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## HILBERT $C^*$ -MODULES IN WHICH ALL CLOSED SUBMODULES ARE COMPLEMENTED

BOJAN MAGAJNA

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ABSTRACT. Let  $B$  be a  $C^*$ -algebra. If there exists a full Hilbert  $B$ -module  $X$  such that  $X = Y \oplus Y^\perp$  for each closed submodule  $Y$ , then  $B$  is  $*$ -isomorphic to a  $C^*$ -algebra of (not necessarily all) compact operators on a Hilbert space.

### 1. INTRODUCTION AND THE MAIN RESULT

It is well known that each closed subspace  $\mathcal{K}$  of a Hilbert space  $\mathcal{H}$  has an orthogonal complement  $\mathcal{K}^\perp$  such that  $\mathcal{K} \oplus \mathcal{K}^\perp = \mathcal{H}$ . (In fact, by [7] a similar property characterizes Hilbert spaces among all Banach spaces.) This property, however, does not hold in general for closed submodules of a Hilbert module over a  $C^*$ -algebra (see [5] or [9]; in [6] Hilbert modules that are complemented in any larger Hilbert module are characterized). In this note we shall characterize all  $C^*$ -algebras  $B$  such that all closed submodules in any Hilbert module over  $B$  are orthogonally complemented. Here a submodule  $Y$  of a Hilbert module  $X$  is called orthogonally complemented if there exists a self-adjoint projection  $p$  on  $X$  such that  $Y = pX$  (see [5, p. 21]). More precisely, we shall prove the following result.

**Theorem 1.** *Let  $B$  be a  $C^*$ -algebra. If there exists a full Hilbert  $B$ -module in which every closed submodule is orthogonally complemented, then  $B$  is  $*$ -isomorphic to a  $C^*$ -algebra of (not necessarily all) compact operators on some Hilbert space. Consequently, all closed submodules in all Hilbert  $B$ -modules are orthogonally complemented.*

### 2. PRELIMINARIES

A (right) Hilbert module over a  $C^*$ -algebra  $B$  is a right  $B$ -module  $X$  equipped with a  $B$ -valued inner product  $\langle \cdot, \cdot \rangle$ , which is linear over  $B$  in the second and conjugate linear in the first variable, such that  $X$  is a Banach space with the norm  $\|x\| = \|\langle x, x \rangle\|^{1/2}$ . Hilbert modules are important for operator  $K$ -theory, and the basic theory of such modules can be found in [4] and [5]. We denote by  $L$  (or by  $L(X)$  if  $X$  is not clear from the context) the  $C^*$ -algebra of all operators (necessarily bounded and linear over  $B$ ) on  $X$  that have an adjoint. For arbitrary  $x, y \in X$  there is an operator  $[y, x] \in L$  defined by  $[y, x]z = y\langle x, z \rangle$  ( $z \in X$ ). The closed linear span of such operators (which is the analogy for compact operators between

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Hilbert spaces) will be denoted by  $K$  (or  $K(X)$ ).  $K$  is a closed two-sided ideal (not necessarily proper) in  $L$ .

For a general Hilbert  $B$ -module  $X$  the closed linear span of the set  $\{\langle x, y \rangle : x, y \in X\}$  is a closed two-sided ideal in  $B$  and will be denoted by  $\langle X, X \rangle$ . If  $\langle X, X \rangle = B$  then  $X$  is called a *full Hilbert  $B$ -module*.

The subset  $\Lambda$  of  $L(X \oplus B)$  consisting of all matrices of the form

$$\begin{bmatrix} a & x \\ y^* & b \end{bmatrix},$$

where  $a \in K$ ,  $b \in B$  and  $x, y \in X$ , is a  $C^*$ -subalgebra. Here each  $y \in X$  is regarded as an operator from  $B$  to  $X$  defined by  $b \mapsto yb$ , with the adjoint given by  $y^*(x) = \langle y, x \rangle$ . This algebra was introduced in [3] and called the *linking algebra of  $X$* .

We shall regard  $B$ ,  $X$  and  $K = K$  as subsets of  $\Lambda$  in the obvious way:

$$B \cong \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad X \cong \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K \cong \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the inner product  $\langle x, y \rangle$  in  $X$  can be expressed as the product  $x^*y$  in  $\Lambda$ .

### 3. PROOF OF THE THEOREM

To prove the theorem we shall relate submodules of  $X$  to right ideals of the linking algebra  $\Lambda$  of  $X$ , but first we need a couple of simple lemmas.

**Lemma 1.** *If  $X$  is a Hilbert  $B$ -module such that every closed submodule of  $X$  is orthogonally complemented, then each closed right ideal  $J$  of  $K = K(X)$  is of the form  $J = pK$  for some projection  $p \in L = L(X)$ .*

*Proof.* Consider the closed submodule  $[JX]$  of  $X$ . By hypothesis  $[JX] = pX$  for some projection  $p \in L$ . Then  $pax = ax$  for all  $x \in X$  and  $a \in J$ , hence  $pa = a$  and consequently  $J \subseteq pK$ . To prove the reverse inclusion, it suffices to show that  $p[x, y] = [px, y]$  is in  $J$  for each ‘rank one’ operator  $[x, y] \in K$  (since  $K$  is spanned by such operators). Since  $px$  can be approximated by finite sums of elements of the form  $az$  ( $a \in J$ ,  $z \in X$ ),  $[px, y]$  can be approximated by finite sums of operators of the form  $[az, y] = a[z, y]$ , which are in  $J$ . This proves that  $J = pK$ .  $\square$

**Lemma 2.** *If every closed right ideal  $J$  in a  $C^*$ -algebra  $A$  is of the form  $J = pA$  for some projection  $p$  in the multiplier algebra  $M(A)$  of  $A$ , then  $A$  is \*-isomorphic to a  $C^*$ -algebra of (not necessarily all) compact operators on some Hilbert space.*

*Proof.* First suppose that  $A$  is commutative, hence  $A \cong C_0(T)$  and  $M(A) \cong C_b(T)$  (= bounded continuous functions on a locally compact Hausdorff space  $T$ ). By hypothesis for each  $t \in T$  there exists a projection  $p \in C_b(T)$  such that

$$\{f \in C_0(T) : f(t) = 0\} = pC_0(T).$$

Clearly such a  $p$  must be the characteristic function of the complement of  $\{t\}$ , hence each singleton in  $T$  is open and  $T$  is discrete. Consequently  $A$  is \*-isomorphic to a  $C^*$ -algebra of (diagonal) compact operators on some Hilbert space.

In general, we shall prove that for each maximal commutative  $C^*$ -subalgebra  $A_0$  of  $A$  every maximal ideal  $J_0$  of  $A_0$  is of the form  $J_0 = pA_0$  for some projection  $p \in M(A_0)$ . By previous paragraph this implies that the maximal ideal space of  $A_0$  is discrete, hence by [2, p. 84]  $A$  is \*-isomorphic to a subalgebra of  $K(\mathcal{H})$  for some

Hilbert space  $\mathcal{H}$ . Given a maximal ideal  $J_0$  of  $A_0$ , consider the closed right ideal  $J = [J_0A]$  of  $A$ . By hypothesis  $J = pA$  for some projection  $p \in M(A)$ . We may assume that  $A \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  such that  $[A\mathcal{H}] = \mathcal{H}$ ; then  $M(A) = \{b \in A'' : bA + Ab \subseteq A\}$ . Note that  $[J_0\mathcal{H}] = [J_0A\mathcal{H}] = [J\mathcal{H}] = [pA\mathcal{H}] = p\mathcal{H}$ ; in particular the range of  $p$  is invariant under  $A'_0$ , therefore  $p \in A''_0$ . From  $p \in M(A)$  it then follows that  $pA_0 \subseteq A''_0 \cap A$ . Since  $A''_0 \cap A$  is a commutative C\*-subalgebra of  $A$  and contains  $A_0$ , while  $A_0$  is maximal commutative in  $A$ ,  $A''_0 \cap A = A_0$ . Thus  $pA_0 \subseteq A_0$ , which implies that  $p \in M(A_0)$ .

Since the range of  $p$  is  $[J_0\mathcal{H}]$ , we have  $J_0 \subseteq pA_0$ ; but, since  $J_0$  is a maximal ideal in  $A_0$ , the required equality  $J_0 = pA_0$  follows if  $pA_0 \neq A_0$ . To see that  $pA_0 \neq A_0$ , let  $\rho_0$  be a character on  $A_0$  with  $\ker \rho_0 = J_0$  and let  $\rho$  be a pure state on  $A$  extending  $\rho_0$ . Then  $J$  is contained in the right kernel of  $\rho$ , hence  $A_0$  is not contained in  $J$ . On the other hand,  $pA_0 \subseteq J$  (since  $J = pA$ ). □

*Proof of Theorem 1.* Let  $X$  be a full Hilbert module over  $B$  such that all closed submodules of  $X$  are orthogonally complemented. Then by Lemmas 1 and 2 there exists a \*-monomorphism  $\pi_K : K \rightarrow K(\mathcal{H}_K)$  for some Hilbert space  $\mathcal{H}_K$  (such that  $[\pi_K(K)\mathcal{H}_K] = \mathcal{H}_K$ ). By a well known result concerning extensions of representations (see [8, p. 162]) there exist a Hilbert space  $\mathcal{H} \supseteq \mathcal{H}_K$  and a representation  $\pi$  of the linking algebra  $\Lambda$  of  $X$  on  $\mathcal{H}$  such that  $\pi(a)|_{\mathcal{H}_K} = \pi_K(a)$  for each  $a \in K$ . Since  $\pi$  is a \*-representation we have

$$(1) \quad \pi(x)^*\pi(x) = \pi(\langle x, x \rangle) \quad \text{and} \quad \pi(x)\pi(x)^* = \pi([x, x])$$

for all  $x \in X$ . Put  $\mathcal{H}_B = [\pi(X)^*\mathcal{H}_K]$  and note that  $\mathcal{H}_B$  is invariant under  $\pi(B)$  since  $BX^* = (XB)^* \subseteq X^*$ . Since  $BK = 0, KB = 0, XK = 0$  and  $X^*X^* = 0$  (the products are computed in  $\Lambda$ ), the subspaces  $\mathcal{H}_K = [\pi(K)\mathcal{H}_K]$  and  $\mathcal{H}_B$  are orthogonal and we have

$$(2) \quad \pi(B)\mathcal{H}_K = 0, \quad \pi(K)\mathcal{H}_B = 0, \quad \pi(X)\mathcal{H}_K = 0 \quad \text{and} \quad \pi(X)^*\mathcal{H}_B = 0.$$

Moreover,

$$[\pi(X)\mathcal{H}_B] = [\pi(X)\pi(X)^*\mathcal{H}_K] = [\pi(K)\mathcal{H}_K] = \mathcal{H}_K.$$

These relations easily imply that  $\mathcal{H}_K \oplus \mathcal{H}_B$  is invariant under  $\pi$ ; hence, replacing  $\pi$  by the corresponding restriction, we may assume that  $\mathcal{H}_K \oplus \mathcal{H}_B = \mathcal{H}$ . Then the second identity in (1) (together with the second identity in (2) and the fact that  $\pi_K(K)$  consists of compact operators) implies that  $\pi(x)$  is compact for all  $x \in X$ . Since  $B = \langle X, X \rangle$ , it follows then from the first identity in (1) that  $\pi(B)$  consists of compact operators only. Moreover, since  $\pi_K$  is one to one, it is easy to see (using (1)) that the restriction of  $\pi$  to  $(X$  and to)  $B$  is one to one. (If  $J \stackrel{\text{def}}{=} \ker \pi|_B \neq 0$ , then  $[XJ, XJ]$  would be a non-zero ideal of  $K$  contained in  $\ker \pi_K$ .) Thus  $B$  is \*-isomorphic to a C\*-subalgebra of  $K(\mathcal{H}_B)$ .

Conversely, suppose that  $B$  is isomorphic to a (nondegenerate) C\*-algebra of compact operators on a Hilbert space  $\mathcal{H}_B$  by a \*-isomorphism  $\pi_B$ . Let  $\pi$  be a representation of  $\Lambda$  on a Hilbert space  $\mathcal{H} \supseteq \mathcal{H}_B$  such that  $\pi(b)|_{\mathcal{H}_B} = \pi_B(b)$  for each  $b \in B$ , and put  $\mathcal{H}_K = [\pi(X)\mathcal{H}_B]$ . As in the previous paragraph, we may assume that  $\mathcal{H}_K \oplus \mathcal{H}_B = \mathcal{H}$ . Since  $\pi_B$  is one to one it follows (using (1)) that the restrictions of  $\pi$  to  $X$  and  $K$  are one to one, but then we see (using (2)) that  $\pi$  is also one to one. Thus,  $\Lambda$  is \*-isomorphic to a C\*-subalgebra of  $K(\mathcal{H})$ , hence by [1, 1.4.5]  $\Lambda$  is \*-isomorphic to a (restricted) direct sum of elementary C\*-algebras of the form  $K(\mathcal{H}_j)$  for some Hilbert spaces  $\mathcal{H}_j$ . Since  $K(\mathcal{H}_j)$  is strongly Morita

equivalent to  $\mathbb{C}$ , it would not be hard to see (using induced representations, see [10] or [5]) that  $K(\mathcal{H}_j)$  and  $\mathbb{C}$  have equivalent categories of Hilbert modules, from which it is possible to conclude the proof, but we shall give here a more elementary approach. Since  $\Lambda$  is  $*$ -isomorphic to a direct sum of algebras of the form  $K(\mathcal{H}_j)$ , each closed right ideal  $J$  of  $\Lambda$  is of the form  $J = p_\Lambda \Lambda$  for some projection  $p_\Lambda$  in the multiplier algebra  $M(\Lambda)$  of  $\Lambda$ . Using the above representation of  $\Lambda$  and the well known fact that the multiplier algebra of  $K$  is  $L$  (see [5]), it is easy to see that  $M(\Lambda)$  is of the form

$$M(\Lambda) = \begin{bmatrix} L & Z \\ Z^* & M(B) \end{bmatrix},$$

where  $Z$  is a certain  $L - M(B)$ -subbimodule of  $B(\mathcal{H}_B, \mathcal{H}_K)$  containing  $X$ . Given a closed submodule  $Y$  of  $X$ , consider the closed right ideal  $J$  of  $\Lambda$  generated by  $Y$ , that is

$$J = \begin{bmatrix} [Y, X] & Y \\ 0 & 0 \end{bmatrix}.$$

If  $p_\Lambda$  is a projection in  $M(\Lambda)$  such that  $J = p_\Lambda \Lambda$ , then a straightforward computation shows that  $p_\Lambda$  is necessarily of the form

$$p_\Lambda = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$$

for some projection  $p \in L$ , and that  $Y = pX$ . □

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