

# A Factorization Problem for Normal Completely Bounded Mappings<sup>1</sup>

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Given an operator space  $X$  and a von Neumann algebra  $A$ , we consider a contractive mapping  $q: A \otimes X \otimes A \rightarrow NCB(X^*, A)$  formally defined by  $q(\sum a_j \otimes x_{jk} \otimes b_k) = \sum x_{jk} \otimes a_j b_k$ , from the extended Haagerup tensor product  $A \otimes X \otimes A$  into the space of  $w^*$ -continuous completely bounded maps from  $X^*$  into  $A$ . We characterize elements of the range space  $Im(q)$  by a factorization property involving decomposable operators and investigate various properties of that space. In the case when  $X = B_*$  is the predual of a von Neumann algebra  $B$ ,  $Im(q)$  is included in the space  $DEC(B, A)$  of decomposable operators from  $B$  into  $A$ . Regarding  $q$  as having values in that space, we show that  $q$  is a quotient map onto its range. Then we prove that  $DEC(B, A)$  is a normal dual operator  $A$ -bimodule and that  $Im(q) \subset DEC(B, A)$  is a strong operator  $A$ -bimodule. © 2001 Academic Press

## 1. INTRODUCTION

Let  $A$  be a  $C^*$ -algebra and let  $X$  be an operator space. We consider

$$q_0: A \otimes X \otimes A \rightarrow X \otimes A \tag{1.1}$$

defined as the unique linear mapping on the algebraic tensor product  $A \otimes X \otimes A$  satisfying  $q_0(a \otimes x \otimes b) = x \otimes ab$  for any  $x \in X$ ,  $a, b \in A$ . If

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$A \otimes X \otimes A$  is equipped with the Haagerup tensor norm of operator spaces, then the quotient norm on  $X \otimes A$  induced by  $q_0$  coincides with Pisier's delta norm  $\delta$  introduced in [28]. Namely, if  $z = q_0(T)$  with  $T \in A \otimes X \otimes A$ , then  $\delta(z)$  equals the norm of the coset of  $T$  in  $A \otimes X \otimes A / \ker(q_0)$ . The delta norm was used in [18, 22, 28] to study matrix space factorizations or approximate factorizations of completely bounded or decomposable maps. We recall that by definition a decomposable map between  $C^*$ -algebras is a linear combination of completely positive maps and we denote by  $\|\cdot\|_{\text{dec}}$  the associated decomposable norm introduced by Haagerup [16]. (See Subsection 2.2 below for some background on this.) We denote by  $\|\cdot\|_{\text{cb}}$  the completely bounded norm of completely bounded maps. We review below three important results concerning  $q_0$  and  $\delta$  which were our main motivations to undertake the study of extensions of  $q_0$ . The first result was established in [28, Corollary 12.5].

(I) Let  $z \in X \otimes A$  and let  $\phi: X^* \rightarrow A$  be the finite rank linear mapping represented by  $z$ . Then the quotient norm  $\delta(z)$  equals  $\inf\{\|u\|_{\text{cb}}\|v\|_{\text{dec}}\}$ , where the infimum runs over all  $n \geq 1$  and all possible factorizations  $\phi = vu$  with mappings  $u: X^* \rightarrow M_n$ ,  $v: M_n \rightarrow A$  such that  $u$  is  $w^*$ -continuous.

Note the simple fact that  $q_0$  extends to a contraction

$$q_1: A \overset{h}{\otimes} X \overset{h}{\otimes} A \rightarrow X \otimes_{\min} A, \quad (1.2)$$

where  $\otimes_{\min}$  denotes the minimal tensor product of operator spaces. In [28, Sect. 12], Pisier proved the following result.

(II) A  $C^*$ -algebra  $A$  is nuclear iff  $q_1$  is onto for any operator space  $X$ .

In the third result we consider the special case when  $X = B_*$  is the predual operator space of a von Neumann algebra  $B$ . The following is due to Junge and Le Merdy [18].

(III) Let  $B$  be a von Neumann algebra, then  $q_0: A \otimes B_* \otimes A \rightarrow B_* \otimes A$  is a quotient map if  $A \otimes B_* \otimes A$  is equipped with the Haagerup tensor norm and  $B_* \otimes A$  is equipped with the decomposable norm of operators from  $B$  into  $A$ . (Equivalently,  $\|\phi\|_{\text{dec}} = \delta(z)$  if  $z \in B_* \otimes A$  and  $\phi: B \rightarrow A$  is the finite rank mapping represented by  $z$ .)

Assume that  $A$  is a von Neumann algebra, and let  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$  be the extended Haagerup tensor product of  $(A, X, A)$  in the sense of [14]. Let  $NCB(X^*, A)$  denote the Banach space of all  $w^*$ -continuous (= normal) completely bounded maps from  $X^*$  into  $A$ , equipped with  $\|\cdot\|_{\text{cb}}$ . In this paper we introduce a natural contractive mapping

$$q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A) \quad (1.3)$$

extending  $q_0$ . We will give in Subsection 2.1 the necessary background to make precise the following formal definition: if  $T = \sum_{j,k \in \mathbb{J}} a_j \otimes x_{jk} \otimes b_k$  with  $(a_j)_j$  (resp.  $(x_{jk})_{j,k}$ , rresp.  $(b_k)_k$ ) a bounded infinite row (resp. matrix, rresp. column) with entries in  $A$  (resp.  $X$ , rresp.  $A$ ), then

$$q(T) = \phi, \quad \text{with} \quad \phi(x^*) = \sum_{j,k \in \mathbb{J}} a_j \langle x^*, x_{j,k} \rangle b_k \quad (x^* \in X^*). \quad (1.4)$$

The purpose of this paper is to give a precise description of  $Im(q)$ , i.e., to characterize the normal completely bounded maps  $\phi: X^* \rightarrow A$  which admit a factorization as in (1.4), and to study various properties of  $q$ . In analogy with (I), (II), and (III), we shall prove the following results.

(I)' A mapping  $\phi: X^* \rightarrow A$  belongs to  $Im(q)$  iff there exists, for some Hilbert space  $\mathcal{H}$ , a factorization  $\phi = vu$  where  $u: X^* \rightarrow B(\mathcal{H})$  is a normal completely bounded map, and  $v: B(\mathcal{H}) \rightarrow A$  is a normal decomposable map. Moreover if  $\phi = q(T)$ , then the infimum  $\inf\{\|u\|_{cb} \|v\|_{dec}\}$  over all  $u, v$  as above coincides with the norm of the coset of  $T$  modulo  $ker(q)$ .

(II)' Given a von Neumann algebra  $A$ , the mapping  $q$  is onto for any operator space  $X$  iff  $A$  is the discrete direct sum of type I factors.

(III)' Let  $B$  be a von Neumann algebra and let  $X = B_*$  be its predual. Then the mapping  $q$  has values in the space  $NDEC(B, A)$  of normal decomposable operators from  $B$  into  $A$  and  $q: A \overset{eh}{\otimes} B_* \overset{eh}{\otimes} A \rightarrow NDEC(B, A)$  is a quotient map onto its range.

These three results will be established in Section 5. In the case when  $X = Y^*$  is a dual operator space, we may also introduce a natural contractive mapping

$$Q: A \overset{sh}{\otimes} X \overset{sh}{\otimes} A \rightarrow CB(Y, A) \quad (1.5)$$

on the so-called normal Haagerup tensor product which extends  $q$  in a suitable way. This will be achieved in Section 3 where we shall prove analogues of (I), (II), (III) for  $Q$  which shed some light on our study of  $q$ . Section 4 is devoted to preparatory results to Section 5 on decomposable operators and their extensions. In particular we give a precise description of normal decomposable operators from some  $B(\mathcal{H})$  into a von Neumann algebra  $A$ . In our last Section 6, we prove that the Banach space of decomposable operators between two von Neumann algebras  $B$  and  $A$  can be equipped with a natural operator space structure which makes it a dual operator space and a normal dual operator  $A$ -bimodule in the sense of [12]. This provides a natural setting to show that in the case  $X = B_*$ , the range space  $Im(q)$  is a strong operator  $A$ -bimodule in the sense of [24, 26].

## 2. DEFINITION OF $Q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A)$ AND PRELIMINARY RESULTS

### 2.1. Notation and Definition of $q$

We will assume that the reader is familiar with the basics of Operator Space Theory, for which we refer to [2–4, 12, 13, 15, 28, 31]. We also refer to [27, 32] for general information on  $C^*$ -algebras, von Neumann algebras, and completely bounded maps. We shall use the following standard notation and terminology. Given two operator spaces  $V$  and  $W$ , we denote by  $CB(V, W)$  the Banach space of all completely bounded maps from  $V$  into  $W$ , equipped with the completely bounded norm  $\|\cdot\|_{cb}$ . If  $V$  and  $W$  are both dual operator spaces, the closed subspace of  $CB(V, W)$  of all  $w^*$ -continuous completely bounded maps (that we shall call normal completely bounded maps as well) will be denoted by  $NCB(V, W)$ . Given a completely bounded map  $\varphi: V \rightarrow W$ , we say that  $\varphi$  is completely contractive (or is a complete contraction) if  $\|\varphi\|_{cb} \leq 1$ . Furthermore we say that  $\varphi$  is completely isometric (or is a complete isometry) if for any integer  $n \geq 1$ , the tensor map  $I_{M_n} \otimes \varphi: M_n(V) \rightarrow M_n(W)$  is an isometry. Concerning tensor products, we will use the notation  $\otimes_{\min}$ ,  $\widehat{\otimes}$  and  $\overset{h}{\otimes}$  for the minimal, the projective, and the Haagerup tensor product of operator spaces respectively. If  $A$  and  $B$  are  $C^*$ -algebras (resp. von Neumann algebras), we shall denote by  $A \otimes_{\max} B$  their maximal  $C^*$ -tensor product (resp. by  $A \widehat{\otimes} B$  their von Neumann tensor product). If  $A$  is a von Neumann algebra and if  $B$  is a  $C^*$ -algebra, we shall use the notation  $A \otimes_{\text{nor}} B$  for their normal tensor product in the sense of [11].

We now recall some well-known facts about operator valued infinite matrices (for which we mainly refer to [4, 12, 13]) and fix some notation that will be used along this paper. Let  $\mathcal{H}$  be a Hilbert space and let  $\mathbb{I}$  be an index set. We denote by  $\mathcal{H}^{\mathbb{I}} = \ell_{\mathbb{I}}^2(\mathcal{H})$  the Hilbertian direct sum of “ $\mathbb{I}$  copies of  $\mathcal{H}$ .” If  $\mathbb{J}$  is another index set, elements of  $B(\mathcal{H}^{\mathbb{J}}, \mathcal{H}^{\mathbb{I}})$  can be canonically represented as infinite matrices  $(x_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}}$ , with each  $x_{ij}$  belonging to  $B(\mathcal{H})$ . The space of all these infinite matrices is denoted by  $M_{\mathbb{I}, \mathbb{J}}(B(\mathcal{H}))$ , and is equipped with the norm of  $B(\mathcal{H}^{\mathbb{J}}, \mathcal{H}^{\mathbb{I}})$ . If  $X \subseteq B(\mathcal{H})$  is an operator space, the set of all elements of  $M_{\mathbb{I}, \mathbb{J}}(B(\mathcal{H}))$  whose entries belong to  $X$  is a subspace denoted by  $M_{\mathbb{I}, \mathbb{J}}(X)$ . It turns out that this construction does only depend on the operator space structure of  $X$  and not on the Hilbert space  $\mathcal{H}$  on which  $X$  acts. We recall the isometric identification

$$M_{\mathbb{I}, \mathbb{J}}(X) = NCB(X^*, M_{\mathbb{I}, \mathbb{J}}(\mathbb{C})) \tag{2.1}$$

obtained by regarding  $(x_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}} \in M_{\mathbb{I}, \mathbb{J}}(X)$  as the mapping which takes  $x^* \in X^*$  to  $(\langle x^*, x_{ij} \rangle)_{i, j}$ . The Banach space  $M_{\mathbb{I}, \mathbb{J}}(X)$  is merely denoted by

$M_{\mathbb{I}}(X)$ , (resp.  $C_{\mathbb{I}}(X)$ , rresp.  $R_{\mathbb{J}}(X)$ ) in the case when  $\mathbb{I} = \mathbb{J}$  (resp.  $\mathbb{J}$  is a singleton, resp.  $\mathbb{I}$  is a singleton).

Let us now assume that  $X = A \subseteq B(\mathcal{K})$  is a von Neumann algebra. In that case, we have a canonical identification  $M_{\mathbb{I}}(A) = M_{\mathbb{I}}(\mathbb{C}) \bar{\otimes} A$ . Furthermore, an  $A$ -valued family  $(a_i)_{i \in \mathbb{I}}$  belongs to  $R_{\mathbb{I}}(A)$  if and only if the series  $\sum_{i \in \mathbb{I}} a_i a_i^*$  is strongly convergent. In that case, the sum  $\sum_i a_i a_i^*$  belongs to  $A$  and  $\|(a_i)_{i \in \mathbb{I}}\|_{R_{\mathbb{I}}(A)} = \|\sum_i a_i a_i^*\|^{1/2}$ . Exchanging  $a_i$  and  $a_i^*$ , we get a similar description of  $C_{\mathbb{I}}(A)$ , with  $\|(a_i)_{i \in \mathbb{I}}\|_{C_{\mathbb{I}}(A)} = \|\sum a_i^* a_i\|^{1/2}$ .

Let  $A \subseteq B(\mathcal{K})$  be a von Neumann algebra, and let  $\mathbb{I}$  and  $\mathbb{J}$  be two index sets. Then the multiplication on  $A$  induces a natural trilinear mapping

$$M_{\mathbb{I}, \mathbb{J}}(A) \times M_{\mathbb{J}}(A) \times M_{\mathbb{J}, \mathbb{I}}(A) \rightarrow M_{\mathbb{I}}(A), \quad (\alpha, c, \beta) \mapsto \alpha c \beta. \tag{2.2}$$

Namely, given  $\alpha \in M_{\mathbb{I}, \mathbb{J}}(A) \subset B(\mathcal{K}^{\mathbb{J}}, \mathcal{K}^{\mathbb{I}})$ ,  $c \in M_{\mathbb{J}}(A) \subset B(\mathcal{K}^{\mathbb{J}})$  and  $\beta \in M_{\mathbb{J}, \mathbb{I}}(A) \subset B(\mathcal{K}^{\mathbb{I}}, \mathcal{K}^{\mathbb{J}})$ , the product  $\alpha c \beta$  is simply defined as the element of  $B(\mathcal{K}^{\mathbb{I}})$  obtained by the composition  $\mathcal{K}^{\mathbb{I}} \xrightarrow{\beta} \mathcal{K}^{\mathbb{J}} \xrightarrow{c} \mathcal{K}^{\mathbb{J}} \xrightarrow{\alpha} \mathcal{K}^{\mathbb{I}}$ . If  $\mathbb{I}$  is a singleton, if  $\alpha = (a_j)_{j \in \mathbb{J}}$ ,  $c = (c_{jk})_{j, k \in \mathbb{J}}$  and  $\beta = (b_k)_{k \in \mathbb{J}}$  are as above, then we use the summation notation

$$\alpha c \beta = \sum_{j, k \in \mathbb{J}} a_j c_{jk} b_k \in A \tag{2.3}$$

to denote the element of  $A$  defined by (2.2).

We now come to the definition of the extended Haagerup tensor product, for which we refer to [14] (see also [10, 24]). We restrict ourselves to the product of three operator spaces. Let  $V_1, V_2, V_3$  be three arbitrary operator spaces. By definition, the extended Haagerup tensor product  $V_1 \overset{eh}{\otimes} V_2 \overset{eh}{\otimes} V_3$  is the closed subspace of all elements of  $(V_1^* \overset{h}{\otimes} V_2^* \overset{h}{\otimes} V_3^*)^*$  which are separately  $w^*$ -continuous in each of the three variables. They have a nice description in terms of infinite matrices, we recall it below in the situation we will be interested in along this paper.

Let  $A$  be a von Neumann algebra, let  $X$  be an operator space, and let  $T \in A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$ . According to [14, Sect. 3], there exist an index set  $\mathbb{J}$  and infinite matrices  $(a_j)_{j \in \mathbb{J}} \in R_{\mathbb{J}}(A)$ ,  $(x_{jk})_{j, k \in \mathbb{J}} \in M_{\mathbb{J}}(X)$ ,  $(b_k)_{k \in \mathbb{J}} \in C_{\mathbb{J}}(A)$ , such that

$$T(\eta_1 \otimes x^* \otimes \eta_2) = \sum_{j, k \in \mathbb{J}} \langle \eta_1, a_j \rangle \langle x^*, x_{jk} \rangle \langle \eta_2, b_k \rangle \quad (x^* \in X^*, \eta_1, \eta_2 \in A^*). \tag{2.4}$$

Moreover the norm of  $T$  in  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$  coincides with

$$\inf\{ \|(a_j)_j\|_{R_{\mathbb{J}}(A)} \|(x_{jk})_{j, k}\|_{M_{\mathbb{J}}(X)} \|(b_k)_k\|_{C_{\mathbb{J}}(A)} \},$$

where the infimum is over all such factorizations. When (2.4) holds, we shall simply write

$$T = \sum_{j, k \in \mathbb{J}} a_j \otimes x_{jk} \otimes b_k. \quad (2.5)$$

We define  $q$  as follows. First recall that the multiplication on  $A$  extends to the mapping  $p: A \overset{eh}{\otimes} A \rightarrow A$  defined by  $p(\sum_{j \in \mathbb{J}} a_j \otimes b_j) = \sum_{j \in \mathbb{J}} a_j b_j$  for all  $\sum_{j \in \mathbb{J}} a_j \otimes b_j \in A \overset{eh}{\otimes} A$ , and that  $p$  is completely contractive. Second we may define a contractive mapping

$$\theta: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow CB(X^*, A \overset{eh}{\otimes} A)$$

by letting  $\theta(T)(x^*) = T(\cdot \otimes x^* \otimes \cdot)$  for all  $T \in A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$ . Then we define  $q$  by the composition formula  $q(T) = p \circ \theta(T)$ . By construction,  $q$  is a linear contractive mapping from  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$  into  $CB(X^*, A)$ . Assume now that  $T$  is defined by (2.5). Then  $\theta(T)(x^*) = \sum_{j, k \in \mathbb{J}} a_j \otimes b_k \langle x^*, x_{jk} \rangle$  for all  $x^* \in X^*$ , hence

$$q(T) = \phi, \quad \text{with} \quad \phi(x^*) = \sum_{j, k \in \mathbb{J}} a_j \langle x^*, x_{jk} \rangle b_k \quad (x^* \in X^*). \quad (2.6)$$

The meaning of that summation is of course given by (2.2) and (2.3). Since the trilinear mapping in (2.2) is separately  $w^*$ -continuous, we deduce from (2.6) that  $q(T) \in NCB(X^*, A)$  for any  $T$ . Summing up, we obtain that (2.5) and (2.6) define a linear mapping

$$q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A), \quad \text{with} \quad \|q\| \leq 1. \quad (2.7)$$

Note the obvious fact that  $q$  extends the mapping  $q_0$  considered in Section 1.

Assume that if  $X = Y^*$  is a dual operator space. Then we have a canonical identification

$$CB(Y, A) = NCB(X^*, A). \quad (2.8)$$

Indeed, given any  $w^*$ -continuous completely bounded map  $\phi: X^* = Y^{**} \rightarrow A$ , let  $j(\phi) = \phi|_Y \in CB(Y, A)$  be its restriction to  $Y$ . Then  $j$  is a linear contraction. On the other hand, given any  $\varphi \in CB(Y, A)$ , let  $\hat{\varphi}: Y^{**} \rightarrow A$  be defined by  $\hat{\varphi} = \pi_A \varphi^{**}$ , where  $\pi_A: A^{**} \rightarrow A$  is the projection defined as the adjoint of the canonical embedding  $A_* \hookrightarrow A^*$ . (Note that  $\pi_A$  is a normal  $*$ -representation.) Then  $\hat{\varphi}$  is a  $w^*$ -continuous completely bounded extension of  $\varphi$ , hence  $\varphi = j(\hat{\varphi})$ . Moreover,  $\|\hat{\varphi}\|_{cb} = \|\varphi\|_{cb}$ , hence we obtain that

$j$  is actually an isometric isomorphism. Taking into account the identification (2.8), we will regard the mapping  $q$  as having values in  $CB(Y, A)$  in the case when  $X = Y^*$ .

2.2. *Background and Preliminary Results on Decomposable Operators*

Let  $A$  and  $B$  be two  $C^*$ -algebras. By definition, a linear mapping  $\phi: B \rightarrow A$  is said to be decomposable if it lies in the linear span of completely positive maps from  $B$  into  $A$ . We denote by  $DEC(B, A)$  the vector space of all such maps. In [16], Haagerup introduced a remarkable norm  $\|\cdot\|_{dec}$  on  $DEC(B, A)$  which is defined as follows. For  $\phi \in DEC(B, A)$ , let  $\phi_*: B \rightarrow A$  be defined by  $\phi_*(y) = \phi(y^*)^*$  for any  $y \in B$ . Then by definition  $\|\phi\|_{dec} = \inf\{\max\{\|\sigma\|, \|\tau\|\}\}$ , where the infimum runs over all possible completely positive maps  $\sigma: B \rightarrow A$  and  $\tau: B \rightarrow A$  such that the mapping  $w: B \rightarrow M_2(A)$  defined by

$$w(y) = \begin{pmatrix} \sigma(y) & \phi(y) \\ \phi_*(y) & \tau(y) \end{pmatrix}$$

is completely positive. It is shown in [16] that  $\|\cdot\|_{dec}$  is a complete norm on  $DEC(B, A)$  and that  $\|\phi\|_{cb} \leq \|\phi\|_{dec}$  for any  $\phi \in DEC(B, A)$ . Moreover the following three important properties hold,

If  $\phi: B \rightarrow A$  is completely positive, then  $\|\phi\|_{dec} = \|\phi\|_{cb} = \|\phi\|$ . (2.9)

If  $A$  is injective, then  $\|\phi\|_{dec} = \|\phi\|_{cb}$  for any  $\phi \in DEC(B, A)$ . (2.10)

Last, given a third  $C^*$ -algebra  $C$  and  $\phi \in DEC(B, C)$ ,  $\psi \in DEC(C, A)$ , we have  $\psi\phi \in DEC(B, A)$  with

$$\|\psi\phi\|_{dec} \leq \|\psi\|_{dec} \|\phi\|_{dec}. \tag{2.11}$$

If  $A$  and  $B$  are both von Neumann algebras, we will denote by  $NDEC(B, A)$  the closed subspace of  $DEC(B, A)$  of normal decomposable maps.

Let  $A \subseteq B(\mathcal{H})$  be a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$  and let  $B$  be a  $C^*$ -algebra. We let  $A'$  denote the commutant of  $A$  in  $B(\mathcal{H})$ . Given any bounded linear mapping  $\phi: B \rightarrow A$ , we introduce the mapping

$$\tilde{\phi}: B \otimes A' \rightarrow B(\mathcal{H}) \tag{2.12}$$

defined by letting

$$\tilde{\phi} \left( \sum_k y_k \otimes a'_k \right) = \sum_k \phi(y_k) a'_k \tag{2.13}$$

for any finite families  $(y_k)_k$  in  $B$  and  $(a'_k)_k$  in  $A'$ . We will use the following characterization of decomposable maps whose main part is due to Kirchberg.

PROPOSITION 2.1. *Let  $A$  and  $B$  be a von Neumann algebra and a  $C^*$ -algebra, respectively. Then let  $\phi: B \rightarrow A$  be a bounded linear mapping and let  $C$  be a constant. The following three assertions are equivalent.*

(i) *The mapping  $\phi$  is decomposable and  $\|\phi\|_{\text{dec}} \leq C$ .*

(ii) *For any  $C^*$ -algebra  $D$ , the tensor map  $\phi \otimes I_D$  extends to a bounded mapping from  $B \otimes_{\max} D$  into  $A \otimes_{\text{nor}} D$ , with*

$$\|\phi \otimes I_D : B \otimes_{\max} D \rightarrow A \otimes_{\text{nor}} D\| \leq C.$$

(iii) *The linear mapping  $\tilde{\phi}$  defined by (2.12) and (2.13) extends to a completely bounded mapping on  $B \otimes_{\max} A'$ , with*

$$\|\tilde{\phi}: B \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{H})\|_{\text{cb}} \leq C.$$

*Proof.* The fact that (i) implies (ii) was proved by Junge and Pisier, see [19, (4.6)]. Assume (ii) and apply it with the  $C^*$ -algebras  $D = M_n(A')$  for  $n \geq 1$ . Then we obtain that the mapping  $\phi \otimes I_{A'}: B \otimes_{\max} A' \rightarrow A \otimes_{\text{nor}} A'$  is actually completely bounded with  $\|\phi \otimes I_{A'}\|_{\text{cb}} \leq C$ . Indeed, we have  $M_n(B \otimes_{\max} A') = B \otimes_{\max} M_n(A')$  and  $M_n(A \otimes_{\text{nor}} A') = A \otimes_{\text{nor}} M_n(A')$  for any  $n \geq 1$ . Now the multiplication mapping  $A \otimes A' \rightarrow \mathbf{B}(\mathcal{H})$  taking  $a \otimes a'$  to  $aa'$  extends to a  $*$ -representation  $\pi: A \otimes_{\text{nor}} A' \rightarrow \mathbf{B}(\mathcal{H})$ . Since  $\tilde{\phi}$  is obtained by composing  $\pi$  and  $\phi \otimes I_{A'}$ , we deduce the assertion (iii). The remaining implication, (iii)  $\Rightarrow$  (i) is due to Kirchberg; we refer to [28, Sect. 14] for a proof. ■

LEMMA 2.2. *Let  $A$  be a  $C^*$ -algebra (resp. a von Neumann algebra), let  $n \geq 1$  and  $m \geq 1$  be two integers (resp.  $\mathbb{J}$  and  $\mathbb{I}$  be two index sets), and let  $\alpha \in M_{m,n}(A)$  (resp.  $M_{\mathbb{I},\mathbb{J}}(A)$ ) and  $\beta \in M_{n,m}(A)$  (resp.  $M_{\mathbb{J},\mathbb{I}}(A)$ ). We define  $\theta: M_n(A) \rightarrow M_m(A)$  (resp.  $\theta: M_{\mathbb{J}}(A) \rightarrow M_{\mathbb{I}}(A)$ ) by letting  $\theta(c) = \alpha\beta$  for any  $c \in M_n(A)$  (resp.  $M_{\mathbb{J}}(A)$ , see (2.2)). Then  $\theta$  is decomposable (resp. normal and decomposable), with  $\|\theta\|_{\text{dec}} \leq \|\alpha\| \|\beta\|$ .*

*Proof.* Since the  $C^*$ -algebra and the von Neumann algebra cases are similar, we only prove the second one. It is plain that  $\theta$  is normal. Changing  $\alpha$  and  $\beta$  to  $\|\beta\|^{1/2} \|\alpha\|^{-1/2} \alpha$  and  $\|\alpha\|^{1/2} \|\beta\|^{-1/2} \beta$ , we may assume that  $\|\beta\| = \|\alpha\|$ . Let  $w: M_{\mathbb{J}}(A) \rightarrow M_2(M_{\mathbb{I}}(A))$  be defined by

$$w(c) = \begin{pmatrix} \alpha & 0 \\ \beta^* & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha^* & \beta \\ 0 & 0 \end{pmatrix} \quad (c \in M_{\mathbb{J}}(A)).$$



Then  $w$  is obviously completely positive and matrix multiplication yields

$$w(c) = \begin{pmatrix} \alpha c \alpha^* & \theta(c) \\ \theta_*(c) & \beta^* c \beta \end{pmatrix}.$$

We deduce that  $\|\theta\|_{\text{dec}} \leq \max\{\|\alpha\|^2, \|\beta\|^2\} = \|\alpha\| \|\beta\|$ . ■

Let  $\mathbb{I}$  and  $\mathbb{J}$  be two index sets and let us consider row and column families  $\alpha = (a_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}} \in \mathbf{R}_{\mathbb{I} \times \mathbb{J}}(A)$  and  $\beta = (b_{ji})_{j \in \mathbb{J}, i \in \mathbb{I}} \in \mathbf{C}_{\mathbb{J} \times \mathbb{I}}(A)$ . As in Lemma 2.2, we define  $\theta: \mathbf{M}_{\mathbb{I} \times \mathbb{J}}(A) \rightarrow A$  by letting  $\theta(c) = \alpha c \beta$ . Then via the identification  $\mathbf{M}_{\mathbb{I} \times \mathbb{J}}(A) = \mathbf{M}_{\mathbb{I}}(\mathbb{C}) \bar{\otimes} \mathbf{M}_{\mathbb{J}}(\mathbb{C}) \bar{\otimes} A$ , we define  $\Phi: \mathbf{M}_{\mathbb{J}}(\mathbb{C}) \rightarrow A$  by letting  $\Phi(s) = \theta(1 \otimes s \otimes 1)$  for  $s \in \mathbf{M}_{\mathbb{J}}(\mathbb{C})$ . In accordance with (2.3), we shall use the notation

$$\Phi(s) = \sum_{i \in \mathbb{I}; j, k \in \mathbb{J}} a_{ij} s_{jk} b_{ki} \quad \text{for } s = (s_{jk}) \in \mathbf{M}_{\mathbb{J}}(\mathbb{C}). \tag{2.14}$$

Since the mapping  $s \mapsto 1 \otimes s \otimes 1$  is a normal  $*$ -representation from  $\mathbf{M}_{\mathbb{J}}(\mathbb{C})$  into  $\mathbf{M}_{\mathbb{I} \times \mathbb{J}}(A)$ , the following is an straightforward consequence of Lemma 2.2 and (2.11).

LEMMA 2.3. *The mapping  $\Phi: \mathbf{M}_{\mathbb{J}}(\mathbb{C}) \rightarrow A$  defined by (2.14) is normal and decomposable, with*

$$\|\Phi\|_{\text{dec}} \leq \left\| \sum_{i \in \mathbb{I}, j \in \mathbb{J}} a_{ij} a_{ij}^* \right\|^{1/2} \left\| \sum_{i \in \mathbb{I}, j \in \mathbb{J}} b_{ji}^* b_{ji} \right\|^{1/2}.$$

### 3. A MAPPING $Q$ DEFINED ON THE NORMAL HAAGERUP TENSOR PRODUCT

We shall introduce and study an analogue of  $q$  on the normal Haagerup tensor product. In this section, we will assume that  $X = Y^*$  is a dual operator space and we give ourselves a von Neumann algebra  $A$ . We recall that by definition (see [9, 14]), the normal Haagerup tensor product is the dual operator space

$$A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A = (A_* \overset{eh}{\otimes} Y \overset{eh}{\otimes} A_*)^*.$$

Regarding  $X \otimes_{\min} A$  as a subspace of  $CB(Y, A)$ , we see that the mapping  $q_1$  considered in (1.2) has values in  $CB(Y, A)$ . Since the latter is the dual

space of the projective tensor product  $A_* \widehat{\otimes} Y$ , we can introduce a mapping  $j: A_* \widehat{\otimes} Y \rightarrow (A \overset{h}{\otimes} X \overset{h}{\otimes} A)^*$  defined as the restriction of  $q_1^*$ . More explicitly,

$$[j(\eta \otimes f)](a \otimes x \otimes b) = \langle f, x \rangle \langle \eta, ab \rangle \quad \text{for any } \eta \in A_*, \\ f \in Y, \quad a, b \in A, \quad x \in X. \quad (3.1)$$

Since the multiplication mapping is separately  $w^*$ -continuous on  $A$ ,  $j$  actually has values in  $A_* \overset{eh}{\otimes} Y \overset{eh}{\otimes} A_*$ , so that we have

$$j: A_* \widehat{\otimes} Y \rightarrow A_* \overset{eh}{\otimes} Y \overset{eh}{\otimes} A_* \subset (A \overset{h}{\otimes} X \overset{h}{\otimes} A)^*$$

We set

$$Q = j^*, \quad Q: A \overset{sh}{\otimes} X \overset{sh}{\otimes} A \rightarrow CB(Y, A).$$

By construction,  $Q$  is  $w^*$ -continuous and  $\|Q\| \leq 1$ .

It was proved by Blecher and Smith [4] that there is a natural embedding

$$A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \hookrightarrow A \overset{sh}{\otimes} X \overset{sh}{\otimes} A \quad (3.2)$$

which turns out to be an isometry. To describe this embedding, we only need to define the action  $\langle T, S \rangle$  of an element  $T \in A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$  on an arbitrary  $S \in A_* \overset{eh}{\otimes} Y \overset{eh}{\otimes} A_*$ . We may assume that  $T$  is given by (2.5), for some index set  $\mathbb{J}$  and some  $(a_j)_{j \in \mathbb{J}} \in \mathbf{R}_{\mathbb{J}}(A)$ ,  $(x_{jk})_{j, k \in \mathbb{J}} \in \mathbf{M}_{\mathbb{J}}(X)$ ,  $(b_k)_{k \in \mathbb{J}} \in \mathbf{C}_{\mathbb{J}}(A)$ . Likewise, we can write  $S = \sum_{p, q \in \mathbb{P}} \eta_p \otimes f_{pq} \otimes v_q$ , for some index set  $\mathbb{P}$  and  $(\eta_p)_{p \in \mathbb{P}} \in \mathbf{R}_{\mathbb{P}}(A_*)$ ,  $(f_{pq})_{p, q \in \mathbb{P}} \in \mathbf{M}_{\mathbb{P}}(Y)$ ,  $(v_q)_{q \in \mathbb{P}} \in \mathbf{C}_{\mathbb{P}}(A_*)$ . Then  $(\langle \eta_p, a_j \rangle)_{j, p} \in \mathbf{R}_{\mathbb{J} \times \mathbb{P}}(\mathbb{C})$ ,  $(\langle f_{pq}, x_{jk} \rangle)_{j, k, p, q} \in \mathbf{M}_{\mathbb{J} \times \mathbb{P}}(\mathbb{C})$ ,  $(\langle v_q, b_k \rangle)_{k, q} \in \mathbf{C}_{\mathbb{J} \times \mathbb{P}}(\mathbb{C})$ , and the duality pairing yielding (3.2) is given by

$$\langle T, S \rangle = \sum_{j, k \in \mathbb{J}; p, q \in \mathbb{P}} \langle \eta_p, a_j \rangle \langle f_{pq}, x_{jk} \rangle \langle v_q, b_k \rangle. \quad (3.3)$$

Via the Blecher–Smith embedding (3.2), consider now  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$  as a subspace of  $A \overset{sh}{\otimes} X \overset{sh}{\otimes} A$ . Then using (2.8), consider the mapping  $q$  defined in Section 2 as having values in  $CB(Y, A)$ . Then from (3.1) and (3.3), we obtain the following.

LEMMA 3.1. *The mapping  $Q$  is an extension of  $q$ .*

Remark 3.2. Restricting the Blecher–Smith embedding (3.2) to the Haagerup tensor product, we obtain an isometric embedding

$$A \overset{h}{\otimes} X \overset{h}{\otimes} A \hookrightarrow A \overset{sh}{\otimes} X \overset{sh}{\otimes} A. \quad (3.4)$$

Of course the restriction of  $Q$  to  $A \overset{h}{\otimes} X \overset{h}{\otimes} A$  coincides with  $q_1$ . Since  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$  is defined as the dual space of a subspace of  $(A \overset{h}{\otimes} X \overset{h}{\otimes} A)^*$ , it is a quotient of  $(A \overset{h}{\otimes} X \overset{h}{\otimes} A)^{**}$  in a canonical way. It turns out that the embedding (3.4) is simply obtained by first embedding  $A \overset{h}{\otimes} X \overset{h}{\otimes} A$  into its bidual and then passing to the quotient. It therefore follows from Goldstine's Theorem that the unit ball of  $A \otimes X \otimes A$  (equipped with  $\| \cdot \|_h$ ) is  $w^*$ -dense in the closed unit ball of  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$ . And our mapping  $Q$  is the unique  $w^*$ -continuous extension of  $q_0$  to  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$ .

We can now give the main result of this section, which is a precise description of the range of  $Q$ . It should be viewed as a normal analogue of property (I) in Section 1.

**THEOREM 3.3.** *Let  $\phi: Y \rightarrow A$  be a completely bounded map. Then  $\phi$  belongs to  $Im(Q)$  if and only if  $\phi$  is semidiscrete in the sense of [22]. More explicitly, let  $C > 0$  be a constant, then the following assertions are equivalent.*

(i)  $\phi = Q(T)$  for some  $T \in A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$  and the norm of the coset of  $T$  modulo  $ker(Q)$  is  $\leq C$ .

(ii) There exists a net  $\phi_t: Y \rightarrow A$  of finite rank operators converging to  $\phi$  in the point- $w^*$  topology such that every  $\phi_t$  admits a factorization  $\phi_t = v_t u_t$ , with

$$Y \xrightarrow{u_t} M_{n_t} \xrightarrow{v_t} A \quad \text{and} \quad \|u_t\|_{cb} \|v_t\|_{dec} < C. \tag{3.5}$$

*Proof.* Assume that  $\phi$  is the point- $w^*$  limit of a net  $\phi_t = v_t u_t$  as in (3.5). For each  $t$ , let  $z_t \in X \otimes A$  be representing  $\phi_t$ . By [28, Corollary 12.5] (i.e., the predual version of (I)), there exists  $T_t \in A \otimes X \otimes A$  such that  $q_0(T_t) = z_t$  and  $\|T_t\|_h < C$ . Let  $T \in A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$  be a cluster point of  $(T_t)_t$  in the  $w^*$ -topology of  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$ . Since  $Q$  is  $w^*$ -continuous and extends  $q_0$ ,  $Q(T)$  is a cluster point of  $(\phi_t)_t$  in the  $w^*$ -topology of  $CB(Y, A)$ . Since the latter is stronger than the point- $w^*$  topology, we deduce that  $\phi = Q(T)$ , hence  $\phi \in Im(Q)$ . Moreover the norm of  $T$  in  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$  is less than  $C$ .

Assume conversely that  $\phi = Q(T)$  with  $\|T\| \leq C$ . By Remark 3.2, we may find a net  $(T_t)_t$  in  $A \otimes X \otimes A$  converging to  $T$  in the  $w^*$ -topology of  $A \overset{\sigma h}{\otimes} X \overset{\sigma h}{\otimes} A$ , with  $\|T_t\|_h < C$ . For each  $t$ , let  $\phi_t: Y \rightarrow A$  be the linear mapping represented by  $Q(T_t) = q_0(T_t)$ . Then  $\phi_t$  admits a factorization  $\phi_t = v_t u_t$  as in (3.5). Arguing as above, we find that  $\phi_t$  converges to  $\phi$  in the point- $w^*$  topology, whence the result. ■

By the equivalence of injectivity and semidiscreteness for von Neumann algebras (see [7]), we easily deduce the following analogue of property (II).

**COROLLARY 3.4.** *Let  $A$  be a von Neuman algebra, then  $Q$  is onto for any dual operator space  $X$  if and only if  $A$  is injective.*

We recall from [22, Definition 4.1 and Theorem 4.3] that condition (ii) in Theorem 3.3 is equivalent to the property that for any  $C^*$ -algebra  $D$ , the tensor map  $\phi \otimes I_D$  extends to a bounded mapping from  $Y \otimes_{\min} D$  into  $A \otimes_{\text{nor}} D$ , with

$$\|\phi \otimes I_D: Y \otimes_{\min} D \rightarrow A \otimes_{\text{nor}} D\| \leq C. \quad (3.6)$$

Relying upon that result, we shall prove the following analogue of (III).

**PROPOSITION 3.5.** *Assume that  $Y=B$  is a  $C^*$ -algebra. Then  $\text{Im}(Q) \subset \text{DEC}(B, A)$  and the induced mapping  $Q: A \overset{\text{sh}}{\otimes} B^* \overset{\text{sh}}{\otimes} A \rightarrow \text{DEC}(B, A)$  is a quotient map onto its range. More explicitly, let  $\phi: B \rightarrow A$  be a semidiscrete mapping. Then the smallest constant for which property (ii) in Theorem 3.3 holds is equal to  $C = \|\phi\|_{\text{dec}}$ .*

*Proof.* Let  $T \in A \overset{\text{sh}}{\otimes} B^* \overset{\text{sh}}{\otimes} A$  and let  $\phi = Q(T)$ . We know from Theorem 3.3 and [22] that  $\phi$  satisfies (3.6) with  $C = \|T\|$  for any  $C^*$ -algebra  $D$ . A fortiori, it satisfies (ii) in Proposition 2.1 with  $C = \|T\|$ , hence by the latter,  $\phi$  is decomposable with  $\|\phi\|_{\text{dec}} \leq \|T\|$ . We obtain that  $Q: A \overset{\text{sh}}{\otimes} B^* \overset{\text{sh}}{\otimes} A \rightarrow \text{DEC}(B, A)$  is a contraction. (Note that this result can be proved without appealing to Proposition 2.1.)

Again, let  $\phi = Q(T)$  and let  $D$  be an arbitrary  $C^*$ -algebra. We let

$$\phi_{\min}: B \otimes_{\min} D \rightarrow A \otimes_{\text{nor}} D$$

denote the bounded extension of  $\phi \otimes I_D$  given by (3.6) and we let

$$\pi: B \otimes_{\max} D \rightarrow B \otimes_{\min} D$$

denote the  $*$ -representation induced by the identity mapping on  $B \otimes D$ . By Proposition 2.1, we have  $\|\phi_{\min} \circ \pi\| \leq \|\phi\|_{\text{dec}}$ . However,  $\pi$  is a  $*$ -representation, hence a quotient map, whence  $\|\phi_{\min}\| = \|\phi_{\min} \circ \pi\|$ . We deduce that  $\|\phi_{\min}\| \leq \|\phi\|_{\text{dec}}$ , i.e., (3.6) holds with  $C = \|\phi\|_{\text{dec}}$ . We thus obtain (by [22, Theorem 4.3]) that property (ii) in Theorem 3.3 holds with  $C = \|\phi\|_{\text{dec}}$ . ■

#### 4. EXTENSION AND REPRESENTATION OF NORMAL DECOMPOSABLE OPERATORS

Let  $\mathcal{H}, \mathcal{K}$  be Hilbert spaces and let  $B \subseteq \mathcal{B}(\mathcal{H})$  be a von Neumann algebra acting on  $\mathcal{H}$ . It was proved in [17, Corollary 2.13] that any

normal completely bounded map  $\phi: B \rightarrow B(\mathcal{H})$  is necessarily of the form

$$\phi(y) = \sum_{i \in \mathbb{I}} \alpha_i y \beta_i \quad (y \in B)$$

for some index set  $\mathbb{I}$ , where  $(\beta_i)_{i \in \mathbb{I}} \in C_{\mathbb{I}}(B(\mathcal{H}), \mathcal{H})$ ,  $(\alpha_i)_{i \in \mathbb{I}} \in R_{\mathbb{I}}(B(\mathcal{H}), \mathcal{H})$ , and

$$\left\| \sum_{i \in \mathbb{I}} \alpha_i \alpha_i^* \right\|^{1/2} \left\| \sum_{i \in \mathbb{I}} \beta_i^* \beta_i \right\|^{1/2} = \|\phi\|_{cb}. \tag{4.1}$$

In particular, any  $\phi \in NCB(B, B(\mathcal{H}))$  extends to an element of  $NCB(B(\mathcal{H}), B(\mathcal{H}))$  of same completely bounded norm. It should be noticed that this extension property is no longer true if  $B$  is replaced by an arbitrary dual operator space (see [12, p. 148]).

Here we shall be interested in the extension property for normal decomposable maps with values in an arbitrary von Neumann algebra. We let  $A \subseteq B(\mathcal{H})$  be a von Neumann algebra acting on  $\mathcal{H}$  and we give ourselves some  $\phi: B \rightarrow A$  in  $NDEC(B, A)$ . In general, such a mapping does not extend to a normal decomposable map from  $B(\mathcal{H})$  into  $A$ , even if  $A$  is injective. See in particular Corollary 4.4 below. In Proposition 4.2, we provide a criterion for such an extension to exist. We shall need the following lemma whose proof is related to that of Proposition 3.5.

**LEMMA 4.1.** *Assume that the mapping  $\tilde{\phi}$  defined by (2.12) and (2.13) admits a normal bounded extension  $\bar{\phi}: B \otimes A' \rightarrow B(\mathcal{H})$ . Then  $\bar{\phi}$  is automatically completely bounded and  $\|\bar{\phi}\|_{cb} = \|\phi\|_{dec}$ .*

*Proof.* Since  $\phi: B \rightarrow A$  is decomposable, it follows from Proposition 2.1 that  $\tilde{\phi}$  extends to a completely bounded mapping  $\tilde{\phi}_{max}: B \otimes_{max} A' \rightarrow B(\mathcal{H})$ , with  $\|\tilde{\phi}_{max}\|_{cb} \leq \|\phi\|_{dec}$ . By assumption,  $\tilde{\phi}$  admits a normal bounded extension  $\bar{\phi}: B \otimes A' \rightarrow B(\mathcal{H})$ . We let  $\tilde{\phi}_{min}: B \otimes_{min} A' \rightarrow B(\mathcal{H})$  denote its restriction to the minimal tensor product. Then we let  $\pi: B \otimes_{max} A' \rightarrow B \otimes_{min} A'$  be the  $*$ -representation induced by the identity mapping on  $B \otimes A'$ , so that  $\tilde{\phi}_{max} = \tilde{\phi}_{min} \circ \pi$ . Being a  $*$ -representation,  $\pi$  is actually a complete quotient map; i.e., for any  $n \geq 1$  and any  $z \in M_n(B \otimes_{min} A')$  with  $\|z\|_{min} < 1$ , there exists  $\hat{z} \in M_n(B \otimes_{max} A')$  such that  $\|\hat{z}\|_{max} < 1$  and  $(I_{M_n} \otimes \pi)(\hat{z}) = z$ . It therefore follows from the equality  $\tilde{\phi}_{max} = \tilde{\phi}_{min} \circ \pi$  that  $\tilde{\phi}_{min}$  is completely bounded, with  $\|\tilde{\phi}_{min}\|_{cb} = \|\tilde{\phi}_{max}\|_{cb}$ , whence  $\|\tilde{\phi}_{min}\|_{cb} \leq \|\phi\|_{dec}$ . Since  $B \otimes_{min} A'$  is weakly dense in  $B \otimes A'$ , an appeal to Kaplansky's density theorem finally shows that  $\bar{\phi}$  is completely bounded as well, with  $\|\bar{\phi}\|_{cb} \leq \|\phi\|_{dec}$ . ■

PROPOSITION 4.2. *Let  $B \subseteq \mathbf{B}(\mathcal{H})$  and  $A \subseteq \mathbf{B}(\mathcal{H})$  be von Neumann algebras and let  $\phi \in \text{NDEC}(B, A)$ . The following three assertions are equivalent.*

(i) *The mapping  $\tilde{\phi}$  defined by (2.12) and (2.13) admits a normal bounded extension*

$$\bar{\phi}: B \bar{\otimes} A' \rightarrow \mathbf{B}(\mathcal{H}).$$

(ii)  *$\phi$  admits a normal decomposable extension  $\Phi: B(\mathcal{H}) \rightarrow A$ , with  $\|\Phi\|_{\text{dec}} = \|\phi\|_{\text{dec}}$ .*

(iii)  *$\phi$  admits a normal decomposable extension  $\Phi: B(\mathcal{H}) \rightarrow A$ .*

*Proof.* Assume (i) and apply Lemma 4.1. We get that  $\|\bar{\phi}\|_{\text{cb}} = \|\phi\|_{\text{dec}}$  hence, regarding  $B \bar{\otimes} A'$  as a von Neumann subalgebra of  $\mathbf{B}(\mathcal{H}) \bar{\otimes} \mathbf{B}(\mathcal{H}) = B(\mathcal{H} \bar{\otimes}^2 \mathcal{H})$ , we know (see (4.1)) that there exist an index set  $\mathbb{I}$  and families  $(\beta_i)_{i \in \mathbb{I}} \in C_{\mathbb{I}}(B(\mathcal{H}), \mathcal{H} \bar{\otimes}^2 \mathcal{H})$  and  $(\alpha_i)_{i \in \mathbb{I}} \in R_{\mathbb{I}}(B(\mathcal{H} \bar{\otimes}^2 \mathcal{H}), \mathcal{H})$  such that  $\|\sum_{i \in \mathbb{I}} \alpha_i \alpha_i^*\|^{1/2} \|\sum_{i \in \mathbb{I}} \beta_i^* \beta_i\|^{1/2} = \|\phi\|_{\text{dec}}$ , and for any  $z \in B \bar{\otimes} A'$ , we have

$$\bar{\phi}(z) = \sum_{i \in \mathbb{I}} \alpha_i z \beta_i. \tag{4.2}$$

Furthermore  $\bar{\phi}$  is an  $A'$ -bimodule map because  $\tilde{\phi}$  is. Hence essentially the same argument as in the proof of [23, Theorem 1.2] shows that the  $\alpha_i$ 's and  $\beta_i$ 's can be chosen to satisfy

$$\beta_i a' = (1 \otimes a') \beta_i \quad \text{and} \quad \alpha_i (1 \otimes a') = a' \alpha_i \quad (a' \in A'). \tag{4.3}$$

Identifying  $\mathcal{H}$  with  $\ell_{\mathbb{J}}^2$  for some index set  $\mathbb{J}$ , we may identify  $\mathbf{B}(\mathcal{H}, \mathcal{H} \bar{\otimes}^2 \mathcal{H})$  with the space of column matrices  $C_{\mathbb{J}}(\mathbf{B}(\mathcal{H}))$  and  $\mathbf{B}(\mathcal{H} \bar{\otimes}^2 \mathcal{H}, \mathcal{H})$  with the space of row matrices  $R_{\mathbb{J}}(\mathbf{B}(\mathcal{H}))$ . For any  $i \in \mathbb{I}$ , let  $(a_{ij})_{j \in \mathbb{J}}$  and  $(b_{ji})_{j \in \mathbb{J}}$  be the components of  $\alpha_i$  and  $\beta_i$  with respect to these identifications. Then (4.3) shows that for any  $i \in \mathbb{I}$  and  $j \in \mathbb{J}$ ,  $a_{ij}$  and  $b_{ji}$  commute with  $A'$ , hence belong to  $A$ . We thus obtain that  $(b_{ji})_{j \in \mathbb{J}, i \in \mathbb{I}} \in C_{\mathbb{J} \times \mathbb{I}}(A)$  and  $(a_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}} \in R_{\mathbb{I} \times \mathbb{J}}(A)$ , with

$$\left\| \sum_{i \in \mathbb{I}, j \in \mathbb{J}} a_{ij} a_{ij}^* \right\|^{1/2} \left\| \sum_{i \in \mathbb{I}, j \in \mathbb{J}} b_{ji}^* b_{ji} \right\|^{1/2} = \|\phi\|_{\text{dec}}.$$

By Lemma 2.3, we can therefore define a normal decomposable mapping  $\Phi: \mathbf{B}(\mathcal{H}) = M_{\mathbb{J}}(\mathbb{C}) \rightarrow A$  given by

$$\Phi(s) = \sum_{i \in \mathbb{I}; j, k \in \mathbb{J}} a_{ij} s_{jk} b_{ki} \quad (s = (s_{jk}) \in M_{\mathbb{J}}(\mathbb{C})), \tag{4.4}$$

and we have  $\|\Phi\|_{\text{dec}} \leq \|\phi\|_{\text{dec}}$ . Clearly we have  $\Phi(y) = \sum_{i \in \mathbb{I}} \alpha_i(y \otimes 1) \beta_i$  for any  $y \in B(\mathcal{H})$ . It therefore follows from (4.2) that if  $y \in B$ , we have

$$\phi(y) = \tilde{\phi}(y \otimes 1) = \bar{\phi}(y \otimes 1) = \Phi(y),$$

whence (ii).

We now assume (iii) and shall prove (i). We let  $\Phi \in NDEC(B(\mathcal{H}), A)$  be an extension of  $\phi$ , and we define  $\tilde{\Phi}: B(\mathcal{H}) \otimes A' \rightarrow B(\mathcal{H})$  by means of (2.12) and (2.13). We let  $K(\mathcal{H})$  denote the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ , and we denote by  $\Phi_0$  the restriction of  $\Phi$  to  $K(\mathcal{H})$ , so that the associated map

$$\tilde{\Phi}_0: K(\mathcal{H}) \otimes A' \rightarrow B(\mathcal{H})$$

is the restriction of  $\tilde{\Phi}$  to  $K(\mathcal{H}) \otimes A'$ . We know from Proposition 2.1 that  $\tilde{\Phi}_0$  extends to a completely bounded map from  $K(\mathcal{H}) \otimes_{\max} A'$  into  $B(\mathcal{H})$ . However,  $K(\mathcal{H})$  is a nuclear  $C^*$ -algebra, hence  $K(\mathcal{H}) \otimes_{\max} A' = K(\mathcal{H}) \otimes_{\min} A'$ , hence we actually obtain a completely bounded mapping

$$\tilde{\Phi}_{0, \min}: K(\mathcal{H}) \otimes_{\min} A' \rightarrow B(\mathcal{H})$$

extending  $\tilde{\Phi}_0$ . By construction,  $\tilde{\Phi}_{0, \min}$  is an  $A'$ -bimodule mapping. It therefore follows from [24, Lemma 6.6] that it admits a (necessarily unique) normal completely bounded extension

$$\bar{\Phi}: B(\mathcal{H}) \bar{\otimes} A' \rightarrow B(\mathcal{H}).$$

The latter necessarily coincides with  $\tilde{\Phi}$  on  $B(\mathcal{H}) \otimes A'$ . Indeed  $\tilde{\Phi}$  and  $\bar{\Phi}$  are  $A'$ -bimodule mappings, normal on  $B(\mathcal{H}) \otimes 1$  and they coincide on  $K(\mathcal{H}) \otimes A'$ . In particular, the restriction of  $\bar{\Phi}$  to  $B \otimes A'$  coincides with  $\tilde{\phi}$ . Regarding  $B \bar{\otimes} A'$  as a von Neumann subalgebra of  $B(\mathcal{H}) \bar{\otimes} A'$ , and using the restriction  $\bar{\phi}: B \bar{\otimes} A' \rightarrow B(\mathcal{H})$  of  $\bar{\Phi}$  to that algebra, we deduce that  $\bar{\phi}$  satisfies (i). ■

Although it is not essential for our purpose, we mention here that the equivalence of (ii) and (iii) in Proposition 4.2 remains true without the normality conditions. Using similar arguments as in Lemma 4.1 and Proposition 4.2, we prove the following result.

**PROPOSITION 4.3.** *Let  $C$  be an arbitrary  $C^*$ -algebra, let  $B \subset C$  be a  $C^*$ -subalgebra, let  $A \subseteq B(\mathcal{H})$  be a von Neumann algebra and let  $\phi: B \rightarrow A$  be a decomposable operator. If  $\phi$  extends to a decomposable operator from  $C$  into  $A$ , then it actually admits a decomposable extension  $\Phi: C \rightarrow A$  with  $\|\Phi\|_{\text{dec}} = \|\phi\|_{\text{dec}}$ .*

*Proof.* By assumption, we have a decomposable operator  $\varphi: C \rightarrow A$  extending  $\phi$ . Applying (2.12), (2.13), and Proposition 2.1 to  $\phi$  and  $\varphi$ , we obtain completely bounded maps  $\tilde{\varphi}_{\max}: C \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{K})$  and  $\tilde{\phi}_{\max}: B \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{K})$  extending  $\tilde{\varphi}$  and  $\tilde{\phi}$ , with

$$\|\tilde{\phi}_{\max}: B \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{K})\|_{\text{cb}} = \|\phi\|_{\text{dec}}. \quad (4.5)$$

Let  $\gamma$  be the  $C^*$ -norm on  $B \otimes A'$  induced by  $C \otimes_{\max} A'$ , so that  $B \otimes_{\gamma} A' \subset C \otimes_{\max} A'$  is a  $C^*$ -algebraic embedding. Then the identity mapping on  $B \otimes A'$  induces a  $*$ -representation

$$\pi: B \otimes_{\max} A' \rightarrow B \otimes_{\gamma} A'$$

and by construction, we have  $\tilde{\varphi}_{\max} \circ \pi = \tilde{\phi}_{\max}$  on  $B \otimes_{\max} A'$ . Since  $\pi$  is a complete quotient map, we deduce from (4.5) that

$$\|\tilde{\phi}_{\max}|_{B \otimes_{\gamma} A'}: B \otimes_{\gamma} A' \rightarrow \mathbf{B}(\mathcal{K})\|_{\text{cb}} = \|\phi\|_{\text{dec}}.$$

Moreover, this mapping  $\tilde{\phi}_{\max}|_{B \otimes_{\gamma} A'}$  is an  $A'$ -bimodule map, hence by Wittstock's extension theorem for bimodule maps [34], we may extend it to an  $A'$ -bimodule completely bounded map  $\theta: C \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{K})$ , with  $\|\theta\|_{\text{cb}} = \|\phi\|_{\text{dec}}$ . We let  $\Phi: C \rightarrow \mathbf{B}(\mathcal{K})$  be defined by  $\Phi(c) = \theta(c \otimes 1)$  for any  $c \in C$ . By the bimodule property of  $\theta$ , we have for any  $c \in C$  and any  $a' \in A'$  that

$$\Phi(c) a' = \theta(c \otimes a') = a' \Phi(c).$$

This shows that  $\Phi$  actually has values in  $A$ , and by construction  $\Phi: C \rightarrow A$  is a completely bounded extension of  $\phi$ . Again we define  $\tilde{\Phi}: C \otimes A' \rightarrow \mathbf{B}(\mathcal{K})$  by means of (2.12) and (2.13). It turns out that  $\tilde{\Phi}$  is simply the restriction of  $\theta$  to the algebraic tensor product  $C \otimes A'$ , hence the equality  $\|\theta\|_{\text{cb}} = \|\phi\|_{\text{dec}}$  means that

$$\|\tilde{\Phi}: C \otimes_{\max} A' \rightarrow \mathbf{B}(\mathcal{K})\|_{\text{cb}} = \|\phi\|_{\text{dec}}.$$

By Proposition 2.1, this yields the result that  $\Phi$  is decomposable with  $\|\Phi\|_{\text{dec}} = \|\phi\|_{\text{dec}}$ . ■

It has been proved by Pisier [29] and, independently, by Christensen and Sinclair [6], that a von Neumann algebra  $A \subseteq \mathbf{B}(\mathcal{K})$  is injective provided that there exists a completely bounded projection from  $\mathbf{B}(\mathcal{K})$  onto  $A$ . The result we just proved yields a similar result in the context of normal projection. Before stating it, observe by (2.10) that if  $A$  is an injective von Neumann algebra, Proposition 4.2 can be rephrased as follows. If



$\phi: B \rightarrow A$  is a normal completely bounded mapping which admits a normal completely bounded extension to  $B(\mathcal{H})$ , then it admits such an extension with a completely bounded norm equal to  $\|\phi\|_{cb}$ .

**COROLLARY 4.4.** *For a von Neumann algebra  $A \subseteq B(\mathcal{H})$ , the following assertions are equivalent.*

- (i) *There exists a normal completely bounded projection from  $B(\mathcal{H})$  onto  $A$ .*
- (ii) *There exists a normal completely contractive projection (= normal conditional expectation) from  $B(\mathcal{H})$  onto  $A$ .*
- (iii)  *$A$  is the discrete direct sum of type I factors.*

*Proof.* The equivalence between (ii) and (iii) is due to Tomiyama, see the proof of [33, Theorem 5]. Since (i) implies that  $A$  is injective by [6, 29], the assertion that (i) implies (ii) follows from Proposition 4.2 applied to the identity operator on  $A$ . ■

We now turn to a description of normal decomposable operators defined on  $B(\mathcal{H})$ . In the next statement, the implication “(ii)  $\Rightarrow$  (i)” is Lemma 2.3 whereas the implication “(i)  $\Rightarrow$  (ii)” follows from the proof of Proposition 4.2 in the case  $B = B(\mathcal{H})$  (see in particular (4.4)).

**PROPOSITION 4.5.** *Let  $\mathcal{H}$  be a Hilbert space, that we identify with  $\ell_{\mathbb{J}}^2$  for some index set  $\mathbb{J}$ , and let  $A$  be an arbitrary von Neumann algebra. Given a linear mapping  $\phi: B(\mathcal{H}) \rightarrow A$ , and a constant  $C \geq 0$ , the following are equivalent.*

- (i)  *$\phi$  is decomposable, normal, and  $\|\phi\|_{dec} \leq C$ .*
- (ii) *There exist an index set  $\mathbb{I}$  and two families  $(a_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}} \in \mathbb{R}_{\mathbb{I} \times \mathbb{J}}(A)$ ,  $(b_{ji})_{j \in \mathbb{J}, i \in \mathbb{I}} \in C_{\mathbb{J} \times \mathbb{I}}(A)$  such that  $\|\sum_{i,j} a_{ij} a_{ij}^*\|^{1/2} \|\sum_{i,j} b_{ji}^* b_{ji}\|^{1/2} \leq C$  and for any  $s = (s_{jk}) \in B(\mathcal{H}) = M_{\mathbb{J}}(\mathbb{C})$ ,*

$$\phi(s) = \sum_{i \in \mathbb{I}; j, k \in \mathbb{J}} a_{ij} s_{jk} b_{ki}.$$

*Remark 4.6.* With  $\mathcal{H}$  and  $A$  as in Proposition 4.5, let  $\phi: B(\mathcal{H}) \rightarrow A$  be a normal completely positive map. Then (ii) holds (for  $C = \|\phi\|$ ) with the additional property that  $b_{ji} = a_{ij}^*$  for any  $i \in \mathbb{I}, j \in \mathbb{J}$ . Indeed writing the proof of Proposition 4.2 for  $\phi$ , we see that  $\tilde{\phi}$  extends to a normal completely positive mapping  $\bar{\phi}: B(\mathcal{H}) \bar{\otimes} A' \rightarrow B(\mathcal{H})$  which is an  $A'$ -bimodule map and satisfies  $\|\bar{\phi}\|_{cb} = \|\phi\|_{dec} = \|\phi\|$ . Then the argument in the proof of [23, Theorem 1.2] yields (4.2) and (4.3), with  $\beta_i = \alpha_i^*$ , whence the result.

The latter result, characterizing normal completely positive maps defined on some  $B(\mathcal{H})$  is due to C. Anantharaman–Delaroche. Indeed, it can be derived from [1, Corollary 4.4]. Note that conversely, the implication “(i)  $\Rightarrow$  (ii)” in Proposition 4.5 can be deduced from the completely positive case. Let us outline the argument for the sake of completeness. We shall apply (2.8) with  $Y = K(\mathcal{H})$  and  $X^* = B(\mathcal{H})$  and use normal extensions yielding that identification. Assume that  $\phi: B(\mathcal{H}) \rightarrow A$  is normal and decomposable. Let  $\phi_0: K(\mathcal{H}) \rightarrow A$  be its restriction to the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ . Using the relative compactness in the point- $w^*$  topology of bounded sets of  $B(K(\mathcal{H}), A)$ , we see that the infimum defining  $\|\phi_0\|_{\text{dec}}$  is attained, that is there exists a completely positive map  $w: K(\mathcal{H}) \rightarrow M_2(A)$  of the form

$$w = \begin{pmatrix} \sigma & \phi_0 \\ \phi_0^* & \tau \end{pmatrix}$$

with  $\|\sigma\| \leq \|\phi\|_{\text{dec}}$  and  $\|\tau\| \leq \|\phi\|_{\text{dec}}$ . We let  $\hat{\sigma}$ ,  $\hat{\tau}$ , and  $\hat{w}$  be the normal extensions of  $\sigma$ ,  $\tau$ ,  $w$  to  $B(\mathcal{H})$ . Since  $\phi$  is the normal extension of  $\phi_0$ , we see that

$$\hat{w} = \begin{pmatrix} \hat{\sigma} & \phi \\ \phi^* & \hat{\tau} \end{pmatrix}.$$

We then apply the Anantharaman–Delaroche result to the normal completely positive map  $\hat{w}$ . We find  $\mathbb{I}$  and  $(\rho_{ki}) \in C_{\mathbb{J} \times \mathbb{I}}(M_2(A))$  so that writing  $\rho_{ki} = \begin{pmatrix} \alpha_{ki} & \beta_{ki} \\ \gamma_{ki} & \delta_{ki} \end{pmatrix}$  for any  $k, i$ , we have

$$\phi(s) = \sum_{i \in \mathbb{I}; j, k \in \mathbb{J}} \alpha_{ji}^* s_{jk} \beta_{ki} + \gamma_{ji}^* s_{jk} \delta_{ki}$$

and  $\hat{\sigma}(s) = \sum_{i, j, k} \alpha_{ji}^* s_{jk} \alpha_{ki} + \gamma_{ji}^* s_{jk} \gamma_{ki}$ ,  $\hat{\tau}(s) = \sum_{i, j, k} \beta_{ji}^* s_{jk} \beta_{ki} + \delta_{ji}^* s_{jk} \delta_{ki}$ . We deduce that  $\phi$  satisfies (ii) in Proposition 4.5 with  $C = \|\hat{\sigma}(1)\|^{1/2} \|\hat{\tau}(1)\|^{1/2}$ , hence with  $C \leq \|\phi\|_{\text{dec}}$ .

*Remark 4.7* (Added in October 2000). It follows from some recent work of Pop *et al.* [30] that our Corollary 4.4 can be improved, as follows. A von Neumann algebra  $A \subseteq B(\mathcal{H})$  is the discrete direct sum of type I factors if (and only if) there exists a normal bounded projection from  $B(\mathcal{H})$  onto  $A$ . The proof uses the Radon–Nikodym property of Banach spaces for which we refer to [8]. Assume that  $\Phi: B(\mathcal{H}) \rightarrow A$  is a normal bounded projection. By [30, Theorem 3.1],  $A$  is type I. Hence  $A$  can be written as a direct sum,  $A = \bigoplus_i L^\infty(\Omega_i, \mu_i) \bar{\otimes} B(H_i)$ , where each  $(\Omega_i, \mu_i)$  is a measure space and each  $H_i$  is a Hilbert space. Given any index  $i$ , let  $p_i$  be a normal state on  $B(H_i)$ . Then  $I \otimes p_i: L^\infty(\Omega_i, \mu_i) \otimes B(H_i) \rightarrow L^\infty(\Omega_i, \mu_i)$  extends to a normal projection from  $L^\infty(\Omega_i, \mu_i) \bar{\otimes} B(H_i)$  onto  $L^\infty(\Omega_i, \mu_i)$ .

Hence there exists a normal bounded projection  $\Phi_i: B(\mathcal{H}) \rightarrow L^\infty(\Omega_i, \mu_i)$ . Then the adjoint mapping  $\Phi_i^*$  induces a Banach space isomorphism between  $L^1(\Omega_i, \mu_i)$  and a closed subspace of  $B(\mathcal{H})_*$ . Now  $B(\mathcal{H})_*$  is isometrically isomorphic to the Banach space of nuclear operators on  $\mathcal{H}$  hence it has the Radon–Nikodym property (see [8; Chap. VII.7]). Therefore  $L^1(\Omega_i, \mu_i)$  has the Radon–Nikodym property as well, hence  $(\Omega_i, \mu_i)$  is purely atomic (see [8; Chap. VII.7]). This shows that each  $L^\infty(\Omega_i, \mu_i) \bar{\otimes} B(H_i)$  is a discrete direct sum of type I factors, hence the same holds for  $A$ .

### 5. DESCRIPTION AND PROPERTIES OF THE RANGE OF $q$

In this section,  $A$  will denote a von Neumann algebra. Given any operator space  $X$ , we consider the contractive mapping  $q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A)$  defined in Subsection 2.1, see in particular (2.6). We recall that when  $X = Y^*$  is a dual operator space, we regard  $q$  as having values in  $CB(Y, A)$ .

**THEOREM 5.1.** (1) *A mapping  $\phi \in NCB(X^*, A)$  belongs to  $Im(q)$  if and only if there exist a Hilbert space  $\mathcal{H}$ , and a factorization*

$$\phi = vu, \quad X^* \xrightarrow{u} B(\mathcal{H}) \xrightarrow{v} A, \tag{5.1}$$

where  $u \in NCB(X^*, B(\mathcal{H}))$  and  $v \in NDEC(B(\mathcal{H}), A)$ . Moreover, if  $\phi = q(T)$ , then the norm of the coset of  $T$  in  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A / \ker(q)$  coincides with

$$\inf\{ \|u\|_{cb} \|v\|_{dec} \}, \tag{5.2}$$

where the infimum is over all possible factorizations  $\phi = vu$  with  $u$  and  $v$  as above.

(2) *Assume that  $X = Y^*$  is a dual operator space. Then a mapping  $\phi \in CB(Y, A)$  belongs to  $Im(q)$  if and only if there exist a Hilbert space  $\mathcal{H}$ , and a factorization  $\phi = vu$  with  $u \in CB(Y, B(\mathcal{H}))$  and  $v \in NDEC(B(\mathcal{H}), A)$ . Moreover if  $\phi = q(T)$ , then the norm of the coset of  $T$  in  $A \overset{eh}{\otimes} X \overset{eh}{\otimes} A / \ker(q)$  coincides with (5.2).*

(3) *Assume that  $X = B_*$  is the operator space predual of a von Neumann algebra  $B$  and let  $\mathcal{H}$  be a Hilbert space so that  $B \subseteq B(\mathcal{H})$ . Then  $Im(q)$  consists of all  $\phi \in NDEC(B, A)$  which extend to an element of  $NDEC(B(\mathcal{H}), A)$ . Moreover, the resulting mapping  $q: A \overset{eh}{\otimes} B_* \overset{eh}{\otimes} A \rightarrow NDEC(B, A)$  is a quotient map onto its range and the latter is closed.*

*Proof.* We first consider the general case (1). Let  $T \in A \overset{eh}{\otimes} X \overset{eh}{\otimes} A$ , let  $\phi = q(T)$ , and let  $C > \|T\|$  be a constant. Then we may write  $T$  in the form

$$T = \sum_{j, k \in \mathbb{J}} a_j \otimes x_{jk} \otimes b_k,$$

where  $(a_j)_j \in R_{\mathbb{J}}(A)$ ,  $(b_k)_k \in C_{\mathbb{J}}(A)$ ,  $(x_{jk})_{j, k} \in M_{\mathbb{J}}(X)$ , and

$$\|(x_{jk})\| < 1, \quad \left\| \sum_j a_j a_j^* \right\|^{1/2} \left\| \sum_k b_k^* b_k \right\|^{1/2} < C. \tag{5.3}$$

Let  $\mathcal{H} = \ell_{\mathbb{J}}^2$  and, using (2.1), let  $u: X^* \rightarrow B(\mathcal{H})$  be the normal completely bounded map induced by  $(x_{jk})$ , i.e.,  $u(x^*) = (\langle x^*, x_{jk} \rangle)$  for any  $x^* \in X^*$ . Then  $\|u\|_{cb} < 1$  by (5.3). We let  $v: B(\mathcal{H}) = M_{\mathbb{J}}(\mathbb{C}) \rightarrow A$  be defined by  $v((s_{jk})) = \sum_{j, k} a_j s_{jk} b_k$ . It follows from Lemma 2.2 and (5.3) that  $v$  is normal and decomposable with  $\|v\|_{dec} < C$ . Applying (2.6), we see that  $\phi = vu$  and we clearly have  $\|u\|_{cb} \|v\|_{dec} < C$ .

Assume conversely that  $\phi: X^* \rightarrow A$  is a normal completely bounded map which admits a factorization of the form (5.1), with  $u \in NCB(X^*, B(\mathcal{H}))$  and  $v \in NDEC(B(\mathcal{H}), A)$ . We identify  $B(\mathcal{H})$  with  $\ell_{\mathbb{J}}^2$  for some index set  $\mathbb{J}$  and apply Proposition 4.5 to  $v$ . We find an index set  $\mathbb{I}$  and two families  $(a_{ij})_{i \in \mathbb{I}, j \in \mathbb{J}} \in R_{\mathbb{I} \times \mathbb{J}}(A)$ ,  $(b_{ji})_{j \in \mathbb{J}, i \in \mathbb{I}} \in C_{\mathbb{J} \times \mathbb{I}}(A)$  such that

$$\left\| \sum_{i, j} a_{ij} a_{ij}^* \right\|^{1/2} \left\| \sum_{i, j} b_{ji}^* b_{ji} \right\|^{1/2} \leq \|v\|_{dec} \tag{5.4}$$

and  $v((s_{jk})) = \sum_{i, j, k} a_{ij} s_{jk} b_{ki}$ . Let  $(x_{jk})_{j, k \in \mathbb{J}}$  be the element of  $M_{\mathbb{J}}(X)$  representing  $u$ , so that  $u(x^*) = (\langle x^*, x_{jk} \rangle)$  for any  $x^* \in X^*$ . Since  $\phi = vu$ , we deduce that we have

$$\phi(x^*) = \sum_{i \in \mathbb{I}; j, k \in \mathbb{J}} a_{ij} \langle x^*, x_{jk} \rangle b_{ki} \quad (x^* \in X^*). \tag{5.5}$$

Defining a new family  $(z_{(i, j)(l, k)})_{(i, j), (l, k) \in \mathbb{I} \times \mathbb{J}}$  in  $M_{\mathbb{I} \times \mathbb{J}}(X)$  by letting  $z_{(i, j)(l, k)} = x_{jk}$  if  $i = l$  and  $z_{(i, j)(l, k)} = 0$  otherwise, we obtain from (5.5) and (2.6) that

$$\phi = q(T) \quad \text{with} \quad T = \sum_{(i, j), (l, k) \in \mathbb{I} \times \mathbb{J}} a_{ij} \otimes z_{(i, j)(l, k)} \otimes b_{kl}.$$

Moreover  $\|(z_{(i, j)(l, k)})\| = \|(x_{jk})\| = \|u\|_{cb}$  hence we deduce from (5.4) that  $\|T\| \leq \|u\|_{cb} \|v\|_{dec}$ . This completes the proof of (1), and (2) is a straightforward consequence of the latter.

We now assume that  $X = B_*$ , with  $B \subseteq B(\mathcal{H})$  a von Neumann algebra and we shall prove (3). Let  $\phi$  be in  $Im(q)$ . By part (1), we have a factorization of the form  $\phi = vu$ ,  $B \xrightarrow{u} B(\mathcal{H}_0) \xrightarrow{v} A$ , with  $u \in NCB(B, B(\mathcal{H}_0))$  and

$v \in NDEC(B(\mathcal{H}_0), A)$ . We know that the mapping  $u$  extends to some  $U \in NCB(B(\mathcal{H}), B(\mathcal{H}_0))$ . Then  $\Phi = vU: B(\mathcal{H}) \rightarrow A$  is a normal extension of  $\phi$ . Moreover it follows from (2.10) and (2.11) that  $\Phi \in NDEC(B(\mathcal{H}), A)$ , hence  $\phi$  admits a normal decomposable extension to  $B(\mathcal{H})$ . (In particular  $\phi$  itself is decomposable.) Conversely, assume that  $\phi$  extends to an element  $\Phi \in NDEC(B(\mathcal{H}), A)$ . Then it follows from Proposition 4.2 that changing the extension if necessary, we may assume  $\|\Phi\|_{dec} = \|\phi\|_{dec}$ . Let  $j: B \rightarrow B(\mathcal{H})$  be the canonical embedding. Applying (1) to the factorization  $\phi = \Phi j$ , we see that  $\phi \in Im(q)$  and  $\|\phi\|_{dec}$  coincides with the norm of the coset of any preimage of  $\phi$  in  $A \otimes_{X^{eh}} X \otimes_{A} A / ker(q)$ . This proves (3). ■

We can now solve the question of characterizing von Neumann algebras  $A$  for which  $q$  is onto for any  $X$ . The following result complements Corollary 3.4.

**COROLLARY 5.2.** *Let  $A$  be a von Neumann algebra, then  $Im(q) = NCB(X^*, A)$  for any operator space  $X$  if and only if  $A$  is a discrete direct sum of type I factors.*

*Proof.* Assume that  $A$  is a discrete direct sum of type I factors, and let  $J: A \rightarrow B(\mathcal{H})$  be a von Neumann embedding for some Hilbert space  $\mathcal{H}$  (i.e., a 1-1 normal  $*$ -representation). Then there exists a normal completely positive projection (of norm 1)  $P: B(\mathcal{H}) \rightarrow A$ , so that  $PJ = I_A$ . Given any  $X$  and any  $\phi \in NCB(X^*, A)$ , write  $\phi = vu$  with  $v = P$  and  $u = J\phi$ . Then we deduce from the first part of Theorem 5.1 that  $\phi$  belongs to  $Im(q)$ , whence  $Im(q) = NCB(X^*, A)$ .

Assume conversely that this property holds for any  $X$  and apply it with  $X = A_*$ . We deduce from the third part of Theorem 5.1 and Corollary 4.4 that  $A$  is indeed a discrete direct sum of type I factors. ■

*Remark 5.3.* We noticed in Section 3 (see Lemma 3.1) that if  $X$  is a dual operator space then  $q$  is the restriction of the mapping  $Q$  defined therein. In view of Corollary 3.4, it is natural to ask if for an injective von Neumann algebra  $A$ , and a dual operator space  $X$ , the mapping  $q$  is necessarily onto. It turns out that the answer is no. Indeed let  $C$  be a nuclear  $C^*$ -algebra and let  $A = C^{**}$  be its second dual (= universal von Neumann algebra). By [11],  $A$  is injective. We take  $X = C^* = A_*$  and assume that the corresponding mapping  $q$  has range equal to  $CB(C, A) = NCB(A, A)$ . Arguing as above, we deduce that  $A$  is a discrete direct sum of type I factors. Thus, if for example we choose  $C$  to be the CAR algebra, we get a contradiction.

Arguing as in Corollary 5.2, it is easy to see that given a von Neumann algebra  $B$ , the mapping  $q: A \otimes_{B_*}^{eh} B_* \otimes_A^{eh} A \rightarrow NDEC(B, A)$  is onto for any  $A$  if and only if  $B$  is a discrete direct sum of type I factors. Another natural question is to characterize the operator spaces  $X$  for which  $Im(q) = NCB(X^*, A)$

for any injective  $A$ . We provide the following partial answer, whose easy proof is left to the reader.

**PROPOSITION 5.4.** *Let  $\mathcal{H}$  be a Hilbert space and let  $Y \subset K(\mathcal{H})$  be a subspace of the  $C^*$ -algebra of compact operators on  $\mathcal{H}$ . Then  $q: A \overset{eh}{\otimes} Y^* \overset{eh}{\otimes} A \rightarrow CB(Y, A)$  is onto for any injective von Neumann algebra  $A$ .*

In the rest of this section, we will establish two properties of the range of  $q$  in the case when  $X$  is the predual of a von Neumann algebra. The first one is a description of  $Im(q)$  as an extended module Haagerup tensor product in the sense of [24] whereas the second one is a stability property of  $Im(q)$  under infinite convex combinations with  $A$ -valued coefficients.

Let  $A \subseteq B(\mathcal{H})$  and  $B \subseteq B(\mathcal{H})$  be two von Neumann algebras and consider the mapping  $q: A \overset{eh}{\otimes} B_* \overset{eh}{\otimes} A \rightarrow NDEC(B, A)$  associated to  $X = B_*$  and  $A$ . We identify  $\mathcal{H}$  with  $\ell_{\mathbb{J}}^2$  for some index set  $\mathbb{J}$ . According to Lemma 2.3, we define a contractive linear mapping

$$\rho: R_{\mathbb{J}}(A) \overset{eh}{\otimes} C_{\mathbb{J}}(A) \rightarrow NDEC(B, A) \quad (5.6)$$

as follows. Let  $(\alpha_i)_{i \in \mathbb{I}}$  and  $(\beta_i)_{i \in \mathbb{I}}$  be in  $R_{\mathbb{I}}(R_{\mathbb{J}}(A))$  and  $C_{\mathbb{I}}(C_{\mathbb{J}}(A))$ , respectively, and let  $(a_{ij})_{j \in \mathbb{J}}$  and  $(b_{ji})_{j \in \mathbb{J}}$  be the components of  $\alpha_i$  and  $\beta_i$ , respectively. Then we let

$$\rho \left( \sum_{i \in \mathbb{I}} \alpha_i \otimes \beta_i \right) = \Phi|_B, \quad (5.7)$$

where  $\Phi$  is defined by (2.14).

**PROPOSITION 5.5.** *The mapping  $\rho$  defined by (5.6) and (5.7) induces an isometric isomorphism*

$$R_{\mathbb{J}}(A) \overset{eh}{\otimes}_{B' \otimes A} C_{\mathbb{J}}(A) = Im(q).$$

*Proof.* It is clear from Theorem 5.1 and Proposition 4.5 that  $\rho$  is a quotient map. Let  $j_0$  be a fixed element of  $\mathbb{J}$  and identify  $R_{\mathbb{J}}(A)$  (resp.  $C_{\mathbb{J}}(A)$ ) with the subspace of  $M_{\mathbb{J}}(A)$  of matrices whose entries are zero except on the row (resp. on the column) indexed by  $j_0$ . Using the following chain of isometries

$$\begin{aligned} R_{\mathbb{J}}(A) \overset{eh}{\otimes} C_{\mathbb{J}}(A) &\subset M_{\mathbb{J}}(A) \overset{eh}{\otimes} M_{\mathbb{J}}(A) \\ &\subset M_{\mathbb{J}}(B(\mathcal{H})) \overset{eh}{\otimes} M_{\mathbb{J}}(B(\mathcal{H})) = NCB(B(\mathcal{H} \overset{2}{\otimes} \mathcal{H})), \end{aligned}$$

we may define

$$\tau: R_{\downarrow}(A) \overset{eh}{\otimes} C_{\downarrow}(A) \rightarrow NCB(B \bar{\otimes} A', B(\mathcal{H} \overset{2}{\otimes} \mathcal{K}))$$

by letting  $\tau(T) = T|_{B \bar{\otimes} A'}$  for any  $T: B(\mathcal{H} \overset{2}{\otimes} \mathcal{K}) \rightarrow B(\mathcal{H} \overset{2}{\otimes} \mathcal{K})$  represented by an element of  $R_{\downarrow}(A) \overset{eh}{\otimes} C_{\downarrow}(A)$ . By definition (see [24, Definition 1.3]), the extended module Haagerup tensor product  $R_{\downarrow}(A) \overset{eh}{\otimes}_{B' \bar{\otimes} A} C_{\downarrow}(A)$  is the range of  $\tau$  equipped with the norm induced by  $NCB(B \bar{\otimes} A', B(\mathcal{H} \overset{2}{\otimes} \mathcal{K}))$ . Moreover, [24, Theorem 3.6] says that  $\tau$  is a quotient map onto its range. Hence to prove our proposition reduces to check that

$$ker(\tau) = ker(\rho). \tag{5.8}$$

Let  $e_{j_0} \in \mathcal{H} = \ell^2_{\downarrow}$  be the unit vector corresponding to  $j_0$  and let  $E = e_{j_0} \otimes e_{j_0} \in B(\mathcal{H})$  be the corresponding rank one projection. Then it is easy to see that for any  $T \in R_{\downarrow}(A) \overset{eh}{\otimes} C_{\downarrow}(A)$ , any  $y \in B$ , and any  $a' \in A'$ , we have

$$\tau(T)(y \otimes a') = E \otimes \rho(T)(y) a'.$$

Then (5.8) follows at once, whence the result. ■

**PROPOSITION 5.6.** *Let  $\mathbb{I}$  be a index set and let  $\varphi: B \rightarrow M_{\mathbb{I}}(A)$  be a normal decomposable map. We let  $\varphi_{i\ell}: B \rightarrow A$  ( $i, \ell \in \mathbb{I}$ ) be the components of  $\varphi$ , so that  $\varphi(y) = (\varphi_{i\ell}(y))$  for any  $y \in B$ . Let  $(a_i)_{i \in \mathbb{I}} \in R_{\mathbb{I}}(A)$ ,  $(b_l)_{l \in \mathbb{I}} \in C_{\mathbb{I}}(A)$ , and let  $\phi: B \rightarrow A$  be defined by*

$$\phi(y) = \sum_{i, \ell \in \mathbb{I}} \alpha_i \varphi_{i\ell}(y) \beta_{\ell}.$$

*Then  $\phi \in Im(q)$  provided that  $\varphi_{i\ell} \in Im(q)$  for each  $(i, \ell) \in \mathbb{I} \times \mathbb{I}$ .*

*Proof.* We assume that each  $\varphi_{i\ell}$  belongs to  $Im(q)$ , and we let

$$q_{\mathbb{I}}: M_{\mathbb{I}}(A) \overset{eh}{\otimes} B_* \overset{eh}{\otimes} M_{\mathbb{I}}(A) \rightarrow NDEC(B, M_{\mathbb{I}}(A))$$

be the mapping studied along the paper, associated to the von Neumann algebra  $M_{\mathbb{I}}(A)$  and the operator space  $X = B_*$ . We shall prove that

$$\phi \in Im(q_{\mathbb{I}}). \tag{5.9}$$

Regarding  $M_{\mathbb{I}}(A) = M_{\mathbb{I}}(\mathbb{C}) \bar{\otimes} A$  as a subalgebra of  $M_{\mathbb{I}}(B(\mathcal{K})) = M_{\mathbb{I}}(\mathbb{C}) \bar{\otimes} B(\mathcal{K})$ , we have  $M_{\mathbb{I}}(A)' = 1 \otimes A'$ . Then the mapping  $\tilde{\varphi}$  associated to  $\varphi$

by (2.12) and (2.13) may be regarded as defined on  $B \otimes A'$  and with this identification, we have

$$\tilde{\varphi}: B \otimes A' \rightarrow M_{\mathbb{I}}(\mathbf{B}(\mathcal{H})),$$

so that for any  $y \in B$  and  $a' \in A'$ ,

$$\tilde{\varphi}(y \otimes a') = \varphi(y)(1 \otimes a') = (\varphi_{il}(y) a')_{i,l \in \mathbb{I}}.$$

Similarly, given any finite subset  $\mathbb{F} \subset \mathbb{I}$ , we let  $\varphi^{\mathbb{F}}: B \rightarrow M_{\mathbb{F}}(A)$  be defined by  $\varphi^{\mathbb{F}}(y) = (\varphi_{il}(y))_{i,l \in \mathbb{F}}$  and we associate  $\widetilde{\varphi^{\mathbb{F}}}: B \otimes A' \rightarrow M_{\mathbb{F}}(\mathbf{B}(\mathcal{H}))$ . By Theorem 5.1, each  $\varphi_{il}$  extends to an element of  $NDEC(\mathbf{B}(\mathcal{H}), A)$ , hence by Proposition 4.2, each  $\widetilde{\varphi_{il}}: B \otimes A' \rightarrow \mathbf{B}(\mathcal{H})$  admits a normal bounded extension  $\overline{\varphi_{il}}: B \overline{\otimes} A' \rightarrow \mathbf{B}(\mathcal{H})$ . Consequently, for any finite  $\mathbb{F} \subset \mathbb{I}$ , the mapping  $\widetilde{\varphi^{\mathbb{F}}}$  admits a normal bounded extension to  $B \overline{\otimes} A'$  obtained by letting  $\overline{\varphi^{\mathbb{F}}}(z) = (\overline{\varphi_{il}}(z))_{i,l \in \mathbb{F}}$ . Since  $\|\varphi^{\mathbb{F}}\|_{\text{dec}} \leq \|\varphi\|_{\text{dec}}$ , our Lemma 4.1 ensures that  $\|\overline{\varphi^{\mathbb{F}}}\|_{\text{cb}} \leq \|\varphi\|_{\text{dec}}$  for any finite  $\mathbb{F}$ . From this uniform bound, we deduce that we may define a completely bounded mapping

$$\overline{\varphi}: B \overline{\otimes} A' \rightarrow M_{\mathbb{I}}(\mathbf{B}(\mathcal{H}))$$

by letting  $\overline{\varphi}(z) = (\overline{\varphi_{il}}(z))_{i,l \in \mathbb{I}}$  for any  $z \in B \overline{\otimes} A'$ . Since each component  $\overline{\varphi_{il}}$  is normal, the mapping  $\overline{\varphi}$  is normal as well. Moreover,  $\overline{\varphi}$  clearly extends  $\tilde{\varphi}$ . By Proposition 4.2, this shows that  $\varphi$  admits a normal decomposable extension to  $\mathbf{B}(\mathcal{H})$ , whence (5.9) by Theorem 5.1.

It is now easy to complete the proof. Let  $\Psi: \mathbf{B}(\mathcal{H}) \rightarrow M_{\mathbb{I}}(A)$  be a normal decomposable extension of  $\varphi$ , with components  $\Psi_{il}$ . Using Lemma 2.2, we obtain that the mapping  $\Phi: B \rightarrow A$  defined by  $\Phi(y) = \sum_{i,l \in \mathbb{I}} a_i \Psi_{il}(y) b_l$  is a normal decomposable mapping extending  $\phi$ , whence the result by Theorem 5.1. ■

## 6. A NORMAL DUAL OPERATOR $A$ -BIMODULE STRUCTURE ON $DEC(B, A)$ AND THE ROLE OF THE $A, A$ -TOPOLOGY

Let  $A$  and  $B$  be two  $C^*$ -algebras. We introduce matrix norms on  $DEC(B, A)$  by letting

$$M_n(DEC(B, A)) = DEC(B, M_n(A)) \quad (6.1)$$

for any  $n \geq 1$ . By this we mean that if  $(\phi_{ij})_{1 \leq i, j \leq n} \in M_n(DEC(B, A))$ , and if  $\phi: B \rightarrow M_n(A)$  is the linear mapping defined by  $\phi(y) = (\phi_{ij}(y))$ , then the



norm of  $(\phi_{ij})$  is defined by  $\|(\phi_{ij})_{1 \leq i, j \leq n}\|_n = \|\phi\|_{\text{dec}}$ . This extends the well-known definition of matrix norms on spaces of completely bounded maps.

Let  $\phi \in \text{DEC}(B, A)$  and let  $a, b \in A$ . Then we may define  $a \cdot \phi \cdot b \in \text{DEC}(B, A)$  by letting  $(a \cdot \phi \cdot b)(y) = a\phi(y)b$  for any  $y \in B$ . This yields a (natural) bimodule action of  $A$  on  $\text{DEC}(B, A)$ . The main result of this section is the following.

**THEOREM 6.1.** (1) *The matrix norms given by (6.1) define an operator space structure on  $\text{DEC}(B, A)$  and the latter is an operator  $A$ -bimodule. That is, there exist a Hilbert space  $H$ , a complete isometry  $J: \text{DEC}(B, A) \rightarrow B(H)$  and a 1-1  $*$ -representation  $\pi: A \rightarrow B(H)$  such that  $J(a \cdot \phi \cdot b) = \pi(a) J(\phi) \pi(b)$  for any  $\phi \in \text{DEC}(B, A)$  and  $a, b \in A$ .*

(2) *Assume that  $A$  is a von Neumann algebra. Then  $\text{DEC}(B, A)$  is a dual operator space. That is, there exists an operator space  $Z$  such that  $Z^* = \text{DEC}(B, A)$  completely isometrically.*

(3) *Assume that  $A$  is a von Neumann algebra. Then  $\text{DEC}(B, A)$  is a normal dual operator  $A$ -bimodule. That is, there exist a Hilbert space  $H$ , a  $w^*$ -continuous complete isometry  $J: \text{DEC}(B, A) \rightarrow B(H)$  and a 1-1 normal  $*$ -representation  $\pi: A \rightarrow B(H)$  such that  $J(a \cdot \phi \cdot b) = \pi(a) J(\phi) \pi(b)$  for any  $\phi \in \text{DEC}(B, A)$  and  $a, b \in A$ .*

*Proof.* According to Ruan’s characterization of operator spaces [31], the identity (6.1) defines an operator space structure on  $\text{DEC}(B, A)$  provided that

$$\forall \phi_1 \in M_n(\text{DEC}(B, A)), \quad \phi_2 \in M_m(\text{DEC}(B, A)),$$

$$\left\| \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \right\|_{n+m} \leq \max\{\|\phi_1\|, \|\phi_2\|\} \tag{6.2}$$

and

$$\forall \phi \in M_n(\text{DEC}(B, A)), \quad \alpha \in M_{m, n}, \quad \beta \in M_{n, m}, \quad \|\alpha\phi\beta\|_m \leq \|\alpha\| \|\phi\|_n \|\beta\|. \tag{6.3}$$

Furthermore it follows from [5] that  $\text{DEC}(B, A)$  is an operator  $A$ -bimodule provided that the bimodule action  $(a, \phi, b) \mapsto a \cdot \phi \cdot b$  extends to a completely contractive trilinear mapping, that is,

$$\forall \phi \in M_n(\text{DEC}(B, A)), \quad \alpha \in M_{m, n}(A), \quad \beta \in M_{n, m}(A),$$

$$\|\alpha\phi\beta\|_m \leq \|\alpha\| \|\phi\|_n \|\beta\|. \tag{6.4}$$

Clearly (6.4) follows from Lemma 2.2 and (6.3) follows from (6.4). To prove (6.2), we let  $\phi_1 \in M_n(DEC(B, A))$ ,  $\phi_2 \in M_m(DEC(B, A))$ , and we let  $C > \max\{\|\phi_1\|, \|\phi_2\|\}$  be a constant. By definition of the decomposable norm, there exist four completely positive maps  $\sigma_1, \tau_1: B \rightarrow M_n(A)$  and  $\sigma_2, \tau_2: B \rightarrow M_m(A)$  such that

$$\max\{\|\sigma_1\|, \|\tau_1\|, \|\sigma_2\|, \|\tau_2\|\} < C \quad (6.5)$$

and the two mappings  $w_1: B \rightarrow M_{2n}(A)$  and  $w_2: B \rightarrow M_{2m}(A)$  defined by

$$w_1(y) = \begin{pmatrix} \sigma_1(y) & \phi_1(y) \\ \phi_1 * (y) & \tau_1(y) \end{pmatrix} \quad \text{and} \quad w_2(y) = \begin{pmatrix} \sigma_2(y) & \phi_2(y) \\ \phi_2 * (y) & \tau_2(y) \end{pmatrix}$$

are completely positive. The mapping  $w: B \rightarrow M_{2n+2m}(A)$  defined by  $w(y) = \begin{pmatrix} w_1(y) & 0 \\ 0 & w_2(y) \end{pmatrix}$  is obviously completely positive and via the canonical identification  $M_{2n+2m}(A) = M_2(M_{n+m}(A))$ , we can rewrite it as

$$w = \begin{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} & \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \\ \begin{pmatrix} \phi_1 * & 0 \\ 0 & \phi_2 * \end{pmatrix} & \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \end{pmatrix}.$$

We deduce that

$$\left\| \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \right\|_{n+m} \leq \max \left\{ \left\| \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \right\|, \left\| \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\| \right\} < C \quad \text{by (6.5).}$$

This proves (6.2) and completes the proof of (1).

Assume that  $A$  is a von Neumann algebra. In proving (2), the first step consists in showing that  $DEC(B, A)$  is a dual Banach space. We consider the usual dual action of  $B(B, A)$  on  $B \otimes A_*$ . (Recall that  $B(B, A)$  is the dual space of the Banach space projective tensor product of  $B$  and  $A_*$ .) Given any  $\theta \in B \otimes A_*$ , we define

$$\gamma(\theta) = \sup\{|\langle \phi, \theta \rangle| : \phi \in DEC(B, A), \|\phi\|_{\text{dec}} \leq 1\}. \quad (6.6)$$

It is plain that  $\gamma$  is a tensor norm on  $B \otimes A_*$  and we let  $B \otimes_{\gamma} A_*$  denote the resulting completion. It follows from the definition of  $\gamma$  that for any  $\phi \in DEC(B, A)$ , we have  $|\langle \phi, \theta \rangle| \leq \|\phi\|_{\text{dec}} \gamma(\theta)$  for every  $\theta \in B \otimes A_*$ . This induces a canonical linear contraction from  $DEC(B, A)$  into  $(B \otimes_{\gamma} A_*)^*$ . We claim that this linear contraction is an isometric isomorphism, so that

$$(B \otimes_{\gamma} A_*)^* = DEC(B, A) \quad \text{isometrically.} \quad (6.7)$$

To prove this identity, we let  $\xi \in (B \otimes_{\gamma} A_*)^*$ . Then we let  $\phi: B \rightarrow A$  be the bounded linear mapping defined by  $\langle \phi(y), \eta \rangle = \langle \xi, y \otimes \eta \rangle$  for any  $y \in B$  and  $\eta \in A_*$ . We will have proved (6.7) if we can show that  $\phi$  is decomposable, with  $\|\phi\|_{\text{dec}} \leq \|\xi\|$ . By Proposition 2.1, it therefore suffices to show that for any  $C^*$ -algebra  $D$ , the tensor product  $\phi \otimes I_D$  extends to a bounded operator from  $B \otimes_{\max} D$  to  $A \otimes_{\text{nor}} D$ , with

$$\|\phi \otimes I_D: B \otimes_{\max} D \rightarrow A \otimes_{\text{nor}} D\| \leq \|\xi\|. \tag{6.8}$$

Let  $w \in B \otimes D$  and let  $(y_k)_{1 \leq k \leq N} \subset B$  and  $(d_k)_{1 \leq k \leq N} \subset D$  be finite families such that  $w = \sum_k y_k \otimes d_k$ . We wish to estimate the norm of

$$(\phi \otimes I_D)(w) = \sum_{1 \leq k \leq N} \phi(y_k) \otimes d_k$$

in  $A \otimes_{\text{nor}} D$ . We let  $\mu$  be a functional of norm 1 on the  $C^*$ -algebra  $A \otimes_{\text{nor}} D$  such that the map  $a \mapsto \langle \mu, a \otimes d \rangle$  is normal on  $A$  for each  $d \in D$ . Then for any  $1 \leq k \leq N$ , let  $\eta_k$  be the functional on  $A$  defined by

$$\langle \eta_k, a \rangle = \langle \mu, a \otimes d_k \rangle \quad (a \in A).$$

By our assumption on  $\mu$ , each  $\eta_k$  belongs to  $A_*$ , hence we may introduce

$$\theta = \sum_k y_k \otimes \eta_k \in B \otimes A_*.$$

Let  $\varphi: B \rightarrow A$  be an arbitrary bounded linear mapping. Then

$$\begin{aligned} \langle \mu, (\varphi \otimes I_D)(w) \rangle &= \left\langle \mu, \sum_k \varphi(y_k) \otimes d_k \right\rangle \\ &= \sum_k \langle \eta_k, \varphi(y_k) \rangle \end{aligned}$$

hence

$$\langle \mu, (\varphi \otimes I_D)(w) \rangle = \langle \varphi, \theta \rangle. \tag{6.9}$$

Since  $\|\mu\| = 1$ , we deduce that in the case when  $\varphi$  is decomposable, we have

$$|\langle \varphi, \theta \rangle| \leq \|(\varphi \otimes I_D)(w)\|_{\text{nor}} \leq \|\varphi\|_{\text{dec}} \|w\|_{\text{max}}$$

by Proposition 2.1. It thus follows from (6.6) that  $\gamma(\theta) \leq \|w\|_{\text{max}}$ .

Let us now apply (6.9) with  $\varphi$  equal to our mapping  $\phi$ . We obtain that

$$\langle \mu, (\phi \otimes I_D)(w) \rangle = \langle \xi, \theta \rangle,$$

hence

$$|\langle \mu, (\phi \otimes I_D)(w) \rangle| \leq \|\xi\| \gamma(\theta) \leq \|\xi\| \|w\|_{\max}.$$

Since the norm of each element  $v \in A \otimes_{\text{nor}} D$  is given by  $\|v\| = \sup |\langle \mu, v \rangle|$ , where the supremum is over all linear functionals  $\mu$  on  $A \otimes_{\text{nor}} D$  with the norm 1 which are normal in the first factor (this follows easily from [11, Lemma 2.4]), we deduce that  $\|(\phi \otimes I_D)(w)\|_{\text{nor}} \leq \|\xi\| \|w\|_{\max}$ , whence (6.8).

We now turn to operator space duality. We let  $Z = B \otimes_{\gamma} A_*$ . We wish to equip  $Z$  with an operator space structure such that (6.7) holds completely isometrically. We choose to equip  $Z$  with the predual operator space structure in the sense of [21, Sect. 3]. Namely, we equip  $DEC(B, A)^*$  with its standard dual operator space structure and using the canonical embedding  $Z \subset Z^{**} = DEC(B, A)^*$ , we equip  $Z$  with the induced structure. By [21, Proposition 3.1], we get  $Z^* = DEC(B, A)$  completely isometrically provided that for any integer  $n \geq 1$ ,

$$\begin{aligned} & \text{the closed unit ball } \text{Ball}(M_n(DEC(B, A))) \\ & \text{is } \sigma(M_n(DEC(B, A)), M_n(B \otimes_{\gamma} A_*))\text{-closed.} \end{aligned} \tag{6.10}$$

To prove (6.10), we let  $(\phi_t)_t$  be a net in  $\text{Ball}(M_n(DEC(B, A)))$  converging to some  $\phi \in M_n(DEC(B, A))$  in the  $w^*$ -topology of  $M_n(DEC(B, A))$ . Via (6.1), we regard  $\phi_t$  and  $\phi$  as decomposable operators from  $B$  into  $M_n(A)$ . Then  $\phi_t: B \rightarrow M_n(A)$  converges to  $\phi: B \rightarrow M_n(A)$  in the point- $w^*$  topology and  $\|\phi_t\|_{\text{dec}} \leq 1$  for any  $t$ . It therefore follows from [22, Lemma 5.2] that  $\|\phi\|_{\text{dec}} \leq 1$ , that is  $\phi \in \text{Ball}(M_n(DEC(B, A)))$ . This shows (6.10) and completes the proof of (2).

We now come to the proof of (3). We still assume that  $A$  is a von Neumann algebra. We already know from (1) that  $DEC(B, A)$  is an operator  $A$ -bimodule and we want to prove that the latter is a normal dual one. Thus by [12, Theorem 4.1], it suffices to show that the trilinear mapping

$$p: A \times DEC(B, A) \times A \rightarrow DEC(B, A), \quad (a, \phi, b) \mapsto a \cdot \phi \cdot b \tag{6.11}$$

is separately  $w^*$ -continuous.

Recall that for any  $\eta \in A_*$  and  $c_1, c_2 \in A$ , the functional  $c_1 \eta c_2$  on  $A$  defined by letting  $\langle c_1 \eta c_2, a \rangle = \langle \eta, c_2 a c_1 \rangle$  for any  $a \in A$  belongs to  $A_*$ . We fix some  $\phi \in DEC(B, A)$ . Then we consider the linear mapping  $\sigma_{\phi}: B \otimes A_* \rightarrow A_*$  defined by letting  $\sigma_{\phi}(y \otimes \eta) = \phi(y) \eta$  for any  $y \in B, \eta \in A_*$ . Then for any  $\sum_k y_k \otimes \eta_k \in B \otimes A_*$ , and any  $a \in A$ , we have

$$\begin{aligned} \left\langle \sigma_\phi \left( \sum_k y_k \otimes \eta_k \right), a \right\rangle &= \sum_k \langle \phi(y_k) \eta_k, a \rangle \\ &= \sum_k \langle \eta_k, a\phi(y_k) \rangle \\ &= \left\langle a \cdot \phi, \sum_k y_k \otimes \eta_k \right\rangle. \end{aligned}$$

Using both (6.6) and Lemma 2.2, we deduce that

$$\left| \left\langle \sigma_\phi \left( \sum_k y_k \otimes \eta_k \right), a \right\rangle \right| \leq \gamma \left( \sum_k y_k \otimes \eta_k \right) \|a\| \|\phi\|_{\text{dec}}.$$

This shows that  $\sigma_\phi$  extends to a bounded mapping (still denoted by)  $\sigma_\phi: B \otimes_\gamma A_* \rightarrow A_*$ , with norm less than or equal to  $\|\phi\|_{\text{dec}}$ . It is now easy to check that its adjoint map is given by  $\sigma_\phi^*(a) = a \cdot \phi$  for any  $a \in A$ . This shows that the mapping  $p$  defined by (6.11) is  $w^*$ -continuous in the first variable. Of course, a similar argument shows that  $p$  is  $w^*$ -continuous in the third variable.

We now fix  $a, b \in A$ , and consider the linear mapping  $\sigma: B \otimes A_* \rightarrow B \otimes A_*$  defined by letting  $\sigma(y \otimes \eta) = y \otimes b\eta a$  for any  $y \in B, \eta \in A_*$ . Arguing as above, one can check that  $\sigma$  extends to a bounded mapping from  $B \otimes_\gamma A_*$  into itself, and for any  $\phi \in \text{DEC}(B, A)$ , we have  $\sigma^*(\phi) = a \cdot \phi \cdot b$ . This shows that  $p$  is  $w^*$ -continuous in the second variable, completing the proof of (3). ■

We shall now connect Theorem 6.1 to results from Section 5. Assume that  $A$  is a von Neumann algebra and let  $\mathbb{I}$  be an index set. It is a simple matter to verify (using, e.g., [22, Lemma 5.2]) that

$$M_{\mathbb{I}}(\text{DEC}(B, A)) = \text{DEC}(B, M_{\mathbb{I}}(A)) \text{ isometrically.} \tag{6.12}$$

Namely, for any  $\varphi \in \text{DEC}(B, M_{\mathbb{I}}(A))$ , let  $\varphi_{i\ell} \in \text{DEC}(B, A)$  be the components of  $\varphi$ , so that  $\varphi(y) = (\varphi_{i\ell}(y))_{i, \ell \in \mathbb{I}}$  for any  $y \in B$ . Then  $(\varphi_{i\ell})_{i, \ell \in \mathbb{I}} \in M_{\mathbb{I}}(\text{DEC}(B, A))$ , with norm equal to  $\|\varphi\|_{\text{dec}}$ . Moreover any  $(\varphi_{i\ell})_{i, \ell \in \mathbb{I}} \in M_{\mathbb{I}}(\text{DEC}(B, A))$  determines a corresponding  $\varphi \in \text{DEC}(B, M_{\mathbb{I}}(A))$ .

Let  $B$  be a von Neumann algebra and let  $q: A \overset{eh}{\otimes} B_* \overset{eh}{\otimes} A \rightarrow \text{NDEC}(B, A) \subset \text{DEC}(B, A)$  be associated to  $A$  and  $X = B_*$ . Note that  $\text{Im}(q)$  is obviously a submodule of  $\text{DEC}(B, A)$ . Since  $\text{DEC}(B, A)$  is a normal dual operator  $A$ -bimodule, the space  $M_{\mathbb{I}}(\text{DEC}(B, A))$  is an operator  $M_{\mathbb{I}}(A)$ -bimodule in a natural way (see [12, Corollary 3.6]). In particular, given any  $\alpha = (a_i)_{i \in \mathbb{I}} \in R_{\mathbb{I}}(A)$ ,  $\varphi = (\varphi_{i\ell})_{i, \ell \in \mathbb{I}} \in M_{\mathbb{I}}(\text{DEC}(B, A))$ , and  $\beta = (b_l)_{l \in \mathbb{I}} \in C_{\mathbb{I}}(A)$ , we may define  $\alpha \cdot \varphi \cdot \beta = \sum_{i, l} a_i \varphi_{i\ell} b_l$ . It is easy to see that this element of  $\text{DEC}(B, A)$  defined by [12, Corollary 3.6] coincides with the mapping taking  $y \in B$  to

$\sum_{i,l} a_i \varphi_{il}(y) b_l$ . Thus Proposition 5.6 implies that  $\alpha \cdot \varphi \cdot \beta$  belongs to  $Im(q)$  provided that  $\varphi$  belongs to  $M_{\mathbb{J}}(Im(q))$ . In the terminology introduced in [24], we obtain the following.

**COROLLARY 6.2.** *If  $X = B_*$  is the predual operator space of a von Neumann algebra, then  $Im(q) \subset DEC(B, A)$  is a strong operator  $A$ -bimodule.*

*Remark 6.3.* The latter result can also be proved using Proposition 5.5. First, we note that by [24, Lemma 3.4 and Lemma 3.5], we have an isometric identification

$$M_n(\mathbb{R}_{\mathbb{J}}(A) \otimes_{B' \otimes A}^{eh} C_{\mathbb{J}}(A)) = \mathbb{R}_{\mathbb{J}}(M_n(A)) \otimes_{B' \otimes M_n(A)}^{eh} C_{\mathbb{J}}(M_n(A))$$

for any integer  $n \geq 1$ . Then applying Proposition 5.5 with  $M_n(A)$  instead of  $A$ , we deduce that the equality proved in that statement is complete, that is,

$$\mathbb{R}_{\mathbb{J}}(A) \otimes_{B' \otimes A}^{eh} C_{\mathbb{J}}(A) = Im(q) \quad \text{completely isometrically.} \quad (6.13)$$

Furthermore  $\mathbb{R}_{\mathbb{J}}(A) \otimes_{B' \otimes A}^{eh} C_{\mathbb{J}}(A)$  is obviously a normal operator  $A$ -bimodule and the completely isometric isomorphism yielding (6.13) is a homomorphism of normal operator  $A$ -bimodules. To prove Corollary 6.2, it therefore suffices, using [24, Proposition 2.1], to show that  $\mathbb{R}_{\mathbb{J}}(A) \otimes_{B' \otimes A}^{eh} C_{\mathbb{J}}(A)$  is a strong operator  $A$ -bimodule. Since  $\mathbb{R}_{\mathbb{J}}(A)$  is a strong operator  $(A, B' \otimes A)$ -bimodule and  $C_{\mathbb{J}}(A)$  is a strong operator  $(B' \otimes A, A)$ -bimodule, the result follows from [24, Proposition 4.1].

We finally come back to our general study of the map  $q: A \otimes X \otimes A \rightarrow NCB(X^*, A) \subset CB(X^*, A)$ , with  $A$  a von Neumann algebra and  $X$  an arbitrary operator space. The operator space  $CB(X^*, A)$  can be regarded as an  $A$ -bimodule in an obvious way ( $(a \cdot \phi \cdot b)(x^*) = a\phi(x^*)b$  for  $\phi \in CB(X^*, A)$ ,  $a, b \in A$ ,  $x^* \in X^*$ ) and it is a simple matter to verify that  $CB(X^*, A)$  is actually a normal dual operator  $A$ -bimodule. Moreover  $Im(q)$  is a (possibly non closed) submodule. We equip  $CB(X^*, A)$  with the  $A, A$ -topology introduced in [25] and [26]. We recall that this topology is defined by the family of all seminorms

$$s_{\eta}^v(\phi) = \inf\{\eta(aa^*)^{1/2} \|\phi\|_{cb} v(b^*b)^{1/2} : \phi = a\phi b, \phi \in CB(X^*, A), a, b \in A\},$$

where  $\eta$  and  $v$  are any normal positive functionals on  $A$ . It follows from [26, Theorem 3.10] that a submodule of  $CB(X^*, A)$  is closed in the  $A, A$ -topology if and only if it is a strong operator  $A$ -bimodule. It is easy to deduce from that criterion that  $NCB(X^*, A) \subset CB(X^*, A)$  is closed in the  $A, A$ -topology. Let  $X \overset{A}{\otimes} A$  denote the closure of  $X \otimes A$  in the  $A, A$ -topology.

We clearly have  $X \overset{A}{\otimes} A \subset NCB(X^*, A)$ . In the next statement we limit the range of  $q$ , and this turns out to be a useful tool to show that  $Im(q) \neq NCB(X^*, A)$  in some special situations.

**PROPOSITION 6.4.** (1)  $Im(q) \subset X \overset{A}{\otimes} A$ .

(2) If  $X \overset{A}{\otimes} A \neq NCB(X^*, A)$ , then the mapping  $q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A)$  is not onto.

(3) If  $A$  is injective and if  $X = B_*$  is the predual operator space of a von Neumann algebra, then  $Im(q) = X \overset{A}{\otimes} A$ .

*Proof.* Clearly (2) follows from (1) and (3) follows from (1), Corollary 6.2 and [26, Theorem 3.10], so we only need to prove (1). Let  $\phi \in Im(q)$ , assume that  $\phi$  is given by (2.6), and write it as  $\phi = \sum_{j, k \in \mathbb{J}} x_{jk} \otimes a_j b_k$  for simplicity. For any finite set  $\mathbb{F} \subset \mathbb{J}$ , we let  $\phi^{\mathbb{F}} = \sum_{j, k \in \mathbb{F}} x_{jk} \otimes a_j b_k \in X \otimes A$ . It clearly suffices to show that

$$\phi^{\mathbb{F}} \rightarrow \phi \quad \text{in the } A, A\text{-topology.} \tag{6.14}$$

We give ourselves two normal positive functionals  $\eta$  and  $\nu$  on  $A$ . We may write

$$\phi - \phi^{\mathbb{F}} = \sum_{j \in \mathbb{J} \setminus \mathbb{F}; k \in \mathbb{J}} x_{jk} \otimes a_j b_k + \sum_{j \in \mathbb{F}; k \in \mathbb{J} \setminus \mathbb{F}} x_{jk} \otimes a_j b_k.$$

Using a polar decomposition argument as in the proof of [25, Theorem 5.3], we deduce that

$$\begin{aligned} s_{\eta}^{\nu}(\phi - \phi^{\mathbb{F}}) &\leq \eta \left( \left( \sum_{j \in \mathbb{J} \setminus \mathbb{F}} a_j a_j^* \right)^{1/2} \right) \| (x_{jk}) \| \left\| \sum_{k \in \mathbb{J}} b_k^* b_k \right\|^{1/2} \\ &\quad + \left\| \sum_{j \in \mathbb{F}} a_j a_j^* \right\|^{1/2} \| (x_{jk}) \| \nu \left( \left( \sum_{k \in \mathbb{J} \setminus \mathbb{F}} b_k^* b_k \right)^{1/2} \right). \end{aligned}$$

Since the two series  $\sum_{j \in \mathbb{J} \setminus \mathbb{F}} a_j a_j^*$  and  $\sum_{k \in \mathbb{J} \setminus \mathbb{F}} b_k^* b_k$  strongly converge to zero when  $\mathbb{F}$  goes to  $\mathbb{J}$ , and  $\| \sum_{j \in \mathbb{F}} a_j a_j^* \| \leq \| \sum_{j \in \mathbb{J}} a_j a_j^* \|$  for any  $\mathbb{F} \subset \mathbb{J}$ , we deduce (6.14). ■

**EXAMPLE 6.5.** Using Proposition 6.4 one can show, for example, that the map  $q: A \overset{eh}{\otimes} B(\mathcal{H}) \overset{eh}{\otimes} A \rightarrow CB(B(\mathcal{H})_*, A)$  is not surjective if  $A$  is an injective separably acting von Neumann algebra without minimal projections and  $\mathcal{H}$  is a separable (infinite dimensional) Hilbert space. This result complements Remark 5.3. Indeed, we may identify  $CB(B(\mathcal{H})_*, A)$  with

$B(\mathcal{H}) \bar{\otimes} A$  and then identify the latter with  $M_\infty(A)$ . Then let  $C$  be a separable  $C^*$ -subalgebra of  $A$  which is dense in  $A$  in the strong operator topology, choose a countable normdense subset  $\{a_n\}$  in the unit ball  $C_1$  of  $C$  and consider the diagonal matrix  $a \in B(\mathcal{H}) \bar{\otimes} A$  with the entries  $a_n$  along the diagonal. We claim that  $a \notin B(\mathcal{H}) \bar{\otimes} A$ , hence  $a \notin \text{Im}(q)$  by Proposition 6.4.

If  $a \in B(\mathcal{H}) \bar{\otimes} A$ , then by [25, Proposition 2.2; 26, Theorem 3.10] there exist two orthogonal families  $\{e_i\}$  and  $\{f_j\}$  of projections in  $A$ , each with the sum 1, such that  $(1 \otimes e_i) a (1 \otimes f_j) \in B(\mathcal{H}) \otimes_{\min} A$  for all  $i, j$ . By [20, 11.5.7] this implies that for each  $i, j$  the norm closure of the set  $S_{i,j} = \{e_i a_n f_j\}$  is compact in the norm topology in  $A$ , hence also in the strong operator topology. Since  $C_1$  is dense in the unit ball of  $A$  in the strong operator topology (by the Kaplansky density theorem) and  $\{a_n\}$  is norm dense in  $C_1$ , the closure of the set  $S_{i,j} \subseteq e_i A f_j$  in the strong operator topology contains the unit ball  $e_i A_1 f_j$  of  $e_i A f_j$ , hence it must be equal to  $e_i A_1 f_j$  and it follows that  $e_i A_1 f_j$  is norm compact. Choose  $i, j$  so that  $e_i A f_j \neq 0$  and then choose two non-zero subprojections  $e \leq e_i$  and  $f \leq f_j$  such that  $e$  and  $f$  are equivalent in  $A$ . Then the unit ball of  $e A f$  is norm compact, hence  $e A f$  is finite dimensional. But, since  $A$  has no non-zero minimal projections and  $e A f \neq 0$ ,  $e A f$  is in fact infinite dimensional. Thus, the assumption that  $a \in B(\mathcal{H}) \bar{\otimes} A$  leads to a contradiction.

*Problem.* Suppose that  $A$  is an injective von Neumann algebra. Is the map  $q: A \overset{eh}{\otimes} X \overset{eh}{\otimes} A \rightarrow NCB(X^*, A)$  a quotient map onto its range for each operator space  $X$ ? Is the range of  $q$  always equal to  $X \bar{\otimes} A$ ?

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