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The Haagerup Norm on the Tensor Product of Operator Modules

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It is proved that the (analogy of the) Haagerup norm on the tensor product of submodules of $\mathscr{B}(\mathscr{H})$ over a von Neumann algebra $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ is injective. If $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ are von Neumann algebras with $\mathscr{S}$ injective and $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$, then the natural map from $\mathscr{S} \otimes_{\mathscr{T}} \mathscr{S}$ equipped with the Haagerup norm to $\mathrm{CB}(\mathscr{R}, \mathscr{S})$ (the space of all completely bounded maps from $\mathscr{R}$ to $\mathscr{S}$) is shown to be an isometry, and from this we deduce the result of Chatterjee and Smith that the natural map from the central Haagerup tensor product $\mathscr{R} \otimes_{\mathscr{C}} \mathscr{R}$ to $\mathrm{CB}(\mathscr{R}, \mathscr{R})$ is an isometry for each von Neumann algebra $\mathscr{R}$. It is also shown that for an elementary operator on a prime C*-algebra with zero socle or on a continuous von Neumann algebra the norm is equal to the completely bounded norm.

INTRODUCTION

If $\mathscr{H}$ is a Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$ and $\mathscr{E}$, $\mathscr{F}$ are vector subspaces of $\mathscr{B}(\mathscr{H})$, then the *Haagerup norm* on $\mathscr{E} \otimes \mathscr{F}$ is defined by

$$\|w\| = \inf\left\{\left\|\sum a_j a_j^*\right\|^{1/2} \left\|\sum b_j^* b_j\right\|^{1/2} : w = \sum a_j \otimes b_j\right\}.$$

The proof that this is indeed a norm can be found in the paper of Effros and Kishimoto [11]. The Haagerup tensor product $\mathscr{E} \otimes^h \mathscr{F}$ is defined as the completion of $\mathscr{E} \otimes \mathscr{F}$ in the Haagerup norm. This norm is very important in the theory of operator spaces and in the last decade it has been studied by many authors (see [8] and the references therein).

Suppose now that $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ is a von Neumann algebra, $\mathscr{E}$ is a right $\mathscr{T}$-submodule of $\mathscr{B}(\mathscr{H})$ and $\mathscr{F}$ is a left $\mathscr{T}$-submodule of $\mathscr{B}(\mathscr{H})$. Then we can consider the Banach space

$$\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F} \overset{\text{def}}{=} \mathscr{E} \otimes^h \mathscr{F} / \bar{\bar{\mathscr{N}}},$$

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where $\bar{\bar{\mathcal{N}}}$ is the closed subspace of $\mathscr{E} \otimes^h \mathscr{F}$ generated by all elements of the form $ax \otimes b - a \otimes xb$ ($a \in \mathscr{E}, b \in \mathscr{F}, x \in \mathscr{T}$). We call this space the *Haagerup tensor product of* $\mathscr{E}$ *and* $\mathscr{F}$ *over* $\mathscr{T}$. Note that $\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$ is (completely isometrically isomorphic to) an operator space, since $\mathscr{E} \otimes^h \mathscr{F}$ is an operator space and the quotient of an operator space by a closed subspace is again an operator space by the Ruan theorem [29]. If $\mathscr{E} = \mathscr{F}$ is a von Neumann algebra $\mathscr{R}$, and $\mathscr{T}$ is the center $\mathscr{C}$ of $\mathscr{R}$, then $\mathscr{R} \otimes^h_{\mathscr{C}} \mathscr{R}$ is the central Haagerup tensor product, studied recently by Chaterjee and Smith [7].

After some preliminaries, all normal completely bounded left $\mathscr{A}$ right $\mathscr{B}$-module homomorphisms from $\mathscr{R}$ to $\mathscr{B}(\mathscr{H})$ will be characterized in Section 1, where $\mathscr{R}$ is a von Neumann algebra on $\mathscr{H}$ and $\mathscr{A}, \mathscr{B}$ are arbitrary subalgebras of $\mathscr{R}$. In the case $\mathscr{R} = \mathscr{B}(\mathscr{H})$ the result was obtained previously by Smith [30, Theorem 3.1] using a different method.

In Section 2 the detailed analysis of the operator identity

$$axb = cxd$$

will be given, where a, b, c, d are fixed elements in a von Neumann algebra $\mathscr{S}$ and the identity is supposed to hold for all x in a von Neumann subalgebra $\mathscr{R}$ of $\mathscr{S}$. Since $\mathscr{R}$ and $\mathscr{S}$ are arbitrary, the usual matrix argument reduces the apparently more general identity $\sum a_j x b_j = \sum c_k x d_k$ to the identity displayed above. The analysis of this identity will enable us to prove the *injectivity* of the product $\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$, where $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ is a von Neumann algebra, $\mathscr{E}$ is a right $\mathscr{T}$-submodule of $\mathscr{B}(\mathscr{H})$, and $\mathscr{F}$ is a left $\mathscr{T}$-submodule of $\mathscr{B}(\mathscr{H})$. Here the injectivity means that the norm in $\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$ is independent of the embedding of $\mathscr{E}$ and $\mathscr{F}$ in larger modules. In the case $\mathscr{T} = \mathbb{C}$ this result reduces to the well known injectivity of the usual Haagerup tensor product [5, 26]. Further, it will be shown that if $\mathscr{R}$ is a von Neumann subalgebra of an injective von Neumann algebra $\mathscr{S}$ and $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$ then the natural map $\theta_{\mathscr{R}, \mathscr{S}}$ from $\mathscr{S} \otimes^h_{\mathscr{T}} \mathscr{S}$ to the space $\mathrm{CB}(\mathscr{R}, \mathscr{S})$ (of all completely bounded maps from $\mathscr{R}$ to $\mathscr{S}$), defined on elementary tensors by $\theta_{\mathscr{R}, \mathscr{S}}(a \otimes_{\mathscr{T}} b)(x) = axb$, is an isometry. In the case $\mathscr{S} = \mathscr{R} = \mathscr{B}(\mathscr{H})$ the isometric character of the map $\theta_{\mathscr{R}, \mathscr{S}}$ is proved by Haagerup in an unpublished manuscript and independently by Smith in [30, Th. 4.3]. (Our proof here will not use this special case.) We shall use the isometry $\theta_{\mathscr{R}, \mathscr{B}(\mathscr{H})}$ to give a short deduction of the recent theorem of Chatterjee and Smith [7] that the natural map $\theta_{\mathscr{C}}: \mathscr{R} \otimes^h_{\mathscr{C}} \mathscr{R} \to \mathrm{CB}(\mathscr{R})$ is an isometry for any von Neumann algebra $\mathscr{R}$.

In Section 3 we consider the problem of computing the norm of an elementary operator on a C*-algebra $\mathscr{A}$. Here by an *elementary operator* we mean an operator $\varphi: \mathscr{A} \to \mathscr{A}$ of the form $\varphi(x) = \sum a_k x b_k$, where (a_k) and (b_k) are two sequences of elements from $\mathscr{A}$ (or from the multiplier algebra of $\mathscr{A}$ if $\mathscr{A}$ has no unit) such that the two series $\sum a_k a_k^*$ and $\sum b_k^* b_k$ are norm convergent. The problem is still open even in the case $\mathscr{A} = \mathscr{B}(\mathscr{H})$ (see [22]), but if $\mathscr{A}$ is prime with zero socle we shall see that $\|\varphi\| = \|\varphi\|_{\rm cb}$ and this is equal to the Haagerup norm of $\sum a_k \otimes b_k$. As a corollary, the norm of an elementary operator on a continuous von Neumann algebra will be determined. Let us remark that when φ is a derivation, $\|\varphi\|$ was computed by Stampfli [31] in the case $\mathscr{A} = \mathscr{B}(\mathscr{H}))$ and by Zsido [35] in the case when $\mathscr{A}$ is a general von Neumann algebra; we refer to the survey article [14] for more complete information concerning norms of special types of elementary operators.

Finally, in Section 4 an extension of the result of Mingo [23], that an automorphism on a von Neumann algebra is inner if and only if it is inner as a completely positive map, will be proved by elementary arguments similar to those used in Section 1.

1. Preliminaries

If $\mathscr{T}$ is a (complex) algebra, $\mathscr{E}$ is a right $\mathscr{T}$-module and $\mathscr{F}$ is a left $\mathscr{T}$-module, the algebraic tensor product $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ of $\mathscr{E}$ and $\mathscr{F}$ over $\mathscr{T}$ is defined as the quotient space of the tensor product $\mathscr{E} \otimes \mathscr{F}$ (over $\mathbb{C}$ by the subspace $\mathscr{N} = \mathscr{N}(\mathscr{E}, \mathscr{F}; \mathscr{T})$ generated by all elements of the form $ax \otimes b - a \otimes xb$ $(a \in \mathscr{E}, b \in \mathscr{F}, x \in \mathscr{T})$. The coset of $a \otimes b$ in $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ is denoted by $a \otimes_{\mathscr{T}} b$. Suppose in addition that $\mathscr{T} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra and $\mathscr{E}, \mathscr{F} \subseteq \mathscr{B}(\mathscr{H})$. Then the Haagerup norm on $\mathscr{E} \otimes \mathscr{F}$ induces the quotient seminorm on $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$. If $\mathscr{N}$ is closed in $\mathscr{E} \otimes \mathscr{F}$ then this seminorm is in fact a norm and, since $\mathscr{E} \otimes^h \mathscr{F}$ is just the completion of $\mathscr{E} \otimes \mathscr{F}$ in the Haagerup norm, the natural map $\mathscr{E} \otimes \mathscr{F}/\mathscr{N} \to \mathscr{E} \otimes^h \mathscr{F}/\bar{\mathscr{N}}$ (where $\bar{\mathscr{N}}$ is the closure of $\mathscr{N}$ in $\mathscr{E} \otimes^h \mathscr{F}$) is isometric with dense range, so the Haagerup tensor product $\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$ over $\mathscr{T}$ can be defined also as the completion of $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$.

For arbitrary cardinal numbers I, J and each subspace $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ we denote by $\mathrm{M}_{I,J}(\mathscr{S})$ the set of all $I \times J$ matrices with entries in $\mathscr{S}$ that represent bounded operators from $\mathscr{H}^J$ (the direct sum of J copies of $\mathscr{H}$) to $\mathscr{H}^I$, and we also use the notation $\mathrm{M}_J(\mathscr{S}) = \mathrm{M}_{J,J}(\mathscr{S})$, $\mathrm{R}_J(\mathscr{S}) = \mathrm{M}_{1,J}(\mathscr{S})$, and $\mathrm{C}_J(\mathscr{S}) = \mathrm{M}_{J,1}(\mathscr{S})$. For $\mathbf{a} \in \mathscr{B}(\mathscr{H}^J, \mathscr{H}) = \mathrm{R}_J(\mathscr{B}(\mathscr{H}))$ and $\mathbf{b} \in \mathscr{B}(\mathscr{H}, \mathscr{H}^J) = \mathrm{C}_J(\mathscr{B}(\mathscr{H}))$ we always denote by a_i and b_i the components of $\mathbf{a}$ and $\mathbf{b}$, respectively. (Basic facts concerning the computations with infinite matrices can be found in [18, Section 2.6].) For two subspaces $\mathscr{E}$ and $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$, any positive integer n and arbitrary $\mathbf{a} \in \mathrm{R}_n(\mathscr{E})$, $\mathbf{b} \in \mathrm{C}_n(\mathscr{F})$ we use the notation

$$\mathbf{a} \odot \mathbf{b} = \sum_{j=1}^{n} a_j \otimes b_j \in \mathscr{E} \otimes \mathscr{F},$$

and if $\mathscr{E}$, $\mathscr{F}$ are a right and a left $\mathscr{T}$-module, respectively, the element $\mathbf{a} \odot_{\mathscr{T}} \mathbf{b}$ is defined similarly. The direct sum of J copies of an operator $a \in \mathscr{B}(\mathscr{H})$ is denoted by $a^{(J)}$ (here again J can be any cardinal), and for a subset $\mathscr{M}$ of $\mathscr{B}(\mathscr{H})$ the set $\{a^{(J)}: a \in \mathscr{M}\}$ is denoted as $\mathscr{M}^{(J)}$.

A linear map $\varphi: \mathscr{M} \to \mathscr{N}$, where $\mathscr{M} \subseteq \mathscr{B}(\mathscr{H})$ and $\mathscr{N} \subseteq \mathscr{B}(\mathscr{K})$, is called *completely bounded* if $\|\varphi\|_{\text{cb}} \overset{\text{def}}{=} \sup_n \|\varphi_n\| < \infty$, where for each positive integer n the map $\varphi_n: \mathrm{M}_n(\mathscr{M}) \to \mathrm{M}_n(\mathscr{N})$ is defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$. (A nice basic reference for such maps is [25], and [8] is a survey containing also more recent results.) For example, for any positive integer n and $\mathbf{a} \in \mathscr{B}(\mathscr{H}^n, \mathscr{H})$, $\mathbf{b} \in \mathscr{B}(\mathscr{H}, \mathscr{H}^n)$ the map $\mu(\mathbf{a}, \mathbf{b}): \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$, $\mu(\mathbf{a}, \mathbf{b})(x) = \mathbf{a}x^{(n)}\mathbf{b}$ $(x \in \mathscr{B}(\mathscr{H}))$ is completely bounded with $\|\mu(\mathbf{a}, \mathbf{b})\|_{\text{cb}} \leqslant \|\mathbf{a}\| \, \|\mathbf{b}\|$. The space of all completely bounded maps from $\mathscr{M}$ to $\mathscr{N}$ is denoted by $\mathrm{CB}(\mathscr{M}, \mathscr{N})$, and $\mathrm{CB}(\mathscr{M}) = \mathrm{CB}(\mathscr{M}, \mathscr{M})$.

It is well known (and not hard to prove) that for arbitrary subspaces $\mathscr{E}$ and $\mathscr{F}$ of $\mathscr{B}(\mathscr{H})$ the natural map $\theta: \mathscr{E} \otimes \mathscr{F} \to \mathrm{CB}(\mathscr{B}(\mathscr{H}))$, defined by $\theta(\mathbf{a} \odot \mathbf{b}) = \mu(\mathbf{a}, \mathbf{b})$ $(\mathbf{a} \in \mathrm{R}_n(\mathscr{E}), \mathbf{b} \in \mathrm{C}_n(\mathscr{F}), n$ any positive integer) is a contraction if $\mathscr{E} \otimes \mathscr{F}$ is equipped with the Haagerup norm, hence θ extends uniquely to a contraction $\theta: \mathscr{E} \otimes^h \mathscr{F} \to \mathrm{CB}(\mathscr{B}(\mathscr{H}))$. Suppose now that $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ are von Neumann algebras, $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$, $\mathscr{E}$ is a right $\mathscr{T}$-submodule of $\mathscr{S}$, and $\mathscr{F}$ is a left $\mathscr{T}$-submodule of $\mathscr{S}$. Then obviously $\theta(w)(\mathscr{R}) \subseteq \mathscr{S}$ for every $w \in \mathscr{E} \otimes^h \mathscr{F}$, and $\theta(w)(\mathscr{R}) = 0$ for each w of the form $w = ax \otimes b - a \otimes xb$ $(a \in \mathscr{E}, b \in \mathscr{F}, x \in \mathscr{T})$, hence θ induces two contractions

$$\theta_{\mathscr{R}, \mathscr{S}}: \mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F} \to \mathrm{CB}(\mathscr{R}, \mathscr{S})$$

and

$$\widetilde{\theta_{\mathscr{R}, \mathscr{S}}}: \mathscr{E} \otimes_{\mathscr{T}} \mathscr{F} \to \mathrm{CB}(\mathscr{R}, \mathscr{S}),$$

where $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F} = \mathscr{E} \otimes \mathscr{F} / \mathscr{N}(\mathscr{E}, \mathscr{F}; \mathscr{T})$ is equipped with the quotient seminorm obtained from the Haagerup norm on $\mathscr{E} \otimes \mathscr{F}$. The following lemma implies that $\widetilde{\theta_{\mathscr{R}, \mathscr{S}}}$ is one to one, hence the natural map $\iota: \mathscr{E} \otimes_{\mathscr{T}} \mathscr{F} \to \mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$ is also one to one (since $\widetilde{\theta_{\mathscr{R}, \mathscr{S}}} = \theta_{\mathscr{R}, \mathscr{S}} \circ \iota$), and in the sequal we shall consider $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ as a subspace of $\mathscr{E} \otimes^h_{\mathscr{T}} \mathscr{F}$. (The fact that ι is one to one implies in particular that the subspace $\mathscr{N}(\mathscr{E}, \mathscr{F}; \mathscr{T})$ of $\mathscr{E} \otimes \mathscr{F}$ is closed.)

LEMMA 1.1. *Let $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ be von Neumann algebras, $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$, J any cardinal, and $\mathbf{a} \in \mathrm{R}_J(\mathscr{S})$, $\mathbf{b} \in \mathrm{C}_J(\mathscr{S})$. Then $\mathbf{a}\mathscr{R}^{(J)}\mathbf{b} = 0$ if and only if there exists a projection $p \in \mathrm{M}_J(\mathscr{T})$ such that $\mathbf{a}p = 0$ and $p\mathbf{b} = \mathbf{b}$.*

Lemma 1.1 has been used already in [19] and its proof is obvious. Namely, let p be the projection onto $[\mathscr{R}^{(J)}\mathbf{b}\mathscr{H}]$. Since the range of p is invariant under $\mathscr{R}^{(J)}$ and under $\mathscr{S}'^{(J)}$, it follows that $p \in \mathscr{R}^{(J)'} \cap (\mathscr{S}'^{(J)})' = \mathrm{M}_J(\mathscr{R}') \cap \mathrm{M}_J(\mathscr{S}) = \mathrm{M}_J(\mathscr{T})$. Obviously $p\mathbf{b} = \mathbf{b}$ and the relation $\mathbf{a}\mathscr{R}^{(J)}\mathbf{b} = 0$ is equivalent to $\mathbf{a}p = 0$. This proves Lemma 1.1.

Suppose now that $w = \mathbf{a} \odot_{\mathscr{T}} \mathbf{b} \in \mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ is in the kernel of $\widetilde{\theta_{\mathscr{R},\mathscr{S}}}$, where $\mathbf{a} \in \mathrm{R}_n(\mathscr{E})\mathscr{E}$ and $\mathbf{b} \in \mathrm{C}_n(\mathscr{F})$ for some positive integer n. Since $\mathscr{E}, \mathscr{F} \subseteq \mathscr{S}$, by Lemma 1.1 there exists a projection $p \in \mathrm{M}_n(\mathscr{T})$ such that $\mathbf{a}p = 0$ and $p\mathbf{b} = \mathbf{b}$. But then $\mathbf{a} \odot_{\mathscr{T}} \mathbf{b} = \mathbf{a} \odot_{\mathscr{T}} p\mathbf{b} = \mathbf{a} \odot_{\mathscr{T}} \mathbf{b} = 0$. This shows that $\widetilde{\theta_{\mathscr{R},\mathscr{S}}}$ is one to one, as claimed above.

The equivalent version of the special case $\mathscr{R} = \mathscr{B}(\mathscr{H})$ of the following theorem is proved in [30, Th. 3.1].

THEOREM 1.2. *Let $\mathscr{R} \subseteq \mathscr{B}(\mathscr{H})$ be a von Neumann algebra, $\mathscr{A}$ and $\mathscr{B}$ arbitrary (not necessarily self-adjoint) subalgebras of $\mathscr{R}$, and $\phi: \mathscr{R} \to \mathscr{B}(\mathscr{H})$ a normal completely bounded left $\mathscr{A}$ right $\mathscr{B}$ module homomorphism. Then there exists a cardinal J and $\mathbf{a} \in \mathrm{R}_J(\mathscr{A}')$, $\mathbf{b} \in \mathrm{C}_J(\mathscr{B}')$ such that $\|\mathbf{a}\|\,\|\mathbf{b}\| = \|\varphi\|_{\mathrm{cb}}$ and*

$$\phi(x) = \mathbf{a}x^{(J)}\mathbf{b} \qquad (x \in \mathscr{R}).$$

Proof. It is known (see [10, Corollary 2.2]) that each normal completely bounded map $\phi: \mathscr{R} \to \mathscr{B}(\mathscr{H})$ can be represented in the form $\phi(x) = \mathbf{a}x^{(J)}\mathbf{b}$, where J is a cardinal, $\mathbf{a} \in \mathscr{B}(\mathscr{H}^J, \mathscr{H})$, $\mathbf{b} \in \mathscr{B}(\mathscr{H}, \mathscr{H}^J)$, and $\|\varphi\|_{\mathrm{cb}} = \|\mathbf{a}\|\,\|\mathbf{b}\|$. We have to show only how to achieve that the components of $\mathbf{a}$ are in $\mathscr{A}'$ and the components of $\mathbf{b}$ are in $\mathscr{B}'$. Let $p \in \mathrm{M}_J(\mathscr{R}')$ be the projection with range $[\mathscr{R}^{(J)}\mathbf{b}\mathscr{H}]$. Then $p\mathbf{b} = \mathbf{b}$ and $\varphi(x) = \mathbf{a}x^{(J)}p\mathbf{b} = \mathbf{a}px^{(J)}\mathbf{b}$ for all $x \in \mathscr{R}$, hence, replacing $\mathbf{a}$ with $\mathbf{a}p$, we may assume that $\mathbf{a}p = \mathbf{a}$ (Since $\|\mathbf{a}p\| \leqslant \|\mathbf{a}\|$, this replacement has no effect on the equality $\|\mathbf{a}\|\,\|\mathbf{b}\| = \|\varphi\|_{\mathrm{cb}}$.)

Since ϕ is a left $\mathscr{A}$-module homomorphism, we have

$$(\mathbf{a}y^{(J)} - y\mathbf{a})\,x^{(J)}\mathbf{b} = 0 \qquad (x \in \mathscr{R})$$

for all $y \in \mathscr{A}$, which is equivalent to $(\mathbf{a}y^{(J)} - y\mathbf{a})p = 0$ by the definition of p. Since $\mathbf{a}p = \mathbf{a}$ and p commutes with $y^{(J)} \in \mathscr{R}^{(J)}$, we have now $\mathbf{a}y^{(J)} - y\mathbf{a} = 0$, which means that the components of $\mathbf{a}$ commute with every $y \in \mathscr{A}$. Thus the components of $\mathbf{a}$ are in $\mathscr{A}'$. Applying this result to the map $x \mapsto \varphi(x^*)^*$ (which is a left $\mathscr{B}^*$ right $\mathscr{A}^*$ module homomorphism), we can achieve that the components of $\mathbf{b}^*$ will be in $(\mathscr{B}^*)'$, hence the components of $\mathbf{b}$ will be in $\mathscr{B}'$.

2. INJECTIVITY AND EMBEDDING INTO COMPLETELY BOUNDED MAPS

The following lemma has been analyzed partially in [19, Lemma 2.3], however, here we shall need a more complete result.

LEMMA 2.1. *Let* $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ *be von Neumann algebras,* $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$ *and* $a, b, c, d \in \mathscr{S}$. *The identity*

$$axb = cxd \tag{2.1}$$

holds for all $x \in \mathscr{R}$ *if and only if there exist projections* $e, e', f, f' \in \mathscr{T}$ *and two sequences* (u_k), (v_k) *in* $\mathscr{T}$ *such that the following conditions* (i), (ii) *and* (ii′) *are satisfied*:

(i) $ee' = e'e$, $ff' = f'f$, *and* $u_k v_k$ *and* $v_k u_k$ *are projections increasing to* ee' *and* ff', *respectively, when* $k \to \infty$;

(ii) $ae' = a$, $eb = b$, $ae = \lim cv_k$, *and* $e'b = \lim u_k d$, *where the convergence is in the strong operator topology*;

(ii′) $cf' = c$, $fd = d$, $cf = \lim au_k$ *and* $f'd = \lim v_k b$ (*again the convergence is in the strong operator topology*).

Moreover, the sequences (u_k) *and* (v_k) *can be chosen so that the following two additional requirements* (iii) *and* (iv) *are satisfied*:

(iii) $\|cv_k\| \leqslant \|a\|$, $\|u_k d\| \leqslant \|b\|$, $\|au_k\| \leqslant \|c\|$, *and* $\|v_k b\| \leqslant \|d\|$ *for all* k;

(iv) *the sequence* $(au_k \otimes_{\mathscr{T}} v_k b)$ *converges to* $a \otimes_{\mathscr{T}} b$ *for any norm on the tensor product* $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ *satisfying the requirement* $\|x \otimes_{\mathscr{T}} y\| \leqslant \|x\| \, \|y\|$ $(x \in \mathscr{E},\ y \in \mathscr{F})$, *where* $\mathscr{E}$ *is an arbitrary right* $\mathscr{T}$-*submodule of* $\mathscr{S}$ *containing* a *and* c, *and* $\mathscr{F}$ *is an arbitrary left* $\mathscr{T}$-*submodule of* $\mathscr{S}$ *containing* b *and* d (*similarly the sequence* $(cv_k \otimes_{\mathscr{T}} u_k d)$ *converges to* $c \otimes_{\mathscr{T}} d$).

Proof. If the projections e, e', f, f' and the sequences (u_k), (v_k) in $\mathscr{T}$ satisfying the conditions (i)–(ii′) do exist, then for each $x \in \mathscr{R}$ we have

$$\begin{aligned} axb &= ae'xeb = aexe'b = \lim cv_k xu_k d \\ &= \lim cxv_k u_k d = cxff' d = cf'xfd = cxd. \end{aligned}$$

Suppose now conversely that the identity (2.1) holds for all $x \in \mathscr{R}$. Then, with $\mathbf{a} =^{\text{def}} (a, c)$ and $\mathbf{b} =^{\text{def}} (b, -d)^T$, we have $\mathbf{a}\mathscr{R}^{(2)}\mathbf{b} = 0$, hence by Lemma 1.1 there exists a projection

$$p = \begin{bmatrix} r & s \\ s^* & t \end{bmatrix}$$

in $\mathrm{M}_2(\mathscr{T})$ such that $\mathbf{a}p = 0$ and $p\mathbf{b} = \mathbf{b}$. The last two identities can be written in components as

$$\begin{aligned} ar &= -cs^*, \qquad & as &= -ct, \\ (1-r)\,b &= -sd, \qquad & (1-t)\,d &= -s^*b, \end{aligned} \tag{2.2}$$

while the fact $p^2 = p$ can be expressed as

$$ss^* = r(1-r), \qquad s^*s = t(1-t), \qquad rs + st = s. \tag{2.3}$$

Let e, e', f, f' be the range projections of r, $1-r$, t, $1-t$, respectively; then clearly $ee' = e'e$ and $ff' = f'f$. Note that $0 \leqslant r, t \leqslant 1$, since $0 \leqslant p \leqslant 1$. We shall now prove the identity $ae' = a$; the identities $eb = b$, $cf' = c$, and $fd = d$ can be proved in the same way and are left to the reader. Let $e_{[\alpha,\beta]}$ be the spectral projection of r corresponding to the interval $[\alpha, \beta]$. Since $1 - e' = e_{[1,1]} \leqslant e_{[1-1/k,1]}$ for all $k = 1, 2, ...$, it suffices to prove that

$$\lim \|ae_{[1-1/k,1]}\| = 0, \tag{2.4}$$

for then we will have $a(1-e') = 0$, hence $ae' = a$. To prove (2.4) let ϑ_k be the function on $[0, 1]$ defined by

$$\vartheta_k(\lambda) = \begin{cases} \lambda^{-1}, & \lambda \in \left[1 - \frac{1}{k}, 1\right] \\ 0, & \lambda \in \left[0, 1 - \frac{1}{k}\right) \end{cases}$$

for each $k = 2, 3, ...$ Then $\|\vartheta_k(r)\| \leqslant 2$ and using the first identity of (2.2) and the first identity of (2.3) we have

$$\begin{aligned} \|ae_{[1-1/k,1]} &= \|ar\vartheta_k(r)\, e_{[1-1/k,1]}\| = \|cs^* e_{[1-1/k,1]} \vartheta_k(r)\| \\ &\leqslant 2\,\|c\|\, \|s^* e_{[1-1/k,1]}\| = 2\,\|c\|\, \|e_{[1-1/k,1]} ss^* e_{[1-1/k,1]}\|^{1/2} \\ &= 2\,\|c\|\, \|e_{[1-1/k,1]} r(1-r)\, e_{[1-1/k,1]}\|^{1/2} \\ &\leqslant 2\,\|c\|\, \|(1-r)\, e_{[1-1/k,1]}\|^{1/2} = 2\,\|c\|\, k^{-1/2}. \end{aligned}$$

This proves (2.4).

For each $k = 1, 2, ...,$ let η_k be the function on $[0, 1]$ defined by

$$\eta_k(\lambda) = \begin{cases} \lambda^{-1}, & \lambda \in \left[\frac{1}{k}, 1\right] \\ 0, & \lambda \in \left[0, \frac{1}{k}\right) \end{cases}.$$

Writing the last identity of (2.3) in the form $st = (1-r)s$, we deduce that $s\eta(t) = \eta(1-r)s$ for each bounded Borel function η on $[0, 1]$ (approximate by polynomials). In particular,

$$u_k \overset{\text{def}}{=} -s\eta_k(t) = -\eta_k(1-r)\, s,$$

and similarly

$$v_k \overset{\text{def}}{=} -s^*\eta_k(r) = -\eta_k(1-t)\, s^*. \tag{2.5}$$

Using the first identity of (2.3) we have now (for $k \geqslant 2$)

$$u_k v_k = \eta_k(1-r) ss^* \eta_k(r) = \eta_k(1-r)(1-r) r \eta_k(r) = e_{[1/k,\, 1-1/k]}, \tag{2.6}$$

and similarly

$$v_k u_k = f_{[1/k,\, 1-1/k]},$$

where $f_{[\alpha, \beta]}$ denotes the spectral projection of t corresponding to the interval $[\alpha, \beta]$. Since the projections $e_{[1/k,\, 1-1/k]}$ and $f_{[1/k,\, 1-1/k]}$ increase to ee' and ff', respectively, as $k \to \infty$, the requirement (i) is satisfied.

From the first identity of (2.5) and (2.2) we have

$$cv_k = -cs^*\eta_k(r) = ar\eta_k(r) = ae_{[1/k,\, 1]},$$

hence the sequence (cv_k) converges strongly to ae and $\|cv_k\| \leqslant \|a\|$ for all $k = 1, 2, \ldots$. Similarly the sequences $(u_k d)$, (au_k) and $(v_k b)$ are bounded and converge to $e'b$, cf, and $f' d$, respectively.

It remains to prove (iv). From (2.6) we have

$$\begin{aligned} a \otimes_{\mathscr{T}} b - au_k \otimes_{\mathscr{T}} v_k b &= a \otimes_{\mathscr{T}} (1 - u_k v_k)\, b \\ &= a \otimes_{\mathscr{T}} (e_{[0,\, 1/k)} + e_{(1-1/k,\, 1]})\, b \\ &= ae_{(1-1/k,\, 1]} \otimes_{\mathscr{T}} b + a \otimes_{\mathscr{T}} e_{[0,\, 1/k)}\, b. \end{aligned}$$

Now it follows from (2.4) and the analogous relation (note that $e_{[0,\, 1/k]}$ is the spectral projection of $1-r$ corresponding to the interval $[1-1/k, 1]$)

$$\lim \|e_{[0,\, 1/k]} b = 0 \tag{2.4'}$$

that $\lim \|a \otimes_{\mathscr{T}} b - au_k \otimes_{\mathscr{T}} v_k b\| = 0$. ∎

In what follows we shall need also the following elementary observation.

Observation. If $\mathscr{T}$ is any (complex) algebra with unit, $\mathscr{E}$ is a right $\mathscr{T}$-module, and $\mathscr{F}$ is a left $\mathscr{T}$-module, then for each positive integer n we have a natural isomorphism of vector spaces

$$\omega \colon R_n(\mathscr{E}) \otimes_{\mathrm{M}_n(\mathscr{T})} C_n(\mathscr{F}) \to \mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$$

defined by $\omega(\mathbf{a} \otimes_{\mathrm{M}_n(\mathscr{T})} \mathbf{b}) = \mathbf{a} \odot_{\mathscr{T}} \mathbf{b}$ ($\mathbf{a} \in \mathrm{R}_n(\mathscr{E})$, $\mathbf{b} \in \mathrm{C}_n(\mathscr{F})$). It is easy to verify that ω is well defined and that a right inverse v to ω is given by

$$v(a \otimes_{\mathscr{T}} b) = (a, 0, \ldots, 0) \otimes_{\mathrm{M}_n(\mathscr{T})} (b, 0, \ldots, 0)^T \qquad (a \in \mathscr{E}, b \in \mathscr{F}).$$

To see that v is also the left inverse of ω, note that, with $\{e_{ij}\}_{i,j=1}^n$ the standard matrix unit in $M_n(\mathbb{C}) \subseteq M_n(\mathcal{T})$,we have for arbitrary $\mathbf{a} \in R_n(\mathcal{E})$ and $\mathbf{b} \in C_n(\mathcal{F})$

$$\begin{aligned}
\mathbf{a} \otimes_{M_n(\mathcal{T})} \mathbf{b} &= \sum_{i=1}^n (0, ..., a_i, ..., 0)\, e_{ii} \otimes_{M_n(\mathcal{T})} \sum_{j=1}^n (0, ..., b_j, ..., 0)^T \\
&= \sum_{j=1}^n (0, ..., a_j, ..., 0)\, e_{jj} \otimes_{M_n(\mathcal{T})} (0, ..., b_j, ..., 0)^T \\
&= \sum_{j=1}^n (0, ..., a_j, ..., 0)\, e_{j1} \otimes_{M_n(\mathcal{T})} e_{1j}(0, ..., b_j, ..., 0)^T \\
&= \sum_{j=1}^n (a_j, 0, ..., 0) \otimes_{M_n(\mathcal{T})} (b_j, 0, ..., 0)^T \\
&= \sum_{j=1}^n v(a_j \otimes_{\mathcal{T}} b_j) = v(\mathbf{a} \odot_{\mathcal{T}} \mathbf{b}) \\
&= v\omega(\mathbf{a} \otimes_{M_n(\mathcal{T})} \mathbf{b}).
\end{aligned}$$

THEOREM 2.2. *If $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, $\mathcal{E} \subseteq \mathcal{E}_0$ are (not necessarily closed) right $\mathcal{T}$-submodules of $\mathcal{B}(\mathcal{H})$ and $\mathcal{F} \subseteq \mathcal{F}_0$ are left $\mathcal{T}$-submodules of $\mathcal{B}(\mathcal{H})$, then the natural map ι: $\mathcal{E} \otimes^h_{\mathcal{T}} \mathcal{F} \to \mathcal{E}_0 \otimes^h_{\mathcal{T}} \mathcal{F}_0$ is an isometry.*

Proof. Clearly ι is a contraction and it suffices to show that the restriction of ι to the dense subspace $\mathcal{E} \otimes_{\mathcal{T}} \mathcal{F}$ of $\mathcal{E} \otimes^h_{\mathcal{T}} \mathcal{F}$ is isometric. Let $w = \mathbf{a} \odot_{\mathcal{T}} \mathbf{b} = \sum_{i=1}^m a_i \otimes_{\mathcal{T}} b_i \in \mathcal{E} \otimes_{\mathcal{T}} \mathcal{F}$, let $\varepsilon > 0$, and choose $\mathbf{c} = (c_1, ..., c_n) \in R_n(\mathcal{E}_0)$ and $\mathbf{d} = (d_1, ..., d_n)^T \in C_n(\mathcal{F}_0)$ so that $\iota(w) = \mathbf{c} \odot_{\mathcal{T}} \mathbf{d}$ and

$$\|\mathbf{c}\|\, \|\mathbf{d}\| \leqslant \|\iota(w)\| + \varepsilon. \tag{2.7}$$

By adding zero entries to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ we may assume that $n \geqslant m$ and $\mathbf{a}, \mathbf{c} \in R_n(\mathcal{E}_0)$, $\mathbf{b}, \mathbf{d} \in C_n(\mathcal{F}_0)$. Since $\mathbf{a} \odot_{\mathcal{T}} \mathbf{b} - \mathbf{c} \odot_{\mathcal{T}} \mathbf{d} = 0$ in $\mathcal{B}(\mathcal{H}) \otimes^h_{\mathcal{T}} \mathcal{B}(\mathcal{H})$, it follows that $\mathbf{a}x^{(n)}\mathbf{b} - \mathbf{c}x^{(n)}\mathbf{d} = 0$ for all $x \in \mathcal{T}'$ or

$$\mathbf{a}x^{(n)}\mathbf{b} = \mathbf{c}x^{(n)}\mathbf{d} \qquad (x \in \mathcal{T}').$$

We regard $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as elements of $M_n(\mathcal{B}(\mathcal{H}))$, where $\mathbf{a}$ and $\mathbf{c}$ are considered as matrices with only the first row non-zero, while $\mathbf{b}$ and $\mathbf{d}$ have only the first column non- zero. Now from Lemma 2.1 (where $\mathcal{R}$ and $\mathcal{S}$ are now replaced by $\mathcal{T}'^{(n)}$ and $M_n(\mathcal{B}(\mathcal{H}))$, respectively) it follows that there exist two sequences (u_k) and (v_k) in $M_n(\mathcal{B}(\mathcal{H})) \cap \mathcal{T}'^{(n)\prime} = M_n(\mathcal{T})$ such that $\|\mathbf{a}u_k\| \leqslant \|\mathbf{c}\|$, $\|v_k\mathbf{b}\| \leqslant \|\mathbf{d}\|$, and

$$\lim_{k \to \infty} \|\mathbf{a}u_k \otimes_{M_n(\mathcal{T})} v_k\mathbf{b} - \mathbf{a} \otimes_{M_n(\mathcal{T})} \mathbf{b}\| = 0, \tag{2.8}$$

where $\| \ \|$ denotes the norm on $\mathrm{R}_n(\mathscr{E}) \otimes_{\mathrm{M}_n(\mathscr{T})} \mathrm{C}_n(\mathscr{F})$ obtained from the Haagerup norm on $\mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$ by the natural isomorphism

$$\mathrm{R}_n(\mathscr{E}) \otimes_{\mathrm{M}_n(\mathscr{T})} \mathrm{C}_n(\mathscr{F}) \cong \mathscr{E} \otimes_{\mathscr{T}} \mathscr{F}$$

(see the Observation above). Since this isomorphism maps $\mathbf{a} \otimes_{\mathrm{M}_n(\mathscr{T})} \mathbf{b}$ to $w = \mathbf{a} \odot_{\mathscr{T}} \mathbf{b}$, it suffices now to prove that $\|\mathbf{a} \otimes_{\mathrm{M}_n(\mathscr{T})} \mathbf{b}\| \leqslant \|\mathbf{c}\| \ \|\mathbf{d}\|$, for then (2.7) implies that $\|w\| = \|\mathbf{a} \otimes_{\mathrm{M}_n(\mathscr{T})} \mathbf{b}\| \leqslant \|\imath(w)\| + \varepsilon$, hence (letting $\varepsilon \to 0$) $\|w\| \leqslant \|\imath(w)\|$. Now from (2.8) we see that it suffices to show that $\|au_k \otimes_{\mathrm{M}_n(\mathscr{T})} v_k \mathbf{b}\| \leqslant \|\mathbf{c}\| \ \|\mathbf{d}\|$ for all $k = 1, 2, \ldots$, but this follows from $\|\mathbf{a}u_k\| \leqslant \|\mathbf{c}\|$ and $\|v_k \mathbf{b}\| \leqslant \|\mathbf{d}\|$. ∎

The following lemma is a special case of Theorem 2.7 below, but it will be used in the proof of Theorem 2.7.

LEMMA 2.3. *For every von Neumann algebra $\mathscr{T} \subseteq \mathscr{B}(\mathscr{H})$ the natural map $\theta_{\mathscr{T}} : \mathscr{B}(\mathscr{H}) \otimes^h_{\mathscr{T}} \mathscr{B}(\mathscr{H}) \to \mathrm{CB}(\mathscr{T}', \mathscr{B}(\mathscr{H}))$ is isometric.*

Proof. Let m be any positive integer and $\mathbf{a} \in \mathrm{R}_m(\mathscr{B}(\mathscr{H}))$, $\mathbf{b} \in \mathrm{C}_m(\mathscr{B}(\mathscr{H}))$. We already know from Section 1 that $\theta_{\mathscr{T}}$ is a contraction, so we have to prove only that $\|\theta_{\mathscr{T}}(\mathbf{a} \odot_{\mathscr{T}} \mathbf{b})_{\mathrm{cb}} \leqslant 1$ implies $\|\mathbf{a} \odot_{\mathscr{T}} \mathbf{b}\| \leqslant 1$. Put $\varphi = \theta_{\mathscr{T}}(\mathbf{a} \odot_{\mathscr{T}} \mathbf{b})$ (thus $\varphi(x) = \mathbf{a}x^{(m)}\mathbf{b}$ for all $x \in \mathscr{T}'$) and assume that $\|\varphi\|_{\mathrm{cb}} \leqslant 1$. Then by Theorem 1.2 there exist a cardinal J and $\mathbf{c} \in \mathrm{R}_J(\mathscr{B}(\mathscr{H}))$, $\mathbf{d} \in \mathrm{C}_J(\mathscr{B}(\mathscr{H}))$ such that $\|\mathbf{c}\| \leqslant 1$, $\|\mathbf{d}\| \leqslant 1$ and

$$\mathbf{a}x^{(m)}\mathbf{b} = \mathbf{c}x^{(J)}\,\mathbf{d} \qquad (\text{for all } x \in \mathscr{T}').$$

As in the proof of Theorem 2.2 we may assume that $J \geqslant m$ and regard $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as elements of $\mathrm{M}_J(\mathscr{B}(\mathscr{H}))$ ($\mathbf{a}$ and $\mathbf{c}$ have only the first row non-zero, while $\mathbf{b}$ and $\mathbf{d}$ have only the first column non-zero). Let (u_k) and (v_k) be the sequences in $\mathrm{M}_J(\mathscr{T})$ constructed as in the proof of Lemma 2.1 (where a, b, c, d and $\mathscr{R}, \mathscr{S}$ are replaced by $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and $\mathscr{T}''^{(J)}$, $\mathrm{M}_J(\mathscr{B}(\mathscr{H}))$, respectively) and let $q \in \mathrm{M}_J(\mathscr{T})$ be the projection with range $\mathscr{H}^m$ (where $\mathscr{H}^m$ is identified with the subspace in $\mathscr{H}^J$ consisting of all elements that have only the first m components non-zero); then $\mathbf{a} = \mathbf{a}q$ and $\mathbf{b} = q\mathbf{b}$. Let us consider the polar decomposition $v_k q = U_k \, |v_k q|$ of $v_k q$. Since $|v_k q|$ and $qu_k U_k q$ considered as matrices in $\mathrm{M}_J(\mathscr{B}(\mathscr{H}))$ can have non-zero entries only in the upper left $m \times m$ corner, the row $\mathbf{a}qu_k U_k q$ has nonzero components at most on the first m positions and the same holds also for the column $|v_k q| \, \mathbf{b}$, thus (since m is finite) we can regard $\mathbf{a}qu_k U_k q \odot_{\mathscr{T}} |v_k q| \, \mathbf{b}$ as an element of $\mathscr{B}(\mathscr{H}) \otimes_{\mathscr{T}} \mathscr{B}(\mathscr{H})$. Since $\|\mathbf{a}qu_k U_k q\| \leqslant \|\mathbf{a}u_k\| \leqslant \|\mathbf{c}\| \leqslant 1$ and similarly $\| \, |v_k q| \, \mathbf{b}\| = \|v_k \mathbf{b}\| \leqslant \|\mathbf{d}\| \leqslant 1$ (by Lemma 2.1 (iii)), it suffices now to prove that

$$\lim_{k \to \infty} \|\mathbf{a}uq_k \, U_k q \odot_{\mathscr{T}} |v_k q| \, \mathbf{b} - \mathbf{a} \odot_{\mathscr{T}} \mathbf{b}\| = 0,$$

for this implies that

$$\|\mathbf{a} \odot_{\mathscr{T}} \mathbf{b}\| \leqslant 1.$$

Transferring the Haagerup norm from $\mathscr{B}(\mathscr{H}) \otimes_{\mathscr{T}} \mathscr{B}(\mathscr{H})$ to

$$\mathrm{R}_m(\mathscr{B}(\mathscr{H})) \otimes_{\mathrm{M}_m(\mathscr{T})} \mathrm{C}_m(\mathscr{B}(\mathscr{H}))$$

through the natural identification of these two spaces (see the Observation just before Theorem 2.2), it suffices to prove that

$$\lim_{k \to \infty} \|\mathbf{a} q u_k U_k q \otimes_{\mathrm{M}_m(\mathscr{T})} |v_k q| \, \mathbf{b} - \mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} \mathbf{b}\| = 0, \tag{2.9}$$

where $\mathbf{a} q u_k U_k q$ and $|v_k q| \, \mathbf{b}$ are considered as elements of $\mathrm{R}_m(\mathscr{B}(\mathscr{H}))$ and $\mathrm{C}_m(\mathscr{B}(\mathscr{H}))$, respectively.

To prove (2.9) we use the spectral projections $e_{[\,.\,]} \in \mathrm{M}_J(\mathscr{T})$ defined as in the proof of Lemma 2.1 so that the relations (2.4), (2.4′), and (2.6) hold (where a and b are replaced by $\mathbf{a}$ and $\mathbf{b}$, respectively). We have

$$\begin{aligned}
&\|\mathbf{a} \otimes_{\mathrm{M}_n(\mathscr{T})} \mathbf{b} - \mathbf{a}(q u_k U_k q) \otimes_{\mathrm{M}_m(\mathscr{T})} |v_k q| \, \mathbf{b}\| \\
&\quad = \|\mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} \mathbf{b} - \mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} (q u_k U_k q) \, |v_k q| \, \mathbf{b}\| \\
&\quad = \|\mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} (1 - q u_k v_k q) \, \mathbf{b}\| \\
&\quad = \|\mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} q(1 - u_k v_k) \, q \mathbf{b}\| \\
&\quad = \|\mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} q(e_{[0,\,1/k)} + e_{(1-1/k,\,1]}) \, q \mathbf{b}\| \qquad \text{(by (2.6))} \\
&\quad = \|\mathbf{a} q e_{(1-1/k,\,1]} q \otimes_{\mathrm{M}_m(\mathscr{T})} \mathbf{b} + \mathbf{a} \otimes_{\mathrm{M}_m(\mathscr{T})} q e_{[0,\,1/k)} q \mathbf{b}\| \\
&\quad \leqslant \|\mathbf{a} e_{(1-1/k,\,1]}\| \, \|\mathbf{b}\| + \|\mathbf{a}\| \, \|e_{[0,\,1/k)} \mathbf{b}\| \qquad (\text{since } \mathbf{a} q = \mathbf{a} \text{ and } q \mathbf{b} = \mathbf{b}).
\end{aligned}$$

Since $\|\mathbf{a} e_{(1-1/k,\,1]}\| \to 0$ by (2.4) and similarly $\|e_{[0,\,1/k)} \mathbf{b}\| \to 0$ by (2.4′), we see that (2.9) is true. ∎

The following result was proved in the case when $\mathscr{R}$ is a factor by Chatterjee and Sinclair [6] and then it was extended to arbitrary von Neumann algebras by Chatterjee and Smith [7, Th 2.4]. Using a technical lemma from [7] Ara and Mathieu [2] have extended the result further to certain C*-algebras.

COROLLARY 2.4. *For every von Neumann algebra $\mathscr{R} \subseteq \mathscr{B}(\mathscr{H})$ the natural map*

$$\theta_{\mathscr{C}} : \mathscr{R} \otimes^h_{\mathscr{C}} \mathscr{R} \to \mathrm{CB}(\mathscr{R}),$$

where $\mathscr{C}$ is the center of $\mathscr{R}$, is an isometry.

Proof. For a C*-algebra $\mathscr{A}$, a positive integer n and $\mathbf{a} \in \mathrm{R}_n(\mathscr{A})$, $\mathbf{b} \in \mathrm{C}_n(\mathscr{A})$ let $\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})$ be the map from $\mathscr{A}$ to $\mathscr{A}$ defined by

$$\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})(x) = \mathbf{a}x^{(n)}\,\mathbf{b}.$$

Then $\theta_{\mathscr{C}}(\mathbf{a} \odot_{\mathscr{C}} \mathbf{b}) = \mu_{\mathscr{R}}(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a} \in \mathrm{R}_n(\mathscr{R})$ and $\mathbf{b} \in \mathrm{C}_n(\mathscr{R})$. By [17, Section 4] there exists a faithful representation π of $\mathscr{R}$ such that $\pi(\mathscr{C}) = \pi(\mathscr{R})'$. (This fact was used by Halpern in the proof of Theorem 4.7 in [17], but can also be deduced from Theorem 4.7 in [17].) Since π is an isometric representation, we have $\|\mu_{\mathscr{R}}(\mathbf{a}, \mathbf{b})\|_{\mathrm{cb}} = \|\mu_{\pi(\mathscr{R})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\mathrm{cb}}$ (where $\pi(\mathbf{a}) = (\pi(a_1), ..., \pi(a_n))$, $\pi(\mathbf{b}) = (\pi(b_1), ..., \pi(b_n))^T$), and $\|\mathbf{a} \odot_{\mathscr{C}} \mathbf{b}\| = \|\pi(\mathbf{a}) \odot_{\pi(\mathscr{R})'} \pi(\mathbf{b})\|$. By Theorem 2.2 the Haagerup norm of $\pi(\mathbf{a}) \odot_{\pi(\mathscr{R})'} \pi(\mathbf{b})$ is the same when computed in $\pi(\mathscr{R}) \otimes^h_{\pi(\mathscr{R})'} \pi(\mathscr{R})$ or in $\overline{\pi(\mathscr{R})} \otimes^h_{\pi(\mathscr{R})'} \overline{\pi(\mathscr{R})}$ (where $\overline{\pi(\mathscr{R})}$ is the weak closure of $\pi(\mathscr{R})$), hence by Lemma 2.3 we have now $\|\mathbf{a} \odot_{\mathscr{C}} \mathbf{b}\| = \|\mu_{\overline{\pi(\mathscr{R})}}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\mathrm{cb}}$. Since for each positive integer m the unit ball of $\mathrm{M}_m(\pi(\mathscr{A}))$ is strongly dense in the unit ball of $\mathrm{M}_m(\overline{\pi(\mathscr{A})})$ by the Kaplansky density theorem, it follows that $\|\mu_{\overline{\pi(\mathscr{R})}}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\mathrm{cb}} = \|\mu_{\pi(\mathscr{R})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\mathrm{bc}} = \|\mu_{\mathscr{R}}(\mathbf{a}, \mathbf{b})\|_{\mathrm{cb}}$, hence we conclude that $\|\mathbf{a} \odot_{\mathscr{C}} \mathbf{b}\| = \|\mu_{\mathscr{R}}(\mathbf{a}, \mathbf{b})\|_{\mathrm{cb}}$. Since n, $\mathbf{a}$, and $\mathbf{b}$ are arbitrary, this proves that $\theta_{\mathscr{C}}$ is isometric. ∎

COROLLARY 2.5. *For every von Neumann algebra $\mathscr{R} \subseteq \mathscr{B}(\mathscr{H})$ with center $\mathscr{C}$ the natural map*

$$i: \mathscr{R} \otimes^h_{\mathscr{C}} \mathscr{R} \to \mathscr{R}(\mathscr{H}) \otimes^h_{\mathscr{R}'} \mathscr{B}(\mathscr{H})$$

is an isometry.

Proof. Let $\theta_{\mathscr{R}'}: \mathscr{B}(\mathscr{H}) \otimes^{\mathrm{h}}_{\mathscr{R}'} \mathscr{B}(\mathscr{H}) \to \mathrm{CB}(\mathscr{R}, \mathscr{B}(\mathscr{H}))$ and $\theta_{\mathscr{C}}: \mathscr{R} \otimes^{\mathrm{h}}_{\mathscr{C}} \mathscr{R} \to \mathrm{CB}(\mathscr{R})$ be the isometries that exist by Lemma 2.3 and Corollary 2.4. Then $\theta_{\mathscr{R}'} \circ i = \theta_{\mathscr{C}}$, hence i must be an isometry. ∎

Recall that a von Neumann algebra $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ is *injective* iff there exists a conditional expectation (=projection of norm 1) from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{S}$. Injective algebras behave nicely with respect to tensor products of operator algebras [12] and Theorem 2.7 below will show that they behave nicely also with respect to the Haagerup tensor product. A special case of the "only if" part of the following lemma (namely the case $\mathscr{R} = \mathscr{S}$ and $\mathscr{H}$ separable) follows also from [7, Th. 4.4].

LEMMA 2.6. *A von Neumann algebra $\mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ is injective if and only if for every von Neumann subalgebra $\mathscr{R} \subseteq \mathscr{S}$ and every map $\phi: \overline{\mathscr{R}\mathscr{S}'} \to \mathscr{B}(\mathscr{H})$ of the form*

$$\varphi(x) = \mathbf{a}x^{(n)}\mathbf{b} \qquad (x \in \overline{\mathscr{R}\mathscr{S}'}),$$

where n is a positive integer and $\mathbf{a} \in \mathrm{R}_n(\mathcal{S})$, $\mathbf{b} \in \mathrm{C}_n(\mathcal{S})$, *the equality* $\|\varphi\|_{\mathrm{cb}} = \|\varphi \mid \mathcal{R}\|_{\mathrm{cb}}$ *holds.*

Proof. We call any map ϕ of the form described in Lemma 2.6 an *elementary map*. Suppose that $\mathcal{S}$ is injective and let us prove that $\|\varphi\|_{\mathrm{cb}} = \|\varphi \mid \mathcal{R}\|_{\mathrm{cb}}$ for every von Neumann subalgebra $\mathcal{R}$ of $\mathcal{S}$ and every elementary map ϕ on $\overline{\mathcal{R}\mathcal{S}'}$. It suffices to show that $\|\varphi\| = \|\varphi \mid \mathcal{R}\|$, for the same argument can then be applied to all matrix algebras $\mathrm{M}_m(\mathcal{R}) \subseteq \mathrm{M}_m(\mathcal{S})$ (m finite). Suppose first that $\mathcal{S}$ is a factor and $\mathcal{H}$ is separable. In this case the argument is essentially the same as in [7, Th. 4.1, (iii) ⇒ (i)]. Namely, since injective factors are semidiscrete (see [9, 34]) the correspondence $xy \mapsto x \otimes y$ can be extended to a *-isomorphism from the C*-algebra generated by $\mathcal{S}$ and $\mathcal{S}'$ to the spatial tensor product $\mathcal{S} \otimes \mathcal{S}'$ (see [12]). This isomorphism maps the C*- algebra $\mathcal{A}$ generated by $\mathcal{R}$ and $\mathcal{S}'$ onto $\mathcal{R} \otimes \mathcal{S}'$ and carries the map $\phi \mid \mathcal{A}$ to the map $(\phi \mid \mathcal{R}) \otimes 1 : \mathcal{R} \otimes \mathcal{S}' \to \mathcal{S} \otimes \mathcal{S}'$ (since ϕ is an $\mathcal{S}'$-module map). Using the dilation theorem for completely bounded maps [25, p. 105] it follows that $\|(\varphi \mid \mathcal{R}) \otimes 1\| = \|\varphi \mid \mathcal{R}\|$, hence $\|\varphi \mid \mathcal{A}\| = \|\varphi \mid \mathcal{R}\|$. Using the Kaplansky density theorem we have that $\|\varphi\| = \|\varphi \mid \mathcal{A}\|$, hence $\|\phi\| = \|\varphi \mid \mathcal{R}\|$.

Suppose now that $\mathcal{S}$ is not necessarily a factor. Suppose that $\mathcal{H}$ is separable and let

$$\mathcal{H} = \int_\Lambda^\oplus \mathcal{H}(\lambda)\, d\mu(\lambda), \qquad \mathcal{S} = \int_\Lambda^\oplus \mathcal{S}(\lambda)\, d\mu(\lambda),$$

and

$$\mathcal{R} \subseteq \int_\Lambda^\oplus \mathcal{R}(\lambda)\, d\mu(\lambda)$$

be the direct integral decompositions along the center $\mathcal{C}$ of $\mathcal{S}$, where Λ is a complete separable metric space and μ is a complete probability Borel measure on Λ (see [18, Ch. 14]). For $\lambda \in \Lambda$ let

$$\phi(\lambda) : \overline{\mathcal{R}(\lambda)\, \mathcal{S}(\lambda)'} \to \mathcal{B}(\mathcal{H}(\lambda))$$

be defined by

$$\varphi(\lambda)(x) = \sum_{j=1}^{n} a_j(\lambda)\, x b_j(\lambda) \qquad (x \in \overline{\mathcal{R}(\lambda)\, \mathcal{S}(\lambda)'}),$$

where $a_j(\ )$ and $b_j(\ )$ ($j = 1, ..., n$) represent the components of $\mathbf{a}$ and $\mathbf{b}$ (so that $a_j(\lambda), b_j(\lambda) \in \mathcal{S}(\lambda)$ for almost all $\lambda \in \Lambda$), let $\psi = \phi \mid \mathcal{R}$ and $\psi(\lambda) = \phi(\lambda) \mid \mathcal{R}(\lambda)$. Choosing a countable strongly dense subset (x_i) in the unit ball of $\mathscr{R}$ (and noting that the set $(x_i(\lambda))$ is strongly dense in the unit ball of $\mathscr{R}(\lambda)$ for almost all $\lambda \in \Lambda$), we have

$$\|\psi(\lambda)\| = \sup_i \|\psi(\lambda)(x_i(\lambda))\| = \sup_i \|(\psi(x_i)(\lambda)\|$$

almost everywhere; in particular, the function $\|\psi(\ )\|$ (and similarly $\|\varphi(\ )\|$) is measurable. From the well known fact that $\|x\| = \text{essup}\{\|x(\lambda)\| : \lambda \in \Lambda\}$ for each decomposable $x \in \mathscr{B}(\mathscr{H})$ it follows immediately that $\|\varphi\| \leqslant \text{essup}\,\|\varphi(\lambda)\|$ and $\|\psi\| \leqslant \text{essup}\,\|\psi(\lambda)\|$, while the displayed equality shows that $\|\psi(\lambda)\| \leqslant \sup_i \|\psi(x_i)\| \leqslant \|\psi\|$ for almost all λ, hence $\text{essup}\,\|\psi(\lambda)\| = \|\psi\|$. Since $\mathscr{S}(\lambda)$ is an injective factor for almost all λ by [9], we have $\|\varphi(\lambda)\| = \|\psi(\lambda)\|$ almost everywhere, hence it follows that $\|\varphi\| = \|\psi\|$.

Assume now that the Hilbert space $\mathscr{H}$ is arbitrary, but $\mathscr{S}$ is countably generated. Choose in $\mathscr{B}(\mathscr{H})$ a net (q_k) of finite rank projections increasing to the identity 1 and for each k denote by p'_k the projection with range $[\mathscr{S} q_k \mathscr{H}]$. Then (p'_k) is a net of projections in $\mathscr{S}'$ with separable ranges increasing to 1. For each $x \in \overline{\mathscr{R}\mathscr{S}'}$ with $\|x\| = 1$ we therefore have

$$\begin{aligned}\|\varphi(x)\| &\leqslant \sup_k \|p'_k \varphi(x)\, p'_k\| = \sup_k \|\varphi(p'_k x p'_k)\| \\ &\leqslant \sup_k \|\varphi \,|\, p'_k \overline{\mathscr{R}\mathscr{S}'}\, p'_k\| = \sup_k \|\varphi \,|\, \overline{\mathscr{R} p'_k \mathscr{S}' p'_k}\|,\end{aligned}$$

hence $\|\varphi\| \leqslant \sup_k \|\varphi \,|\, \overline{\mathscr{R} p'_k \mathscr{S}' p'_k}\|$. Since $\overline{\mathscr{R} p'_k \mathscr{S}' p'_k} \subseteq \overline{\mathscr{R}\mathscr{S}'}$, the equality $\|\varphi\| = \sup_k \|\varphi \,|\, \overline{\mathscr{R} p'_k \mathscr{S}' p'_k}\|$ holds. Since each element of the unit ball of $\mathscr{R} p'_k$ is of the form $x p'_k$, where x is from the unit ball of $\mathscr{R}$, and since $\varphi(x p'_k) = \varphi(x)\, p'_k$, it follows that $\|\varphi \,|\, \mathscr{R}\| = \sup_k \|\varphi \,|\, \mathscr{R} p'_k\|$. Since the space $p'_k \mathscr{H}$ is separable, we have $\|\varphi \,|\, \mathscr{R} p'_k\| = \|\varphi \,|\, \overline{\mathscr{R} p'_k \mathscr{S}' p'_k}\|$ for each k, and it follows that $\|\varphi \,|\, \mathscr{R}\| = \|\varphi\|$.

Consider now the case when $\mathscr{S}$ is countably decomposable ($=\sigma$-finite) and $\mathscr{R}$ is countably generated. Then there exists a countably generated injective von Neumann subalgebra $\mathscr{S}_0$ of $\mathscr{S}$ which contains $\mathscr{R}$ and the components of **a** and **b** (then $\mathscr{S}_0$ contains also $\varphi(\mathscr{R})$). Indeed, if $\mathscr{S}_0$ is the smallest von Neumann subalgebra of $\mathscr{S}$ that contains $\mathscr{R}$ and the components of **a** and **b** and is invariant under the modular group of $\mathscr{S}$ corresponding to some faithful normal state, then by [32] there exists a (normal) conditional expectation from $\mathscr{S}$ to $\mathscr{S}_0$, hence $\mathscr{S}_0$ is injective (since $\mathscr{S}$ is). Let φ_0 be the extension of φ to $\overline{\mathscr{R}\mathscr{S}'_0}$ as an $\mathscr{S}'_0$-module homomorphism. By the previous paragraph we have $\|\varphi_0\| = \|\varphi \,|\, \mathscr{R}\|$, but from $\mathscr{S}_0 \subseteq \mathscr{S}$ it follows that $\overline{\mathscr{R}\mathscr{S}'} \subseteq \overline{\mathscr{R}\mathscr{S}'_0}$, hence we conclude that $\|\varphi\| = \|\varphi \,|\, \mathscr{R}\|$.

Let now $\mathscr{S}$ be arbitrary (injective) but $\mathscr{R}$ still countably generated and denote here by $\mathscr{S}_0$ the von Neumann subalgebra of $\mathscr{S}$ generated by $\mathscr{R}$ and the components of $\mathbf{a}$ and $\mathbf{b}$. There exists a net (f_k) of countably decomposable projections in $\mathscr{S}$ increasing to 1. For each k let e_k be the projection with range $[\mathscr{S}_0 f_k \mathscr{H}]$. Then $e_k \in \mathscr{S}'_0 \cap \mathscr{S} \subseteq \mathscr{R}' \cap \mathscr{S}$ and each e_k is countably decomposable in $\mathscr{S}$. (To see the last property, note that f_k is an orthogonal sum of a countable family of cyclic projections in $\mathscr{S}$ and $\mathscr{S}_0$ is countably generated.) For each k consider the map $\varphi_k: \overline{\mathscr{R}\mathscr{S}'}\, e_k \to \mathscr{B}(e_k \mathscr{H})$ defined by

$$\varphi_k(x) = \sum_{j=1}^{n} (a_j e_k)\, x(b_j e_k) \qquad (x \in \overline{\mathscr{R}\mathscr{S}'}\, e_k).$$

Note that $a_j e_k$ and $b_j e_k$ are in $e_k \mathscr{S} e_k$ since e_k commutes with all a_j and b_j. If $\pi_k: \overline{\mathscr{R}\mathscr{S}'} \to \overline{\mathscr{R}\mathscr{S}'}\, e_k$ is the epimorphism defined by $\pi_k(x) = x e_k$, then we have $\varphi_k(\pi_k(x)) = \varphi(x)\, e_k$ for each $x \in \overline{\mathscr{R}\mathscr{S}'}$, and this implies that $\|\varphi_k\| \leqslant \|\varphi\|$ for all k (since the unit ball of $\overline{\mathscr{R}\mathscr{S}'}\, e_k$ is the image under π_k of the unit ball of $\overline{\mathscr{R}\mathscr{S}'}$). Since the projections e_k converge strongly to 1, it follows that $\|\varphi\| = \sup_k \|\varphi_k\|$. Similarly $\|\varphi \mid \mathscr{R}\| = \sup_k \|\varphi_k \mid \mathscr{R} e_k\|$. Since $\|\varphi\| = \|\varphi_k \mid \mathscr{R} e_k\|$ by the countably decomposable case, it follows that $\|\varphi\| = \|\varphi \mid \mathscr{R}\|$.

Finally, in the general case, we let $\mathscr{R}_0$ be a countably generated von Neumann subalgebra of $\mathscr{R}$ such that $\|\varphi \mid \mathscr{R}_0\| = \|\varphi \mid \mathscr{R}\|$ and $\|\varphi \mid \overline{\mathscr{R}_0 \mathscr{S}'}\| = \|\varphi\|$, and then we conclude from the previous paragraph that $\|\varphi \mid \mathscr{R}\| = \|\varphi\|$. This proves the theorem in one direction.

Suppose now conversely that $\|\varphi\|_{\mathrm{cb}} = \|\varphi \mid \mathscr{R}\|_{\mathrm{cb}}$ for each von Neumann subalgebra $\mathscr{R}$ of $\mathscr{S}$ and each elementary map $\varphi: \overline{\mathscr{R}\mathscr{S}'} \to \mathscr{B}(\mathscr{H})$. We shall show then that the assumption that $\mathscr{S}$ is not injective leads to a contradiction. Since $\mathscr{S}$ is not injective, the continuous central part $p_c \mathscr{S}$ of $\mathscr{S}$ (where p_c is a central projection in $\mathscr{S}$) is non-zero and satisfies the condition $\|\varphi\|_{\mathrm{cb}} = \|\varphi \mid \mathscr{R}\|_{\mathrm{cb}}$ for all von Neumann subalgebras $\mathscr{R} \subseteq p_c \mathscr{S}$ and all elementary maps $\varphi: \overline{\mathscr{R} p_c \mathscr{S}'} \to \mathscr{B}(p_c \mathscr{H})$. To simplify the notation we may assume that $p_c = 1$, hence $\mathscr{R}$ is continuous. Then for each positive integer n the identity operator 1 can be expressed as a sum of n equivalent mutually orthogonal projections p_i in $\mathscr{S}$; we denote by $\mathscr{R}_n$ the linear span of such a family of projections $\{p_1, \ldots, p_n\}$, so that $\mathscr{R}_n$ is a commutative von Neumann algebra of dimension n. Then no nontrivial linear combination of the projections p_i can be contained in a proper two-sided ideal of $\mathscr{S}$, hence it follows from [20, Proposition 1.1] that each linear map ψ from $\mathscr{R}_n$ to $\mathscr{S}$ is a restriction of an elementary map $\varphi: \overline{\mathscr{R}_n \mathscr{S}'} \to \mathscr{B}(\mathscr{H})$ with coefficients in $\mathscr{S}$ (if $\mathscr{S}$ is a simple factor, this follows also from the Jacobson density theorem [28, p. 220]), and by hypothesis we have $\|\psi\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$. But every such $\mathscr{S}'$-bimodule homomorphism φ can be extended to a completely bounded $\mathscr{S}'$-bimodule homomorphism $\tilde{\varphi}: \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ so that $\|\tilde{\varphi}\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$, and $\tilde{\varphi}$ can be expressed as a linear combination of four completely positive $\mathscr{S}'$-bimodule maps $\tilde{\varphi}_j: \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ with

$\|\tilde{\varphi}_j\|_{cb} \leq \|\tilde{\varphi}\|_{cb}$ (see [25, p. 118]). Since $\tilde{\varphi}_j$ are $\mathscr{S}'$-bimodule maps, it follows that $\tilde{\varphi}_j(\mathscr{S}) \subseteq \mathscr{S}$, hence ψ can be expressed as a linear combination of four completely positive maps from $\mathscr{R}_n$ to $\mathscr{S}$ that have the completely bounded norm dominated by $\|\psi\|_{cb}$. But every finite dimensional commutative C*-algebra is isomorphic to $\mathscr{R}_n$ for some n, hence it follows from [16, Theorem 2.1] that $\mathscr{S}$ is injective, which is in contradiction with our assumption. ∎

THEOREM 2.7. *Let $\mathscr{R} \subseteq \mathscr{S} \subseteq \mathscr{B}(\mathscr{H})$ be von Neumann algebras, where $\mathscr{S}$ is injective, and put $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$. Then the natural map $\theta_{\mathscr{R},\mathscr{S}}: \mathscr{S} \otimes^h_{\mathscr{T}} \mathscr{S} \to$ $\mathrm{CB}(\mathscr{R}, \mathscr{S})$ is an isometry.*

Proof. From $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$ and $\mathscr{R} \subseteq \mathscr{S}$ we have $\mathscr{T}' = \overline{\mathscr{R}\mathscr{S}'}$. The map $\theta_{\mathscr{R},\mathscr{S}}$ can be expressed as the composition

$$\mathscr{S} \otimes^h_{\mathscr{T}} \mathscr{S} \xrightarrow{\iota} \mathscr{B}(\mathscr{H}) \otimes^h_{\mathscr{T}} \mathscr{B}(\mathscr{H}) \xrightarrow{\theta_{\mathscr{T}}} \mathrm{CB}(\mathscr{T}', \mathscr{B}(\mathscr{H}))$$
$$\xrightarrow{\rho} \mathrm{CB}(\mathscr{R}, \mathscr{S}),$$

where the first two maps are isometries by Theorem 2.2 and Lemma 2.3 respectively, and ρ is induced by the restriction. Since the restriction of ρ to the range of $\theta_{\mathscr{T}}\,\iota$ is isometric by Lemma 2.6, $\theta_{\mathscr{R},\mathscr{S}}$ is also isometric. ∎

Remark. The assumption in Theorem 2.7, that $\mathscr{S}$ is injective, is not redundant. To see this, note that in the above proof the restriction of ρ to the range of $\theta_{\mathscr{T}}\,\iota$ is isometric if $\theta_{\mathscr{R},\mathscr{S}}$ is isometric, hence $\|\varphi\|_{cb} = \|\varphi \mid \mathscr{R}\|_{cb}$ for each map $\varphi \in \mathrm{CB}(\overline{\mathscr{R}\mathscr{S}'}, \mathscr{B}(\mathscr{H}))$ of the form $\varphi(x) = \mathbf{a}x^{(n)}\mathbf{b}$, where n is finite and $\mathbf{a} \in \mathrm{R}_n(\mathscr{S})$ and $\mathbf{b} \in \mathrm{C}_n(\mathscr{S})$. If this is true for every von Neumann subalgebra $\mathscr{R}$ of $\mathscr{S}$, then $\mathscr{S}$ must be injective by Lemma 2.6.

3. THE NORM OF ELEMENTARY OPERATORS ON PRIME C^*-ALGEBRAS WITH ZERO SOCLE

Let $\mathscr{A}$ be any C*-algebra. A map $\varphi: \mathscr{A} \to \mathscr{A}$ of the form

$$\varphi(x) = \sum_{i=1}^{\infty} a_i x b_i \qquad (x \in \mathscr{A}),$$

where $\mathbf{a} = (a_1, a_2, \ldots) \in \mathrm{R}_\infty(\mathscr{A})$ and $\mathbf{b} = (b_1, b_2, \ldots)^T \in \mathrm{C}_\infty(\mathscr{A})$ are such that the two series $\sum_i a_i a_i^*$ and $\sum_i b_i^* b_i$ converge in norm, is called in this section an *elementary operator*. (Usually it is required in the definition of elementary operator that the sum is finite, but this restriction would be unnecessary here. Such operators have attracted much attention in the past, see for example [14].) We denote such an operator also by $\mu(\mathbf{a}, \mathbf{b})$ or (in the case when we deal simultaneously with more algebras) $\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})$. In the case when $\mathscr{A}$ has no unit, we allow that a_i and b_i are in the multiplier algebra of $\mathscr{A}$.

Recall that a (complex) algebra $\mathscr{A}$ is *prime* if for each $a, b \in \mathscr{A}$ the identity $a\mathscr{A}b = 0$ implies that $a = 0$ or $b = 0$. An equivalent requirement is that $\mathscr{A}$ has no ideal divisors of zero. A C*-algebra is called primitive if it has a faithful irreducible representation. It is easy to verify that every primitive C*-algebra is prime, and for separable algebras the converse is also true [27, p. 102].

An idempotent e in a C*-algebra $\mathscr{A}$ is called *minimal* if $e\mathscr{A}e = \mathbb{C}e$ (equivalently, the left ideal $\mathscr{A}e$ is a minimal non-zero left ideal with respect to the inclusion). The two-sided ideal generated by all minimal idempotents in $\mathscr{A}$ is called the *socle* of $\mathscr{A}$ and is denoted by $\operatorname{soc}(\mathscr{A})$. (For example, the socle of $\mathscr{B}(\mathscr{H})$ is the ideal of all finite rank operators.) More about the socle in C*-algebras can be found in [4]. We can now state the main result of this section.

THEOREM 3.1. *If $\mathscr{A}$ is a prime C*-algebra with zero socle, then $\|\varphi\| = \|\varphi\|_{\text{cb}}$ for every elementary operator φ on $\mathscr{A}$.*

It is known that for an elementary operator $\varphi = \mu(\mathbf{a}, \mathbf{b})$ on a prime C*-algebra $\mathscr{A}$ the completely bounded norm is equal to the Haagerup norm of $\mathbf{a} \odot \mathbf{b}$ in $\mathscr{A} \otimes^h \mathscr{A}$ (using Lemma 3.2 below this can be proved by a reduction to separable irreducible C*-algebras, but in fact a more general result is proved in [2, Th. 3.7]). Thus, Theorem 3.1 answers in principle the question of how to compute the norm of an elementary operator on a prime C*-algebra with zero socle. The basic example of an algebra to which Theorem 3.1 applies is the Calkin algebra $\mathscr{C}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, where $\mathscr{K}(\mathscr{H})$ is the ideal of all compact operators on $\mathscr{H}$. It was proved by Apostol and Fialkow in [1] that for elementary operators on $\mathscr{C}(\mathscr{H})$ ($\mathscr{H}$ separable) the norm is equal to the essential norm.

Theorem 3.1 will be first proved for the Calkin algebra of a separable Hilbert space by using the Voiculescu non-commutative Weyl–von Neumann theorem and the proof for general $\mathscr{A}$ will be reduced to this special case. In this reduction the following lemma is needed, the first part of which was proved by Elliott and Zsido [13, Proposition 3.1] using a less elementary method.

LEMMA 3.2. *For each separable subalgebra $\mathscr{B}_0$ in a prime C*-algebra $\mathscr{A}$ there exists a separable prime C*-subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing $\mathscr{B}_0$. Moreover, if $\operatorname{soc}(\mathscr{A}) = 0$, then there exists $\mathscr{B}$ such that (in addition to being separable and prime) $\operatorname{soc}(\mathscr{B}) = 0$.*

Proof. Assume first that $\mathscr{A}$ has a unit 1. We shall construct by an induction a sequence $\mathscr{B}_0 \subseteq \mathscr{B}_1 \subseteq \mathscr{B}_2 \subseteq \cdots$ of separable C*-subalgebras of $\mathscr{A}$ such that

$$\|\mu(a, b) | \mathscr{B}_{i+1}\| = \|a\| \, \|b\|$$

for every $a, b \in \mathscr{B}_i$ and, if $\operatorname{soc}(\mathscr{A}) = 0$,

$$\dim e\mathscr{B}_{i+1} e > 1$$

for every idempotent $e \in \mathscr{B}_i$. Then we shall show that $\mathscr{B} \stackrel{\text{def}}{=} \overline{\bigcup_{i=1}^{\infty} \mathscr{B}_i}$ satisfies the requirements of Lemma 3.2. Suppose that for some i the algebras $\mathscr{B}_1, ..., \mathscr{B}_i$ have been already constructed. Let $\mathscr{G}$ be a countable dense set in $\mathscr{B}_i$. Since $\mathscr{A}$ is prime, we have $\|\mu(a, b)\| = \|a\| \, \|b\|$ for each $a, b \in \mathscr{A}$ by [21, Proposition 2.3]. It follows that there exists a separable C*-subalgebra $\mathscr{D}$ of $\mathscr{A}$ such that $\mathscr{D} \supseteq \mathscr{B}_i$ and $\|\mu(a, b) | \mathscr{D}\| = \|a\| \, \|b\|$ for every a and b in the countable set $\mathscr{G}$. Since $\mathscr{G}$ is dense in $\mathscr{B}_i$, we then have $\|\mu(a, b) | \mathscr{D}\| = \|a\| \, \|b\|$ for all $a, b \in \mathscr{B}_i$.

If $\operatorname{soc}(\mathscr{A}) = 0$, choose a countable set $\mathscr{E}$ of projections which is dense in the set of all projections in $\mathscr{B}_i$. Since $\mathscr{A}$ has no non-zero minimal projections, there exists a separable C*-subalgebra $\mathscr{B}_{i+1}$ of $\mathscr{A}$ containing $\mathscr{D}$ such that $\dim f\mathscr{B}_{i+1} f > 1$ for every non-zero projection f in the countable set $\mathscr{E}$. (In the case $\operatorname{soc}(\mathscr{A}) \neq 0$ we simply put $\mathscr{B}_{i+1} = \mathscr{D}$.) Since any two projections e, f in a C*-algebra that satisfy the relation $\|e - f\| < 1$ are unitarily equivalent in the algebra [24, p. 178], it follows that $\dim e\mathscr{B}_{i+1} e > 1$ for each non-zero projection $e \in \mathscr{B}_i$.

By this construction we have now that $\|\mu(a, b) | \mathscr{B}\| = \|a\| \, \|b\|$ for every $a, b \in \bigcup_{i=1}^{\infty} \mathscr{B}_i$, hence for every $a, b \in \mathscr{B}$. So $\mathscr{B}$ is a prime C*-algebra. Moreover, if $\operatorname{soc}(\mathscr{A}) \neq 0$, then $\dim f\mathscr{B}f > 1$ for each non-zero projection f in $\bigcup_{i=1}^{\infty} \mathscr{B}_i$. Since each projection $e \in \mathscr{B}$ can be approximated arbitrarily closely by a projection f from $\bigcup_{i=1}^{\infty} \mathscr{B}_i$ (see [24, p. 226]) and e, f are then unitarily equivalent in $\mathscr{B}$, it follows that $\dim e\mathscr{B}e > 1$ for each non-zero projection $e \in \mathscr{B}$. This shows that $\mathscr{B}$ has no non-zero minimal projections, hence $\operatorname{soc}(\mathscr{B}) = 0$.

If $\mathscr{A}$ has no unit, then we embed $\mathscr{A}$ into $\tilde{\mathscr{A}} = \mathscr{A} + \mathbb{C}1$, which is again a prime C*-algebra (with zero socle if $\operatorname{soc}(\mathscr{A}) = 0$). Let $\tilde{\mathscr{B}}$ be a separable prime C*-subalgebra of $\tilde{\mathscr{A}}$ (with zero socle if $\operatorname{soc}(\mathscr{A}) = 0$) such that $\tilde{\mathscr{B}} \supseteq \mathscr{B}_0$ and put $\mathscr{B} = \mathscr{A} \cap \tilde{\mathscr{B}}$. Since $\mathscr{B}$ is an ideal in the prime algebra $\tilde{\mathscr{B}}$, $\mathscr{B}$ is prime. If $\mathscr{A}$ has no non-zero minimal projections then $\mathscr{B}$ has no non-zero minimal projections, for if e were such a projection, then from $\dim e\mathscr{B}e = 1$ and $\dim \tilde{\mathscr{B}}/\mathscr{B} \leqslant \dim \tilde{\mathscr{A}}/\mathscr{A} = 1$ it would follow that $\dim e\tilde{\mathscr{B}}e \leqslant 2$, and this would imply that $\tilde{\mathscr{B}}$ contains non-zero minimal projections. ∎

If $\mathscr{A}$ is a prime C*-algebra then its multiplier algebra is also prime [21]; moreover, using the Kaplansky density theorem it is easy to see that the norm of an elementary operator φ on $\mathscr{A}$ is the same as on the multiplier algebra of $\mathscr{A}$, and the same conclusion holds also for completely bounded norms, hence in proving Theorem 3.1 we may assume that $\mathscr{A}$ has a unit. We may also assume that **a** and **b** have only finitely many non-zero components, since the general case follows then easily by approximation.

Proof of Theorem 3.1. Throughout the proof $\varphi=\mu(\mathbf{a},\mathbf{b})=\mu_{\mathscr{A}}(\mathbf{a},\mathbf{b})$ will denote a fixed elementary operator on $\mathscr{A}$, where $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_n)^T$, with all a_i, b_i in $\mathscr{A}$, and n is a positive integer. If ρ is any map defined on $\mathscr{A}$ we denote $\rho(\mathbf{a})=(\rho(a_1),...,\rho(a_n))$ and $\rho(\mathbf{b})$ is defined similarly. The proof is divided in the three steps.

Step I. *Assume that $\mathscr{A}$ is the Calkin algebra $\mathscr{C}(\mathscr{H})$ of a separable Hilbert space.* Let $\mathscr{B}$ be a separable C*-subalgebra of $\mathscr{C}(\mathscr{H})$ such that $\mathscr{B}$ contains all a_i, b_i and such that $\|\mu(\mathbf{a},\mathbf{b})\,|\,\mathscr{B}\|_{\text{cb}}=\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}$. Let $\pi:\mathscr{B}(\mathscr{H})\to\mathscr{C}(\mathscr{H})$ be the natural map, $\tilde{\mathscr{B}}=\pi^{-1}(\mathscr{B})$, and let $\tilde{a}_i$, $\tilde{b}_i\in\tilde{\mathscr{B}}$ be such that $\pi(\tilde{a}_i)=a_i$, $\pi(\tilde{b}_i)=b_i$ for all i. Let σ be a faithful representation of $\mathscr{B}$ on a separable Hilbert space $\mathscr{H}_\sigma$ and let $\rho=(\sigma\pi)^{(\infty)}\oplus\text{id}$ (the direct sum of countably many copies of the representation $\sigma\pi$ and the identity representation of $\tilde{\mathscr{B}}$). Since the weak closure of $\sigma(\mathscr{B})^{(\infty)}$ in $\mathscr{B}(\mathscr{H}_\sigma^\infty)$ has a separating vector, the commutant $\mathscr{D}=^{\text{def}}(\sigma(\mathscr{B})^{(\infty)})'$ has a cyclic vector. Since the elementary operator $\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))$ on $\mathscr{B}(\mathscr{H}_\sigma^\infty)$ is obviously a homomorphism of $\mathscr{D}$-bimodules, it follows from [30, Th. 2.1] that

$$\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))\|=\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))\|_{\text{cb}}.$$

Since $\sigma^{(\infty)}$ is completely isometric, we have $\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))\|_{\text{cb}}\geqslant\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))\,|\,\sigma(\mathscr{B})^{(\infty)}\|_{\text{cb}}=\|\mu(\mathbf{a},\mathbf{b})\,|\,\mathscr{B}\|_{\text{cb}}=\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}$, hence

$$\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))\|\geqslant\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}.$$

Therefore for each $\varepsilon>0$ there exists $y\in\mathscr{B}(\mathscr{H}_\sigma^\infty)$, with $\|y\|=1$, such that

$$\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))(y)\|\geqslant\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}-\varepsilon$$

Replacing y by $y^{(\infty)}$ and $\sigma^{(\infty)}$ by $(\sigma^{(\infty)})^{(\infty)}$, we can achieve that

$$\|\mu(\sigma^{(\infty)}(\mathbf{a}),\sigma^{(\infty)}(\mathbf{b}))(y)-c\|\geqslant\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}-\varepsilon$$

for every compact operator c on $\mathscr{H}_\sigma^\infty$, hence a fortiori

$$\|\mu(\rho(\tilde{\mathbf{a}}),\rho(\tilde{\mathbf{b}}))(y\oplus 0)-d\|\geqslant\|\mu(\mathbf{a},\mathbf{b})\|_{\text{cb}}-\varepsilon \tag{3.1}$$

for every compact operator d on $\mathscr{H}_\sigma^\infty\oplus\mathscr{H}$. (Here, of course, $\tilde{\mathbf{a}}$ denotes $(\tilde{a}_1,...,\tilde{a}_n)$ and $\tilde{\mathbf{b}}=(\tilde{b}_1,...,\tilde{b}_n)^T$.)

By Voiculescu's theorem (see [3] or [33]) the representation ρ is approximately equivalent to the identity representation of $\tilde{\mathscr{B}}$, which implies in particular that there exists a unitary operator $U: \mathscr{H} \to \mathscr{H}_\rho$ such that the operator $U^*\rho(x)U - x$ is compact for each $x \in \tilde{\mathscr{B}}$. Put now $x = U^*(y \oplus 0)U$. Then $\|x\| = 1$, and from (3.1) we have (since $U^*\rho(\tilde{a}_i)U - \tilde{a}_i$ and $U^*\rho(\tilde{b}_i)U - \tilde{b}_i$ are compact for all i)

$$\|\mu_{\mathscr{B}(\mathscr{H})}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})(x) - g\| \geqslant \|\mu(\mathbf{a}, \mathbf{b})\|_{\text{cb}} - \varepsilon$$

for each compact operator g on $\mathscr{H}$. This means that $\|\mu(\mathbf{a}, \mathbf{b})(\pi(x))\| \geqslant \|\mu(\mathbf{a}, \mathbf{b})\|_{\text{cb}} - \varepsilon$, hence $\|\mu(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu(\mathbf{a}, \mathbf{b})\|_{\text{cb}} - \varepsilon$. Letting $\varepsilon \to 0$, it follows that $\|\mu(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu(\mathbf{a}, \mathbf{b})\|_{\text{cb}}$. The reverse inequality is obvious.

Step II. *Assume that $\mathscr{A}$ is an irreducible C*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing no non-zero compact operators and that $\mathscr{H}$ is separable.* Since $\mathscr{A}$ is irreducible, the Kaplansky density theorem implies that

$$\|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\| = \|\mu_{\mathscr{B}(\mathscr{H})}(\mathbf{a}, \mathbf{b})\|.$$

It is straightforward that $\|\mu_{\mathscr{B}(\mathscr{H})}(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu_{\mathscr{C}(\mathscr{H})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|$, hence

$$\|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu_{\mathscr{C}(\mathscr{H})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|$$

or (by Step I)

$$\|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu_{\mathscr{C}(\mathscr{H})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\text{cb}}.$$

But, since the restriction of π to $\mathscr{A}$ is an isometric isomorphism, we have

$$\|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\|_{\text{cb}} = \|\mu_{\pi(\mathscr{A})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\text{cb}} \leqslant \|\mu_{\mathscr{C}(\mathscr{H})}(\pi(\mathbf{a}), \pi(\mathbf{b}))\|_{\text{cb}},$$

hence it follows that $\|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\| \geqslant \|\mu_{\mathscr{A}}(\mathbf{a}, \mathbf{b})\|_{\text{cb}}$.

Step III. *The general case.* Let now $\mathscr{A}$ be as in the statement of Theorem 3.1 and let $\mathscr{B}_0$ be a separable subalgebra of $\mathscr{A}$ such that $\|\varphi \mid \mathscr{B}_0\| = \|\varphi\|$ and $\|\varphi \mid \mathscr{B}_0\|_{\text{cb}} = \|\varphi\|_{\text{cb}}$. By Lemma 3.2 there exists a separable prime C*-subalgebra $\mathscr{B}$ of $\mathscr{A}$, containing $\mathscr{B}_0$, with zero socle. By [27, p. 102] $\mathscr{B}$ is primitive, hence we may assume that $\mathscr{B}$ is an irreducible C*-subalgebra of $\mathscr{B}(\mathscr{H})$ for some (necessarily separable) Hilbert space $\mathscr{H}$. Since $\operatorname{soc}(\mathscr{B}) = 0$, $\mathscr{B}$ contains no non- zero (finite rank hence) compact operators, hence by Step II we conclude that $\|\varphi \mid \mathscr{B}\| = \|\varphi \mid \mathscr{B}\|_{\text{cb}}$. Therefore $\|\varphi\| = \|\varphi\|_{\text{cb}}$. ∎

COROLLARY 3.3. *If $\mathscr{R}$ is a continuous von Neumann algebra, then $\|\varphi\| = \|\varphi\|_{\text{cb}}$ for every elementary operator φ on $\mathscr{R}$.*

Proof. Let X be the maximal ideal space of the center $\mathscr{C}$ of $\mathscr{R}$ and for each $x \in X$ let $\mathscr{M}_x$ be the closed ideal in $\mathscr{R}$ generated by x. By [15, Lemma 11] the quotient C*-algebra $\mathscr{R}(x) =^{\text{def}} \mathscr{R}/\mathscr{M}_x$ is prime for each x

and, since each projection in $\mathscr{R}$ can be expressed as a sum of any finite number of equivalent orthogonal subprojections (see [18, p. 426]), it follows also that $\mathscr{R}(x)$ has no non-zero minimal projections. Let φ_x be the elementary operator induced on $\mathscr{R}(x)$ by φ for each $x \in X$. Then $\|\varphi_x\| = \|\varphi_x\|_{cb}$ by Theorem 3.1. Using the fact that $\|a\| = \sup_{x \in X} \|a(x)\|$ for each $a \in \mathscr{R}$, where $a(x)$ denotes the coset of a in $\mathscr{R}(x)$ (see [15, p. 262]), it is easy to conclude that $\|\varphi\| = \sup_{x \in X} \|\varphi_x\|$. Now for each positive integer n the center of $M_n(\mathscr{R})$ is (isomorphic to) $\mathscr{C}$, hence by the same reasoning we also have $\|\varphi\|_{cb} = \sup_{x \in X} \|\varphi_x\|_{cb}$. It follows that $\|\varphi\| = \|\varphi\|_{cb}$. ∎

4. Spatial Epimorphisms

Using his theory of correspondences Mingo proved in [23] that an automorphism φ of a σ-finite von Neumann algebra of the form $\varphi(x) = \mathbf{a}x^{(\infty)}\mathbf{b}$, where $\mathbf{a} \in R_\infty(\mathscr{R})$ and $\mathbf{b} \in C_\infty(\mathscr{R})$, is necessarily inner. (Here, as usual, ∞ denotes the countably infinite cardinal.) In this section we prove a generalization of this result by elementary methods. First we need a simple lemma.

Lemma 4.1. *Let $\mathscr{R}$ be a von Neumann algebra, J any cardinal number, and $\mathscr{R}^J$ the algebra of all diagonal matrices in $M_J(\mathscr{R})$. Then each projection $e \in M_J(\mathscr{R})$ is equivalent to some projection in $\mathscr{R}^J$ (in the Murray von Neumann sense).*

Proof. Note that $\mathscr{R}^J$ contains the center of $M_J(\mathscr{R})$. We may assume that the central carrier C_e of e is 1, otherwise we consider the von Neumann algebra $\mathscr{R}C_e$ instead of $\mathscr{R}$. It suffices to prove that for each central projection $q \neq 0$ in $\mathscr{R}$ there exists a non-zero central projection $p \leqslant q$ in $\mathscr{R}$ and a projection $f \in \mathscr{R}^J$ such that $ep^{(J)} \sim fp^{(J)}$ (for then a standard maximality argument completes the proof). Replacing $\mathscr{R}$ with $q\mathscr{R}$, we may assume that $q = 1$. Then it suffices to find a non-zero central projection $p \in \mathscr{R}$ such that $ep^{(J)}$ is equivalent to some projection in $\mathscr{R}^J$.

Let $\{e_i\}$ be a maximal orthogonal family of non-zero subprojections of e in $M_J(R)$ such that $\tilde{e} =^{\text{def}} \sum_i e_i$ is equivalent to some projection $f \in \mathscr{R}^J$. If $f = 1$, then it follows from $\tilde{e} \sim 1$ and $\tilde{e} \leqslant e$ that $e \sim 1$ (by [18, p. 406]), so we may assume that $f < 1$. If $\tilde{e} = e$, the proof is ended, so assume that $\tilde{e} < e$.

Let $\{E_i\}$ be the projections from the natural matrix unit of $M_J(\mathscr{R})$ (so E_i are mutually orthogonal equivalent projections with sum 1 that commute with all elements of $\mathscr{R}^J$). Then $(1 - f)\, E_i \neq 0$ for some $i < J$. By comparison theorem [18, p. 409] there exists a central projection $P \in M_J(\mathscr{R})$ (necessarily of the form $P = p^{(J)}$ for some central projection $p \in \mathscr{R}$) such that $P(e-\tilde{e}) \preccurlyeq P(1-f)\, E_i$ and $(1-P)(1-f)\, E_i \prec (1-P)(e-\tilde{e})$. If $P \neq 0$, and g is a subprojection of $P(1-f)\, E_i$ equivalent to $P(e-\tilde{e})$, then we have $Pe = P\tilde{e} + P(e-\tilde{e}) \sim Pf + Pg$. Since each subprojection of E_i in $\mathrm{M}_J(\mathscr{R})$ is in $\mathscr{R}^J$, g must be in $\mathscr{R}^J$, hence $Pf + Pg$ is also in $\mathscr{R}^J$ and the proof is ended in this case. So we may assume that $P = 0$, hence $(1-f)\, E_i \prec e - \tilde{e}$. But then, with e_0 a subprojection of $e - \tilde{e}$ equivalent to $(1-f)\, E_i$, we have $\tilde{e} + e_0 \sim f + (1-f)\, E_i$, which contradicts the maximality of the family $\{e_i\}$. ∎

PROPOSITION 4.2. *Let $\mathscr{R} \subseteq \mathscr{S}$ be von Neumann algebras, $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$, J a cardinal number, and $\varphi: \mathscr{R} \to \mathscr{S}$ an epimorphism of the form $\varphi(x) = \mathbf{a}^* x^{(J)} \mathbf{a}$ for some $\mathbf{a} \in \mathrm{C}_J(\mathscr{S})$. Then there exists an isometry $u \in \mathscr{S}$ with final projection e $(=uu^*)$ in $\mathscr{T}$ such that $\varphi(x) = u^* x u$ for all $x \in \mathscr{R}$. Moreover, if $\mathscr{S} = \mathscr{R}$, then u is a unitary.*

Proof. As in the proof of Lemma 1.1 we consider the projection $p \in \mathrm{M}_J(\mathscr{T})$ with range $[\mathscr{R}^{(J)} \mathbf{a} \mathscr{H}]$. Clearly $p\mathbf{a} = \mathbf{a}$. Since $\mathbf{a}^*\mathbf{a} = \varphi(1) = 1$, the element $e =^{\mathrm{def}} \mathbf{a}\mathbf{a}^*$ is a projection in $\mathrm{M}_J(\mathscr{S})$. We shall show that $e = p$, so that $e \in \mathrm{M}_J(\mathscr{T})$. Indeed, since φ is a homomorphism, we have $\mathbf{a}^*(xy)^{(J)}\mathbf{a} = \mathbf{a}^* x^{(J)} \mathbf{a}\mathbf{a}^* y^{(J)} \mathbf{a}$ for all $x, y \in \mathscr{R}$, hence $\mathbf{a}^* \mathscr{R}^{(J)}(1-e)\, \mathscr{R}^{(J)}\mathbf{a} = 0$ or $\mathbf{a}^* \mathscr{R}^{(J)}(1-e)\, p = 0$. By taking adjoints in the last identity we obtain $p(1-e)\, \mathscr{R}^{(J)}\mathbf{a} = 0$, hence $p(1-e)\, p = 0$, which means that $p \leqslant e$. But, from $e = \mathbf{a}\mathbf{a}^*$ and $p\mathbf{a} = \mathbf{a}$ it follows that $pe = e$, hence $e = p$.

Let $\mathscr{T}^J$ be the algebra of all diagonal matrices in $\mathrm{M}_J(\mathscr{T})$. By Lemma 4.1 there exists a partial isometry $v \in \mathrm{M}_J(\mathscr{T})$ such that $e = v^*v$ and the projection $f =^{\mathrm{def}} vv^*$ is in $\mathscr{T}^J$. Since $\varphi(x) = \mathbf{a}^* x^{(J)} e \mathbf{a} = \mathbf{a}^* v^* x^{(J)} v \mathbf{a}$ for all $x \in \mathscr{R}$, we may replace $\mathbf{a}$ with $v\mathbf{a}$ and so we achieve that $\mathbf{a}\mathbf{a}^* = f$. If $f = \mathrm{diag}(f_i)$, where f_i $(i < J)$ are projections in $\mathscr{T}$, then we have $a_i a_j^* = \delta_{ij} f_j$ for all i, j. This implies that $p_i =^{\mathrm{def}} a_i^* a_i$ $(i < J)$ are projections in $\mathscr{S}$ with mutually orthogonal ranges, and $\sum_j p_j = 1$ (since $\sum_j a_j^* a_j = \varphi(1) = 1$). Since for $i \neq j$ non-zero elements of the form $p_i y p_j$ $(y \in \mathscr{S})$ can not be in the range of φ (to see this, note that $\varphi(x) = \sum_i p_i a_i^* x a_i p_i$) and φ is onto, it follows that $p_i \mathscr{S} p_j = 0$, hence the central carriers C_{p_i} of p_i in $\mathscr{S}$ are mutually orthogonal. From $\sum_i p_i = 1$ we now conclude that $C_{p_i} = p_i$ for each i, hence $p_i \in \mathscr{T}$ (since the center of $\mathscr{S}$ is contained in $\mathscr{T} = \mathscr{R}' \cap \mathscr{S}$).

For $i \neq j$ we have now $a_i^* \mathscr{S} a_j = p_i a_i^* \mathscr{S} a_j p_j \subseteq p_i \mathscr{S} p_j = 0$. In particular $a_i^* a_j = 0$, hence the projections $f_i = a_i a_i^* (i < J)$ also have mutually orthogonal ranges. It follows that the element $u =^{\mathrm{def}} \sum_j a_j = \sum_j f_j a_j p_j$ is an isometry in $\mathscr{S}$ and $u^* x u = \varphi(x)$ for all $x \in \mathscr{R}$.

If $\mathscr{R} = \mathscr{S}$, then $\mathscr{T}$ is the center of $\mathscr{R}$, so for each i we have that p_i and f_i are equivalent central projections in $\mathscr{R}$, hence $p_i = f_i$ and consequently u is a unitary. ∎

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