = a\sigma a$, and since also $\lim \varphi_k = \varphi = a\psi a$, we conclude that

$$a\sigma a = a\psi a.$$

Now choose any $\theta \in S$ and consider the maps

$$\tau_k := a_k \psi_k a_k + b_k \theta b_k = \varphi_k + b_k \theta b_k.$$

Since $\tau_k \in S$ and $\tilde{f}|S = f$ is weak* continuous and $\mathcal{A}$-affine, and since τ_k weak* converges to $a\psi a + b\theta b =: \tau$, it follows from the definition of $\tilde{f}$ that

$$\tilde{f}(\varphi_k) + b_k \tilde{f}(\theta) b_k = a_k f(\psi_k) a_k + b_k f(\theta) b_k = f(\tau_k) \tag{3.5}$$
$$\xrightarrow{\text{weak}^*} f(\tau) = a f(\psi) a + b f(\theta) b = \tilde{f}(\varphi) + b\tilde{f}(\theta) b.$$

Since b_k converges to b in the s.o.t., $b_k\tilde{f}(\theta)b_k$ weak* converges to $b\tilde{f}(\theta)b$, hence it follows from (3.5) that $\tilde{f}(\varphi_k) \to \tilde{f}(\varphi)$ in the weak* topology. This contradicts $\tilde{f}(\varphi_k) \to d \neq \tilde{f}(\varphi)$. Thus $\tilde{f}$ is weak* continuous.

Finally, since $\tilde{f}$ is weak* continuous, it is bounded. Moreover, since by hypothesis $\mathcal{H}$ is W*-universal as a normal Hilbert $\mathcal{C}$-module, all normal states on $\mathcal{C}$ are vector states. Hence $\tilde{f}$ (as an $\mathcal{A}$-bimodule map, where $\mathcal{A} = \mathcal{C}'$) is completely bounded with $\|\tilde{f}\|_{\mathrm{cb}} = \|\tilde{f}\|$ by [24, 2.3, 2.2, 2.1]. ∎

Together with Proposition 2.1 the following theorem characterizes strong operator systems. The smaller class of dual systems has been characterized in [4].

THEOREM 3.2. *Let $\mathcal{H}$ be a W*-universal Hilbert $\mathcal{C}$-module, X a faithful strong operator $\mathcal{C}$-system, $\mathcal{A} = \mathbb{B}_{\mathcal{C}}(\mathcal{H})$ and $S = \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$. Then S is a weak* compact $\mathcal{A}$-convex subset of $X^{\natural} = \mathrm{CB}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and each $x \in X$ defines an $\mathcal{A}$-affine weak* continuous function $\hat{x} : S \to \mathbb{B}(\mathcal{H})$ by $\hat{x}(\rho) := \rho(x)$. The map*

$$\kappa : X \to \mathbb{A}_{\mathcal{A}}^{\mathrm{w}^*}(S, \mathbb{B}(\mathcal{H})), \qquad \kappa(x) = \hat{x},$$

is a unital, surjective, completely contractive homomorphism of operator $\mathcal{C}$-bimodules, and the restriction of κ to the self-adjoint part X_h of X is isometric.

Proof. Since κ is evidently involution preserving (because c.p. maps $\rho \in S$ are) and $X = \mathrm{NCB}_{\mathcal{A}}(X^{\natural}, \mathbb{B}(\mathcal{H}))$ by (2.1), it follows from Lemma 3.1 that κ is surjective. We will prove here only that $\kappa|X_h$ is isometric since the other assertions are easy to verify. Clearly κ is a contraction. Let $\mathcal{K}$ be a normal Hilbert $\mathcal{C}$-module such that $X \subseteq \mathbb{B}(\mathcal{K})$. Given $x \in X_h$ and $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{K}$ such that $|\langle x\xi, \xi\rangle| \geq \|x\| - \varepsilon$. Since $\mathcal{H}$ is W*-universal, we may assume that $[\mathcal{C}\xi] \subseteq \mathcal{H}$. Denoting by p the projection of $\mathcal{H}$ onto $[\mathcal{C}\xi]$, we see that the map

$$\rho_0 : X \to \mathbb{B}(p\mathcal{K}), \qquad \rho_0(x) = pxp,$$

is in $\mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(p\mathcal{K}))$ since $p \in \mathcal{A}$. Now choose any $\theta \in \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(p^{\perp}\mathcal{H}))$ (say, by extending to X the map $\mathcal{C} \to \mathbb{B}(p^{\perp}\mathcal{H})$, $c \mapsto p^{\perp}c$ using [23, Ex. 8.6]) and set $\rho = \rho_0 + \theta$. Then $\rho \in \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ and

$$\|\rho(x)\| \geq \|\rho_0(x)\| = \|pxp\| \geq |\langle x\xi, \xi\rangle| \geq \|x\| - \varepsilon.$$

As this holds for all $\varepsilon > 0$, we get $\|\kappa(x)\| = \|\hat{x}\| \geq \|x\|$ for all $x \in X_h$. ∎

REMARK 3.3. Using, instead of S, all the

$$S_n = \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}^n))$$

as the target spaces, it is not hard to modify Theorem 3.2 so that the map κ becomes (completely) isometric on X, not just on X_h, but we will not need such a modification here.

For a Hilbert space $\mathcal{L}$ let us now show the equivalence of the category of strong operator $\mathbb{B}(\mathcal{L})$-systems (with u.c.p. $\mathbb{B}(\mathcal{L})$-bimodule maps as morphisms) and the category of usual operator systems (with u.c.p. maps as morphisms).

THEOREM 3.4. *Each strong operator $\mathbb{B}(\mathcal{L})$-system X is of the form $\mathbb{M}_{\mathbb{I}}(V)$ for an operator system V, where $\mathbb{I} = \dim \mathcal{L}$. Moreover, each c.p. $\mathbb{B}(\mathcal{L})$-bimodule map between two such systems $\mathbb{M}_{\mathbb{I}}(V_j)$, $j = 1, 2$, is the amplification of a map $\phi \in \mathrm{CP}_{\mathbb{C}}(V_1, V_2)$. The categories of operator systems and strong operator $\mathbb{B}(\mathcal{L})$-systems are equivalent.*

Sketch of proof. Let $(e_{i,j})_{i,j\in\mathbb{I}}$ be matrix units in $\mathbb{B}(\mathcal{L}) = \mathbb{M}_{\mathbb{I}}(\mathbb{C})$ and denote $e_i = e_{i,i}$, so that the e_i are mutually orthogonal minimal equivalent projections with sum 1. Since X is a normal operator $\mathbb{B}(\mathcal{L})$-bimodule, each $x \in X$ can be expressed as $x = \sum_{i,j} e_i x e_j$ (where the convergence is, say, in the weak* topology of an ambient $\mathbb{B}(\mathcal{K}) \supseteq X$). Set $V = e_1 X e_1$ and for each finite subset $F \subseteq \mathbb{I}$ let $e_F = \sum_{i\in F} e_i$. Then $e_F \mathbb{B}(\mathcal{L}) e_F \cong \mathbb{M}_{n_F}(\mathbb{C})$, where n_F is the cardinality of F. Elementary arguments show that $B_0 := \bigcup_{F\in\mathcal{F}} e_F \mathbb{B}(\mathcal{L}) e_F$ is an algebra consisting of finite rank operators (with $\overline{\overline{B}}_0 = \mathbb{B}_0(\mathcal{L})$, the algebra of all compact operators on $\mathcal{L}$) and that the algebraic tensor product $B_0 \odot V$ is isomorphic to the subspace $X_0 = \bigcup_{F\in\mathcal{F}} e_F X e_F$ of X by means of the map

$$\sum_{i,j\in F} \beta_{i,j} e_{i,j} \otimes v \overset{f}{\mapsto} \sum_{i,j\in F} \beta_{i,j} e_{i,1} v e_{1,j},$$

where $\beta_{i,j} \in \mathbb{C}$. Here $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{I}$. This map sends $\mathbb{M}_{n_F}(V) \cong e_F \mathbb{B}(\mathcal{L}) e_F \otimes V$ isomorphically onto $e_F X e_F$, so it can be used to transfer the norm from $e_F X e_F$ to $\mathbb{M}_{n_F}(V)$. For two subsets F and G of the same cardinality the subspaces $e_F X e_F$ and $e_G X e_G$ are unitarily equivalent, so it can be verified that the norms on $\mathbb{M}_n(V)$ are unambiguously defined and satisfy the Ruan conditions for an operator space structure on V. Then, extending f by continuity, we obtain a complete isometry from $\mathbb{B}_0(\mathcal{L}) \otimes V$ onto $\overline{\overline{X}}_0 = \mathbb{B}_0(\mathcal{L}) X \mathbb{B}_0(\mathcal{L})$ which is also an isomorphism of $\mathbb{B}(\mathcal{L})$-bimodules. By [20, 2.2] this can be extended to a completely isometric isomorphism from the strong $\mathbb{B}(\mathcal{L})$-bimodule generated by $\mathbb{B}_0(\mathcal{L}) \otimes V$, which is easily seen to be equal to $\mathbb{M}_{\mathbb{I}}(V)$, onto the strong $\mathbb{B}(\mathcal{L})$-bimodule generated by $\mathbb{B}_0(\mathcal{L}) X \mathbb{B}_0(\mathcal{L})$, which is X since X is strong and $\mathbb{B}_0(\mathcal{L})$ is dense in $\mathbb{B}(\mathcal{L})$ in the s.o.t.

Further, for any operator spaces V_1, V_2 each c.b. $\mathbb{M}_{\mathbb{I}}(\mathbb{C})$-bimodule map $\psi : \mathbb{M}_{\mathbb{I}}(V_1) \to \mathbb{M}_{\mathbb{I}}(V_2)$ is of the form $\psi([v_{i,j}]) = [\phi(v_{i,j})]$ $(v_{i,j} \in V_1)$ for a c.b. map $\phi : V_1 \to V_2$. (This follows easily from the bimodule property of ψ by using matrix units.) Finally, the assignment $V \mapsto \mathbb{M}_{\mathbb{I}}(V)$ (together with the amplification of maps) is categorical equivalence, the appropriate functor in the reverse direction is just the compression $X \mapsto e_1 X e_1$. ■

4. C*-convex sets. It is shown in [20] that for every von Neumann algebra $\mathcal{A}$ each weak* compact $\mathcal{A}$-convex set K of Hilbert space operators is naturally $\mathcal{A}$-affinely weak* homeomorphic to a set S of a special form, namely

$$K \cong S := \mathrm{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H})), \tag{4.1}$$

where $\mathcal{H}$ is a standard Hilbert $\mathcal{A}$-module,

$$\mathcal{C} := \mathbb{B}_{\mathcal{A}}(\mathcal{H}) \quad \text{and} \quad X = \mathbb{A}_{\mathcal{A}}^{\mathrm{w}^*}(K, \mathbb{B}(\mathcal{H})).$$

By Proposition 2.1, X is a strong $\mathcal{C}$-bimodule. If $\mathcal{A} = \mathbb{B}(\mathcal{L})$, then it is known that the standard Hilbert $\mathcal{A}$-module is $\mathcal{H} := \mathcal{L} \otimes \mathcal{L}$. We will identify $\mathcal{A}$ with $\mathcal{A} \otimes 1_{\mathbb{B}(\mathcal{L})}$ inside $\mathbb{B}(\mathcal{L} \otimes \mathcal{L})$, and $\mathcal{C}$ with $1_{\mathbb{B}(\mathcal{L})} \otimes \mathbb{B}(\mathcal{L})$. Since by Theorem 3.4 every strong operator $\mathbb{B}(\mathcal{L})$-system is of the form $X = \mathbb{M}_{\mathbb{I}}(V)$ for an operator system V, where $\mathbb{I} = \dim \mathcal{L}$, identifying $\mathbb{B}(\mathcal{L})$ with $\mathbb{M}_{\mathbb{I}}(\mathbb{C})$, by (4.1) we may assume that $K = \mathrm{UCP}_{\mathcal{C}}(\mathbb{M}_{\mathbb{I}}(V), \mathbb{M}_{\mathbb{I}}(\mathbb{M}_{\mathbb{I}}(\mathbb{C})))$, where $\mathcal{C} \cong \mathbb{M}_{\mathbb{I}}(\mathbb{C})$. Thus from Theorem 3.4 we deduce the following corollary.

Corollary 4.1. *Each weak* compact $\mathbb{B}(\mathcal{L})$-convex subset K of $\mathbb{B}(\mathcal{H}_0)$, where $\mathcal{H}_0$ is a normal Hilbert $\mathbb{B}(\mathcal{L})$-module, is equivalent to one of the form*

$$S := \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L})). \tag{4.2}$$

If $\mathcal{H}_0$ is separable, then V is separable.

Proof. We still need to prove the last sentence of the corollary. By hypothesis we have the inclusion $S \cong K \subset \mathbb{B}(\mathcal{H}_0)$. Composing this with the injection $x \mapsto \left(\frac{1}{2}(x + x^*), \frac{1}{2i}(x - x^*), 1\right)$ of $\mathbb{B}(\mathcal{H}_0)$ into $\mathbb{B}(\mathcal{K})$, where $\mathcal{K} = \mathcal{H}_0^3$, we obtain a $\mathbb{B}(\mathcal{L})$-affine weak* continuous injection $\kappa_0 : S \to \mathbb{B}(\mathcal{K})_h$. By the same arguments as in the proof of Lemma 3.1 we can extend κ_0 to a weak* continuous involution preserving $\mathbb{B}(\mathcal{L})$-bimodule map

$$\kappa : \mathrm{CB}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L})) \to \mathbb{B}(\mathcal{K}).$$

To show that κ is injective, it suffices to prove that $\kappa(\psi_2 - \psi_1) = 0$ implies $\psi_2 - \psi_1 = 0$, where $\psi_i \in \mathrm{CB}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L}))$ are completely positive contractions (since κ is involution preserving and each self-adjoint map in $\mathrm{CB}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L}))$ is a difference of two completely positive maps). By Lemma 2.2, ψ_i is of the form $\psi_i = a_i y_i a_i$ for positive contractions $a_i \in \mathbb{B}(\mathcal{L})$ and

$$y_i \in \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L})) = S.$$

Then from $a_2\kappa(y_2)a_2 = \kappa(\psi_2) = \kappa(\psi_1) = a_1\kappa(y_1)a_1$ we see (by noting that the third components of $\kappa_0(y_2)$ and $\kappa_0(y_1)$ are 1) that $a_2^2 = a_1^2$, hence $a_2 = a_1$. Now choose any $y_0 \in \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L}))$ and denote $a := a_1 = a_2$ and $b := \sqrt{1 - a^2}$. By the $\mathbb{B}(\mathcal{L})$-affinity of κ_0 we have $\kappa_0(ay_2a + by_0b) = a\kappa_0(y_2)a + b\kappa_0(y_0)b = \kappa(\psi_2) + b\kappa_0(y_0)b = \kappa(\psi_1) + b\kappa_0(y_0)b = \kappa_0(ay_1a + by_0a)$. By the injectivity of κ_0 this implies that $ay_2a + by_0b = ay_1a + by_0b$, hence $ay_2a = ay_1a$, that is, $\psi_2 = \psi_1$. This proves that κ is injective. Since κ is injective, its pre-adjoint $\mathbb{T}(\mathcal{K}) \to \mathbb{T}(\mathcal{L}) \hat{\otimes} V$ has a norm dense range (see [3, (1.51)] for an explanation of $\hat{\otimes}$). So V must be separable since $\mathbb{T}(\mathcal{K})$ is separable. ∎

Such a description of convex sets can be used to study generalized extreme points.

DEFINITION 4.2. A point $y \in K$ is called an A-*extreme* point of K if the condition

$$y = \sum_{j=1}^{n} a_j^* y_j a_j, \quad y_j \in K, \sum_{j=1}^{n} a_j^* a_j = 1 \ (n \text{ finite}), \tag{4.3}$$

where $a_j \in A$ are invertible, implies that there exist unitary elements $u_j \in A$ such that $y_j = u_j^* y u_j$. If (4.3), where $a_j \in A$ are assumed to be positive and invertible, implies that $y_j = y$ and $a_j y = y a_j$ for all j, then y is called a *strong A-extreme point* of K.

Using the polar decomposition we easily deduce that strong A-extreme points are A-extreme. Although in a general A-convex weak* compact set there may not be enough strong A-extreme points (even if $A = \mathbb{B}(\mathcal{H})$), this turns out to be a useful auxiliary notion.

A-extreme points have been introduced by Loebl and Paulsen [16] for subsets of A under the name of C*-extreme points. Later, similar concepts for matrix state spaces of operator systems and matricially convex sets have been studied e.g. in [10], [11], [12], [13], [22], [28], and more recently in [14] in the context of quantum information theory. In particular, when the coefficients in the definition of convexity are taken from finite-dimensional matrix algebras A, appropriate versions of the non-commutative Krein–Milman theorem have already been proved in [22], [28], [10], [11]. In [12] Farenick and Zhou obtained a precise characterization of $\mathbb{B}(\mathcal{L})$-extreme points of sets of the form (4.2) in the case when V is a C*-algebra and $\mathcal{L}$ is finite-dimensional. In the case $A = \mathbb{B}(\ell^2_{\mathbb{N}})$ we can use Corollary 4.1 and recent theory ([1], [7], [8]) of completely positive maps with the unique extension property to show that weak* compact A-convex sets have enough A-extreme points of a special kind.

Let B be the C*-algebra generated by an operator system V. Then, as defined in [1, 2.1], a map $\varphi \in S = \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L}))$ has the *unique extension property* if it can be extended to a representation $\pi : B \to \mathbb{B}(\mathcal{L})$ and π is the only c.p. extension of φ to B. If in addition π is irreducible, then π is called a *boundary representation for V*. By [7], boundary representations of B for V *completely norm* the system V in the sense that

$$\|[v_{i,j}]\| = \sup_{\pi} \|[\pi(v_{i,j})]\|$$

for each matrix $[v_{i,j}] \in \mathbb{M}_n(V)$ $(n \in \mathbb{N})$, where the supremum is over all boundary representations π.

If V (hence also B) is separable, then the Hilbert spaces of all boundary representations are also separable, hence we may assume that they are all contained in a fixed separable Hilbert space $\mathcal{L}$. The infinite-dimensional ones among such representations can be regarded as unital maps. On the other hand, if π is a finite-dimensional boundary representation, then the direct sum $\pi^{(\aleph_0)}$ can be regarded as a unital map into $\mathbb{B}(\mathcal{L})$, and $\pi^{(\aleph_0)}|V$ still has the unique extension property by [2, 2.2]. Thus, we have the following statement:

(S) *For a separable V, boundary representations of $B = C^*(V)$ into $\mathbb{B}(\ell^2_{\mathbb{N}})$ completely norm V.*

DEFINITION 4.3. A subset K_0 of a weak* compact A-convex set K *generates K over A* if K is equal to the weak* closure of the A-convex hull of K_0.

The above statement (S) can be used to prove that each set of the form (4.2) is generated by its $\mathbb{B}(\mathcal{L})$-extreme points, if V is separable, since maps in S with the unique extension property are strong $\mathbb{B}(\mathcal{L})$-extreme points of S; they are in fact a special kind of such points, called Choquet $\mathbb{B}(\mathcal{L})$-points in [20]. However, to deduce that the set K_0 of all such points generates S, we need to consider S in the form $S_{\mathbb{N}} := \mathrm{UCP}_{\mathbb{M}_{\mathbb{N}}(\mathbb{C})}(\mathbb{M}_{\mathbb{N}}(V), \mathbb{M}_{\mathbb{N}}(\mathbb{B}(\mathcal{L})))$. This is because $\mathcal{L}^{\mathbb{N}} \cong \mathcal{L} \otimes \mathcal{L}$ is the standard Hilbert module over $\mathbb{B}(\mathcal{L})$, which allows us to apply [20, Theorem 7.2] and conclude that the copy of K_0 in $S_{\mathbb{N}}$ generates $S_{\mathbb{N}}$ since it norms $\mathbb{M}_{\mathbb{N}}(V)$. If $\varphi \in S$ has the unique extension property and $\pi : B := C^*(V) \to \mathbb{B}(\mathcal{L})$ is the representation extending φ, then the amplification $\varphi_{\mathbb{N}} : \mathbb{M}_{\mathbb{C}}(V) \to \mathbb{M}_{\mathbb{N}}(\mathbb{B}(\mathcal{L}))$ has an extension $\pi_{\mathbb{N}}$ defined on $\mathbb{M}_{\mathbb{N}}(B)$.

However, since the space $\mathbb{M}_{\mathbb{N}}(B)$ is not necessarily an algebra, we will use the universal von Neumann envelope B^{**} of B. If $\bar{\pi}$ is the unique normal extension of π [15, Vol. 2, Section 10.1], then the amplification $\bar{\pi}_{\mathbb{N}} : \mathbb{M}_{\mathbb{N}}(B^{**}) \to \mathbb{M}_{\mathbb{N}}(\mathbb{B}(\mathcal{L}))$ is the only weak* continuous extension of $\varphi_{\mathbb{N}}$ to a c.c.p. map from $\mathbb{M}_{\mathbb{N}}(B^{**})$ into $\mathbb{M}_{\mathbb{N}}(\mathbb{B}(\mathcal{L}))$. Indeed, any such extension fixes elements of $\mathbb{M}_{\mathbb{N}}(\mathbb{C})$, hence must be an $\mathbb{M}_{\mathbb{N}}(\mathbb{C})$-bimodule map by the multiplicative domain argument (see [23, 3.18] or [25, 2.1.5]) and is therefore of the form $\psi_{\mathbb{N}}$ for a c.p. map $\psi : B^{**} \to \mathbb{B}(\mathcal{L})$. Since ψ must extend φ, and φ has the unique extension property, we see that $\psi|B = \pi$ and then $\psi = \bar{\pi}$ by weak* continuity. Thus, under the identification of S with $S_{\mathbb{N}}$, maps with the unique extension property correspond to maps that have unique weak* continuous extensions to $\mathbb{M}_{\mathbb{N}}(B^{**})$, and these extensions are multiplicative. Since maps in S with the unique extension property completely norm V, the corresponding maps in $S_{\mathbb{N}}$ norm $\mathbb{M}_{\mathbb{N}}(V)$ and it follows from [20, 7.2] that they generate $S_{\mathbb{N}}$. Because of the equivalence $S_{\mathbb{N}} \cong S$ as weak* compact $\mathbb{B}(\mathcal{L})$-convex sets we have thus deduced the following proposition:

PROPOSITION 4.4. *If V is a separable operator system, then the set*

$$S = \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{B}(\ell^2_{\mathbb{N}}))$$

is generated by maps with the unique extension property, hence also by its strong $\mathbb{B}(\mathcal{L})$-extreme points.

The following example shows that statement (S) does not hold if $\ell^2_{\mathbb{N}}$ is replaced by $\mathbb{C}^n$ and V is an operator system inside $\mathbb{M}_n(\mathbb{C})$ for a finite n.

EXAMPLE 4.5. Let $D_0 = \mathbb{M}_{n-1}(\mathbb{C})$ ($n \geq 3$), V_0 an irreducible operator subsystem of D_0, and $\tau : V_0 \to \mathbb{C}$ a state that has more than one extension to a state on D_0 and is such that $\ker \tau$ generates D_0 as a C*-algebra. Let $V = \{(v, \tau(v)) : v \in V_0\} \subseteq D_0 \oplus \mathbb{C}$. (As a concrete example, V_0 may be the linear span of the identity and the matrices $e_{1,j}$, $e_{j,1}$ ($j = 2, \ldots, n-1$) from the set of the usual matrix units $e_{i,j}$ of $\mathbb{M}_{n-1}(\mathbb{C})$. For τ we can take the map $v \mapsto \langle v\varepsilon_j, \varepsilon_j\rangle = \langle v\varepsilon_1, \varepsilon_1\rangle$, where $\{\varepsilon_1, \ldots, \varepsilon_{n-1}\}$ is the standard basis of $\mathbb{C}^{n-1}$.) Then $D := C^*(V)$ contains $C^*(\ker\tau \oplus 0) = D_0 \oplus 0$ and also 1. Since $D \subseteq D_0 \oplus \mathbb{C}$, it follows that $D = D_0 \oplus \mathbb{C}$. Now the only two irreducible representations of D are the projections π_j on the two summands. Since τ has more than one extension to a state on D_0, the projection $\pi_2 : D = D_0 \oplus \mathbb{C} \to \mathbb{C}$ cannot be a boundary representation for V. (Indeed, if $\theta : D_0 \to \mathbb{C}$ is any state extending τ, then the u.c.p. map $\theta\pi_1 : D \to \mathbb{C}$ extends $\pi_2|V$ and $\theta\pi_1|(D_0 \oplus 0) = \theta$, hence there are at least as many c.p. extensions of $\pi_2|V$ as there are extensions θ of τ.) Since there exist boundary representations for V, it follows that π_1 must be a boundary representation for V. Moreover, since each representation of D is a direct sum of copies of π_1 and π_2, and a direct sum of maps has the unique extension property if and only if all summands do, we deduce that the only maps on V with the unique extension property are the restrictions to V of multiples of π_1. But since $\dim \pi_1 = n-1$, no such multiple can be a unital map into $\mathbb{M}_n(\mathbb{C})$. Consequently, there are no maps in $\mathrm{UCP}_{\mathbb{C}}(V, \mathbb{M}_n(\mathbb{C}))$ with the unique extension property.

If V is a C*-algebra, the $\mathbb{M}_n(\mathbb{C})$-extreme points of

$$S_n(V) := \mathrm{UCP}_{\mathbb{C}}(V, \mathbb{M}_n(\mathbb{C}))$$

have been characterized in [12, 2.1]. Using this, one can prove that the set $\mathrm{UCP}_{\mathbb{C}}(\mathbb{M}_2(\mathbb{C}), \mathbb{M}_3(\mathbb{C}))$ is not generated by its strong $\mathbb{M}_3(\mathbb{C})$-extreme points, although it is generated by its usual $\mathbb{M}_3(\mathbb{C})$-extreme points.

We will now briefly consider A-extreme points when A has large socle. By definition, a C*-algebra A has *large socle* if A contains as an essential ideal a C*-algebra isomorphic to a c_0-direct sum

$$A_0 = \bigoplus_{j\in\mathbb{J}} \mathbb{B}_0(\mathcal{L}_j), \tag{4.4}$$

where $\mathbb{B}_0(\mathcal{L}_j)$ denotes the algebra of all compact operators on a Hilbert space $\mathcal{L}_j$.

Proposition 4.6. *Suppose that A has large socle A_0, where A_0 is as in* (4.4). *Let K be a weak* compact A-convex subset of $\mathbb{B}(\mathcal{K})$ for some non-degenerate Hilbert A_0-module $\mathcal{K}$. Let $\mathcal{A}_0 = \overline{\bigoplus}_{j\in\mathbb{J}} \mathbb{B}(\mathcal{L}_j)$ (an ℓ^∞-direct sum). Then there exists an $\mathcal{A}_0$-affine weak* homeomorphism of K onto a subset of $\ell^\infty_{\mathbb{L}}(\mathcal{A}_0)$ for an index set $\mathbb{L}$, which can be taken to be countable if $\mathcal{K}$ is separable.*

Proof. Note that $\mathcal{K}$ is a Hilbert A-module since A_0 is an ideal in A [6, I.9.14], hence the notion of A-convexity makes sense for subsets of $\mathbb{B}(\mathcal{K})$. Further, since A_0 is an essential ideal in A, we can regard A as a C*-subalgebra of the universal von Neumann envelope $\mathcal{A}_0$ of A_0.

Let $\pi_0 : A_0 \to \mathbb{B}(\mathcal{K})$ be the representation through which $\mathcal{K}$ is the A_0-module, and $\tilde{\mathcal{A}}_0 := \overline{\pi_0(A_0)}$ (a direct summand in $\mathcal{A}_0$). Since every representation of A_0 (a C*-algebra of compact operators) is equivalent to a direct sum of copies of the irreducible subrepresentation of the identity [6, I.10.7], it is clear what $\mathcal{K}$ looks like as a Hilbert A_0-module. (For example, if $A_0 = \mathbb{B}_0(\mathcal{L}_1)$, then $\mathcal{K}$ is just a direct sum of copies of $\mathcal{L}_1$.) Therefore there exists a normal $\tilde{\mathcal{A}}_0$-bimodule projection $\Psi : \mathbb{B}(\mathcal{K}) \to \tilde{\mathcal{A}}_0$ (namely the projection onto certain diagonal blocks), and (using suitable matrix units) it is not hard to show that the family of all maps $\Psi_{a',b'} : \mathbb{B}(\mathcal{K}) \to \tilde{\mathcal{A}}_0$ of the form $\Psi_{a',b'}(x) := \Psi(a'xb')$ ($x \in \mathbb{B}(\mathcal{K})$), where $a', b' \in \tilde{\mathcal{A}}_0' := \mathbb{B}_{\tilde{\mathcal{A}}_0}(\mathcal{K})$, is a *faithful family* (that is, if $\Psi_{a',b'}(x) = 0$ for all $a', b' \in \tilde{\mathcal{A}}_0'$, then $x = 0$) of weak* continuous $\tilde{\mathcal{A}}_0$-bimodule maps. From the definitions of $\mathcal{A}_0$ and $\tilde{\mathcal{A}}_0$ it follows that these are $\mathcal{A}_0$-bimodule (hence also A-bimodule) maps. Let B_1 be the unit ball of $\tilde{\mathcal{A}}_0'$. The map

$$f : K \to \ell^\infty_{B_1\times B_1}(\tilde{\mathcal{A}}_0), \qquad f(y) := \bigoplus_{a',b'\in B_1\times B_1} \Psi_{a',b'}(y),$$

is an $\mathcal{A}_0$-affine weak* homeomorphism of K to $f(K) \subseteq \ell^\infty_{B_1\times B_1}(\tilde{\mathcal{A}}_0) \subseteq \ell^\infty_{B_1\times B_1}(\mathcal{A}_0)$. Here we may replace $B_1 \times B_1$ by any of its weak* dense subsets $\mathbb{L}$, hence $\mathbb{L}$ can be taken to be countable if $\mathcal{K}$ is separable. ∎

Remark 4.7. If K is a weak* compact $\mathcal{A}$-convex subset of a dual normal Banach $\mathcal{A}$-bimodule, then it follows from [20, 4.4] and the representation theorem for operator bimodules that K is isomorphic to a subset of $\mathbb{B}(\mathcal{K})$ for some normal Hilbert $\mathcal{A}$-module $\mathcal{K}$. Thus, if $\mathcal{A}$ is a factor of type I (or a direct sum of such factors), then Proposition 4.6 implies that K can be realized in $\ell^\infty_{\mathbb{I}}(\mathcal{A})$ for some $\mathbb{I}$.

THEOREM 4.8. *Suppose that A contains an essential ideal A_0 of the form* (4.4). *Let K be a weak* compact A-convex subset of $\mathbb{B}(\mathcal{K})$ for some faithful non-degenerate separable Hilbert A_0-module $\mathcal{K}$. Then K is generated by its A-extreme points.*

Proof. Since by Proposition 4.6, K can be realized as a subset of $\ell^\infty_{\mathbb{N}}(\mathcal{A}_0)$, it follows as a special case of [21, Theorem 5.3] that K is generated by its $\mathcal{A}_0$-extreme points. Moreover, in the special case when there is only one summand in (4.4), more precisely, when $A_0 = \mathbb{B}_0(\ell^2)$, it follows from Proposition 4.4 and Corollary 4.1 that K is generated by its strong $\mathcal{A}_0$-extreme points. We will use this special case to prove that in general (where there may be many summands in (4.4)) K is generated by its A-extreme points. (Note that if a subset K_0 of K generates K over $\mathcal{A}_0$, then K_0 also generates K over A since $\overline{A} = \mathcal{A}_0$; see [20, proof of 3.3] for details.)

First we note that K contains an $\mathcal{A}_0$-convex $\mathcal{A}_0$-extreme point z (that is, an element of $\ell^\infty_{\mathbb{N}}(\mathcal{Z})$, where $\mathcal{Z}$ is the center of $\mathcal{A}_0$, which is $\mathcal{A}_0$-extreme in K); this follows by an argument from the last paragraph of [21, proof of Theorem 5.3] for properly infinite algebras. The translation by z is an $\mathcal{A}_0$-affine bijection and, replacing K with $K-z$, we may assume that $0 \in K$ is an $\mathcal{A}_0$-extreme point of K. Then $a^*Ka \subseteq K$ for each $a \in \mathcal{A}_0$ with $\|a\| \leq 1$. Denote $\mathcal{D} := \ell^\infty_{\mathbb{N}}(\mathcal{A}_0) \cong \overline{\bigoplus}_{j\in\mathbb{J}} \ell^\infty_{\mathbb{N}}(\mathbb{B}(\mathcal{L}_j))$. Let $\mathcal{H}_{\mathcal{D}} = \mathcal{L}^{\mathbb{N}}$ be the direct sum of countably many copies of the Hilbert space $\mathcal{L} := \bigoplus_{j\in\mathbb{J}} \mathcal{L}_j$, so that $\mathcal{D}$ acts naturally on $\mathcal{H}_{\mathcal{D}}$. Denote by $p_j : \mathcal{L} \to \mathcal{L}_j$ the projection; note that p_j is a central projection in $\mathcal{A}_0$ and $p_j^{(\mathbb{N})} : \mathcal{H}_{\mathcal{D}} \to \mathcal{L}_j^{\mathbb{N}}$ is a central projection in $\mathcal{D}$. In particular we now have $p_jK = p_jKp_j \subseteq K$ (where p_jx means $p_j^{(\mathbb{N})}x$ for $x \in \mathbb{B}(\mathcal{L}^{\mathbb{N}})$). So K decomposes into the direct sum $K = \sum_{j\in\mathbb{J}} Kp_j$, where Kp_j is a $\mathbb{B}(\mathcal{L}_j)$-convex weak* compact subset of $\ell^\infty_{\mathbb{N}}(\mathbb{B}(\mathcal{L}_j))$. Since 0 is an $\mathcal{A}_0$-extreme point of K, each of the sets Kp_j has 0 as a $\mathbb{B}(\mathcal{L}_j)$-extreme point.

Let K_0 be the subset of K consisting of all elements of the form (y_j), where y_j is either 0 or a strong $\mathbb{B}(\mathcal{L}_j)$-extreme point of Kp_j if $\mathcal{L}_j$ is infinite-dimensional, while y_j is a $\mathbb{B}(\mathcal{L}_j)$-extreme point of Kp_j if $\mathcal{L}_j$ is finite-dimensional, and only finitely many y_j are not 0. Since Kp_j is generated by its strong $\mathbb{B}(\mathcal{L}_j)$-extreme points if $\mathbb{B}(\mathcal{L}_j)$ is infinite-dimensional (by Proposition 4.4 and Corollary 4.1) and in any case by its $\mathbb{B}(\mathcal{L}_j)$-extreme points (by [21, 5.3]), it is clear that K_0 generates K. It only remains to verify that all points y of K_0 are A-extreme in K. For simplicity of notation we may assume that for some $k \in \mathbb{N}$ and all $j \leq k$ the components y_j of y are contained in $\mathbb{B}(\mathcal{L}_j)$ for a finite-dimensional $\mathcal{L}_j$, while for every $j > k$ the component y_j is either 0 or a strong $\mathbb{B}(\mathcal{L}_j)$-extreme point of Kp_j. Then for $j \leq k$ we have $\mathbb{B}_0(\mathcal{L}_j) = \mathbb{B}(\mathcal{L}_j)$, hence $A_1 := \bigoplus_{j=0}^k \mathbb{B}(\mathcal{L}_j) \subseteq A$ since by assumption A contains A_0. In particular, for $j \le k$ all projections p_j are in A, hence so is the sum $e := \sum_{j=0}^k p_j$. Since $1 \in A$, it follows that $A_2 := e^\perp A \subseteq A$, and $A = A_1 \oplus A_2$. Accordingly $y = v \oplus w$, where $v = (y_0, \ldots, y_k)$ and $w = (y_{k+1}, y_{k+2}, \ldots)$. Since for $j > k$ each y_j is a strong $\mathbb{B}(\mathcal{L}_j)$-extreme point of Kp_j, w is a strong $\overline{\bigoplus}_{j>k} \mathbb{B}(\mathcal{L}_j)$-extreme point of $K_2 := \sum_{j>k} Kp_j$ (by a direct verification), hence also a strong A_2-extreme point of K_2 since A_2 is a C*-subalgebra of $\overline{\bigoplus}_{j>k} \mathbb{B}(\mathcal{L}_j)$. On the other hand, for $j \le k$ each y_j is a $\mathbb{B}(\mathcal{L}_j)$-extreme point of Kp_j, hence v is an A_1-extreme point of $K_1 := \sum_{j\le k} Kp_j$. Since $A = A_1 \oplus A_2$ and $K = K_1 \oplus K_2$, it follows that y is an A-extreme point of K. ∎

Remark 4.9. The separability assumption on $\mathcal{K}$ in Theorem 4.8 can be removed. In the case $A = \mathbb{B}(\mathcal{L})$ this can be proved by using the fact that there exists a net of conditional expectations $E_k : \mathbb{B}(\mathcal{L}) \to \mathcal{A}_k \subset \mathbb{B}(\mathcal{L})$, where $\mathcal{A}_k \cong \mathbb{M}_{n_k}(\mathbb{C})$ $(n_k \in \mathbb{N})$, weak* converging to the identity map on $\mathbb{B}(\mathcal{L})$. In this case the method is similar to the proof of [21, 5.5], so we will not present it here.

5. A characterization of operator-valued quasi-state spaces. Example 4.5 suggests that when considering A-extreme points arising from maps with the unique extension property, it would be easier to deal with operator-valued quasi-state spaces, that is, sets of the form

$$\mathrm{CCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H})),$$

instead of $\mathrm{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$. Arguments from the previous section can be used to show that sets of the form $\mathrm{CCP}_{\mathbb{C}}(V, \mathbb{B}(\mathcal{L}))$ (for a separable operator system V) are generated by maps with the unique extension property even if V is finite-dimensional. In this section we will characterize such sets among weak* compact $\mathcal{A}$-convex sets.

Definition 5.1. An A-convex subset K of an operator A-bimodule Y is said to be *in basic position* if $axa = byb$ implies that $a = b$ whenever $x, y \in K$ and $a, b \in A_+$. If K is in basic position, let $\tilde{K} = \mathrm{co}_A(K \cup \{0\})$, the smallest A-convex set containing K and 0.

Each K is A-affinely isomorphic to a set in basic position (for example, to $K \times \{1\} \subseteq Y \times A$). Clearly a set in basic position cannot contain 0. A prototypical example of a set in basic position is $K = \mathrm{UCP}_C(X, \mathbb{B}(\mathcal{H}))$; then $\tilde{K} = \mathrm{CCP}_C(X, \mathbb{B}(\mathcal{H}))$.

Proposition 5.2. *Let $\mathcal{A}$ be a von Neumann algebra and K an $\mathcal{A}$-convex set of operators in basic position. Then:*

(i) $\mathrm{co}_A(K \cup \{0\}) = \tilde{K} = \{aya : y \in K,\ a \in \mathcal{A}_+,\ \|a\| \le 1\}$.

(ii) *Each $\mathcal{A}$-affine weak* homeomorphism $g : K \to H$ between two weak* compact $\mathcal{A}$-convex sets in basic position extends uniquely to an $\mathcal{A}$-affine weak* homeomorphism $\tilde{g} : \tilde{K} \to \tilde{H}$. In particular each $\tilde{K}$ is $\mathcal{A}$ affinely weak* homeomorphic to a set of the form $\mathrm{CCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$, where $\mathcal{H}$ is a standard Hilbert $\mathcal{A}$-module, $\mathcal{C} = \mathbb{B}_{\mathcal{A}}(\mathcal{H})$ and $\mathcal{X}$ is a faithful operator $\mathcal{C}$-system.*

(iii) *If K is weak* compact (in a dual normal operator $\mathcal{A}$-bimodule), then $\tilde{K}$ is also weak* compact.*

Proof. (i) This is a special case of [20, Lemma 4.2].

(ii) The uniqueness of the extension is obvious from (i), but the proof of weak* continuity is somewhat indirect. By [20] there exists an $\mathcal{A}$-affine weak* homeomorphism $f : S = \mathrm{UCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H})) \to K$. We may assume that $K \subseteq \mathbb{B}(\mathcal{K})$ for a normal Hilbert $\mathcal{A}$-module $\mathcal{K}$. By the proof of Lemma 3.1 (see (3.2)), f can be uniquely extended to an $\mathcal{A}$-affine map $\tilde{f} : \tilde{S} = \mathrm{CCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H})) \to \overline{\tilde{K}}$ such that $\tilde{f}(0) = 0$, and $\tilde{f}$ is weak* continuous. Clearly $\tilde{f}(a\varphi a) = af(\varphi)a$ for all $\varphi \in S$ and $a \in \mathcal{A}_+$ with $\|a\| \leq 1$, hence it follows from part (i) that $\tilde{f}(\tilde{S}) = \tilde{K}$. Since $\tilde{S}$ is compact, so is $\tilde{K}$. To prove that $\tilde{f}$ is injective, suppose that $\tilde{f}(\varphi) = \tilde{f}(\psi)$ and write φ and ψ as $\varphi = a\varphi_1 a$ and $\psi = b\psi_1 b$, where $\varphi_1, \psi_1 \in S$ and a, b are positive contractions in $\mathcal{A}$. Then $af(\varphi_1)a = bf(\psi_1)b$, and since $K = f(S)$ is in basic position by hypothesis, this implies that $a = b$. Then writing the identity $af(\varphi_1)a = af(\psi)a$ in the form of $\mathcal{A}$-convex combinations, $af(\varphi_1)a + df(\theta)d = af(\psi)a + df(\theta)d$, where $\theta \in S$ and $d = (1 - a^2)^{1/2}$, it follows from $\mathcal{A}$-affinity and injectivity of f that $a\varphi_1 a = a\psi_1 a$, thus $\varphi = \psi$. By compactness of $\tilde{S}$ and weak* continuity of $\tilde{f}$, we see that $\tilde{f}$ must be a weak* homeomorphism of $\tilde{S}$ onto $\tilde{K}$. In the same way the composition $gf : S \to H$ can be uniquely extended to an $\mathcal{A}$-affine weak* homeomorphism $\widetilde{gf} : \tilde{S} \to \tilde{H}$ and then $\tilde{g} := \widetilde{gf}\tilde{f}^{-1}$ is an A-affine weak* homeomorphic extension of g.

(iii) This has already been observed in the proof of (ii). ∎

Part (ii) of the above proposition tells us that operator-valued quasi-state spaces $\mathrm{CCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ are just sets of the form $\tilde{K}$, where K is any $\mathcal{A}$-convex weak* compact set in basic position.

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